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# Exactly soluble local bosonic cocycle models, statistical transmutation, and simplest time-reversal symmetric topological orders in 3+1D

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We propose a generic construction of exactly soluble *local bosonic models* that realize various topological orders with gappable boundaries. In particular, we construct an exactly soluble bosonic model that realizes a 3+1D  $Z_2$  gauge theory with emergent fermionic Kramer doublet. We show that the emergence of such a fermion will cause the nucleation of certain topological excitations in space-time without  $\text{pin}^+$  structure. The exactly soluble model also leads to a statistical transmutation in 3+1D. In addition, we construct exactly soluble bosonic models that realize 2 types of time-reversal symmetry enriched  $Z_2$ -topological orders in 2+1D, and 20 types of simplest time-reversal symmetry enriched topological (SET) orders which have only one non-trivial point-like and string-like topological excitations. Many physical properties of those topological states are calculated using the exactly soluble models. We find that some time-reversal SET orders have point-like excitations that carry Kramer doublet – a fractionalized time-reversal symmetry. We also find that some  $Z_2$  SET orders have string-like excitations that carry anomalous (non-on-site)  $Z_2$  symmetry, which can be viewed as a fractionalization of  $Z_2$  symmetry on strings. Our construction is based on cochains and cocycles in algebraic topology, which is very versatile. In principle, it can also realize emergent topological field theory beyond the twisted gauge theory.

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## I. INTRODUCTION

A sign of a comprehensive understanding of a type of phases of matter is being able to classify all of them. We understand that the crystal orders are due to spontaneous symmetry breaking<sup>1</sup> of the translation and the rotation symmetry. This leads to the classification of all 230 crystal orders in 3-dimensions using group theory. Now we realized that the phases of matter beyond symmetry breaking theory are due to long range entanglement<sup>2-4</sup> for topologically ordered phases<sup>5-7</sup>, and due to symmetry-protected short-range entanglement<sup>8,9</sup> for symmetry-protected trivial (SPT) phases<sup>8,10,11</sup>. This leads to complete classification of many topological phases. Using projective representations<sup>9</sup>, we can classify all 1+1D gapped phases for bosonic and fermionic systems with any symmetry<sup>12-15</sup>. We can also classify all 2+1D gapped liquid<sup>16,17</sup> phases for bosonic and fermionic systems with any finite unitary symmetry using unitary modular tensor categories<sup>18,19</sup>,  $G$ -crossed unitary modular tensor categories<sup>20</sup>, and/or unitary braided fusion categories over  $\text{Rep}(G)$  or  $\text{sRep}(G^f)$ <sup>21,22</sup>. Those

phases are symmetry breaking phases, topologically ordered phases, SPT phases (such as odd-integer-spin Haldane phase<sup>23,24</sup> and topological insulators<sup>25-30</sup>), symmetry enriched topological (SET) orders, *etc*. So far, we still do not have a classification of 3+1D gapped liquid phases, although we know that it is closely related to unitary 4-category theory with one object.<sup>31,32</sup>

With those powerful classification results, we would like to have a systematic construction of those topological phases. Ideally, we would like to have a universal construction that can realize any given topological phases. There are very systematic ways to construct exactly soluble models<sup>32-41</sup> based on tensor network<sup>31</sup>. Using unitary fusion categories as input, Turaev-Viro state-sum<sup>34</sup> and Levin-Wen string-net models allow us to realize all 2+1D bosonic topological orders with gappable boundary. Using finite group  $G$  and group 4-cohomology classes  $\omega_4 \in \mathcal{H}^4(G; \mathbb{R}/\mathbb{Z})$  as input, Dijkgraaf-Witten models allow us to realize all 3+1D bosonic topological orders whose point-like excitations are all bosons<sup>42</sup>. Using premodular categories as input, Walker-Wang models can also realize a large class of 3+1D bosonic topological orders. But Walker-Wang models cannot realize all Dijkgraaf-Witten models. A further generalization of Walker-Wang models in Ref. 32 and 39 allow us to include all Dijkgraaf-Witten models as well. Such systematic construction were also generalized to fermion systems<sup>37,40,41,43,44</sup>.

The above constructions are very systematic, but also very complicated and hard to use. Despite their complexity, it is still not clear if they can realize all 3+1D topological orders or not. (We already know that they cannot realize all 2+1D topological orders.) In this paper, we are going to develop a simpler systematic construction. Our constructed models are not a subset of any one of the above mentioned tensor network constructions. But our construction also does include any one of the above mentioned tensor network constructions, as a subset.

We will start with topological invariants for topological orders. Then, we will use cochain theory and cohomology theory<sup>33,45,46</sup> to construct exactly soluble local bosonic models whose ground states have topological orders described by the corresponding topological invariants. In other words, the low energy effective field theory of those local bosonic models are the desired topological field theory. (Here a *local bosonic model* is defined as a quantum model whose total Hilbert space has a tensor product decomposition  $\mathcal{H}_{\text{tot}} = \bigotimes_i \mathcal{H}_i$  where  $\mathcal{H}_i$  is a finite dimensional local Hilbert space for site- $i$ , and the Hamiltonian is local respect to such a tensor product decomposition.) Many mathematical techniques developed for cohomology theory and algebraic topology will help us to do concrete calculations with our models.

One class of topological invariants are given by volume-independent partition function  $Z^{\text{top}}(M^d)$  on manifolds with vanishing Euler number and Pontryagin number<sup>31</sup>  $\chi(M^d) = P(M^d) = 0$ . For invertible topological orders<sup>31,47</sup> and for SPT orders<sup>11,48</sup> (which have no non-

trivial bulk topological excitations), such topological invariants are pure phases<sup>31,47,49–52</sup>

$$Z^{\text{top}}(M^d, a^{\text{sym}}) = e^{i2\pi \int_{M^d} W(w_i, a^G) + k\omega_d} \quad (1)$$

where  $w_i$  is the  $i^{\text{th}}$  Stiefel-Whitney class,  $a^G$  the flat connection that describes symmetry  $G$  twist<sup>53–56</sup>, and  $\omega_d$  the gravitational Chern-Simons term. For example, a 2+1D  $Z_n$ -SPT state labeled by  $k \in \mathcal{H}^3(Z_n, \mathbb{R}/\mathbb{Z}) = \mathbb{Z}_n$  is characterized by its SPT invariant<sup>49–51,54,55</sup> (see Section VB)

$$Z^{\text{top}}(M^{2+1}, a^{Z_n}) = e^{ik \frac{2\pi}{n} \int_{M^{2+1}} a^{Z_n} \cup \mathcal{B}_n a^{Z_n}} \quad (2)$$

where  $a^{Z_2}$  becomes an 1-cochain and  $\mathcal{B}_n$  is Bockstein homomorphism eqn. (43).

For other non-invertible topological orders (which have non-trivial bulk topological excitations), their topological invariants can be sums of phases

$$Z^{\text{top}}(M^d, a^{\text{sym}}) = \sum_{c \in H^*(M^d; \mathbb{M})} e^{i2\pi \int_{M^d} W(c, w_i, a^{\text{sym}}) + k\omega_d},$$

where  $c$  are cohomology classes. Our constructed local bosonic model is designed to produce such form of topological invariants. The construction is very versatile and many exactly soluble local bosonic models can be constructed systematically to produce all the topological invariants of the above form (with  $k = 0$ ). Some of those models have emergent gauge theories or emergent Dijkgraaf-Witten theories<sup>33</sup>. Other models have emergent “twisted” gauge theories beyond Dijkgraaf-Witten type.

In this paper, we will discuss many different types of gauge theories. To avoid confusion, here we will explain the terminology that will be used in this paper. We will use untwisted (UT) gauge theory to refer to the usual lattice gauge theories (without any twist)<sup>57</sup>. We will use all-boson (AB) gauge theory to refer to the lattice gauge theories (may be twisted) where all the pure gauge charges are bosons. We will use emergent-fermion (EF) gauge theory to refer to the lattice gauge theory (may be twisted) where some pure gauge charges are fermions. We will use the term  $G$ -gauge theory to refer gauge theory with  $G$  gauge group. The Dijkgraaf-Witten theories<sup>33</sup> are AB gauge theories. This is because the Dijkgraaf-Witten theories can be viewed as the  $G$ -SPT states with the gauged symmetry  $G$ <sup>53</sup>, all the gauge charge are bosonic in Dijkgraaf-Witten theories.

We will also discuss 3+1D topological theories beyond Dijkgraaf-Witten theories. Many of those theories do not contain gauge fields, and it is hard to call them gauge theories. However, the point-like topological excitations in those theories have the same fusion rule as 3+1D gauge theories, *i.e.* fuse like the irreducible representations of a group  $G$ . So we will still call those 3+1D topological theories as gauge theory, which include EF gauge theories. Certainly, the EF gauge theories are not Dijkgraaf-Witten theories in 3+1D.

We would like to mention that there are many related constructions of topological field theories using 1-form, 2-form gauge fields *etc*<sup>46,58–64</sup>. In contrast to those work, the cocycle models constructed in this paper are defined on lattice instead of continuous manifold. Also cocycle models are not gauge theories. They are local bosonic models without any gauge redundancy. In other words, the emergent topological field theories studied in this paper are free of all anomalies. In comparison, some 1-form, 2-form gauge field theories defined on continuous manifold can be anomalous since they may not be emergable from local lattice theories<sup>31,50,65</sup>.

In this paper, we will use  $\stackrel{n}{=}$  to mean equal up to a multiple of  $n$ , and use  $\stackrel{d}{=}$  to mean equal up to  $df$  (*i.e.* up to a coboundary). We will use  $[f]_n$  to mean  $\text{mod}(f, n)$  and  $\langle l, m \rangle$  to mean the greatest common divisor of  $l$  and  $m$  ( $\langle 0, m \rangle \equiv m$ ). We also introduce some modified  $\delta$ -functions

$$\delta_n(x) = \begin{cases} 1 & \text{if } x \stackrel{n}{=} 0, \\ 0 & \text{otherwise} \end{cases}, \quad \bar{\delta}(x) = \begin{cases} 1 & \text{if } x \stackrel{d}{=} 0, \\ 0 & \text{otherwise} \end{cases}.$$

$$\bar{\delta}_n(x) = \begin{cases} 1 & \text{if } x \stackrel{n, d}{=} 0, \\ 0 & \text{otherwise} \end{cases}.$$

## II. A SUMMARY OF RESULTS

The cocycle models introduced in this paper not only can realize many types of topological orders, SPT orders, and SET orders, they are exactly soluble in the sense that their partition function can be calculated exactly on any space-time manifold<sup>34</sup>. Those models are realizable by commuting projectors. Because the models are exactly soluble, we can use them to compute many physical properties of those topological phases, such as ground state degeneracies, fractional quantum numbers on point-like and string-like topological excitations, braiding statistics, topological partition functions, dimension reduction, *etc*.

### A. Symmetry fractionalization on string-like defects in SPT states

One way to probe SPT order is to measure fractional quantum number carried by symmetry twist defect (see Fig. 1). For example, consider a 2+1D  $Z_n$ -SPT state which is labeled by  $k$ . In Ref. 55 it was shown that a symmetry-twist defect can carry a  $Z_n$  quantum number  $2k$  (*i.e.* each defect will carry a fractional  $Z_n$  quantum number  $\frac{2k}{n}$ ). We can use this property to measure the 2+1D  $Z_n$ -SPT order.

Similar results also appear in higher dimensions. Consider a 3+1D  $Z_n \times \bar{Z}_n$ -SPT state which is labeled by

TABLE I. The 2+1D time-reversal ( $T$ ) symmetric topological orders from four 1-cocycle models in eqn. (6). They have 3 or 4 types of point-like topological excitations.  $d_i$ 's and  $s_i$ 's are the quantum dimensions and spins of those excitations. A quantum dimension  $d = 2$  means that the excitation has 2 internal degrees of freedom.  $2_{\pm}$  means that the 2 internal degrees of freedom form a  $T^2 = 1$  time-reversal doublet or a  $T^2 = -1$  Kramer doublet. Spin  $s = \frac{1}{2}$  corresponds to a fermion, and  $s = \frac{1}{4}$  a semion. Spin  $s = \frac{3}{4}$  is the time-reversal conjugate of a semion. The fourth and fifth columns are volume-independent partition functions  $Z_{M^3}^{\text{top}}$  with  $M^3 = S^1 \times \Sigma_g, S^1 \times \Sigma_g^{\text{non}}$ , where  $\Sigma_g$  is the genus  $g$  Riemannian surface and  $\Sigma_g^{\text{non}}$  is the genus  $g$  non-orientable surface.

$k_0 k_1 k_2$	$(d_1, d_e, \dots)$	$(s_1, s_e, \dots)$	$Z_{S^1 \times \Sigma_g}^{\text{top}}$	$Z_{S^1 \times \Sigma_g^{\text{non}}}^{\text{top}}$	Comments
000	(1, 1, 1, 1)	$(0, 0, 0, \frac{1}{2})$	$4^g$	$2^g$	$Z_2$ -gauge theory (three bosons and one fermion)
001	(1, 1, $2_{-}, 2_{-}$ )	$(0, 0, 0, \frac{1}{2})$	$4^g$	$2^{g-1}[1 + (-)^g]$	A boson and a fermion are Kramer doublets
10*	(1, $2_{-}, 1, 2_{-}$ )	$(0, 0, 0, \frac{1}{2})$	$4^g$	$2^{g-1}[1 + (-)^g]$	The same SET order as above
010	(1, 1, $2_{+}$ )	$(0, 0, [\frac{1}{4}, \frac{3}{4}])$	$4^g$	$2^{g-1}$	Two semions form a $T^2 = 1$ time-reversal doublet
011	(1, 1, $2_{-}$ )	$(0, 0, [\frac{1}{4}, \frac{3}{4}])$	$4^g$	$2^{g-1}$	Two semions form a $T^2 = -1$ Kramer doublet
11*	(1, $2_{-}, 2_{-}, 2_{+}$ )	$(0, 0, [\frac{1}{4}, \frac{3}{4}], [\frac{1}{4}, \frac{3}{4}])$	$4^g$	$2^{g-2}[1 + (-)^g]$	A boson is Kramer doublet

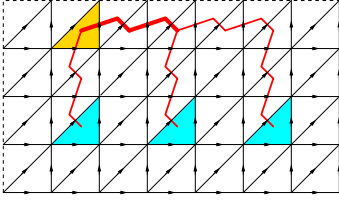


FIG. 1. (Color online) Three *identical*  $Z_3$ -symmetry twist defects (blue triangles) on a torus. The red line are the symmetry twist line. A symmetry twist defect is an end of symmetry twist line.

$k_1 = 0, \dots, n-1$  and  $k_2 = 0, \dots, n-1$ . A  $Z_n$ -symmetry-twist defect will be a line defect in 3+1D. We show that such a line defect must be gapless or symmetry breaking, which behave just like the edge state of some 2+1D SPT state. This phenomena can be viewed as symmetry fractionalization on defect lines.

To see the edge state of which 2+1D SPT state that the defect line carries, we need to specify that the 3+1D  $Z_n \times \tilde{Z}_n$ -SPT state is described by the following SPT invariant  $Z^{\text{top}} = e^{i \frac{2\pi}{n} \int_{M^{3+1}} k_1 a^{Z_n} \cup a^{\tilde{Z}_n} \cup \mathcal{B}_n a^{\tilde{Z}_n} + k_2 a^{Z_n} \cup a^{Z_n} \cup \mathcal{B}_n a^{Z_n}}$ . Then a  $Z_n$ -symmetry-twist defect line will carry the edge state of a 2+1D  $Z_n \times \tilde{Z}_n$ -SPT state characterized by the SPT invariant  $Z^{\text{top}} = e^{i \frac{2\pi}{n} \int_{M^{2+1}} k_1 a^{\tilde{Z}_n} \cup \mathcal{B}_n a^{\tilde{Z}_n} - k_2 a^{Z_n} \cup \mathcal{B}_n a^{\tilde{Z}_n}}$  (see Section III D 3).

To be more precise, the  $Z_n$ -symmetry-twist defect line in 3+1D has a non-on-site (anomalous)  $Z_n \times \tilde{Z}_n$ -symmetry<sup>10,11,65</sup> along the defect line. This 1+1D anomalous symmetry makes the defect line to be either gapless or symmetry breaking.<sup>10</sup> This result generalizes the one in Ref. 55. This 1+1D anomalous symmetry can be viewed as the symmetry fractionalization on the strings.

