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# Chiral anomaly enhancement and photo-irradiation effects in multi-band touching fermion systems

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Multi-band touchings together with the emergence of fermions exhibiting linear dispersions have recently been predicted and realized in various materials. We first investigate the Adler-Bell-Jackiw chiral anomaly in these multi-band touching semimetals when they are described by the pseudospin operator in high dimensional representation. By evaluating the Chern number, we show that the anomalous Hall effect is enhanced depending on the magnitude of the pseudospin. It is also confirmed by the analysis of the Landau levels when magnetic field is applied. Namely, charge pumping occurs from one multi-band touching point to another through multi-channel Landau levels in the presence of parallel electric and magnetic fields. We also show a pair annihilation of two multi-band touching points by photo-irradiation. Furthermore, we propose generalizations of Dirac semimetals, multiple-Weyl semimetals and loop-nodal semimetals to those composed of fermions carrying pseudospins in high dimensional representation. Finally we investigate the 3-band touching protected by the  $C_3$  symmetry. We show that the 3-band touching point is broken into two Weyl points by photo-irradiation.

# I. INTRODUCTION

Weyl semimetal has created one of the most active fields of modern condensed matter physics<sup>1,2</sup>. Experimental observations<sup>3,4</sup> accelerate the study. It is described by a two-band theory with a linear dispersion. It is topologically protected by the monopole charge<sup>5</sup>. A prominent feature is the emergence of the Adler-Bell-Jackiew (ABJ) anomaly<sup>6</sup>, where the chiral charge is not preserved in the presence of the parallel magnetic and electric fields<sup>7–12</sup>. This anomalous phenomenon has experimentally been observed<sup>13–17</sup>. The pair creation or annihilation of Weyl semimetals by photo-irradiation has also been discussed<sup>18–22</sup>.

Recently, there are reports on new types of fermions which have no counterparts in particle physics. They appear at multi-band touching points with linear dispersions in crystals, which we may call multi-band touching fermions. Examples have been found at 3-band<sup>23–27</sup>, 4-band<sup>28</sup>, 6-band<sup>23</sup> and 8band<sup>23,29</sup> touching points. Among them, there are proposals that 3-band<sup>23</sup> and 4-band<sup>28</sup> touching fermions are described by the pseudospin operator in *J*-dimensional representation with J = 1 and 3/2, respectively. Let us refer to this kind of multi-band touching fermions as *J*-fermions. Their experimental observations are yet anticipated. On the other hand, there is another kind of 3-band touching fermions appearing along the C<sub>3</sub> symmetry invariant line, which is protected by the C<sub>3</sub> symmetry<sup>24–27</sup>. We refer to them as the C<sub>3</sub>-protected fermions. They have experimentally been observed<sup>30</sup>.

In this paper, motivated by these discoveries, we propose and explore various models composed of multi-band touching fermions. The minimal models are obtained by replacing the Pauli matrices with the pseudospin operators having a high dimensional representation. In this minimal scheme we may easily generalize Weyl semimetals, multiple Weyl semimetal, Dirac semimetals, and loop-nodal semimetals to those composed of *J*-fermions for a general value of *J*. We analyze the Landau levels (LLs) and photo-irradiation effects to reveal underlying physics. We also study the photo-irradiation effects of the  $C_3$ -protected fermions.

This paper is composed as follows. In Section II, we investigate a system made of fermion multiplets described by the pseudospin operator J appearing at a multi-band touching point. Each band, indexed by the eigenvalue j of the operator J, is shown to have a monopole with the charge 2j in the momentum space. We introduce the Chern number by counting the monopole charges associated with the bands below the Fermi energy. The Chern number contributes to the anomalous Hall conductivity in the 3-dimensional (3D) space. Then, we analyze the Landau levels (LLs) by applying external magnetic field to the multi-band touching system.

In Section III, we introduce a Hamiltonian describing a pair of J-fermions. Here we recall that in a crystal multiband touching points always appear in pairs, whose chiralities are opposite<sup>31</sup>. When electric field is applied additionally in parallel with the magnetic field, certain LLs convey electric charges from one multi-band touching point to another, as is a manifestation of the ABJ anomaly. The ABJ anomaly is closely related to the Chern number. Next, we study the effect of photo-irradiation applied to the multi-band touching semimetal. We show that a pair-annihilation of two J-fermions is induced by photo-irradiation.

In Sec IV, V and VI, we generalize Dirac fermions, multiple-Weyl semimetals and loop-nodal semimetals by replacing the Pauli matrices with the pseudospin operators. Especially, we find various transitions to occur in the generalized Dirac-like *J*-fermions by photo-irradiation.

In Section VII, we show that the  $C_3$ -protected fermion is broken by photo-irradiation. We have prepared Supplementary Material, where various formulas used in this work are derived in details.