The 1+1D anomalous symmetry also appear on the edge of 2+1D SPT state. The 1+1D anomalous sym-

metry on the  $Z_n$ -symmetry-twist defect line happen to be the same 1+1D anomalous symmetry on the edge of a 2+1D  $Z_n \times \tilde{Z}_n$ -SPT state characterized by the SPT invariant  $Z^{\text{top}} = e^{i \frac{2\pi}{n} \int_{M^{2+1}} k_1 a^{\tilde{Z}_n} \cup \mathcal{B}_n a^{\tilde{Z}_n} - k_2 a^{Z_n} \cup \mathcal{B}_n a^{\tilde{Z}_n}}$ .

Point-like and string-like symmetry-twist defects are extrinsic defects in the SPT states. The above results indicates that extrinsic defects in the SPT states can carry fractional quantum numbers or anomalous symmetry. We would like to remark here we need to distinguish extrinsic defects from excitations which are intrinsic. The point-like or string-like excitations, by definition, can all be trapped by potential traps of the same dimension. For example a point-like excitation at  $\mathbf{x}_0$  can be trapped by a potential  $V(\mathbf{x})$ , which is non zero only near  $\mathbf{x}_0$ . Those point-like or string-like excitations in SPT states do not have symmetry fractionalization. In contrast, extrinsic defects cannot be trapped by potentials of the same dimension. For example, a point-like symmetry-twist defect in 2+1D can only be trapped by a “potential” (a change of Hamiltonian) that is non-zero along a line, where the point-like defect is trapped at an end of the line.

## B. Statistical transmutation in 3+1D

We have constructed a 3+1D exactly soluble local bosonic model

$$Z(M^{3+1}) = \sum_{\substack{b \in C^2(M^{3+1}; \mathbb{Z}_2) \\ db \equiv 0}} e^{i\pi \int_{M^{3+1}} b \cup b}, \quad (3)$$

where  $b$  is a 2-cochain field (see Section III A for a definition of cochain field) and  $\cup$  the cup product of cochains. The model has an emergent fermion, and its low energy effective theory is a EF  $Z_2$ -gauge theory. Such kind of EF  $Z_2$ -gauge theory has been constructed in terms of strings in 3+1D<sup>36,66</sup>. Here we give a construction in terms of membranes (see Section III E)<sup>60,63</sup>.



As a corollary of the above construction, we find a statistical transmutation in 3+1D lattice  $M^{3+1}$  (expressed in terms of partition function):

$$Z(M^{3+1}) = \sum_{\substack{b \in C^2(M^{3+1}; \mathbb{Z}_2) \\ db \stackrel{2}{=} *j}} e^{i\pi \int_{M^{3+1}} b \cup b}, \quad (4)$$

where  $j$  is a cycle corresponding to the world-line of a bosonic scalar particle, and  $*j$  is the 3-cocycle corresponding to the Poincaré dual of  $j$ . The term  $\pi \int_{M^{3+1}} b \cup b$  changes the statistics of the particle from bosonic to fermionic. This is similar to the statistical transmutation in 2+1D by Chern-Simons term. Note that the condition  $db \stackrel{2}{=} *j$  means  $db = *j \bmod 2$  which can be enforced using energy penalty  $e^{-U \int_{M^{3+1}} |db - *j|^2}$ .

Does the transmuted fermion a spin-up-spin-down doublet? To address this issue, we like to mention that the term  $\pi \int_{M^{3+1}} b \cup b$  is compatible with time-reversal symmetry. If the total model has a time-reversal symmetry, then the particle dressed by the  $b$  field, *i.e.* the fermion, will be a time-reversal singlet, which corresponds to a scalar fermion. However, this behavior can be adjusted by changing the topological term to  $\pi \int_{M^{3+1}} (d\tilde{g} \cup d\tilde{g} + b) \cup b$ , where  $\tilde{g}_i$  is a  $\mathbb{Z}_2$ -0-cochain field which is a pseudo scalar. So the new statistical transmutation is given by

$$Z(M^{3+1}) = \sum_{\substack{\tilde{g} \in C^0(M^{3+1}; \mathbb{Z}_2) \\ b \in C^2(M^{3+1}; \mathbb{Z}_2), db \stackrel{2}{=} *j}} e^{i\pi \int_{M^{3+1}} [d\tilde{g} \cup d\tilde{g} + b] \cup b}. \quad (5)$$

The second type of statistical transmutation can still changes the statistics of the particle from bosonic to fermionic, but now the fermion, dressed by  $b$  and  $\tilde{g}$  fields, will be a Kramer doublet which corresponds to a spin-1/2 fermion (see Section [VIC 4](#)).

### C. 2+1D time-reversal symmetric topological orders

We have constructed  $2^3 = 8$  time-reversal symmetric local bosonic models in 2+1D (see eqn. (203)).

$$\begin{aligned} & Z_{k_0 k_1 k_2; t \mathbb{Z}_2 a T}(M^3) \\ &= \sum_{\{\tilde{g}_i^{\mathbb{Z}_2}, a_{ij}^{\mathbb{Z}_2}, b_{ijk}^{\mathbb{Z}_2}\}} e^{i\pi \int_{M^3} b^{\mathbb{Z}_2} \cup (da^{\mathbb{Z}_2} - k_0 \mathcal{B}_2 d\tilde{g}^{\mathbb{Z}_2})} \times \\ & \quad e^{i\pi \int_{M^3} k_1 a^{\mathbb{Z}_2} \cup a^{\mathbb{Z}_2} \cup a^{\mathbb{Z}_2} + k_2 d\tilde{g}^{\mathbb{Z}_2} \cup d\tilde{g}^{\mathbb{Z}_2} \cup a^{\mathbb{Z}_2}}, \end{aligned} \quad (6)$$

where  $\tilde{g}_i^{\mathbb{Z}_2}, a_{ij}^{\mathbb{Z}_2}, b_{ijk}^{\mathbb{Z}_2}$  are  $\mathbb{Z}_2$ -valued 0-cochain, 1-cochain, and 2-cochain fields (see Section [III A](#)), and  $k_{0,1,2} = 0, 1$ . Also the time-reversal symmetry is described by group  $Z_2^T$  with  $T^2 = 1$ , whose action is given by  $(\tilde{g}_i^{\mathbb{Z}_2}, a_{ij}^{\mathbb{Z}_2}, b_{ijk}^{\mathbb{Z}_2}) \rightarrow (\text{mod}(\tilde{g}_i^{\mathbb{Z}_2} + 1, 2), a_{ij}^{\mathbb{Z}_2}, b_{ijk}^{\mathbb{Z}_2})$  plus the complex conjugation. (The above model also has an additional  $Z_2'$  symmetry generated by  $(\tilde{g}_i^{\mathbb{Z}_2}, a_{ij}^{\mathbb{Z}_2}, b_{ijk}^{\mathbb{Z}_2}) \rightarrow$

$(\text{mod}(\tilde{g}_i^{\mathbb{Z}_2} + 1, 2), a_{ij}^{\mathbb{Z}_2}, b_{ijk}^{\mathbb{Z}_2})$  without the complex conjugation.) We see that  $\tilde{g}_i^{\mathbb{Z}_2}$  is a pseudo scalar field. The above eight models realize five types of time-reversal SET orders.

The four constructed models (labeled by  $k_0 0 k_2$ ) reduce to the  $Z_2$ -topological order described by UT  $Z_2$  gauge theory after we break the time-reversal symmetry (see top three rows in Table [I](#)). But three of them have identical topological orders. Thus the four models only give us two types of time-reversal symmetric  $Z_2$  gauge theories.<sup>67</sup> They correspond to two types of time-reversal symmetric  $Z_2$  gauge theories. Those four models are obtained by gauging the  $Z_2$  subgroup in two of the four  $Z_2 \times Z_2^T$  SPT states and by gauging the  $Z_2$  subgroup of  $Z_4^T$  SPT states ( $Z_4^T$  has  $T^2 = -1$ ).

There is another type of time-reversal symmetric  $Z_2$  gauge theory where the time-reversal transformation exchange the  $Z_2$ -charge and  $Z_2$ -vortex.<sup>68</sup> Such a theory is missing from the table.

The other three of five constructed time-reversal SET orders correspond to three types of time-reversal symmetric double-semion theories<sup>35,36</sup> (see bottom three rows in Table [I](#)). Those theories are obtained by gauging the  $Z_2$  subgroup in two of the four  $Z_2 \times Z_2^T$  SPT states. Two of four constructed models (labeled by  $k_0 1 k_2$ ) have identical topological orders. They give us three types of time-reversal symmetric double-semion theories.

It is interesting to note that one of the time-reversal symmetric double-semion topological order (the last row in Table [I](#)) contain four types of point-like excitations: (1) a trivial type which is a time-reversal singlet; (2) a bosonic Kramer doublet (denoted by quantum dimension, *i.e.* internal degrees of freedom,  $d = 2_-$ ); (3) a  $T^2 = 1$  time-reversal doublet formed by two semions with spin  $\frac{1}{4}$  and  $\frac{3}{4}$  (denoted by quantum dimension  $d = 2_+$ ); (4) a  $T^2 = -1$  Kramer doublet formed by two semions with spin  $\frac{1}{4}$  and  $\frac{3}{4}$  (denoted by quantum dimension  $d = 2_-$ ).

### D. 3+1D time-reversal symmetric $Z_2$ gauge theories

We also have constructed  $2^6 = 64$  local bosonic models in 3+1D which can realize 20 types of simplest topological orders with time-reversal symmetry (see the black rows in Table [II](#)). Those topological orders are simplest since they have only one type of non-trivial point-like topological excitations and one type of non-trivial string-like topological excitations. The point-like topological excitations in those 3+1D SET orders can be Kramer doublet (which corresponds to the fractionalization<sup>69,70</sup> of time-reversal symmetry) and can be fermionic. If we break the time-reversal symmetry, 16 of the 20 SET orders reduce to the 3+1D  $Z_2$ -topological order described by the UT  $Z_2$ -gauge theory, and other 4 of the 20 SET orders reduce to the 3+1D topological order described by the EF  $Z_2$ -gauge theory.

TABLE II. The 3+1D time-reversal ( $T$ ) symmetric  $Z_2$ -gauge theories emerged from lattice bosonic models  $Z_{k_1 k_2 k_3 k_4 k_5 k_6}$  in eqn. (7). Each row corresponds to a root family which contains a few ( $N_{\text{dis}}$ )  $T$ -symmetric topological orders labeled by  $k_5, k_6 = 0, 1$ . Topological orders in the same root family differ only by  $Z_2^T$ -SPT states.  $d_i$ 's and  $s_i$ 's are the quantum dimensions and spins of point-like excitations. A quantum dimension  $d = 2$  means that the excitation has 2 internal degrees of freedom.  $2_-$  means that the 2 internal degrees of freedom form a Kramer doublet with  $T^2 = -1$ . The fourth column is the volume-independent partition function on space-time  $M^4$ , where  $w_i$  is the  $i^{\text{th}}$  Stiefel-Whitney class.

$k_1 k_2 k_3 k_4 k_5 k_6$	$(d_1, d_2)$	$(s_1, s_2)$	$Z^{\text{top}} = \frac{ H^1(M^4; \mathbb{Z}_2) }{ H^0(M^4; \mathbb{Z}_2) } e^{i\pi \int_{M^4} k_5 w_1^4} \times e^{i\pi \int_{M^4} k_6 w_2^2}$	$N_{\text{dis}}$	As gauged SPT state
0000 $k_5 k_6$	(1, 1)	(0, 0)	1	4	Bosonic $Z_2 \times Z_2^T$ trivial state
0100 * $k_6$	(1, 1)	(0, 0)	$\delta(w_1^3)$	2	Bosonic $Z_2 \times Z_2^T$ -SPT state
1000 $k_5 k_6$	(1, 1)	(0, 0)	$\delta(w_3)$	4	Bosonic $Z_2 \times Z_2^T$ -SPT state
1100 * $k_6$	(1, 1)	(0, 0)	$\delta(w_3 + w_1^3)$	2	Bosonic $Z_2 \times Z_2^T$ -SPT state
0001 $k_5$ *	(1, 2 $_-$ )	(0, $\frac{1}{2}$ )	$\delta(w_2)$	2	Free fermion $Z_4^T$ -SPT state
0101 * *	(1, 2 $_-$ )	(0, $\frac{1}{2}$ )	$\delta(w_1^3)\delta(w_2)$	1	?
0010 * $k_6$	(1, 2 $_-$ )	(0, 0)	$\delta(w_1^2)$	2	Bosonic $Z_4^T$ -SPT state
0110 * $k_6$	(1, 2 $_-$ )	(0, 0)	$\delta(w_1^3)\delta(w_1^2) = \delta(w_1^5)$	2	
1010 * $k_6$	(1, 2 $_-$ )	(0, 0)	$\delta(w_3)\delta(w_1^2)$	2	Bosonic $Z_4^T$ -SPT state
1110 * $k_6$	(1, 2 $_-$ )	(0, 0)	$\delta(w_3 + w_1^3)\delta(w_1^2) = \delta(w_3)\delta(w_1^5)$	2	
0011 * *	(1, 1)	(0, $\frac{1}{2}$ )	$\delta(w_1^2 + w_2)$	1	Fermionic $Z_2^f \times Z_2^T$ trivial state
0111 * *	(1, 1)	(0, $\frac{1}{2}$ )	$\delta(w_1^3)\delta(w_1^2 + w_2) = \delta(w_1^5 + w_2)$	1	

Those 64 bosonic models are given by (see eqn. (218)):

$$\begin{aligned}
 & Z_{k_1 k_2 k_3 k_4 k_5 k_6}(M^4) \\
 &= \sum_{\{\tilde{g}_i^{\mathbb{Z}_2}, a_{ij}^{\mathbb{Z}_2}, b_{ijk}^{\mathbb{Z}_2}\}} e^{i\pi \int_{M^4} a^{\mathbb{Z}_2} \cup [b \tilde{g}^{\mathbb{Z}_2} + k_1 a^{\mathbb{Z}_2} \cup a^{\mathbb{Z}_2} \cup a^{\mathbb{Z}_2} + (k_1 + k_2) d\tilde{g} \cup d\tilde{g} \cup d\tilde{g}] \\
 & e^{i\pi \int_{M^4} [k_4 b^{\mathbb{Z}_2} + (k_3 + k_4) d\tilde{g} \cup d\tilde{g}] \cup b^{\mathbb{Z}_2} + k_5 d\tilde{g} \cup d\tilde{g} \cup d\tilde{g} \cup d\tilde{g} + k_6 w_2 \cup w_2},
 \end{aligned} \tag{7}$$

where  $k_I = 0, 1$ ,  $b^{\mathbb{Z}_2}$  is a  $\mathbb{Z}_2$ -2-cocycle field,  $a^{\mathbb{Z}_2}$  a  $\mathbb{Z}_2$ -1-cocycle field, and  $\tilde{g}_i$  a pseudo scalar field which changes under the time-reversal transformation  $\tilde{g}_i \rightarrow \text{mod}(\tilde{g}_i + 1, 2)$ . The above local bosonic models have a time-reversal symmetry: the action amplitude is invariant under the combined transformation of  $\tilde{g}_i \rightarrow \text{mod}(\tilde{g}_i + 1, 2)$  and complex conjugation. The models also have a  $Z_2^T$  symmetry: the action amplitude is invariant under  $\tilde{g}_i \rightarrow \text{mod}(\tilde{g}_i + 1, 2)$  (without the complex conjugation).

But the above model is exactly soluble only when  $k_1 k_4 = 0$ . Those 48 exactly soluble models produce the rows in Table II. The models described by the green rows in Table II produce topological orders that are identical to some black rows. Those identities come from the relations between the Stiefel-Whitney classes on 4-dimensional space-time (see eqn. (D14)):

$$\begin{aligned}
 w_1 \cup w_2 &= 0, \quad w_1 \cup w_3 = 0, \\
 w_1 \cup w_1 \cup w_1 \cup w_1 + w_2 \cup w_2 + w_4 &= 0.
 \end{aligned} \tag{8}$$

We see that  $w_1 \cup w_1 + w_2 = 0$  implies  $w_1 \cup w_1 \cup w_1 + w_2 \cup w_1 = w_1 \cup w_1 \cup w_1 = 0$ . Thus  $\delta(w_1^3)\delta(w_1^2 + w_2) = \delta(w_1^5 + w_2)$ , which implies that the first and the second rows in the fourth block in Table II have the same partition function and thus correspond to the same theory.

We note that the four types of 3+1D  $Z_2^T$ -SPT states<sup>11,71</sup> can be labeled by  $k_5, k_6 = 0, 1$  and are characterized by the SPT invariant  $Z(M^4) = e^{i\pi \int_{M^4} k_5 w_1^4 + k_6 w_2^2}$ . The  $Z_2^T$ -SPT state  $(k_5 k_6) = (10)$  is the one described by group-cohomology  $\mathcal{H}^4(Z_2^T; (\mathbb{R}/\mathbb{Z})_T)$ ,<sup>11</sup> and has a time-reversal symmetric boundary described by an anomalous  $Z_2$  gauge theory where the  $Z_2$ -charge  $e$  and the  $Z_2$ -vortex  $m$  are both Kramer doublet, while the  $e$  and  $m$  bound state  $\varepsilon$  is a time-reversal singlet fermion.<sup>71</sup> The  $Z_2^T$ -SPT state  $(k_5 k_6) = (01)$  is beyond  $\mathcal{H}^4(Z_2^T; (\mathbb{R}/\mathbb{Z})_T)$ , and has a time-reversal symmetric boundary described by an anomalous  $Z_2$  gauge theory where  $e$ ,  $m$ , and  $\varepsilon$  are all fermions.

The model with the same  $k_1 k_2 k_3 k_4$  but different  $k_5 k_6$  only differ by stacking those four  $Z_2^T$ -SPT states. We call two time-reversal SET order that differ only by stacking of  $Z_2^T$ -SPT states as to have the same root, since those SETs have identical bulk point-like and string-like excitations. We find that the 20 SET orders belong to 9 root families. This is because stacking the four  $Z_2^T$ -SPT states does not always produce four distinct time-reversal SET phases, since the partition function may vanish on space-time with non-trivial  $w_1 \cup w_1 \cup w_1 \cup w_1$ ,  $w_2 \cup w_2$ , or  $w_1 \cup w_1 \cup w_1 \cup w_1 + w_2 \cup w_2$ . The number  $N_{\text{dis}}$  of distinct time-reversal SET phases in each root family is given in Table II. The 9 root families correspond to 9 types of 3+1D time-reversal symmetric  $Z_2$ -gauge theories.

From the Table II and from the discussions in Section VIC5, we also see the physical meaning of each topological term labeled by  $k_1 k_2 k_3 k_4$ :

1.  $k_4 = 1$  makes the point-like excitations to be fermions.

2.  $k_4 + k_3 = 1 \bmod 2$  makes the point-like excitations to be Kramer doublet.
3.  $k_1 + k_2 = 1 \bmod 2$  makes the string-like excitations to carry an anomalous  $Z'_2$  symmetry that appear on the boundary of a 2+1D  $Z'_2$ -SPT state. Such an anomalous (non-on-site)  $Z'_2$  symmetry is given by  $U' = \prod_I \sigma_I^x \prod_I \sigma_I^z \frac{1+\sigma_I^z+\sigma_{I+1}^z-\sigma_I^z\sigma_{I+1}^z}{2}$ , where  $\sigma_i^z = (-)^{\tilde{g}_i}$  and  $\frac{1+\sigma_I^z+\sigma_{I+1}^z-\sigma_I^z\sigma_{I+1}^z}{2} = CZ(\sigma_I^z, \sigma_{I+1}^z)$  is the controlled- $Z$  gate acting on the two qubits  $\sigma_I$  and  $\sigma_{I+1}$ .

Certainly, when  $k_1 + k_2 = 0 \bmod 2$ , the string will not have anomalous symmetry, and are in general gapped and symmetric.

There are also many other ways to realize time-reversal symmetric  $Z_2$  gauge theories. For example, one can use non-linear  $\sigma$ -model field theory to realize many of the above time-reversal SET's with bosonic point-like excitations.<sup>72</sup> More generally, one may start with  $Z_2 \times Z_2^T$  bosonic SPT states. There are 8 such states since  $\mathcal{H}^4(Z_2 \times Z_2^T; (\mathbb{R}/\mathbb{Z})_T) = \mathbb{Z}_2^{\oplus 3}$ .<sup>11</sup> Gauging the  $Z_2$  symmetry give us 8 time-reversal symmetric  $Z_2$  topological orders. But some of them only differ by a  $Z_2^T$  SPT state. We only obtain 4 root states (*i.e.* 4 time-reversal symmetric  $Z_2$ -gauge theories), that correspond to the first four rows in the Table II. We can also start with  $Z_4^T$  bosonic SPT states. There are 2 such states since  $\mathcal{H}^4(Z_4^T; (\mathbb{R}/\mathbb{Z})_T) = \mathbb{Z}_2$ . After gauging the unitary  $Z_2$  subgroup of  $Z_4^T$ , we obtain two time-reversal symmetric  $Z_2$  gauge theories (see the two black rows in the third block in Table II). Those two root states have a property that stacking with the  $(k_5 k_6) = (10)$   $Z_2^T$ -SPT state gives us the same root states back. The other two root states (the green rows) are identical to the two black rows in the third block.

The second block in the Table II contains two root states. The first one can be obtained by gauging  $Z_4^T$  fermionic SPT states, which is also known as the  $T^2 = -1$  fermionic topological superconductor.<sup>73,74</sup> There are at least 16  $Z_4^T$  fermionic SPT states labeled by  $\nu = 0, 1, \dots, 15$ .<sup>75-77</sup> Gauging the fermion-parity  $Z_2^f$  subgroup in  $Z_4^T$  fermionic SPT states will produce several time-reversal symmetric topological orders that contain Kramer-doublet fermions. The string-like excitations (*i.e.* the  $Z_2^f$  vortex lines or the  $Z_2^f$ -symmetry-twist defect line) in those topological orders must be gapless unless  $\nu = \text{even}$ , if the time-reversal symmetry is not broken.<sup>78</sup> In comparison, the strings in the three time-reversal symmetric topological orders described by the two rows in the second block do not carry any anomalous time-reversal symmetry. In other words, the excitations on the strings can be gapped even if we do not break the time reversal symmetry.

All the states in the second block have a property that stacking with the  $(k_5 k_6) = (01)$  bosonic  $Z_2^T$ -SPT state (characterized by the SPT invariant  $e^{i\pi \int_{M^4} k_5 w_2 \cup w_2}$ ) does not change their  $Z_2^T$  SET orders. Similarly, the

$Z_4^T$  fermionic topological superconductors also have the property that stacking with the  $(k_5 k_6) = (01)$   $Z_2^T$ -SPT state does not change the SPT order. For example, the  $\nu = 0$   $Z_4^T$  fermionic topological superconductor has a boundary with two types of quasiparticles  $\{1, c\}$ , where 1 is the trivial type and  $c$  is a Kramer-doublet fermion. The  $(k_5 k_6) = (01)$   $Z_2^T$ -SPT state has boundary with four types of quasiparticles  $\{1, f_1, f_2, \varepsilon\}$ , where  $f_1$  and  $f_2$  are Kramer-doublet fermions and  $\varepsilon$  is a time-reversal singlet fermion. Also  $f_{1,2}$  and  $\varepsilon$  have  $\pi$ -mutual statistics among them. The stacking of the two states have a boundary with quasiparticles  $\{1, f_1, f_2, \varepsilon\} \times \{1, c\}$ . We may condense the time-reversal singlet boson  $f_2 c$ . Then, the new boundary state will have quasiparticles  $\{1, c\}$ . The quasiparticle  $f_2$  is also not confined, but it is equivalent to  $c$ , since the two only differ by a condensed boson. Thus the stacking of the  $\nu = 0$  state and the  $(k_5 k_6) = (01)$  state can have the same boundary as the  $\nu = 0$  state. The stacking of  $(k_5 k_6) = (01)$  state does not change the SPT order in  $Z_4^T$  fermionic topological superconductor.

The two states that correspond to the first row in the second block differ by stacking with the  $(k_5 k_6) = (10)$   $Z_2^T$ -SPT state characterized by the SPT invariant  $e^{i\pi \int_{M^4} k_5 w_1 \cup w_1 \cup w_1 \cup w_1}$ . For  $Z_4^T$  fermionic topological superconductors, stacking with the  $(k_5 k_6) = (10)$  state will shift  $\nu$  by 8.<sup>75,78,79</sup> This suggests that the two states are the  $\nu = 0$  and the  $\nu = 8$   $Z_4^T$  fermionic topological superconductor with gauged  $Z_2^f$  symmetry.<sup>75,78,79</sup>

On the other hand, for the time-reversal symmetric topological order described by the second row in the second block, stacking with any  $Z_2^T$ -SPT states does not change its  $Z_2^T$  SET order. It is not clear if the  $Z_2^T$  SET order can be viewed as the  $Z_2^f$ -gauged  $\nu = \pm 4$   $Z_4^T$  fermionic topological superconductor or not.

### E. Vanishing of the volume-independent partition function

We have calculated many volume-independent partition functions, and find they vanish some times. In general, a partition function may have a form

$$Z(M^d) = e^{-c_d L^d - c_{d-1} L^{d-1} - \dots - c_0 L^0 - c_{-1} L^{-1} - \dots}, \quad (9)$$

where  $L$  is the linear size of  $M^d$ . If the ground state does not contain point-like, string-like, *etc* defects, then  $c_1 = c_2 = \dots = c_{d-1} = 0$ . In this case,

$$Z^{\text{top}}(M^d) \equiv \lim_{L \rightarrow \infty} \frac{Z(M^d)}{e^{-c_d L^d}} = e^{-c_0 L^0} \quad (10)$$

is the volume-independent partition function. When the calculated volume-independent partition function vanishes, it does not mean the partition function to vanish nor the theory to be anomalous. It just means that  $c_i > 0$ , for some  $0 < i < d$ . This implies that the given space-time topology  $M^d$  induces point-like, string-like, *etc* topological excitations.



We have calculated volume-independent partition functions for many constructed systems and for many space-time manifolds (see Table I, II, and III). From those results, we conjecture that:

*A local bosonic model with emergent fermion always has vanishing volume-independent partition function  $Z^{\text{top}}(M^d) = 0$  if the orientable  $M^d$  is not spin.*

In the presence of time-reversal symmetry:

(1) *A local bosonic model with emergent Kramer doublet fermions always has vanishing volume-independent partition function  $Z^{\text{top}}(M^d) = 0$  if  $M^d$  is not  $\text{pin}^+$  (i.e.  $w_2 \neq 0$ ).*

(2) *A local bosonic model with emergent time-reversal singlet fermions always has vanishing volume-independent partition function  $Z^{\text{top}}(M^d) = 0$  if  $M^d$  is not  $\text{pin}^-$  (i.e.  $w_2 + w_1^2 \neq 0$ ).*

(3) *A local bosonic model with emergent Kramer doublet bosons always has vanishing volume-independent partition function  $Z^{\text{top}}(M^d) = 0$  if  $w_1^2 \neq 0$  on  $M^d$ .*

(See Appendix E for a brief introduction of spin,  $\text{pin}^+$ , and  $\text{pin}^-$  manifolds.) Those properties has been used to develop cobordism theory for fermionic SPT states<sup>80</sup>.

In the rest of this paper, we will present detailed constructions and calculation.

### III. A GENERIC CONSTRUCTION OF EXACTLY SOLUBLE BOSONIC LATTICE MODELS ON SPACE-TIME LATTICE

In this section, we are going to introduce a general way to construct exactly soluble local bosonic models. Those models are written in terms of path integral on space-time lattice. Those models are also designed to have topologically ordered ground states. In other words, those models have emergent topological field theory at low energies.

First, we will briefly review the related mathematics. Then we will construct models that realize some well known topological orders, such as those described by discrete gauge theories, and by Dijkgraaf-Witten theories. After that we will construct models that realize more general topological orders whose low energy effective theories are beyond Dijkgraaf-Witten theories. We will also compute the volume-independent partition functions for those constructed models on several choices of space-time manifolds. The results are summarized in Table III.

#### A. Space-time complex, cochains, and cocycles

Our local bosonic models will be defined on a space-time lattice. A space-time lattice is a triangulation of the  $d$ -dimensional space-time, which is denoted as  $M_{\text{latt}}^d$ . We will also call the triangulation  $M_{\text{latt}}^d$  as a space-time complex. A cell in the complex is called a simplex. We will use  $i, j, \dots$  to label vertices of the space-time complex. The links of the complex (the 1-simplices) will be labeled

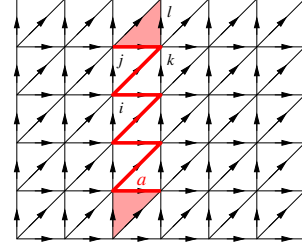


FIG. 2. (Color online) A 1-cochain  $a$  has a value 1 on the red links:  $a_{ik} = a_{jk} = 1$  and a value 0 on other links:  $a_{ij} = a_{kl} = 0$ .  $da$  is non-zero on the shaded triangles:  $(da)_{jkl} = a_{jk} + a_{kl} - a_{jl}$ . For such 1-cochain, we also have  $a \cup a = 0$ . So when viewed as a  $\mathbb{Z}_2$ -valued cochain,  $\mathcal{B}_2 a \neq a \cup a \pmod{2}$ .

by  $(i, j), (j, k), \dots$ . Similarly, the triangles of the complex (the 2-simplices) will be labeled by  $(i, j, k), (j, k, l), \dots$ .

A cochain  $f_n$  is an assignment of values in  $\mathbb{M}$  to each  $n$ -simplex, for example a value  $f_{n;i,j,\dots,k} \in \mathbb{M}$  for  $n$ -simplex  $(i, j, \dots, k)$ . So a cochain  $f_n$  can be viewed as a bosonic field on the space-time lattice. In this paper, we will use such cochain bosonic field to construct our models.

In this paper, we will assume  $\mathbb{M}$  to be a ring which support addition and multiplication operations, as well as scaling by an integer:

$$\begin{aligned} x + y &= z, & x * y &= z, & mx &= y, \\ x, y, z &\in \mathbb{M}, & m &\in \mathbb{Z}. \end{aligned} \quad (11)$$

We see that  $\mathbb{M}$  can also be viewed a  $\mathbb{Z}$ -module (i.e. a vector space with integer coefficient) that also allows a multiplication operation. In this paper we will view  $\mathbb{M}$  as a  $\mathbb{Z}$ -module. The direct sum of two modules  $\mathbb{M}_1 \oplus \mathbb{M}_2$  (as vector spaces) is equal to the direct product of the two modules (as sets):

$$\mathbb{M}_1 \oplus \mathbb{M}_2 \stackrel{\text{as set}}{=} \mathbb{M}_1 \times \mathbb{M}_2 \quad (12)$$

We like to remark that a simplex  $(i, j, \dots, k)$  can have two different orientations. We can use  $(i, j, \dots, k)$  and  $(j, i, \dots, k) = -(i, j, \dots, k)$  to denote the same simplex with opposite orientations. The value  $f_{n;i,j,\dots,k}$  assigned to the simplex with opposite orientations should differ by a sign:  $f_{n;i,j,\dots,k} = -f_{n;j,i,\dots,k}$ . So to be more precise  $f_n$  is a linear map  $f_n : n\text{-simplex} \rightarrow \mathbb{M}$ . We can denote the linear map as  $\langle f_n, n\text{-simplex} \rangle$ , or

$$\langle f_n, (i, j, \dots, k) \rangle = f_{n;i,j,\dots,k} \in \mathbb{M}. \quad (13)$$

More generally, a cochain  $f_n$  is a linear map of  $n$ -chains:

$$f_n : n\text{-chains} \rightarrow \mathbb{M}, \quad (14)$$

or (see Fig. 2)

$$\langle f_n, n\text{-chain} \rangle \in \mathbb{M}, \quad (15)$$

where a chain is a composition of simplices. For example, a 2-chain can be a 2-simplex:  $(i, j, k)$ , a sum of two 2-simplices:  $(i, j, k) + (j, k, l)$ , a more general composition

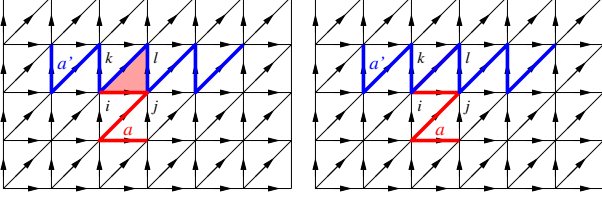


FIG. 3. (Color online) A 1-cochain  $a$  has a value 1 on the red links, Another 1-cochain  $a'$  has a value 1 on the blue links. On the left,  $a \cup a'$  is non-zero on the shade triangles:  $(a \cup a')_{ijl} = a_{ij}a'_{jl} = 1$ , while on the right,  $a' \cup a$  is zero. Thus  $a \cup a' + a' \cup a$  is not a coboundary.

of 2-simplices:  $(i, j, k) - 2(j, k, l)$ , etc. The map  $f_n$  is linear respect to such a composition. For example, if a chain is  $m$  copies of a simplex, then its assigned value will be  $m$  times that of the simplex.  $m = -1$  correspond to an opposite orientation.