# II. J-FERMIONS

**Hamiltonian:** The emergence of multi-band touching is assured at high-symmetry points by crystalline symmetry in some lattice models<sup>23,28</sup>. In the vicinity of a multi-band touching point the effective Hamiltonian is well described by

$$H = \hbar v \mathbf{k} \cdot \mathbf{J},\tag{1}$$

where examples are known for J = 1 and 3/2. Here,  $\mathbf{J} = (J_x, J_y, J_z)$  is the generator of the pseudospin with mag-

nitude  $J, J \ge 1/2$ . Note that J-fermions emerge at a touching point of 2J + 1 bands at the zero energy. It can be viewed as the generalization of the Weyl fermion described by the Hamiltonian  $H = \hbar v \mathbf{k} \cdot \sigma$  with the Pauli matrix  $\sigma$ . We investigate the Hamiltonian (1) for general **J**.

The Hamiltonian (1) can be fermionic or bosonic in general. However, in the bosonic case, the bosons are condensed around the bottom of the band. Namely, the occupancy in the vicinity of the multi-band touching point is scarce and the multi-band touching point becomes inactive. On the other hand, in the fermionic case, the Fermi sea can be set at the multi-band touching point due to the Fermi degeneracy and the multi-band touching point plays an important role. In the following, we only consider multi-band touching fermions.

The Lie algebra reads  $[J_{\mu}, J_{\nu}] = i\varepsilon_{\mu\nu\rho}J_{\rho}$ . Using the polar coordinate,  $k_x = k\sin\theta\cos\phi$ ,  $k_y = k\sin\theta\sin\phi$ ,  $k_z = k\cos\theta$ , the Hamiltonian is diagonalized as  $UHU^{-1} = \hbar v k J_z$  with

$$U = \exp[i\theta J_x] \exp[i\left(\phi + \frac{\pi}{2}\right) J_z].$$
 (2)

Using the relation

$$U^{-1}J_z U = J_x \sin\theta \cos\phi + J_y \sin\theta \sin\phi + J_z \cos\theta, \quad (3)$$

the energy spectrum is given by

$$E = j\hbar vk, \tag{4}$$

where j labels the band with  $j = -J, -J + 1, \dots, J - 1, J$ . The eigenstates functions are given by  $|\psi_j\rangle = U^{-1} |j\rangle$ .

**Monopole charges:** With the use of the eigenstate  $|\psi_j\rangle$ , the Berry curvature is rewritten as

$$-i\Omega_{j} = \frac{\mathbf{k}}{k^{3}\sin\theta} \left[ \left\langle \frac{\partial\psi_{j}}{\partial\theta} \middle| \frac{\partial\psi_{j}}{\partial\phi} \right\rangle - \left\langle \frac{\partial\psi_{j}}{\partial\phi} \middle| \frac{\partial\psi_{j}}{\partial\theta} \right\rangle \right] \\ = \frac{\mathbf{k}}{k^{3}\sin\theta} \left\langle j \middle| \frac{\partial U}{\partial\theta} \frac{\partial U^{-1}}{\partial\phi} - \frac{\partial U}{\partial\phi} \frac{\partial U^{-1}}{\partial\theta} \middle| j \right\rangle.$$
(5)

Using the relations

$$\frac{\partial U}{\partial \theta} \frac{\partial U^{-1}}{\partial \phi} - \frac{\partial U}{\partial \phi} \frac{\partial U^{-1}}{\partial \theta} = J_z \sin \theta - J_y \cos \theta,$$

the Berry curvature is explicitly calculated for each band as

$$\mathbf{\Omega}_j = i \nabla \times \langle \psi_j | \nabla \psi_j \rangle = j \frac{\mathbf{k}}{k^3}.$$
 (6)

Since  $\frac{1}{2\pi} \int d^3k \, \nabla \cdot \mathbf{\Omega}_j = 2j$ , the Berry curvature of the band indexed by j describes a monopole carrying the monopole charge 2j in the momentum space. This result is already known for Weyl semimetals<sup>5</sup>, 3-band<sup>23</sup> and 4-band<sup>28</sup> touching cases. We have shown that it is a generic result in the system (1).

**Chern numbers:** We may associate the Chern number to the multi-band touching points. Since it is defined only in the 2D space, we treat  $k_z$  as a parameter and analyze the system of 2D fermions on the  $k_x$ - $k_y$  space for each  $k_z$ . Namely we



FIG. 1: Landau Levels for a multi-band touching fermion multiplet. The horizontal axis represents the momentum  $k_z$ , while the vertical axis represents the energy E. The energy spectrum of the LLs is determined as a function of  $k_z$ . The spectrum resembles that of a nanoribbon, which consists of the bulk spectrum and the edge modes (colored curves). We refer to the LLs corresponding to edge modes as the mid-gap LLs. Mid-gap LLs connecting the bands with  $\Delta j = 1$  are marked in magenta, those with  $\Delta j = 2$  are marked in cyan, those with  $\Delta j = 3$  are marked in green and those with  $\Delta j = 4$  are marked in orange. The mid-LLs contributing to the ABJ anomaly are emphasized in thick curves, while the other mid-LLs are indicated by dotted curves. (a) Weyl semimetal with J = 1/2, (b) 3-band touching semimetal with J = 3/2 and (d) 5-band touching semimetal with J = 2.

evaluate the Chern number for the band j as a function of  $k_{z} \\ \mathrm{as}$ 

$$C_{j}\left(k_{z}\right) = \frac{1}{2\pi} \iint dk_{x} dk_{y} \Omega_{j}^{z} = -j \operatorname{sgn}\left(k_{z}\right).$$
(7)

The  $k_z$ -dependent total Chern number is given by taking the sum under the Fermi energy,

$$C(k_z) = \sum_{j \le 0} C_j(k_z) = \frac{1}{2} N \operatorname{sgn}(k_z), \qquad (8)$$

where

$$N = -2\sum_{j\leq 0} j = \begin{cases} (J+1/2)^2 & \text{for } J \in \text{half integer,} \\ J(J+1) & \text{for } J \in \text{integer.} \end{cases}$$
(9)

For examples, N = 1 for J = 1/2, N = 2 for J = 1, N = 4 for J = 3/2 and N = 6 for J = 2. The Chern number  $C(k_z)$  is quantized. The sign changes at  $k_z = 0$ , where a multi-band touching point exists.

**Landau levels of J=1 fermions:** We proceed to include a homogeneous magnetic field  $\mathbf{B} = \nabla \times \mathbf{A} = (0, 0, -B)$ with B > 0 along the z axis into the multi-band touching fermion system. Landau-levels for the J = 1 fermion are discussed in Ref.<sup>23</sup>. Let us summarize it in order to generalize it to the J-fermions. By making the minimal substitution to the Hamiltonian (1), we obtain

$$\hat{H} = \hbar\omega_{\rm c} \left( \hat{a}^{\dagger} J_{-} + \hat{a} J_{+} \right) + \hbar v k_z J_z \tag{10}$$

with the covariant momentum  $P_i \equiv \hbar k_i + eA_i$ . We have introduced a pair of Landau-level ladder operators,

$$\hat{a} = \ell_B (P_x + iP_y) / \sqrt{2}\hbar, \quad \hat{a}^{\dagger} = \ell_B (P_x - iP_y) / \sqrt{2}\hbar,$$
(11)

satisfying  $[\hat{a}, \hat{a}^{\dagger}] = 1$ , where  $\ell_B = \sqrt{\hbar/eB}$  is the magnetic length and  $\omega_c = \sqrt{2\hbar v/\ell_B}$ .

There are two types of spectra; the bulk spectrum and the mid-gap spectrum. See the caption of Fig.1. First, we obtain the bulk spectrum by setting

$$\psi = (u_n | n \rangle, u_{n+1} | n+1 \rangle, u_{n+2} | n+2 \rangle)^t$$
 (12)

for  $n \ge 0$  and solving the eigenvalue problem  $\hat{H}\psi = E\psi$ . There are three solutions since there are three variables  $u_n$ ,  $u_{n+1}$  and  $u_{n+2}$ . They produce three LLs. Second, we obtain the mid-gap spectrum as follows. There are two solutions by setting  $\psi = (0, u_0 | 0 \rangle, u_1 | 1 \rangle)^t$ , yielding two LLs, as illustrated by two magenta curves in Fig.1(b). One connects the bands between j = 1 and j = 0 and the other connects the bands between j = 0 and j = -1. We also have one LL by setting  $\psi = (0, 0, u_0 | 0 \rangle)^t$ , which connects the band between j = 1 and j = -1, as illustrated by one cyan curve in Fig.1(b). This LL has zero energy at  $k_z = 0$ . These LLs connects two bulk bands in all combinations, and then the number of the mid-gap LLs is given by  $N_{\text{mid}} = {}_{3}C_{2} = 3$ .

Landau levels of J=3/2 fermions: We may carry out a similar study for the case with J = 3/2. First, we obtain the bulk spectrum by setting

$$\psi = (u_n |n\rangle, u_{n+1} |n+1\rangle, u_{n+2} |n+2\rangle, u_{n+3} |n+3\rangle)^t$$
(13)

for  $n \geq 0$ . Next, we obtain the mid-gap spectrum: There are three LLs by solving  $\psi = (0, u_0 | 0 \rangle, u_1 | 1 \rangle, u_2 | 2 \rangle)^t$  connecting the bands with  $\Delta j = 1$ , two LLs by solving  $\psi = (0, 0, u_0 | 0 \rangle, u_1 | 1 \rangle)^t$  connecting the bands with  $\Delta j = 2$  and one LL by solving  $\psi = (0, 0, 0, u_0 | 0 \rangle)^t$  connecting the bands with  $\Delta j = 3$ . There are  $N_{\text{mid}} = 4C_2 = 6$  LLs. We show the LLs as a function of  $k_z$  in Fig.1(c).

On the other hand, the number of the LLs contributing to the ABJ anomaly is  $N_{ABJ} = 4$  since the two bands connecting (j = 3/2 and j = 1/2) and (j = -3/2 and j = -1/2) do not cross the Fermi energy. This is consistent with the result that the anomalous Hall conductance becomes N times larger than that of the Weyl semimetal with N = 4 for J = 3/2: See (20). Namely,  $N_{ABJ} = N$ .

**Landau levels** of *J*-fermions: We generalize the above scheme to the system of *J*-fermions. The number of the midgap LLs is  $N_{\text{mid}} = \sum_{i=1}^{2J} = 2J(2J+1)/2$ . It is also derived as  $N_{\text{mid}} = {}_{2J+1}C_2$  by the fact that the mid-gap LLs connect all the bands.