The total space-time lattice  $M_{\text{latt}}^d$  correspond to a  $d$ -chain. We will use the same  $M_{\text{latt}}^d$  to denote it. Viewing  $f_d$  as a linear map of  $d$ -chains, we can define an “integral” over  $M_{\text{latt}}^d$ :

$$\int_{M_{\text{latt}}^d} f_d \equiv \langle f_d, M_{\text{latt}}^d \rangle. \quad (16)$$

In this paper, we usually take  $\mathbb{M}$  to be integer  $\mathbb{Z}$  or mod  $n$  integer  $\mathbb{Z}_n = \{0, 1, \dots, n-1\}$ . So not only the field  $f_{n;i,j,\dots,k}$  is defined on a discrete space-time lattice, even the value of the field is discrete. We will use  $C^n(M_{\text{latt}}^d; \mathbb{M})$  to denote the set of all  $n$ -cochains on  $M_{\text{latt}}^d$ .  $C^n(M_{\text{latt}}^d; \mathbb{M})$  can also be viewed as a set all  $\mathbb{M}$ -values fields (or paths) on  $M_{\text{latt}}^d$ . Note that  $C^n(M_{\text{latt}}^d; \mathbb{M})$  is an abelian group under the  $+$ -operation.

We can define a derivative operator  $d$  acting on an  $n$ -cochain  $f_n$ , which give us an  $n+1$ -cochain (see Fig. 2):

$$\begin{aligned} & \langle df_n, (i_0 i_1 i_2 \dots i_{n+1}) \rangle \\ &= \sum_{m=0}^{n+1} (-)^m \langle f_n, (i_0 i_1 i_2 \dots \hat{i}_m \dots i_{n+1}) \rangle \end{aligned} \quad (17)$$

where  $i_0 i_1 i_2 \dots \hat{i}_m \dots i_{n+1}$  is the sequence  $i_0 i_1 i_2 \dots i_{n+1}$  with  $i_m$  removed, and  $i_0, i_1, i_2 \dots i_{n+1}$  are the ordered vertices of the  $(n+1)$ -simplex  $(i_0 i_1 i_2 \dots i_{n+1})$ .

A cochain  $f_n \in C^n(M_{\text{latt}}^d; \mathbb{M})$  is called a *cocycle* if  $df_n = 0$ . The set of cocycles is denoted as  $Z^n(M_{\text{latt}}^d; \mathbb{M})$ . A cochain  $f_n$  is called a *coboundary* if there exist a cochain  $f_{n-1}$  such that  $df_{n-1} = f_n$ . The set of coboundaries is denoted as  $B^n(M_{\text{latt}}^d; \mathbb{M})$ . Both  $Z^n(M_{\text{latt}}^d; \mathbb{M})$  and  $B^n(M_{\text{latt}}^d; \mathbb{M})$  are abelian groups as well. Since  $d^2 = 0$ , a coboundary is always a cocycle:  $B^n(M_{\text{latt}}^d; \mathbb{M}) \subset Z^n(M_{\text{latt}}^d; \mathbb{M})$ . We may view two cocycles differ by a coboundary as equivalent. The equivalence classes of cocycles,  $[f_n]$ , form the so called cohomology group denoted

as

$$H^n(M_{\text{latt}}^d; \mathbb{M}) = Z^n(M_{\text{latt}}^d; \mathbb{M}) / B^n(M_{\text{latt}}^d; \mathbb{M}), \quad (18)$$

$H^n(M_{\text{latt}}^d; \mathbb{M})$ , as a group quotient of  $Z^n(M_{\text{latt}}^d; \mathbb{M})$  by  $B^n(M_{\text{latt}}^d; \mathbb{M})$ , is also an abelian group.

From two cochains  $f_m$  and  $h_n$ , we can construct a third cochain  $p_{m+n}$  via the cup product (see Fig. 3):

$$\begin{aligned} p_{m+n} &= f_m \cup h_n, \\ \langle p_{m+n}, (i_0 \dots i_{m+n}) \rangle &= \langle f_m, (i_0 i_1 \dots i_m) \rangle \times \\ & \quad \langle h_n, (i_m i_{m+1} \dots i_{m+n}) \rangle \end{aligned} \quad (19)$$

The cup product has the following property (see Fig. 3):

$$d(f_m \cup h_n) = (dh_n) \cup f_m + (-)^n h_n \cup (df_m) \quad (20)$$

We see that  $f_m \cup h_n$  is a cocycle if both  $f_m$  and  $h_n$  are cocycles. If both  $f_m$  and  $h_n$  are cocycles, then  $f_m \cup h_n$  is a coboundary if one of  $f_m$  and  $h_n$  is a coboundary. So the cup product is also an operation on cohomology groups  $\cup : H^m(M^d; \mathbb{M}) \times H^n(M^d; \mathbb{M}) \rightarrow H^{m+n}(M^d; \mathbb{M})$ . When both  $f_m$  and  $h_n$  are cocycles, we also have

$$f_m \cup h_n = (-)^{mn} h_n \cup f_m + \text{coboundary}. \quad (21)$$

In the rest of this paper, we abbreviate the cup product  $a \cup b$  as  $ab$  by dropping  $\cup$ . Also, we will use  $\mathbb{Z}_n = \{0, 1, \dots, n-1\}$  and  $Z_n = \{1, e^{i\frac{2\pi}{n}}, e^{i2\frac{2\pi}{n}}, \dots, e^{i(n-1)\frac{2\pi}{n}}\}$  to denote the same abelian group. In  $\mathbb{Z}_n$ , the group multiplication is mod- $n$  “+” and in  $Z_n$ , the group multiplication is “\*”.

## B. $\mathbb{Z}_n$ -1-cocycle model and emergent $\mathbb{Z}_n$ gauge theory

### 1. Model construction

Using the above mathematical formalism, let us construct a local bosonic model on a space-time lattice  $M_{\text{latt}}^{d+1}$ , where the local degrees of freedom live on the links. The possible values on each link are  $a_{ij}^{\mathbb{Z}_n} = 0, 1, \dots, n-1 \in \mathbb{Z}_n$ .

The action amplitude  $e^{-S_{\text{cell}}}$  for a  $d+1$ -simplex  $(ij \dots l)$  is a complex function of  $a_{ij}^{\mathbb{Z}_n}$ :  $e^{-L_{ij \dots l}(\{a_{ij}^{\mathbb{Z}_n}\})}$ . The total action amplitude  $e^{-S}$  for a configuration (or a path) is given by

$$e^{-S(\{a_{ij}^{\mathbb{Z}_n}\})} = \prod_{(ij \dots l)} e^{-L_{ij \dots l}(\{a_{ij}^{\mathbb{Z}_n}\})} \quad (22)$$

where  $\prod_{(ij \dots l)}$  is the product over all the  $d+1$ -simplices  $(ijkl)$ . Our local bosonic model is defined by the following imaginary-time path integral (or partition function)

$$\begin{aligned} Z_{\mathbb{Z}_n, a} &= \sum_{\{a_{ij}^{\mathbb{Z}_n}\}} e^{-S(\{a_{ij}^{\mathbb{Z}_n}\})} \\ &= \sum_{\{a_{ij}^{\mathbb{Z}_n}\}} e^{-\sum_{(ij \dots l)} L_{ij \dots l}(\{a_{ij}^{\mathbb{Z}_n}\})} \end{aligned} \quad (23)$$

TABLE III. Volume independent partition function  $Z^{\text{top}}(M^4)$  for the constructed local bosonic models, on closed 4-dimensional space-time manifolds. The space-time  $M^4$  considered here satisfy  $\chi(M^4) = P_1(M^4) = 0$ , which makes  $Z^{\text{top}}(M^4)$  to be a topological invariant<sup>31</sup>. The topological invariants listed below are also the ground state degeneracy on the corresponding spatial manifold  $M^3_{\text{space}}$ . Here  $L^3(p)$  is the 3-dimensional lens space and  $F^4 = (S^1 \times S^3) \# (S^1 \times S^3) \# \mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$ .  $F^4$  is not spin. The different models are labeled by  $k_I$  which all have a range  $k_I = 0, 1, \dots, n-1$ .

Models \ $M^4$ :	$T^4$	$T^2 \times S^2$	$S^1 \times L^3(p)$	$F^4$	Low energy effective theory
$Z^{\text{top}}_{\mathbb{Z}_n \text{a}}(M^4)$	$n^3$	$n$	$\langle n, p \rangle$	$n$	UT $\mathbb{Z}_n$ gauge theory
$Z^{\text{top}}_{k; b^2 \mathbb{Z}_n}(M^4)$	$\langle 2k, n \rangle^3$	$\langle 2k, n \rangle$	$\langle 2k, n, p \rangle$	$\langle 2k, n \rangle$ if $\frac{2kn}{\langle 2k, n \rangle^2} = \text{even}$ 0 if $\frac{2kn}{\langle 2k, n \rangle^2} = \text{odd}$	$Z_{\langle 2k, n \rangle}$ gauge theory with fermions iff $\frac{2kn}{\langle 2k, n \rangle^2} = \text{odd}$
$Z^{\text{top}}_{k_1 k_2; aa' Ba' \mathbb{Z}_n}(M^4)$	$n^6$	$n^2$	$\langle n, p \rangle \langle n, p, k_1, k_2 \rangle$ if $p$ has no repeated prime factors.	$n^2$	$\mathbb{Z}_n \times \mathbb{Z}_n$ Dijkgraaf-Witten theory
$Z^{\text{top}}_{k_1 k_2; b Ba-bb \mathbb{Z}_n}(M^4)$	$n^3 \langle 2k_2, n \rangle^3$	$n \langle 2k_2, n \rangle$	$\langle n, p \rangle \langle 2k_2, n, p, \frac{k_1 p}{\langle n, p \rangle} \rangle$	$n \langle 2k_2, n \rangle$ if $\frac{2k_2 n}{\langle 2k_2, n \rangle^2} = \text{even}$ 0 if $\frac{2k_2 n}{\langle 2k_2, n \rangle^2} = \text{odd}$	$Z_{\frac{n \langle 2k_2, n \rangle}{\langle k_1, 2k_2, n \rangle}} \times Z_{\langle k_1, 2k_2, n \rangle}$ gauge theory with fermions iff $\frac{2k_2 n}{\langle 2k_2, n \rangle^2} = \text{odd}$

where  $\sum_{\{a_{ij}^{\mathbb{Z}_n}\}}$  is a sum over all paths (*i.e.* the path integral).

We may view  $a_{ij}^{\mathbb{Z}_n}$  as  $\mathbb{Z}_n$ -valued 1-cochain on the space-time complex  $M^3_{\text{latt}}$ :

$$a_{ij}^{\mathbb{Z}_n} = \langle a^{\mathbb{Z}_n}, (ij) \rangle, \quad a^{\mathbb{Z}_n} \in C^1(M^3_{\text{latt}}, \mathbb{Z}_n) \quad (24)$$

The Lagrangian  $L_{ij \dots l}(\{a_{ij}^{\mathbb{Z}_n}\})$  will produce an emergent low energy  $\mathbb{Z}_2$ -gauge theory (*i.e.* have a  $\mathbb{Z}_2$  topological order) if we choose it to be

$$\begin{aligned} L_{ij \dots l}(\{a_{ij}^{\mathbb{Z}_n}\}) &= +\infty, & \text{if } (da^{\mathbb{Z}_n}) \neq 0 \text{ on } (ij \dots l), \\ L_{ij \dots l}(\{a_{ij}^{\mathbb{Z}_n}\}) &= 0, & \text{if } (da^{\mathbb{Z}_n}) = 0 \text{ on } (ij \dots l). \end{aligned} \quad (25)$$

So the action amplitude  $e^{-L_{ij \dots l}(\{a_{ij}^{\mathbb{Z}_n}\})}$  is non-zero only when  $a^{\mathbb{Z}_n}$  is a cocycle, and the non-zero value is always 1. In other words, our local bosonic model is described by an action  $S(a^{\mathbb{Z}_n}) = 0$  when  $a^{\mathbb{Z}_n}$  is a cocycle, and  $S(a^{\mathbb{Z}_n}) = +\infty$  when  $a^{\mathbb{Z}_n}$  is not a cocycle. We see that the configurations described by non-cocycles cost an infinity energy. We will call the local bosonic model described by the above  $L_{ij \dots l}$  as a  $\mathbb{Z}_n$ -1-cocycle model.

## 2. Topological partition functions

The partition function  $Z_{\mathbb{Z}_n \text{a}}(M^4_{\text{latt}})$  of the  $\mathbb{Z}_n$ -1-cocycle model can be calculated exactly, which is given by the number of 1-cocycles  $|Z^1(M^4_{\text{latt}}; \mathbb{Z}_n)|$ , where  $|S|$  denotes that number elements in set  $S$ . The number of 1-cochains is given by  $|H^1(M^4_{\text{latt}}; \mathbb{Z}_n)|$  times the number of 0-cochains whose derivatives is non-zero. The number of 0-cochains whose derivatives is non-zero is the number of 0-cochains,  $|C^0(M^4_{\text{latt}}; \mathbb{Z}_n)|$ , divide by  $|H^0(M^4_{\text{latt}}; \mathbb{Z}_n)|$ . Since  $|C^0(M^4_{\text{latt}}; \mathbb{Z}_n)| = 2^{N_v}$ , where  $N_v$  is the number of vertices (the “volume” of space-time), we find that the

partition function is

$$\begin{aligned} Z_{\mathbb{Z}_n \text{a}}(M^{d+1}_{\text{latt}}) &= |Z^1(M^{d+1}_{\text{latt}}; \mathbb{Z}_n)| \\ &= |H^1(M^{d+1}_{\text{latt}}; \mathbb{Z}_n)| \frac{|C^0(M^{d+1}_{\text{latt}}; \mathbb{Z}_n)|}{|H^0(M^{d+1}_{\text{latt}}; \mathbb{Z}_n)|} \\ &= 2^{N_v} \frac{|H^1(M^{d+1}_{\text{latt}}; \mathbb{Z}_n)|}{|H^0(M^{d+1}_{\text{latt}}; \mathbb{Z}_n)|}. \end{aligned} \quad (26)$$

According to Ref. 31, the topological information is given by the volume-independent part of partition function, which is obtained by taking the limit  $N_v \rightarrow 0$ :

$$Z^{\text{top}}_{\mathbb{Z}_n \text{a}}(M^{d+1}) = \frac{|H^1(M^{d+1}; \mathbb{Z}_n)|}{|H^0(M^{d+1}; \mathbb{Z}_n)|}, \quad (27)$$

The volume-independent of partition function can be a topological invariant<sup>31</sup> if the Euler number and the Pontryagin number vanish:  $\chi(M^{d+1}) = P(M^{d+1}) = 0$ . Such topological invariant characterizes the topological order realized by the model. Since the  $\mathbb{Z}_2$  gauge theory will produce the same volume-independent partition function  $Z^{\text{top}}_{\mathbb{Z}_n \text{a}}(M^{d+1})$  in large system size and low energy limit, this allows us to determine that the  $\mathbb{Z}_n$ -1-cocycle model realizes the  $\mathbb{Z}_2$ -topological order<sup>81,82</sup> – the topological order described UT  $\mathbb{Z}_n$ -gauge theory.

In 2+1D and for  $n = 2$ , the  $\mathbb{Z}_2$ -topological order has two bosonic topological quasiparticles,  $\mathbb{Z}_2$ -charge  $e$  and  $\mathbb{Z}_2$ -vortex  $m$ , and one fermionic topological quasiparticles  $f$  which is a bound state of  $e$  and  $m$ . In higher dimensions, the  $\mathbb{Z}_n$ -topological order has  $n$  types of bosonic point-like excitations – the  $\mathbb{Z}_n$ -charge  $q = 0, 1, \dots, n-1$ . It also has  $n$  types of  $(d-2)$ -dimensional brane-like excitations – the  $\mathbb{Z}_n$ -flux  $m = 0, \frac{2\pi}{n}, \dots, \frac{2\pi(n-1)}{n}$ .

We note that the volume-independent partition function on space-time  $S^1 \times S^1 \times S^{d-1} = T^2 \times S^{d-1}$  is given by

$$Z^{\text{top}}_{\mathbb{Z}_n \text{a}}(T^2 \times S^{d-1}) = n, \quad (28)$$

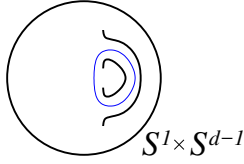


FIG. 4. (Color online) A particle-hole tunneling process is a process where we create a particle-hole pair, move the particle around a non-contractible loop, and then annihilate the particle and the hole. The GSD on a  $d$ -dimensional space  $S^1 \times S^{d-1}$  is generated by the particle-hole tunneling process described by the blue loop. Thus, each degenerate ground state correspond to a type of particle, and  $\text{GSD}(S^1 \times S^{d-1}) = \text{number of types of point-like excitations}$ . Similarly,  $\text{GSD}(S^k \times S^{d-k}) = \text{number of types of } (k-1)\text{-dimensional excitations}$ .