On the other hand, the LLs contributing to the ABJ anomaly are counted as

$$N_{\rm ABJ} = N_{\rm mid} - 2_{J+1/2}C_2 = (J+1/2)^2$$
(14)

for half integer J, and

$$N_{\rm ABJ} = N_{\rm mid} - 2_{J+1}C_2 - J = J(J+1)$$
(15)



FIG. 2: Landau Levels for a pair of multi-band touching fermion multiplets. (a) 3-band touching semimetal with J = 1, where  $N_{\text{ABJ}} = 2$ . (b) 4-band touching semimetal with J = 3/2, where  $N_{\text{ABJ}} = 4$ . See the caption of Fig.1.

for integer J, where we have subtracted the LLs which are away from the Fermi energy. Comparing these with (9), we find the relation  $N_{ABJ} = N$ . See Fig.1(d) for the case of J =2. Recall that  $N_{ABJ} = 1$  for the Weyl semimetal with J =1/2 [Fig.1(a)]. Consequently the ABJ anomaly is N times enhanced compared with the Weyl semimetal.

# **III. PAIR OF J-FERMIONS**

In crystals, multi-band touching points always appear in pairs, whose chiralities are opposite<sup>31</sup>. A simplest model which has a pair of multi-band touching points is described by the Hamiltonian

$$H = \hbar v k_x J_x + \hbar v k_y J_y + \frac{\hbar v}{c\sqrt{1-m^2}} (\cos ck_z - m) J_z,$$
(16)

where the Brillouin zone is taken as  $-\pi/c \le k_z \le \pi/c$  with c being the lattice constant. A pair of multi-band touching points exist at  $ck_z = \pm \arccos m$  for |m| < 1.

In the vicinity of these points, the Hamiltonian is expanded as

$$H = \hbar v k_x J_x + \hbar v k_y J_y \pm \hbar v (k_z \pm \frac{\arccos m}{c}) J_z.$$
(17)

The  $k_z$ -dependent total Chern number now reads

$$C(k_z) = \frac{N}{2} \operatorname{sgn} \left( \cos ck_z - m \right).$$
(18)

Integrating this over  $k_z$ , we obtain

$$C = \int_{-\pi/c}^{\pi/c} C(k_z) \, dk_z = \frac{1}{2} N b_z, \tag{19}$$

where  $b_z = (\arccos m - \pi) / c$  is the distance between the two multi-band touching points. Note that C is no longer a topological charge and not quantized.

**Anomalous Hall effect:** The Thouless-Kohmoto-Nightingale-den Nijs formula formula is valid also in the present 3D case. Indeed, we first apply it in the 2D space



FIG. 3: Pair annihilation of a pair of multi-band touching multiplets. The horizontal plane is spanned by the  $k_z$  and  $k_x$  axes. The vertical axis is the energy E as a function of  $k_z$  and  $k_x$ , where we have set  $k_y = 0$ . (a1) 3-band touching semimetal with J = 1 without photo-irradiation. (a2) That with the critical photo-irradiation. (a3) That above the critical point. (b1), (b2), (b3) The corresponding ones for 4-band touching semimetal with J = 3/2.

indexed by  $k_z$ , and then we integrate it over  $k_z$  to obtain the anomalous Hall effect as

$$\mathbf{j} = N \frac{e^2}{2\pi^2 \hbar^2} \mathbf{b} \times \mathbf{E}, \quad |\mathbf{b}| = \frac{\arccos m - \pi}{c}, \tag{20}$$

where the rotational invariance has been restored. The anomalous Hall conductance becomes N times larger than that of the Weyl semimetal.

The Adler-Bell-Jackiew anomaly: The analysis of the LLs is important to reveal the ABJ anomaly in the system. In crystals multi-bands touching points emerge in pairs. Making the minimum substituting into the Hamiltonian (16) we obtain the LLs for such a pair and show them in Fig.2(a). When we apply electric field along the z direction additionally, charge pumping occurs via the mid-gap LLs as in the Weyl semimetal<sup>1,6</sup>, which is a manifestation of the ABJ anomaly. Namely, the chiral charge, which is defined by the difference of the charge between the two multi-band touching points, is not conserved,

$$\frac{\partial \rho_{\chi}}{\partial t} = \chi N_{\text{ABJ}} \frac{e^3}{h^2} \mathbf{E} \cdot \mathbf{B},$$
(21)

where  $\chi = \pm 1$  denotes the chirality and  $\rho_{\chi}$  is the charge of each multi-band touching point, while  $N_{ABJ}$  is the number of channels. The mid-gap LLs which contribute to the ABJ anomaly are those crossing the Fermi energy. The number of such LLs is  $N_{ABJ}$ . Here,  $N_{ABJ} = 2$  for J = 1. It will be observed experimentally as an enhancement of the negative magnetoresistance. This is consistent with the result that the anomalous Hall conductance becomes N times larger than that of the Weyl semimetal with N = 2 for J = 1: See (20). Namely,  $N_{ABJ} = N$ .

**Pair annihilation induced by photo-irradiation:** Creation of Weyl semimetals by photo-irradiation has been discussed intensively<sup>18–22</sup>. We generalize the analysis to the multi-band touching semimetals. We consider a beam of cir-



FIG. 4: Band structure of double *J*-fermions. The horizontal plane is spanned by the  $k_x$  and  $k_y$  axes. The vertical axis is the energy *E* as a function of  $k_x$  and  $k_y$ , where we have set  $k_z = 0$ . Multiband touching occurs quadratically for N = 2. (a) J = 1 and (b) J = 3/2. We have set  $\hbar v = c = 1$ .

cularly polarized light irradiated onto the multi-band touching semimetals. We take the electromagnetic potential as  $\mathbf{A}(t) = (A \cos \omega t, A \sin (\omega t + \phi), 0)$ , where  $\omega$  is the frequency of light with  $\omega > 0$  for the right circulation and  $\omega < 0$ for the left circulation. The choice  $\phi = 0$  and  $\phi = \pi$  corresponds to the right-handed and left-handed circularly polarized light, respectively.

We discuss the effect of photo-irradiation. The effective Hamiltonian is given by  $^{32-38}$ 

$$\Delta H_{\text{eff}} = \frac{1}{\hbar\omega} \sum_{n \ge 1} \frac{[H_{-n}, H_{+n}]}{n}, \qquad (22)$$

with

$$H_{+n} = \frac{1}{T} \int_0^T H e^{in\omega t} dt.$$
 (23)

By using

$$H_{+1} = \hbar v e A \left(\frac{1 + e^{-i\phi}}{2}\right) J_+, \qquad (24)$$

$$H_{-1} = \hbar v e A\left(\frac{1+e^{i\phi}}{2}\right) J_{-}, \qquad (25)$$

and  $H_{\pm n} = 0$  for  $n \ge 2$ , the effective Hamiltonian is calculated as

$$\Delta H_{\text{eff}} = \frac{1}{\hbar\omega} \left[ H_{-1}, H_{+1} \right] = -\frac{2\left(\hbar v e A\right)^2}{\hbar\omega} J_z \cos\phi, \quad (26)$$

which only shifts the crossing point in the z direction  $k_z \mapsto k_z - 2e^2 A^2 / (\hbar^2 \omega)$  for the Hamiltonian (1). By adding the term (26) to the Hamiltonian (16), we find the parameter m is renormalized to be  $m + \frac{2e^2 A^2}{\hbar \omega}$ . We investigate the energy spectrum by changing A. When  $\left|m + \frac{2e^2 A^2}{\hbar \omega}\right| = 1$ , a pair annihilation occurs between two multi-band touching points, as illustrated in Fig.3(a) and (b) for the cases of J = 1 and 3/2. A detached perfect flat band appears for integer J after the pair annihilation, as in Fig.3(a).



FIG. 5: Landau Levels for a multiple *J*-fermions with J = 1. (a) double *J*-fermions, (b) triple *J*-fermions, See the caption of Fig.1.

#### **IV. MULTIPLE J-FERMIONS**

**Hamiltonian:** Multiple-Weyl semimetals are described by<sup>39,40,42,43</sup>

$$H = \begin{pmatrix} \hbar v k_z & \frac{c}{2} k_-^N \\ \frac{c}{2} k_+^N & -\hbar v k_z \end{pmatrix} = \frac{c}{2} k_+^N \sigma_- + \frac{c}{2} k_-^N \sigma_+ + \hbar v k_z \sigma_z$$
(27)

with an integer N. It is known in crystals that only double-Weyl semimetals with N = 2 and triple-Weyl semimetals with N = 3 are possible due to the crystal symmetry restriction<sup>39,40</sup>. The Landau levels have been investigated in multiple-Weyl semimetals<sup>41,42</sup>.

We propose to generalize it for J-fermions as

$$H = \frac{c}{2}k_{+}^{N}J_{-} + \frac{c}{2}k_{-}^{N}J_{+} + \hbar vk_{z}J_{z}.$$
 (28)

We call them double *J*-fermions for N = 2 and triple *J*-fermions for N = 3, as in the case of double Weyl semimetals and triple Weyl semimetals.

Using the polar coordinate, the Hamiltonian is rewritten as

$$H = ck^{N} \left( J_{x} \cos N\phi + J_{y} \sin N\phi \right) + \hbar v k_{z} J_{z}, \qquad (29)$$

the Hamiltonian is diagonalized as

$$UHU^{-1} = \sqrt{c^2 \left(k_x^2 + k_y^2\right)^N + \left(\hbar v k_z\right)^2} J_z, \qquad (30)$$

where

$$U = \exp[i\theta J_x] \exp[i\left(N\phi + \frac{\pi}{2}\right)J_z], \qquad (31)$$

with  $\tan \theta = \frac{ck^N}{\hbar v k_z}$ . The energy dispersion is obtained as

$$E = j\sqrt{c^2 \left(k_x^2 + k_y^2\right)^N + \left(\hbar v k_z\right)^2}.$$
 (32)

We show the band structures of double J-fermion with J = 1and J = 3/2 in Fig.4. The band touching is not linear but quadratic.