Since the volume-independent partition function on  $S^1 \times M^d$  equal to the ground state degeneracy (GSD) on space  $M^d$ :

$$\text{GSD}(M^d) = Z^{\text{top}}(S^1 \times M^d), \quad (29)$$

we find that the GSD of our  $\mathbb{Z}_n$ -1-cocycle model on space  $S^1 \times S^{d-1}$  is given by  $\text{GSD}(S^1 \times S^{d-1}) = n$ .

It turns out that,

for any topological order,  $\text{GSD}(S^1 \times S^{d-1})$  is always equal to the number of types of point-like topological excitations.

Such a result can be understood by the particle-hole tunneling process in Fig. 4. Such a particle tunneling process changes one ground state to another degenerate one, and relate the number types of point-like topological excitations to  $\text{GSD}(S^1 \times S^{d-1})$ . It is also true that,

for any topological order,  $\text{GSD}(S^1 \times S^{d-1})$  is always equal to the number of types of  $(d-2)$ -dimensional brane-like topological excitations<sup>83</sup>.

(The notion of types of topological excitations, in particular high dimensional topological excitations was discussed in Ref. 31. It is very tricky to define the types of high dimensional topological excitations.) This can be understood by a similar brane tunneling process around  $S^{d-1}$ .

In general,

in  $d$ -dimensional space, the number of types of  $k-1$ -dimensional brane-like excitations is equal to the number of types of  $d-k-1$ -dimensional brane-like excitations, and they both equal to  $\text{GSD}(S^k \times S^{d-k})$ .

### 3. Boundary effective theory

Using the cocycle model, we can also easily study the properties of the boundary. Consider a space-time  $M^{d+1}$

whose boundary is  $N^d = \partial M^d$ . What is the low energy effective theory of our  $\mathbb{Z}_n$ -1-cocycle model on the boundary  $N^d$ ? To be more concrete, what is the partition function for the boundary effective theory? Here, we propose that the partition function for the boundary effective theory is simply given by

$$Z_{\mathbb{Z}_n \text{a}}^{\text{bndr}}(N^d) = Z_{\mathbb{Z}_n \text{a}}(M^{d+1}). \quad (30)$$

However, the above definition has a problem: the same  $N^d$  can be viewed as boundary of different space-time manifolds  $N^d = \partial M^{d+1} = \partial \tilde{M}^{d+1}$ . In general

$$Z_{\mathbb{Z}_n \text{a}}(M^{d+1}) \neq Z_{\mathbb{Z}_n \text{a}}(\tilde{M}^{d+1}) \quad (31)$$

so the above definition of  $Z_{\mathbb{Z}_n \text{a}}^{\text{bndr}}(N^d)$  is not self consistent.

In order for the definition to be self consistent, we require that

$$Z_{\mathbb{Z}_n \text{a}}(M^{d+1}) = Z_{\mathbb{Z}_n \text{a}}(\tilde{M}^{d+1}) \quad (32)$$

for all  $M^{d+1}$  and  $\tilde{M}^{d+1}$  with  $\partial M^{d+1} = \partial \tilde{M}^{d+1}$ . This implies that the bulk model on  $M^{d+1}$  has no topological order. So the boundary effective theory is well defined by itself iff the bulk theory on  $M^{d+1}$  has no topological order. This is exactly the gravitational-anomaly-free condition discussed in Ref. 31, 50, and 65.

Since the bulk  $\mathbb{Z}_n$ -1-cocycle model has a non-trivial topological order, the boundary effective theory is anomalous. This implies that the boundary effective partition function  $Z_{\mathbb{Z}_n \text{a}}^{\text{bndr}}(N^d)$  not only depends on  $N^d$ , it also depends on how  $N^d$  is extended to one higher dimension, *i.e.* depend on  $M^{d+1}$ . The definition eqn. (30) correctly reflects such anomaly effect, and thus is a proper definition. However, to stress the dependence on the extension, we rewrite eqn. (30) as

$$Z_{\mathbb{Z}_n \text{a}}^{\text{bndr}}(N^d, M^{d+1}) = Z_{\mathbb{Z}_n \text{a}}(M^{d+1}). \quad (33)$$

Even though the boundary partition function depend on the bulk extension, it is still very useful in determine boundary low energy properties, such as if the boundary gapped or not. Let us first choose  $N^d = S^d$  and choose its extension to be  $M^{d+1} = B^{d+1}$ , where  $B^{d+1}$  is a  $d+1$ -dimensional ball. We find the boundary partition function to be

$$Z_{\mathbb{Z}_n \text{a}}^{\text{bndr}}(S^d, B^{d+1}) = \frac{1}{n} n^{N_v^{\text{bndr}}} n^{N_v^{\text{blk}}}. \quad (34)$$

where  $N_v^{\text{bndr}}$  is the number of vertices on the boundary  $S^d$  and  $N_v^{\text{blk}}$  is the number of vertices *inside* the ball  $B^{d+1}$ . The partition function only depend on the “volume” of the boundary and does not depend on the shape of the boundary. This implies that the boundary theory is gapped.

Next, let us choose  $N^d = S_t^1 \times S^{d-1}$ , where we use  $S_t^1$  to represent the closed time direction. We choose its extension to be  $M^{d+1} = S_t^1 \times B^d$ . We find the boundary partition function to be

$$Z_{\mathbb{Z}_n \text{a}}^{\text{bndr}}(S_t^1 \times S^{d-1}, S_t^1 \times B^d) = n^{N_v^{\text{bndr}}} n^{N_v^{\text{blk}}}. \quad (35)$$

We see that the volume-independent boundary partition function is

$$Z_{\mathbb{Z}_n \text{a}}^{\text{bndr, top}}(S_t^1 \times S^{d-1}, S_t^1 \times B^d) = 1. \quad (36)$$

This implies that the gapped boundary has no ground state degeneracy (for the boundary  $S^{d-1}$ ). For example there is no symmetry breaking.

To see if the boundary carries an anomalous topological order, let us choose  $N^d = S_t^1 \times S^{k+1} \times S^{d-2-k}$  and choose its extension to be  $M^d = S_t^1 \times S^{k+1} \times B^{d-1-k}$ . Since the tunneling process of  $k$ -dimensional brane-like topological excitations around  $S^{k+1}$  on the boundary corresponds to a non-contractible loop in the bulk  $S_t^1 \times S^{k+1} \times B^{d-1-k}$ , the tunneling process will generate a map between different degenerate ground states. In contrast, the brane tunnel process around  $S^{d-2-k}$  on the boundary corresponds to a contractible “loop” in the bulk  $S_t^1 \times S^{k+1} \times B^{d-1-k}$  and does not generate non-trivial map between degenerate ground states. Therefore, similar to the bulk case,  $Z_{\mathbb{Z}_n \text{a}}^{\text{bndr}}(S_t^1 \times S^{k+1} \times S^{d-2-k}, S_t^1 \times S^{k+1} \times B^{d-1-k})$  can tell us the number of types of  $k$ -dimensional brane-like topological excitations on the boundary.

For our  $\mathbb{Z}_n$ -1-cocycle model, we found that the volume-independent partition function to be

$$\begin{aligned} & Z_{\mathbb{Z}_n \text{a}}^{\text{bndr, top}}(S_t^1 \times S^{k+1} \times S^{d-2-k}, S_t^1 \times S^{k+1} \times B^{d-1-k}) \\ &= \frac{|H^1(S_t^1 \times S^{k+1} \times B^{d-1-k}; \mathbb{Z}_n)|}{|H^0(S_t^1 \times S^{k+1} \times B^{d-1-k}; \mathbb{Z}_n)|} = \begin{cases} 1, & k > 0 \\ n, & k = 0 \end{cases} \end{aligned} \quad (37)$$

Thus the boundary theory contains  $n$  types of point-like excitations, and no non-trivial brane-like excitations of dimensions greater than 0.

The  $n$  types of point-like topological excitations on the boundary contain a trivial type and  $n-1$  non-trivial type. When  $n > 1$ , the existence of non-trivial topological excitations on the boundary implies that the boundary carries a non-trivial topological order (which is anomalous). This agrees with the previous known result<sup>31,84</sup>.

A given bulk model can have many types of boundaries. For our  $\mathbb{Z}_n$ -1-cocycle model, the bulk contain  $n$  types of point-like topological excitations and  $n$  types of  $(d-2)$ -dimensional brane-like topological excitations. One type of the boundary is formed by the brane condensation. Such a boundary has  $n$  types of point-like topological excitations only. Another type of boundary is formed by the particle condensation. Such a boundary has  $n$  types of  $(d-2)$ -dimensional brane-like topological excitations only. We see that our boundary of  $\mathbb{Z}_n$ -1-cocycle model is the first type induced by the condensation of branes. We will call such boundary as “free boundary” since the 1-cocycle field has a free boundary condition on the boundary.

To realize the second type of the boundary, we need to use the fixed boundary condition by setting the 1-cocycle field to be  $a_{ij}^{\mathbb{Z}_n} = 0$  on the boundary. Again  $Z_{\mathbb{Z}_n \text{a}}^{\text{bndr}}(S_t^1 \times S^{k+1} \times S^{d-2-k}, S_t^1 \times S^{k+1} \times B^{d-1-k})$  can tell us the number of types of  $k$ -dimensional brane-like topological excitations on the boundary. To compute such

partition function, we notice that when  $k < d-2$ , the 1-cocycle  $a^{\mathbb{Z}_n}$  can be written as  $a^{\mathbb{Z}_n} = dg^{\mathbb{Z}_n}$  is a  $\mathbb{Z}_n$ -valued 0-cochain which vanishes on the boundary. The correspondence between  $a_{ij}^{\mathbb{Z}_n}$  and  $g_i^{\mathbb{Z}_n}$  is one-to-one. This is because even when  $k=0$ ,  $\oint_{S^{k+1}} a^{\mathbb{Z}_n} = 0$  since we have fixed  $a^{\mathbb{Z}_n} = 0$  on the boundary. Thus

$$\begin{aligned} & Z_{\mathbb{Z}_n \text{a}}^{\text{bndr}}(S_t^1 \times S^{k+1} \times S^{d-2-k}, S_t^1 \times S^{k+1} \times B^{d-1-k}) \\ &= n^{N_v^{\text{bik}}}. \end{aligned} \quad (38)$$

The volume-independent partition function is

$$\begin{aligned} & Z_{\mathbb{Z}_n \text{a}}^{\text{bndr, top}}(S_t^1 \times S^{k+1} \times S^{d-2-k}, S_t^1 \times S^{k+1} \times B^{d-1-k}) \\ &= 1, \quad \text{for } k < d-2. \end{aligned} \quad (39)$$

Thus, there is no non-trivial  $k$ -dimensional brane-like excitations on the boundary for  $k < d-2$ . There is no non-trivial point-like excitations on the boundary which is the  $k=0$  case included above. When  $k=d-2$ ,  $S^{d-2-k} = S^0$  is a set of two points. In this case, the boundary contains two disconnected pieces. We may set the 0-cochain field  $g^{\mathbb{Z}_n} = 0$  on one piece. But we need to set  $g^{\mathbb{Z}_n} = \text{const.}$  on the other piece. We find that

$$\begin{aligned} & Z_{\mathbb{Z}_n \text{a}}^{\text{bndr}}(S_t^1 \times S^{d-1} \times S^0, S_t^1 \times S^{d-1} \times B^1) \\ &= nn^{N_v^{\text{bik}}}. \end{aligned} \quad (40)$$

Or the volume-independent one

$$\begin{aligned} & Z_{\mathbb{Z}_n \text{a}}^{\text{bndr, top}}(S_t^1 \times S^{d-1} \times S^0, S_t^1 \times S^{d-1} \times B^1) \\ &= n. \end{aligned} \quad (41)$$

There is  $n$  types of  $(d-2)$ -dimensional brane-like excitations on the boundary. The  $a^{\mathbb{Z}_n} = 0$  boundary gives us the second type of boundary formed by condensing the point-like excitations.

### C. Twisted 2+1D $\mathbb{Z}_n$ -1-cocycle model and emergent Dijkgraaf-Witten theory

#### 1. Model construction

To construct another local bosonic model that realize a different topological order, we may choose  $L_{ijkl}$  to be

$$\begin{aligned} & L_{ijkl} = +\infty, \quad \text{if } (da^{\mathbb{Z}_n}) \neq 0, \\ & L_{ijkl} = -ik \frac{2\pi}{n} (a^{\mathbb{Z}_n} \mathcal{B}_n a^{\mathbb{Z}_n})(i, j, k, l), \quad \text{if } (da^{\mathbb{Z}_n}) = 0. \end{aligned} \quad (42)$$

Here we have use Bockstein homomorphism  $\mathcal{B}_n : H^m(M^d; \mathbb{Z}_n) \rightarrow H^{m+1}(M^d; \mathbb{Z}_n)$ ,

$$\begin{aligned} & \mathcal{B}_n x \stackrel{n}{=} \frac{1}{n} dx, \\ & x \in H^m(M^d; \mathbb{Z}_n), \quad \mathcal{B}_n x \in H^{m+1}(M^d; \mathbb{Z}_n). \end{aligned} \quad (43)$$



To understand the Bockstein homomorphism, we note that  $x$  in the above is a cocycle with  $\mathbb{Z}_n$ . If we view it as a cochain with integer coefficient  $\mathbb{Z}$ , then  $dx$  is an cochain whose values are always multiples of  $n$ . Thus  $\frac{1}{n}dx$  is a valid cochain with integer coefficient. In fact, it is  $(m+1)$ -cocycle with integer coefficient. After a mod  $n$  reduction,  $\frac{1}{n}dx \bmod n$  becomes a  $(m+1)$ -cocycle with  $\mathbb{Z}_n$  coefficient. This is why  $\mathcal{B}_n$  is a map from  $H^m(M^d; \mathbb{Z}_n)$  to  $H^{m+1}(M^d; \mathbb{Z}_n)$ . Therefore  $\mathcal{B}_n a^{\mathbb{Z}_n}$  is a 2-cocycle and  $a^{\mathbb{Z}_n} \mathcal{B}_n a^{\mathbb{Z}_n}$  is a 3-cocycle. Here, we use such a 3-cocycle to construct the action  $L_{ijkl}$

The total action amplitude  $e^{-S(\{a_{ij}^{\mathbb{Z}_n}\})}$  is given by

$$e^{-S(\{a_{ij}^{\mathbb{Z}_n}\})} = e^{ik \frac{2\pi}{n} \int_{M_{\text{Latt}}^3} a^{\mathbb{Z}_n} \mathcal{B}_n a^{\mathbb{Z}_n}} \quad (44)$$

for  $da^{\mathbb{Z}_n} = 0$ , and  $e^{-S(\{a_{ij}^{\mathbb{Z}_n}\})} = 0$  for  $da^{\mathbb{Z}_n} \neq 0$ . The partition function is given by

$$Z_{k;a\mathcal{B}a\mathbb{Z}_n}(M_{\text{Latt}}^3) = \sum_{\{a_{ij}^{\mathbb{Z}_n}\}, da^{\mathbb{Z}_n}=0} e^{ik \frac{2\pi}{n} \int_{M_{\text{Latt}}^3} a^{\mathbb{Z}_n} \mathcal{B}_n a^{\mathbb{Z}_n}}. \quad (45)$$

Such a partition function defines the twisted 2+1D  $\mathbb{Z}_n$ -1-cocycle model.

The volume-independent part of partition function eqn. (45) is given by

$$Z_{k;a\mathcal{B}a\mathbb{Z}_n}^{\text{top}}(M^3) = \frac{\sum_{a^{\mathbb{Z}_n} \in H^1(M^3; \mathbb{Z}_n)} e^{ik \frac{2\pi}{n} \int_{M^3} a^{\mathbb{Z}_n} \mathcal{B}_n a^{\mathbb{Z}_n}}}{|H^0(M^3; \mathbb{Z}_n)|} \quad (46)$$

Since the Euler number on odd-dimensional closed manifolds vanishes, the above volume-independent partition function is a topological invariant.