**Monopole charges:** It is known that the monopole charges for double and triple Weyl semimetals are given by  $\pm 2$  and  $\pm 3$  respectively. We generalize the results to the case of general *J*-fermions. Using the relations

$$\frac{\partial U}{\partial \theta} \frac{\partial U^{-1}}{\partial \phi} - \frac{\partial U}{\partial \phi} \frac{\partial U^{-1}}{\partial \theta} = N J_z \sin \theta - N J_y \cos \theta, \quad (33)$$



FIG. 6: Band structure of Dirac-like J-fermions with J = 1 under photo-irradiation. The horizontal plane is spanned by the  $k_z$  and  $k_x$ axes. The vertical axis is the energy E as a function of  $k_z$  and  $k_x$ , where we have set  $k_y = 0$ . (a) h = 0, (b) h = m and (c) h = 2m. (b) represents a critical point of a Lifshitz transition.

the Berry curvature is explicitly calculated for each band as

$$\mathbf{\Omega}_j = N j \frac{\mathbf{k}}{k^3}.\tag{34}$$

Since  $\frac{1}{2\pi} \int d^3k \, \nabla \cdot \mathbf{\Omega}_j = 2Nj$ , the Berry curvature of the band indexed by j describes a monopole with the monopole charge 2Nj in the momentum space.

Landau levels: The Hamiltonian under magnetic field is given by

$$\hat{H} = \frac{\hbar\omega_{\rm c}}{2} \left(\hat{a}^{\dagger}\right)^N J_- + \frac{\hbar\omega_{\rm c}}{2} \hat{a}^N J_+ + \hbar v k_z J_z.$$
(35)

The bulk spectrum is obtained by setting

$$\psi = (u_n | n \rangle, u_{n+N} | n+N \rangle, u_{n+2N} | n+2N \rangle)^t$$
 (36)

with J = 1 and

$$\psi = \left(u_n \left|n\right\rangle, \cdots, u_{n+JN} \left|n+JN\right\rangle\right)^t \tag{37}$$

with general J for  $n \ge 0$  and solving the eigenvalue problem  $\hat{H}\psi = E\psi$ . We show the LLs in Fig.5. The number of the mid-gap LLs is  $N_{\text{mid}} = 2JN(2J+1)/2$ .

**Photo-irradiation:** The effective Hamiltonian with photoirradiation is given by

$$\Delta H_{\rm eff} = -2 \frac{(c/2)^2}{\hbar \omega} \sum_{n=0}^{N} \frac{\left({}_N C_n\right)^2}{n} \left(eA\right)^{2n} \left(k_x^2 + k_y^2\right)^{N-n} J_z,$$
(38)

where we have used

$$H_{+n} = {}_{N}C_{n}\frac{c}{2} (eA)^{n} k_{-}^{N-n} J_{+}, \qquad (39)$$

$$H_{-b} = {}_{N}C_{n}\frac{c}{2} \left(eA\right)^{n} k_{+}^{N-n} J_{-}.$$
 (40)

The band degeneracy is not resolved but only the dispersion shifts and changes.

#### V. DIRAC-LIKE J-FERMIONS

**Hamiltonian:** In the chiral representation, the Dirac fermion is described by the Hamiltonian

$$H = \hbar v \mathbf{k} \cdot \sigma \tau_z + m \tau_x. \tag{41}$$



FIG. 7: Band structure of Dirac-like J-fermions with J = 3/2 under photo-irradiation. The horizontal plane is spanned by the  $k_z$  and  $k_x$  axes. The vertical axis is the energy E as a function of  $k_z$  and  $k_x$ , where we have set  $k_y = 0$ . (a) h = 0, (b) h = m/2, (c) h = 2m/3, (d) h = m, (e) h = 2m and (f) h = 5m/2. (c) and (e) represent critical points of Lifshitz transitions.

We propose to generalize it to J-fermions as

$$H = \hbar v \mathbf{k} \cdot \mathbf{J} \tau_z + m \tau_x, \tag{42}$$

which yields the energy spectrum

$$E = j\sqrt{\left(\hbar vk\right)^2 + m^2}.$$
(43)

When m = 0, this Hamiltonian is split into two independent Hamiltonians describing the *J*-fermions whose monopole charges are opposite.

Photo-irradiation: Especially, the perfect flat bands emerge at  $E = \pm m$  for the even J. The effective Hamiltonian by photo-irradiation is calculated as

$$\Delta H_{\text{eff}} = \frac{1}{\hbar\omega} \left[ H_{-1}, H_{+1} \right] = -\frac{2\left(\hbar v e A\right)^2}{\hbar\omega} J_z \cos\phi, \quad (44)$$

where we have used

$$H_{+1} = \hbar v e A \left(\frac{1+e^{-i\phi}}{2}\right) J_+ \tau_z, \qquad (45)$$

$$H_{-1} = \hbar v e A \left(\frac{1+e^{i\phi}}{2}\right) J_{-}\tau_z, \qquad (46)$$

and  $H_{\pm n} = 0$  for  $n \ge 2$ . The Hamiltonian is summarized as

$$H = \hbar v \mathbf{k} \cdot \mathbf{J} \tau_z + m \tau_x + h J_z \tag{47}$$

with  $h = -\frac{2(\hbar v e A)^2}{\hbar \omega} \cos \phi$ . It is hard to obtain the energy dispersion for general **J**. However, we can diagonalize the Hamiltonian along the  $k_z$ axis, for which  $k_x = k_y = 0$ . Wit the use of the unitary transformation

$$U = \exp[i\theta\tau_y], \quad \tan\theta = \frac{m}{\hbar v k_z}, \tag{48}$$

we obtain

$$U^{-1}\tau_z U = \tau_x \sin\theta + \tau_z \cos\theta, \tag{49}$$

and hence

$$UH(0,0,k_z)U^{-1} = \sqrt{(\hbar v k_z J_z)^2 + m^2 \tau_z + h J_z}.$$
 (50)