## 2. Topological partition functions

In this section, we are going to calculate some topological invariants. On  $M^3 = S^3$ ,  $S^1 \times S^2$ , or  $T^3 = S^1 \times S^1 \times S^1$ ,  $\mathcal{B}_n a^{\mathbb{Z}_n} = 0$  and the topological term  $k \frac{2\pi}{n} \int_{M^3} a^{\mathbb{Z}_n} \mathcal{B}_n a^{\mathbb{Z}_n}$  vanishes. We find

$$\begin{aligned} Z_{k;a\mathcal{B}a\mathbb{Z}_n}^{\text{top}}(S^3) &= \frac{1}{n}, \\ Z_{k;a\mathcal{B}a\mathbb{Z}_n}^{\text{top}}(S^1 \times S^2) &= 1, \\ Z_{k;a\mathcal{B}a\mathbb{Z}_n}^{\text{top}}(T^3) &= n^2. \end{aligned} \quad (47)$$

From  $Z_{1;a\mathcal{B}a\mathbb{Z}_n}^{\text{top}}(M^2 \times S^1)$ , we can determine the ground state degeneracy (GSD) on  $M^2$ :

$$Z_{1;a\mathcal{B}a\mathbb{Z}_n}^{\text{top}}(M^2 \times S^1) = \text{GSD}_{1;a\mathcal{B}a\mathbb{Z}_n}(M^2). \quad (48)$$

Using

$$\begin{aligned} Z_{1;a\mathcal{B}a\mathbb{Z}_n}^{\text{top}}(S^2 \times S^1) &= 1, \\ Z_{1;a\mathcal{B}a\mathbb{Z}_n}^{\text{top}}(T^2 \times S^1) &= n^2, \end{aligned} \quad (49)$$

we find that the GSD on a sphere  $S^2$  is 1 and the GSD on a torus  $T^2 = S^1 \times S^1$  is  $n^2$ .

To obtain the topological invariant that see the topological term, we put the system on the lens space  $L^3(p)$  (see Appendix F 4). We find from

$$\begin{aligned} H_1(L^3(p), \mathbb{Z}) &= \mathbb{Z}_p, \\ H_2(L^3(p), \mathbb{Z}) &= 0, \\ H_3(L^3(p), \mathbb{Z}) &= \mathbb{Z}. \end{aligned} \quad (50)$$

that (using eqn. (A8))

$$\begin{aligned} H^1(L^3(p), \mathbb{Z}_n) &= \mathbb{Z}_{\langle p, n \rangle} = \{a\}, \\ H^2(L^3(p), \mathbb{Z}_n) &= \mathbb{Z}_{\langle p, n \rangle} = \{b\}, \\ H^3(L^3(p), \mathbb{Z}_n) &= \mathbb{Z}_n = \{c\}, \end{aligned} \quad (51)$$

where we have also listed the generators  $\{a, b, c\}$ . Here,  $\langle l, m \rangle$  is the greatest common divisor of  $l$  and  $m$ , and  $\langle 0, m \rangle \equiv m$ . In Appendix F 4, we have computed the cohomology ring  $H^*(L^3(p), \mathbb{Z}_n)$  (see eqn. (F32)):

$$a^2 = \frac{n^2 p(p-1)}{2\langle p, n \rangle^2} b, \quad ab = \frac{n}{\langle p, n \rangle} c, \quad b^2 = ac = 0. \quad (52)$$

We have also computed the Bockstein homomorphism

$$\mathcal{B}_n a = \frac{p}{\langle p, n \rangle} b. \quad (53)$$

We can parametrize  $a^{\mathbb{Z}_n}$  as

$$a^{\mathbb{Z}_n} = \alpha a, \quad \alpha \in \mathbb{Z}_{\langle n, p \rangle}. \quad (54)$$

and find that

$$Z_{k;a\mathcal{B}a\mathbb{Z}_n}^{\text{top}}(L^3(p)) = \frac{1}{n} \sum_{\alpha=0}^{\langle n, p \rangle - 1} e^{i 2\pi \alpha^2 \frac{kp}{\langle n, p \rangle^2}}. \quad (55)$$

We find the above topological invariant is identical to the topological invariant of 2+1D  $\mathbb{Z}_n$  Dijkgraaf-Witten theory on lens space  $L^3(p)$  for any  $p$  (see eqn. (168)). In fact, one can show that the  $\mathbb{Z}_n$ -1-cocycle model realize the 2+1D  $\mathbb{Z}_n$  Dijkgraaf-Witten theory<sup>33</sup> (see discussions below eqn. (169)). In other words, the above topological invariant is the topological invariant of a  $\mathbb{Z}_n$ -gauge theory twisted by a quantized topological term<sup>85</sup>  $k \frac{2\pi}{n} \int_{M^3} a^{\mathbb{Z}_n} \mathcal{B}_n a^{\mathbb{Z}_n}$ . The quantized topological term correspond to a group-cocycle in  $\mathcal{H}^3(\mathbb{Z}_n, \mathbb{R}/\mathbb{Z}) = \mathbb{Z}_n$ . It is the simplest Dijkgraaf-Witten theory. Such a Dijkgraaf-Witten theory can be obtained by gauging the  $\mathbb{Z}_n$ -symmetry of a  $\mathbb{Z}_n$ -SPT state<sup>53</sup>. When  $k=0$ , our model realizes the  $\mathbb{Z}_n$  topological order described by UT  $\mathbb{Z}_n$  gauge theory. For  $(n, k) = (2, 1)$ , our model realizes the double-semion topological order<sup>35,36,53</sup>.

We like to remark that the twisted 2+1D  $\mathbb{Z}_n$ -1-cocycle model and Dijkgraaf-Witten theory are different. Dijkgraaf-Witten theory is a gauge theory where two  $a_{ij}^{\mathbb{Z}_n}$  configurations differ by a  $\mathbb{Z}_n$  gauge transformation are regard as the same configuration. In other words,

two  $a_{ij}^{\mathbb{Z}_n}$  configurations differ by a coboundary are regarded as the same configuration. Thus the Dijkgraaf-Witten  $\mathbb{Z}_n$ -gauge theory may be called twisted  $\mathbb{Z}_n$ -1-cohomology model. In our twisted 2+1D  $\mathbb{Z}_n$ -1-cocycle model, different  $a_{ij}^{\mathbb{Z}_n}$  configurations are always different with no gauge redundancy. So the cocycle model is not a gauge theory but a local bosonic system. However, the cocycle model has an emergent gauge theory at low energies which is described by Dijkgraaf-Witten theory.

#### D. Twisted 3+1D $\mathbb{Z}_n \oplus \mathbb{Z}_n$ -1-cocycle model and emergent Dijkgraaf-Witten theory

##### 1. Model construction

In this section, we like to design a 3+1D local bosonic model that realizes the Dijkgraaf-Witten twisted gauge theory at low energies. Since  $\mathcal{H}^4(Z_n, \mathbb{R}/\mathbb{Z}) = 0$ , there is no  $\mathbb{Z}_n$  Dijkgraaf-Witten gauge theory in 3+1D. So here we try to realize the  $\mathbb{Z}_n \times \mathbb{Z}_n$  Dijkgraaf-Witten gauge theory. Such theory exists since  $\mathcal{H}^4(Z_n \times \mathbb{Z}_n, \mathbb{R}/\mathbb{Z}) = \mathbb{Z}_n \oplus \mathbb{Z}_n$ .

To realize  $\mathbb{Z}_n \times \mathbb{Z}_n$  gauge theory, we construct a  $\mathbb{Z}_n \oplus \mathbb{Z}_n$ -1-cocycle theory on 3+1D space-time lattice. The local degrees of freedom of the model correspond to two 1-cochains  $a_1^{\mathbb{Z}_n}, a_2^{\mathbb{Z}_n} \in C^2(M_{\text{Latt}}^4; \mathbb{Z}_n)$  (*i.e.* the local degrees of freedom are described by  $\mathbb{Z}_n \oplus \mathbb{Z}_n$  on each 1-simplex). The partition function on an oriented space-time  $M_{\text{Latt}}^4$  is given by<sup>59,63</sup>

$$Z_{k_1 k_2; aa' \mathcal{B} a' \mathbb{Z}_n}(M_{\text{Latt}}^4) = \sum_{\{a_{I; ij}^{\mathbb{Z}_n}\}, da_I^{\mathbb{Z}_n}=0} e^{i \frac{2\pi}{n} \int_{M_{\text{Latt}}^4} k_1 a_1^{\mathbb{Z}_n} a_2^{\mathbb{Z}_n} \mathcal{B}_n a_2^{\mathbb{Z}_n} + k_2 a_2^{\mathbb{Z}_n} a_1^{\mathbb{Z}_n} \mathcal{B}_n a_1^{\mathbb{Z}_n}}, \quad (56)$$

where  $k_1, k_2 = 0, 1, \dots, n-1$ . We have assumed that the configuration with  $da_I^{\mathbb{Z}_n} \neq 0$ ,  $I = 1, 2$ , have infinite energy and do not contribute to the partition function. The term  $\frac{1}{n} \int_{M_{\text{Latt}}^4} k_1 a_1^{\mathbb{Z}_n} a_2^{\mathbb{Z}_n} \mathcal{B}_n a_2^{\mathbb{Z}_n} + k_2 a_2^{\mathbb{Z}_n} a_1^{\mathbb{Z}_n} \mathcal{B}_n a_1^{\mathbb{Z}_n}$  corresponds to a cocycle  $(k_1, k_2) \in \mathcal{H}^4(Z_n \times \mathbb{Z}_n, \mathbb{R}/\mathbb{Z}) = \mathbb{Z}_n \oplus \mathbb{Z}_n$ .

There are other possible choices of the action amplitude, such as

$$e^{i k \frac{2\pi}{n} \int_{M_{\text{Latt}}^4} \mathcal{B}_n a_1^{\mathbb{Z}_n} \mathcal{B}_n a_2^{\mathbb{Z}_n}} \quad (57)$$

But

$$\int_{M_{\text{Latt}}^4} \mathcal{B}_n a_1^{\mathbb{Z}_n} \mathcal{B}_n a_2^{\mathbb{Z}_n} = \int_{M_{\text{Latt}}^4} \frac{1}{n} da_1^{\mathbb{Z}_n} \frac{1}{n} da_2^{\mathbb{Z}_n} = 0, \quad (58)$$

if  $M_{\text{Latt}}^4$  is orientable. So such a term always vanishes. Yet another possible choice is  $\int_{M_{\text{Latt}}^4} a_1^{\mathbb{Z}_n} (a_2^{\mathbb{Z}_n})^3$ . But when  $n = 2$ , it is the same as  $\int_{M_{\text{Latt}}^4} a_1^{\mathbb{Z}_n} a_2^{\mathbb{Z}_n} \mathcal{B}_n a_2^{\mathbb{Z}_n}$ , and when  $n$  is odd, it vanishes. So here we do not discuss it further.

##### 2. Topological partition functions

When  $k_1, k_2 = 0$ , the partition function is given by the square of the number of 1-cocycles,  $|Z^1(M_{\text{Latt}}^4; \mathbb{Z}_n)|^2$ .  $|Z^1(M_{\text{Latt}}^4; \mathbb{Z}_n)|$  is  $|H^1(M_{\text{Latt}}^4; \mathbb{Z}_n)|$  times the number of 0-cochains whose derivatives is non-zero. The number of 0-cochains whose derivatives is non-zero is the number of 0-cochains ( $|C^0(M_{\text{Latt}}^4; \mathbb{Z}_n)| = n^{N_v}$ ) divide by  $|H^0(M_{\text{Latt}}^4; \mathbb{Z}_n)|$ . Thus the partition function is

$$\begin{aligned} Z_{0,0; aa' \mathcal{B} a' \mathbb{Z}_n}(M_{\text{Latt}}^4) &= |Z^1(M_{\text{Latt}}^4; \mathbb{Z}_n)|^2 \\ &= |H^1(M_{\text{Latt}}^4; \mathbb{Z}_n)|^2 \frac{|C^0(M_{\text{Latt}}^4; \mathbb{Z}_n)|^2}{|H^0(M_{\text{Latt}}^4; \mathbb{Z}_n)|^2} \\ &= n^{N_v} \frac{|H^1(M_{\text{Latt}}^4; \mathbb{Z}_n)|^2}{|H^0(M_{\text{Latt}}^4; \mathbb{Z}_n)|^2}. \end{aligned} \quad (59)$$

The volume-independent topological partition function is given by

$$Z_{0,0; aa' \mathcal{B} a' \mathbb{Z}_n}(M_{\text{Latt}}^4) = \frac{|H^1(M_{\text{Latt}}^4; \mathbb{Z}_n)|^2}{|H^0(M_{\text{Latt}}^4; \mathbb{Z}_n)|^2} \quad (60)$$

When  $k_1, k_2 \neq 0$ , the volume-independent topological partition function is given by

$$\begin{aligned} Z_{k_1 k_2; aa' \mathcal{B} a' \mathbb{Z}_n}(M_{\text{Latt}}^4) &= \sum_{a_I^{\mathbb{Z}_n} \in H^1(M^4; \mathbb{Z}_n)} \frac{e^{i \frac{2\pi}{n} \int_{M_{\text{Latt}}^4} k_1 a_1^{\mathbb{Z}_n} a_2^{\mathbb{Z}_n} \mathcal{B}_n a_2^{\mathbb{Z}_n} + k_2 a_2^{\mathbb{Z}_n} a_1^{\mathbb{Z}_n} \mathcal{B}_n a_1^{\mathbb{Z}_n}}}{|H^0(M^4; \mathbb{Z}_n)|^2}, \end{aligned} \quad (61)$$

where  $|H^1(M_{\text{Latt}}^4; \mathbb{Z}_n)|^2$  is replaced by the summation of phase factors.

Now, let us compute  $Z_{k_1 k_2; aa' \mathcal{B} a' \mathbb{Z}_n}(M^4)$  on several  $M^4$ . On  $M^4 = S^1 \times S^1 \times S^1 \times S^1 = T^4$  or  $M^4 = S^2 \times S^1 \times S^1 = S^2 \times T^2$ ,  $\mathcal{B}_n a_I^{\mathbb{Z}_n} = 0$ . Thus  $Z_{k_1 k_2; aa' \mathcal{B} a' \mathbb{Z}_n}(M^4) = Z_{0,0; aa' \mathcal{B} a' \mathbb{Z}_n}(M^4)$  on those manifolds. Using

$$H^1(T^4; \mathbb{Z}_n) = 4\mathbb{Z}_n, \quad H^1(T^2 \times S^2; \mathbb{Z}_n) = \mathbb{Z}_n^{\oplus 2}, \quad (62)$$

we find that (see Table III)

$$\begin{aligned} Z_{k_1 k_2; aa' \mathcal{B} a' \mathbb{Z}_n}(T^4) &= n^6, \\ Z_{k_1 k_2; aa' \mathcal{B} a' \mathbb{Z}_n}(S^2 \times T^2) &= n^2. \end{aligned} \quad (63)$$

On  $M^4 = S^1 \times L^3(p)$ , from

$$H_1(L^3(p), \mathbb{Z}) = \mathbb{Z}_p, \quad H_2(L^3(p), \mathbb{Z}) = 0, \quad H_3(L^3(p), \mathbb{Z}) = \mathbb{Z}. \quad (64)$$

we find that (using eqn. (A4))

$$\begin{aligned} H_1(S^1 \times L^3(p), \mathbb{Z}) &= \mathbb{Z} \oplus \mathbb{Z}_p, \\ H_2(S^1 \times L^3(p), \mathbb{Z}) &= \mathbb{Z}_p, \\ H_3(S^1 \times L^3(p), \mathbb{Z}) &= \mathbb{Z}, \\ H_4(S^1 \times L^3(p), \mathbb{Z}) &= \mathbb{Z}. \end{aligned} \quad (65)$$

This allows us to obtain (using eqn. (A8))

$$\begin{aligned} H^1(S^1 \times L^3(p), \mathbb{Z}_n) &= \mathbb{Z}_n \oplus \mathbb{Z}_{\langle p, n \rangle} = \{a_1, a\}, \\ H^2(S^1 \times L^3(p), \mathbb{Z}_n) &= \mathbb{Z}_{\langle p, n \rangle} \oplus \mathbb{Z}_{\langle p, n \rangle} = \{a_1 a, b\}, \\ H^3(S^1 \times L^3(p), \mathbb{Z}_n) &= \mathbb{Z}_n \oplus \mathbb{Z}_{\langle p, n \rangle} = \{c, a_1 b\}, \\ H^4(S^1 \times L^3(p), \mathbb{Z}_n) &= \mathbb{Z}_n = \{a_1 c\}. \end{aligned} \quad (66)$$

where we have also listed the generators, where  $a_1$  comes from  $S^1$  and  $a, b, c$  from  $L^3(p)$ . Here,  $\langle l, m \rangle$  is the greatest common divisor of  $l$  and  $m$ , and  $\langle 0, m \rangle \equiv m$ .