The energy along the  $k_z$  axis is found to be

$$E = t\sqrt{\left(\hbar v k_z J_z\right)^2 + m^2} + hj,$$
(51)

where  $t = \pm 1$  and  $j = -J, -J + 1, \cdots, J - 1, J$ . There are several Lifshitz transitions at

$$h = \frac{t}{j}m,\tag{52}$$

where the band gap closes at  $\mathbf{k} = \mathbf{0}$ . For example, there are transitions at  $h = \pm m$  for J = 1 and  $h = \pm 2m/3$  and  $h = \pm 2m/3$  $\pm 2m$  for J = 3/2. We show the numerically obtained band structure for various h in Figs.6 and 7.

#### VI. LOOP-NODAL J-FERMIONS

Hamiltonian: In the 3D space, it is known that another type of semimetal called loop-nodal semimetals is possible. Loop-nodal semimetals are described by the Hamiltonian<sup>44-63</sup>

$$H = \left[a\left(k_x^2 + k_y^2\right) - m\right]\sigma_x + ck_z\sigma_z.$$
(53)

We propose to generalize it to J-fermions as

$$H = \left[a\left(k_x^2 + k_y^2\right) - m\right]J_x + ck_z J_z.$$
(54)

Using the relation

$$U^{-1}J_z U = J_x \sin \theta + J_z \cos \theta, \quad \tan \theta = \frac{a\left(k_x^2 + k_y^2\right) - m}{ck_z}$$
(55)

with  $U = \exp[i\theta J_y]$ , the Hamiltonian is diagonalized as  $UHU^{-1} = EJ_z$  with the energy spectrum

$$E = j\sqrt{\left[a\left(k_x^2 + k_y^2\right) - m\right]^2 + (ck_z)^2}.$$
 (56)

We show the band structure of loop-nodal *J*-fermions in Fig.8. The gap closes at the nodal loop

$$k_x^2 + k_y^2 = \frac{m}{a} \tag{57}$$

for m/a > 0. This loop node is protected by the mirror symmetry

$$MH(k_x, k_y, k_z) M^{-1} = H(k_x, k_y, -k_z)$$
(58)



FIG. 8: Band structure of loop-nodal *J*-fermions with (a) J = 1 and (b) J = 3/2. The horizontal plane is spanned by the  $k_x$  and  $k_y$  axes. The vertical axis is the energy *E* as a function of  $k_x$  and  $k_y$ , where we have set  $k_z = 0$ . A loop node exists along the  $k_x$ - $k_y$  plane for m/a > 0. We have set  $\hbar v = m = a = 1$ .

with the mirror operator  $M = iJ_x$ .

**Photo-irradiation:** It is shown that the Weyl semimetal emerges when we apply photo-irradiation along the x direction to the loop-nodal semimetals<sup>21</sup>. We generalize it to loop-nodal J-fermions. For the photo-irradiation along the x direction with  $\mathbf{A}(t) = (0, A \cos \omega t, A \sin (\omega t + \phi))$ , the first order effective Hamiltonian for (63) is obtained as

$$H^{(1)} = -ac (eA)^2 k_y J_y \cos \phi.$$
 (59)

The energy is modified as

$$E = j\sqrt{\left[a\left(k_x^2 + k_y^2\right) - m\right]^2 + (ck_z)^2 + \left(ac\left(eA\right)^2 k_y \cos\phi\right)}$$
(60)

The gap opens except for the two points

$$k_x^{\pm} = \pm \sqrt{\frac{m}{a}}, \quad k_y = 0, \quad k_z = 0.$$
 (61)

In the vicinity of these points, the Hamiltonian is expanded as

$$H = \pm 2\sqrt{am} \left(k_x \pm \sqrt{\frac{m}{a}}\right) J_x - ac \left(eA\right)^2 k_y J_y \cos\phi + ck_z J_z$$
(62)

which shows that anisotropic *J*-fermions are realized by photo-irradiation. We show the band structure in Fig.9.