In Appendix F 4, we have computed the cohomology ring  $H^*(S^1 \times L^3(p), \mathbb{Z}_n)$  (see eqn. (F32)):

$$a_1^2 = 0, \quad a^2 = \frac{n^2 p(p-1)}{2\langle p, n \rangle^2} b, \quad ab = \frac{n}{\langle p, n \rangle} c, \quad b^2 = ac = 0. \quad (67)$$

We have also computed the Bockstein homomorphism

$$\mathcal{B}_n a = \frac{p}{\langle p, n \rangle} b, \quad \mathcal{B}_n a_1 = 0. \quad (68)$$

We see that for  $\langle n, p \rangle = 1$ ,  $a = b = 0$ , and thus  $\mathcal{B}_n a_I^{\mathbb{Z}_n} = 0$ . Therefore  $\int_{M_{\text{Latt}}^4} k_1 a_1^{\mathbb{Z}_n} a_2^{\mathbb{Z}_n} \mathcal{B}_n a_2^{\mathbb{Z}_n} + k_2 a_2^{\mathbb{Z}_n} a_1^{\mathbb{Z}_n} \mathcal{B}_n a_1^{\mathbb{Z}_n} = 0$ . So  $Z_{k_1 k_2; aa' \mathcal{B} a' \mathbb{Z}_n}(S^1 \times L^3(p)) = 1$ .

For  $\langle n, p \rangle \neq 1$ , we can parametrize  $a_I^{\mathbb{Z}_n}$  as

$$a_I^{\mathbb{Z}_n} = \alpha_I a_1 + \tilde{\alpha}_I a, \quad \alpha_I \in \mathbb{Z}_n, \quad \tilde{\alpha}_I \in \mathbb{Z}_{\langle n, p \rangle}. \quad (69)$$

Using eqn. (66), eqn. (67), and eqn. (68), we find that

$$\begin{aligned} Z_{k_1 k_2; aa' \mathcal{B} a' \mathbb{Z}_n}(S^1 \times L^3(p)) &= \frac{1}{n^2} \sum_{\alpha_{1,2} \in \mathbb{Z}_n, \tilde{\alpha}_{1,2} \in \mathbb{Z}_{\langle n, p \rangle}} e^{i \frac{2\pi p}{\langle p, n \rangle^2} [k_1(\alpha_1 \tilde{\alpha}_2^2 - \tilde{\alpha}_1 \alpha_2 \tilde{\alpha}_2) + k_2(\alpha_2 \tilde{\alpha}_1^2 - \tilde{\alpha}_2 \alpha_1 \tilde{\alpha}_1)]} \\ &= s^2 \sum_{\tilde{\alpha}_{1,2}=0}^{m-1} \delta_m(k_1 \tilde{\alpha}_2^2 - k_2 \tilde{\alpha}_1 \tilde{\alpha}_2) \times \\ &\quad \delta_m(k_2 \tilde{\alpha}_1^2 - k_1 \tilde{\alpha}_1 \tilde{\alpha}_2), \\ s &= \langle \frac{p}{\langle n, p \rangle}, \langle n, p \rangle \rangle, \quad m = \langle n, p \rangle / s. \end{aligned} \quad (70)$$

When  $p$  has no repeated prime factor, the above sum has a simple expression:

$$\begin{aligned} Z_{k_1 k_2; aa' \mathcal{B} a' \mathbb{Z}_n}(S^1 \times L^3(p)) &= \langle n, p \rangle \langle n, p, k_1, k_2 \rangle \\ &= s^2 m \langle m, k_1, k_2 \rangle \end{aligned} \quad (71)$$

On  $M^4 = F^4 \equiv (S^1 \times S^3) \# (S^1 \times S^3) \# \mathbb{C}P^2 \# \overline{\mathbb{C}P}^2$ , we note that the cup product of 1-cocycles are always zero (see Appendix F 5). Thus  $Z_{k_1 k_2; aa' \mathcal{B} a' \mathbb{Z}_n}(F^4) = Z_{0,0; aa' \mathcal{B} a' \mathbb{Z}_n}(F^4) = n^2$ .

### 3. Dimension reduction

Last, let us consider  $M^4 = M^3 \times S^1$ , where  $M^4$  and  $M^3$  are assumed to be closed manifolds. We write  $a_I^{\mathbb{Z}_n}$

as

$$a_I^{\mathbb{Z}_n} = a_{I, M^3}^{\mathbb{Z}_n} + a_{I, S^1}^{\mathbb{Z}_n}, \quad (72)$$

where  $a_{I, M^3}^{\mathbb{Z}_n}$  lives on  $M^3$  and  $a_{I, S^1}^{\mathbb{Z}_n}$  on  $S^1$ . We also fix  $\oint_{S^1} a_{I, S^1}^{\mathbb{Z}_n} = \alpha_I \in \mathbb{Z}$ . The partition function now has a form

$$\begin{aligned} Z_{k_1 k_2; aa' \mathcal{B} a' \mathbb{Z}_n}(M^3 \times S^1, \alpha_1, \alpha_2) &= \frac{1}{|H^0(M^3; \mathbb{Z}_n)|^2} \times \\ &\sum_{a_{I, M^3}^{\mathbb{Z}_n} \in H^1(M^3; \mathbb{Z}_n)} e^{i \frac{2\pi}{n} \int_{M^3} (k_1 \alpha_2 - k_2 \alpha_1) a_{1, M^3}^{\mathbb{Z}_n} \mathcal{B}_n a_{2, M^3}^{\mathbb{Z}_n} \times} \\ &\quad e^{i \frac{2\pi}{n} \int_{M^3} k_1 \alpha_1 a_{2, M^3}^{\mathbb{Z}_n} \mathcal{B}_n a_{1, M^3}^{\mathbb{Z}_n} + k_2 \alpha_2 a_{1, M^3}^{\mathbb{Z}_n} \mathcal{B}_n a_{2, M^3}^{\mathbb{Z}_n}}. \end{aligned} \quad (73)$$

In fact,  $\alpha_I$  in the above happen to label the different sectors. We find the topological theory in each sector from the partition function  $Z_{k_1 k_2; aa' \mathcal{B} a' \mathbb{Z}_n}(M^3 \times S^1, \alpha_1, \alpha_2)$ . As we can see that they are 2+1D Dijkgraaf-Witten theories.

Since the Dijkgraaf-Witten theories can be viewed as gauged SPT states<sup>53</sup>, the dimension reduction of the Dijkgraaf-Witten theories implies a similar the dimension reduction of SPT states: If we compact a 3+1D  $Z_n^{(1)} \times Z_n^{(2)}$ -SPT state to 2+1D via a circle  $S^1$ , and add a symmetry twist around  $S^1$  described by  $e^{i2\pi\alpha_1/n}$  for the  $Z_n^{(1)}$  and  $e^{i2\pi\alpha_2/n}$  for the  $Z_n^{(2)}$ , then the resulting 2+1D SPT state is a stacking of a  $Z_n^{(1)}$ -SPT state labeled by  $k_2 \alpha_2 \in \mathcal{H}^3(Z_n^{(1)}, \mathbb{R}/\mathbb{Z}) = \mathbb{Z}_n$ , a  $Z_n^{(2)}$ -SPT state labeled by  $k_1 \alpha_1 \in \mathcal{H}^3(Z_n^{(2)}, \mathbb{R}/\mathbb{Z}) = \mathbb{Z}_n$ , and a  $Z_n^{(1)} \times Z_n^{(2)}$ -SPT state labeled by  $k_1 \alpha_2 - k_2 \alpha_1 \in \mathcal{H}^3(Z_n^{(1)} \times Z_n^{(2)}, \mathbb{R}/\mathbb{Z})$ <sup>51,59</sup>.

This implies that the symmetry-twist defect line (twisted by  $e^{i2\pi\alpha_1/n}$  for the  $Z_n^{(1)}$  and  $e^{i2\pi\alpha_2/n}$  for the  $Z_n^{(2)}$ ) in the 3+1D  $Z_n^{(1)} \times Z_n^{(2)}$ -SPT state (labeled by  $(k_1, k_2) \in \mathcal{H}^4(Z_n^{(1)} \times Z_n^{(2)}, \mathbb{R}/\mathbb{Z}) = \mathbb{Z}_n \oplus \mathbb{Z}_n$ ), will carry gapless 1+1D excitations along the symmetry-twist defect line described by the boundary of the  $k_2 \alpha_2$ th  $Z_n^{(1)}$ -SPT state, the  $k_1 \alpha_1$ th  $Z_n^{(2)}$ -SPT state, and the  $(k_1 \alpha_2 - k_2 \alpha_1)$ th  $Z_n^{(1)} \times Z_n^{(2)}$ -SPT state, provided that the  $Z_n^{(1)} \times Z_n^{(2)}$  symmetry is not broken. This result generalized the one in Ref. 55.

## E. Twisted 3+1D $\mathbb{Z}_n$ -2-cocycle model and emergence of fermions

### 1. Model construction

In this section, we will study  $\mathbb{Z}_n$ -2-cocycle theory on 3+1D space-time lattice. The local degrees of freedom of the model correspond to 2-cochains  $b^{\mathbb{Z}_n} \in C^2(M_{\text{Latt}}^4; \mathbb{Z}_n)$  (i.e. the local degrees of freedom are described by  $\mathbb{Z}_n$  on each 2-simplex). The partition function is given by, for

$k = 0, 1, \dots, n-1$ <sup>60</sup>,

$$Z_{k;b^2\mathbb{Z}_n}(M_{\text{Latt}}^4) = \sum_{\{b_{ij}^{\mathbb{Z}_n}\}, \text{db}^{\mathbb{Z}_n}=0} e^{ik\frac{2\pi}{n} \int_{M_{\text{Latt}}^4} (b^{\mathbb{Z}_n})^2}, \quad (74)$$

(i.e. the configuration with  $\text{db}^{\mathbb{Z}_n} \neq 0$  have infinite energy.) Note that the source (or “charge”) of the 2-cocycle field  $b$  is a  $\mathbb{Z}_n$  string. When  $k = 0$ , it describes a  $\mathbb{Z}_n$ -2-cocycle theory. When  $k \neq 0$ , it describes a twisted  $\mathbb{Z}_n$ -2-cocycle theory.

When  $k = 0$ , the partition function is given by the number of 2-cocycles  $|Z^2(M_{\text{Latt}}^4; \mathbb{Z}_n)|$ , which is  $|H^2(M_{\text{Latt}}^4; \mathbb{Z}_n)|$  times the number of 1-cochains whose derivatives is non-zero. The number of 1-cochains whose derivatives is non-zero is the number of 1-cochains ( $|C^1(M_{\text{Latt}}^4; \mathbb{Z}_n)| = n^{N_e}$ ) divide by  $|H^1(M_{\text{Latt}}^4; \mathbb{Z}_n)|$  and by the number of number of 0-cochains whose derivatives is non-zero. The number of 0-cochains whose derivatives is non-zero is the number of 0-cochains ( $|C^0(M_{\text{Latt}}^4; \mathbb{Z}_n)| = n^{N_v}$ ) divide by  $|H^0(M_{\text{Latt}}^4; \mathbb{Z}_n)|$ . Thus the partition function is

$$\begin{aligned} Z_{0;b^2\mathbb{Z}_n}(M_{\text{Latt}}^4) &= |Z^2(M_{\text{Latt}}^4; \mathbb{Z}_n)| \\ &= |H^2(M_{\text{Latt}}^4; \mathbb{Z}_n)| \frac{|C^1(M_{\text{Latt}}^4; \mathbb{Z}_n)|}{|H^1(M_{\text{Latt}}^4; \mathbb{Z}_n)|} \frac{|H^0(M_{\text{Latt}}^4; \mathbb{Z}_n)|}{|C^0(M_{\text{Latt}}^4; \mathbb{Z}_n)|} \\ &= n^{N_e - N_v} \frac{|H^2(M_{\text{Latt}}^4; \mathbb{Z}_n)| |H^0(M_{\text{Latt}}^4; \mathbb{Z}_n)|}{|H^1(M_{\text{Latt}}^4; \mathbb{Z}_n)|}. \end{aligned} \quad (75)$$

The volume-independent topological partition function is given by

$$Z_{0;b^2\mathbb{Z}_n}^{\text{top}}(M^4) = \frac{|H^2(M^4; \mathbb{Z}_n)| |H^0(M^4; \mathbb{Z}_n)|}{|H^1(M^4; \mathbb{Z}_n)|} \quad (76)$$

When  $k \neq 0$ , The volume-independent topological partition function is given by

$$\begin{aligned} Z_{k;b^2\mathbb{Z}_n}^{\text{top}}(M^4) &= \frac{|H^0(M^4; \mathbb{Z}_n)|}{|H^1(M^4; \mathbb{Z}_n)|} \sum_{b^{\mathbb{Z}_n} \in H^2(M^4; \mathbb{Z}_n)} e^{ik\frac{2\pi}{n} \int_{M^4} (b^{\mathbb{Z}_n})^2} \end{aligned} \quad (77)$$

where  $\sum_{b^{\mathbb{Z}_n} \in H^2(M^4; \mathbb{Z}_n)} e^{ik\frac{2\pi}{n} \int_{M^4} (b^{\mathbb{Z}_n})^2}$  replaces  $|H^2(M^4; \mathbb{Z}_n)|$ .

## 2. Topological partition functions

Now, let us compute topological invariants (see Table III). On  $M^4 = T^4$ , the cohomology ring  $H^*(T^4; \mathbb{Z}_n)$  is generated by  $a_I$ ,  $I = 1, 2, 3, 4$ , where  $a_I \in H^1(T^4; \mathbb{Z}_n) = 4\mathbb{Z}_n$ . Using the cohomology ring eqn. (F5) in Appendix F, we can parametrize  $b^{\mathbb{Z}_n}$  as

$$b^{\mathbb{Z}_n} = \beta_{IJ} a_I a_J, \quad \beta_{IJ} = -\beta_{JI} \in \mathbb{Z}_n. \quad (78)$$

Thus

$$\begin{aligned} Z_{k;b^2\mathbb{Z}_n}^{\text{top}}(T^4) &= \frac{1}{n^3} \sum_{\beta_{IJ} \in \mathbb{Z}_n} e^{ik\frac{2\pi}{n} (\beta_{12}\beta_{34} - \beta_{13}\beta_{24} + \beta_{14}\beta_{23})} \end{aligned} \quad (79)$$

Using  $\sum_{\beta_1, \beta_2 \in \mathbb{Z}_n} e^{ik\frac{2\pi}{n} 2\beta_1\beta_2} = \langle 2k, n \rangle n$ , we find that

$$Z_{k;b^2\mathbb{Z}_n}^{\text{top}}(T^4) = \langle 2k, n \rangle^3. \quad (80)$$

On  $M^4 = S^2 \times T^2$ , the cohomology ring  $H^*(T^2 \times S^2; \mathbb{Z}_n)$  is generated by  $a_I$ ,  $I = 1, 2$  and  $b$ , where  $a_I \in H^1(T^2 \times S^2; \mathbb{Z}_n) = \mathbb{Z}_n^{\oplus 2}$  and  $b \in H^2(T^2 \times S^2; \mathbb{Z}_n) = \mathbb{Z}_n^{\oplus 2}$ . Using the cohomology ring eqn. (F7) in Appendix F, we can parametrize  $b^{\mathbb{Z}_n}$  as

$$b^{\mathbb{Z}_n} = \beta_1 a_1 a_2 + \beta_2 b, \quad \beta_1, \beta_2 \in \mathbb{Z}_n. \quad (81)$$

Thus

$$\begin{aligned} Z_{k;b^2\mathbb{Z}_n}^{\text{top}}(S^2 \times T^2) &= \frac{1}{n} \sum_{\beta_1, \beta_2 \in \mathbb{Z}_n} e^{ik\frac{2\pi}{n} 2\beta_1\beta_2} = \langle 2k, n \rangle. \end{aligned} \quad (82)$$

On  $M^4 = S^1 \times L^3(p)$ , we need to use the cohomology ring  $H^*(S^1 \times L^3(p); \mathbb{Z}_n)$  as described in eqn. (66), eqn. (67), and eqn. (68). For  $\langle n, p \rangle = 1$ ,  $Z_{k;b^2\mathbb{Z}_n}^{\text{top}}(S^1 \times L^3(p)) = 1$ . For  $\langle n, p \rangle \neq 1$ , we can parametrize  $b^{\mathbb{Z}_n}$  as

$$b^{\mathbb{Z}_n} = \beta_1 a a_1 + \beta_2 b, \quad \beta_1, \beta_2 \in \mathbb{Z}_{\langle n, p \rangle}. \quad (83)$$

Using  $a a_1 b = \frac{n}{\langle n, p \rangle} a_1 c$  and  $(a a_1)^2 = b^2 = 0$ , we find that

$$\begin{aligned} Z_{k;b^2\mathbb{Z}_n}^{\text{top}}(S^1 \times L^3(p)) &= \frac{1}{\langle n, p \rangle} \sum_{\beta_1, \beta_2=0}^{\langle n, p \rangle-1} e^{i2k\frac{2\pi}{\langle n, p \rangle} \beta_1\beta_2} = \langle 2k, n, p \rangle. \end{aligned} \quad (84)$$

On  $M^4 = F^4$ , we need to use the cohomology ring  $H^*(F^4; \mathbb{Z}_n)$  as described in Appendix F5. We can parametrize  $b^{\mathbb{Z}_n}$  as

$$b^{\mathbb{Z}_n} = \beta_1 b_1 + \beta_2 b_2, \quad \beta_1, \beta_2 \in \mathbb{Z}_n, \quad (85)$$

where  $b_1, b_2$  are generators of  $H^2(F^4; \mathbb{Z}_n)$ . Using  $b_1^2 = -b_2^2 = v$  and  $b_1 b_2 = 0$ , we find that

$$\begin{aligned} Z_{k;b^2\mathbb{Z}_n}^{\text{top}}(F^4) &= \frac{1}{n} \sum_{\beta_1, \beta_2=0}^{n-1} e^{ik\frac{2\pi}{n} (\beta_1^2 - \beta_2^2)} \\ &= \begin{cases} \langle 2k, n \rangle, & \text{if } \frac{2kn}{(2k, n)^2} = \text{even}; \\ 0, & \text{if } \frac{2kn}{(2k, n)^2} = \text{odd}. \end{cases} \end{aligned} \quad (86)$$

The above results are summarized in Table III.