# VII. C<sub>3</sub>-PROTECTED FERMIONS

**Hamiltonian:** It has been proposed<sup>24,25</sup> that a three-fold degeneracy is protected by the C<sub>3</sub> symmetry and the mirror symmetry  $M_y$  of the Hamiltonian,

$$H = \begin{pmatrix} \hbar v k_z & \lambda_1 k_+^2 & \lambda_2 k_+ \\ \lambda_1 k_-^2 & \hbar v k_z & \lambda_2 k_- \\ \lambda_2 k_- & \lambda_2 k_+ & -\hbar v k_z \end{pmatrix}$$
(63)

with  $k_{\pm} = k_x \pm i k_y$ . The energy spectrum is given by

$$E_{0} = \hbar v k_{z} - \lambda_{1} k^{2},$$
  

$$E_{\pm} = \frac{\lambda_{1}}{2} k^{2} \pm \sqrt{\left(2\hbar v k_{z} + \lambda_{1} k^{2}\right)^{2} + 8\lambda_{2}^{2} k^{2}}.$$
 (64)



FIG. 9: Band structure of loop-nodal J-fermions with photoirradiation. The horizontal plane is spanned by the  $k_x$  and  $k_y$  axes. The vertical axis is the energy E as a function of  $k_x$  and  $k_y$ , where we have set  $k_z = 0$ . Two J-fermions emerge by photo-irradiation along the x direction. The gap opens except for two points. (a) J = 1and (b) J = 3/2. We have set  $\hbar v = m = ac (eA)^2 \cos \phi = 1$ .



FIG. 10: Breaking of C<sub>3</sub>-protected fermion by photo-irradiation. The vertical axis is the energy E as a function of  $k_z$  and  $k_x$  by setting  $k_y = 0$ . (a) C<sub>3</sub>-protected fermion without photo-irradiation. The 3-fold degenerate point is marked by a magenta circle. (b) That with photo-irradiation along the z direction and (c) along the x direction. The 3-fold degenerate point is split into two Weyl points marked by magenta circles.

A 3-band touching point exists at  $\mathbf{k} = 0$ . In addition, there is a line degeneracy along the line  $k_x = k_y = 0$ . Thus the Chern number is ill-defined. On the other hand, the Berry phase is nonzero for the  $E_0$  band,

$$\Gamma_{\rm B} = -i \int d\theta \left\langle \psi_0 \right| \frac{\partial}{\partial \theta} \left| \psi_0 \right\rangle = 2\pi, \tag{65}$$

while it is zero for the  $E_{\pm}$  bands.

**Photo-irradiation:** We study photo-irradiation effects. We show that the three-band touching is resolved since photo-irradiation breaks the mirror symmetry.

For the photo-irradiation along the z direction with  $\mathbf{A}(t) = (A \cos \omega t, A \sin (\omega t + \phi), 0)$ , the first order effective Hamiltonian for (63) is obtained as

$$H^{(1)} = \cos\phi \begin{pmatrix} -F & 0 & -G \\ 0 & F & G^* \\ -G^* & G & 0 \end{pmatrix}$$
(66)

with

$$F = \frac{1}{\hbar\omega} (eA)^2 \left(\lambda_2^2 + 4\lambda_1^2 q^2\right),$$
  

$$G = \frac{2}{\hbar\omega} (eA)^2 \lambda_1 \lambda_2 q_+.$$
(67)

On the other hand, for the second order term we obtain

$$H^{(2)} = \text{diag.} (F', -F', 0) \cos \phi$$
(68)

with

$$F' = \frac{1}{\hbar\omega} \left(eA\right)^2 \lambda_1^2,\tag{69}$$

and  $H_{\pm n} = 0$  for  $n \ge 3$ . The terms proportional to F breaks the 3-fold degeneracy along the C<sub>3</sub> invariant line  $k_x = k_y =$ 0, while the terms proportional to G renormalize the velocity. As a result, the 3-band touching point is split into two Weyl points,

$$H = \varepsilon_{\pm} + \lambda_2 \sigma_x \pm \lambda_2 \sigma_y + v \sigma_z, \tag{70}$$

by the photo-irradiation as shown in Fig.10(b), which is highly contrasted to the case of the J-fermions.

For the photo-irradiation along the x direction with

$$\mathbf{A}(t) = (0, A\cos\omega t, A\sin(\omega t + \phi)), \tag{71}$$

the first order effective Hamiltonian for (63) is obtained as

$$H^{(1)} = F' \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 1 \\ -1 & 1 & 0 \end{pmatrix},$$
 (72)

with

$$F'' = \cos\phi \frac{4}{\hbar\omega} \hbar v \lambda_2, \tag{73}$$

and  $H_{\pm n} = 0$  for  $n \ge 2$ . The 3-band touching is resolved as shown in Fig.10(c).

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#### VIII. CONCLUSION

We have studied two typical types of multi-band touching fermions. One is described by *J*-fermions described by the pseudospin operators in the *J*-dimensional representation, where the chiral anomaly is induced by the presence of monopoles. As experimental evidences the enhancement of the anomalous Hall effect and the negative magnetoresistance are expected. Photo-irradiation does not resolve the degeneracy but shifts the position of the multi-band touching points, which results in the pair annihilation of them. The other is a 3-band touching fermion protected by the C<sub>3</sub> symmetry. Photo-irradiation breaks the 3-band touching point into two Weyl points. We have also proposed generalizations of Dirac semimetals, multiple-Weyl semimetals and loop-nodal semimetals to those composed of *J*-fermions.

There are already some reports<sup>23,64</sup> that J-fermions with J = 1 and 3/2 have the realization of real materials by firstprinciples calculations. Furthermore, they are protected by non-symmorphic symmetries<sup>23</sup>. On the other hand, there are so far no physical systems to materialize J-fermions with  $J \ge$ 2, and the mechanisms of symmetry protection is yet unclear for them. The search of these systems is a future problem.

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