### 3. Point-like and string-like topological excitations

When  $k \neq 0$ , the twisted 3+1D  $\mathbb{Z}_n$ -2-cocycle theory realizes a topological order that is not described by  $\mathbb{Z}_n$ -gauge theory, nor by the group-cocycle-twisted Dijkgraaf-Witten theory, since the group-cohomology  $\mathcal{H}^4(\mathbb{Z}_n, \mathbb{R}/\mathbb{Z}) = 0$ . Here, we will show that

the 3+1D twisted  $\mathbb{Z}_n$ -2-cocycle theory realizes a 3+1D  $Z_{\langle 2k, n \rangle}$ -gauge theory. The  $Z_{\langle 2k, n \rangle}$ -gauge theory is a EF  $Z_{\langle 2k, n \rangle}$ -gauge theory if  $2kn/\langle 2k, n \rangle^2 = \text{odd}$ , and it is a UT  $Z_{\langle 2k, n \rangle}$ -gauge theory if  $2kn/\langle 2k, n \rangle^2 = \text{even}$ .

The reduction from  $\mathbb{Z}_n$  to  $Z_{\langle 2k, n \rangle}$  by the twist can be seen from the GSD of the model. The GSD on  $S^1 \times S^2$  count the number of types of point-like topological excitations, and the number of types of string-like topological excitations. From eqn. (82), we see that twisted  $\mathbb{Z}_n$ -2-cocycle model gives rise to a topological order with  $\langle 2k, n \rangle$  types of point-like topological excitations and  $\langle 2k, n \rangle$  types of string-like topological excitations.

It is interesting to see that the twisted model describes an invertible topological order when  $\langle 2k, n \rangle = 1$ . Since all 3+1D invertible topological orders are trivial topological orders, thus

the twisted  $\mathbb{Z}_n$ -2-cocycle model describes an trivial product state when  $\langle 2k, n \rangle = 1$ .

Naively, the twisted  $\mathbb{Z}_n$ -2-cocycle model should have  $n$  types of point-like topological excitations and  $n$  types of string-like topological excitations. But actually, there are only  $\langle 2k, n \rangle$  types of point-like topological excitations and  $\langle 2k, n \rangle$  types of string-like topological excitations. Other excitations are confined.

To understand the unconfined topological excitations in level- $k$   $\mathbb{Z}_n$ -2-cocycle model, we note that we can view  $b^{\mathbb{Z}_n}$  as the field strength 2-form of a  $U(1)$  gauge theory

$$2\pi b^{\mathbb{Z}_n} = f, \quad (87)$$

where the  $2\pi$  factor comes from the different quantization convention  $\int_{M_{\text{closed}}^2} b^{\mathbb{Z}_n} = \text{integer}$  and  $\int_{M_{\text{closed}}^2} f = 2\pi \times \text{integer}$ . In this case, the point-like topological excitations correspond to the monopoles in the  $U(1)$  gauge theory. Such a  $U(1)$  gauge theory is described by the partition function

$$Z_{k, U(1)}(M^4) = \int D[a] e^{i \frac{\Theta}{8\pi^2} \int_{M^4} f f + \dots}, \quad (88)$$

where  $\Theta = \frac{4\pi k}{n}$ , and  $\dots$  represents additional interactions. Without the additional interactions, the particle like excitation in the  $U(1)$  gauge theory are labeled by two integers  $(q, m)$  where  $m = M$  is the magnetic charge. The  $U(1)$  charge of  $(q, m)$  is given by  $Q_{q, m} = q + \frac{\Theta}{2\pi} m$ . The statistics of particle  $(q, m)$  is

determined by  $e^{i\theta} = (-)^{mq}$ , where  $e^{i\theta} = 1$  correspond to boson and  $e^{i\theta} = -1$  correspond to fermion. Let us express the statistics in terms of physical quantities  $(Q, M)$ :  $e^{i\theta} = (-)^{MQ - \frac{\Theta}{2\pi} M^2}$ . We see that when  $\Theta = 0$  or  $\Theta = 2\pi$ , both  $Q$  and  $M$  are integers, but the statistics of particles with charge  $(Q, M)$  are different for  $\Theta = 0$  and  $\Theta = 2\pi$ . Thus changing  $\Theta$  by  $2\pi$  will lead to a different  $U(1)$  gauge theory. Changing  $\Theta$  by  $4\pi$  will give us the same  $U(1)$  gauge theory. This is consistent with the mod  $n$  periodicity of  $k$ .

For  $\Theta = \frac{4\pi k}{n}$ , we note that the  $(q, m) = (-2k, n)$  particle has a vanishing  $U(1)$  charge and is a boson. We can use the additional interactions to condense such a dyon<sup>86</sup>. Such a condensation will make the  $U(1)$  gauge theory to be our  $\mathbb{Z}_n$ -2-cocycle theory. This is because a change of  $\int_{M_{\text{closed}}^2} b^{\mathbb{Z}_n}$  by  $n$  is a trivial change, which means a change of  $\int_{M_{\text{closed}}^2} f$  by  $2\pi n$  should also be a trivial change in the  $U(1)$  gauge theory. This is achieved by condensing  $n$  unit of magnetic charge that is carried by  $(q, m) = (-2k, n)$  particle.

Since the condensing particles have a non-zero magnetic charge, in the condensed phase, all the particles with non-zero  $U(1)$  charge,  $Q_{q, m} \neq 0$ , are confined. Thus the unconfined point-like topological excitations are given by  $(q, m) = l(\frac{-2k}{\langle 2k, n \rangle}, \frac{n}{\langle 2k, n \rangle})$ , with  $l = 0, 1, \dots, \langle 2k, n \rangle - 1$ . We see that the GSD on  $S^1 \times S^2$  corresponds to the number of types of point-like topological excitations. We also note that when  $2kn/\langle 2k, n \rangle^2 = \text{odd}$  (such as  $k = 1, n = 2$ ), some point-like topological excitations are fermions. When  $2kn/\langle 2k, n \rangle^2 = \text{even}$ , all point-like topological excitations are bosons.

In the condensed phase, the electric flux-lines are quantized as  $\int_{M_{\text{closed}}^2} d\mathbf{S} \cdot \mathbf{E} = \frac{1}{n} \times m$ ,  $m \in \mathbb{Z}$ . They are the string-like topological excitations. Moving a point-like excitation labeled by  $l$  around a string-like excitation labeled by  $m$  give rise a phase  $e^{i \frac{lm}{\langle 2k, n \rangle}}$ . So the string labeled by  $m$  and  $m + \langle 2k, n \rangle$  are indistinguishable. This suggests that we have  $\langle 2k, n \rangle$  type of string-like excitations.

We find that the point-like and string-like topological excitations in the level- $k$   $\mathbb{Z}_n$ -2-cocycle model are very similar to those in  $Z_{\langle 2k, n \rangle}$ -gauge theory, except that the odd  $Z_{\langle 2k, n \rangle}$  charges are fermions when  $2kn/\langle 2k, n \rangle^2 = \text{odd}$ . The emergence of fermions is supported by the vanishing of volume-independent partition function on a non-spin manifold  $F^4 = (S^1 \times S^3) \# (S^1 \times S^3) \# \mathbb{C}P^2 \# \overline{\mathbb{C}P}^2$  (see eqn. (86)), which happens exactly at  $2kn/\langle 2k, n \rangle^2 = \text{odd}$ .

It was first pointed out in the string-net theory<sup>36</sup> that a 3+1D gauge theory can be twisted which makes some gauge charge described by “odd” representations to be fermionic. But when we use cocycles in  $\mathcal{H}^4(G, \mathbb{R}/\mathbb{Z})$  to twist a  $G$ -gauge theory<sup>33</sup>, the point-like topological excitations are always boson<sup>87</sup>. Thus the level- $k$   $\mathbb{Z}_n$ -2-cocycle model is a different realization of the twist discussed in the 3+1D string-net theory.



#### 4. Including excitations in the path integral

We know that the point-like excitations are described by the world-lines  $M_{\text{WL}}^1$  in space-time. A world-line  $M_{\text{WL}}^1$  can be viewed as a  $\mathbb{Z}_n$ -valued 1-cycle, which is dual to a 3-coboundary  $C_{\text{WL}}^{\mathbb{Z}_n}$ . In the twisted  $\mathbb{Z}_n$ -2-cocycle model, such a point-like excitation is described by the 2-cochain field  $b^{\mathbb{Z}_n}$  that satisfies  $db^{\mathbb{Z}_n} = \tilde{p}C_{\text{WL}}^{\mathbb{Z}_n}$ , where  $\tilde{p}$  is the charge of the point-like excitation. The world-sheet can be viewed as  $\mathbb{Z}_n$ -valued 2-cycles  $M_{\text{WS}}^2$  in the space-time lattice. Therefore, in the presence of point-like topological excitations described by  $C_{\text{WL}}^{\mathbb{Z}_n}$  and string-like topological excitations described by  $M_{\text{WS}}^2$ , the partition function becomes

$$Z_{k;b^2\mathbb{Z}_n}(M^4; \tilde{p}C_{\text{WL}}^{\mathbb{Z}_n}, sM_{\text{WS}}^2) \quad (89)$$

$$= \sum_{\{b_{ijk}^{\mathbb{Z}_n}\}, db^{\mathbb{Z}_n} = \tilde{p}C_{\text{WL}}^{\mathbb{Z}_n}} e^{ik\frac{2\pi}{n} \int_{M^4_{\text{Latt}}} (b^{\mathbb{Z}_n})^2 + is\frac{2\pi}{n} \int_{M_{\text{WS}}^2} b^{\mathbb{Z}_n}},$$

where  $s$  is the charge of the string-like excitation.

We first solve  $db^{\mathbb{Z}_n} = \tilde{p}C_{\text{WL}}^{\mathbb{Z}_n} \bmod n$  as

$$b^{\mathbb{Z}_n} \stackrel{n}{=} \tilde{p}b_{\text{WL}}^{\mathbb{Z}_n} + b_0^{\mathbb{Z}_n} + da^{\mathbb{Z}_n}, \quad (90)$$

where  $b_{\text{WL}}^{\mathbb{Z}_n}$  is a fixed 2-cochain field that satisfies  $db_{\text{WL}}^{\mathbb{Z}_n} = C_{\text{WL}}^{\mathbb{Z}_n}$  and  $b_0^{\mathbb{Z}_n} \in H^2(M^4; \mathbb{Z}_n)$ . We can rewrite the partition function as

$$Z_{k;b^2\mathbb{Z}_n}(M^4; \tilde{p}C_{\text{WL}}^{\mathbb{Z}_n}, sM_{\text{WS}}^2) \propto e^{i\frac{2\pi}{n} \int_{M^4} k\tilde{p}^2 (b_{\text{WL}}^{\mathbb{Z}_n})^2} \quad (91)$$

$$e^{i\frac{2\pi}{n} \int_{M_{\text{WS}}^2} s\tilde{p}b_{\text{WL}}^{\mathbb{Z}_n}} \sum_{\{a_{ij}^{\mathbb{Z}_n}\}, b_0^{\mathbb{Z}_n} \in H^2(M^4; \mathbb{Z}_n)} e^{ik\frac{2\pi}{n} \int_{M^4} 2\tilde{p}(b_0^{\mathbb{Z}_n} + da^{\mathbb{Z}_n})b_{\text{WL}}^{\mathbb{Z}_n}}$$

$$= e^{i\frac{2\pi k\tilde{p}^2}{n} \int_{M^4} (b_{\text{WL}}^{\mathbb{Z}_n})^2} e^{i\frac{2\pi s\tilde{p}}{n} \int_{M_{\text{WS}}^2} b_{\text{WL}}^{\mathbb{Z}_n}} \times$$

$$\sum_{b_0^{\mathbb{Z}_n} \in H^2(M^4; \mathbb{Z}_n)} e^{ik\frac{2\pi}{n} \int_{M^4} b_0^{\mathbb{Z}_n} (2\tilde{p}b_{\text{WL}}^{\mathbb{Z}_n} + b_0^{\mathbb{Z}_n})} \sum_{\{a_{ij}^{\mathbb{Z}_n}\}} e^{i2k\tilde{p}\frac{2\pi}{n} \int_{M^4} a^{\mathbb{Z}_n} C_{\text{WL}}^{\mathbb{Z}_n}}.$$

Let  $D_{\text{WS}}^3$  be the extension of  $M_{\text{WS}}^2$ , i.e.  $\partial D_{\text{WS}}^3 = M_{\text{WS}}^2$ . Then we can rewrite  $\int_{M_{\text{WS}}^2} b_{\text{WL}}^{\mathbb{Z}_n} = \int_{D_{\text{WS}}^3} db_{\text{WL}}^{\mathbb{Z}_n} = \int_{D_{\text{WS}}^3} C_{\text{WL}}^{\mathbb{Z}_n}$ . In fact  $\int_{D_{\text{WS}}^3} C_{\text{WL}}^{\mathbb{Z}_n} = \text{Int}(D_{\text{WS}}^3, M_{\text{WL}}^1)$ , is the intersection number between  $D_{\text{WS}}^3$  and  $M_{\text{WL}}^1$ , which in turn is the linking number between  $M_{\text{WS}}^2$  and  $M_{\text{WL}}^1$ :  $\text{Lnk}(M_{\text{WS}}^2, M_{\text{WL}}^1)$ .

Using the Poincaré duality we can also rewrite  $\int_{M^4} a^{\mathbb{Z}_n} C_{\text{WL}}^{\mathbb{Z}_n}$  as  $\int_{M_{\text{WL}}^1} a^{\mathbb{Z}_n}$ . Then

$$\sum_{\{a_{ij}^{\mathbb{Z}_n}\}} e^{i2k\tilde{p}\frac{2\pi}{n} \int_{M^4} a^{\mathbb{Z}_n} C_{\text{WL}}^{\mathbb{Z}_n}} = \sum_{\{a_{ij}^{\mathbb{Z}_n}\}} e^{i2k\tilde{p}\frac{2\pi}{n} \int_{M_{\text{WL}}^1} a^{\mathbb{Z}_n}} \neq 0 \quad (92)$$

only when  $[2k\tilde{p}]_n = 0$ , i.e. when  $\tilde{p}$  is quantized as  $\tilde{p} = p\frac{n}{2k}$ ,  $p \in \mathbb{Z}_{2k,n}$ . If  $\tilde{p}$  is not quantized as the above, the correspond point-like excitation is confined.

Thus the above partition function for unconfined excitations can be rewritten as

$$Z_{k;b^2\mathbb{Z}_n}(M^4; \frac{pn}{2k,n} C_{\text{WL}}^{\mathbb{Z}_n}, sM_{\text{WS}}^2) \quad (93)$$

$$\propto e^{isp\frac{2\pi}{2k,n} \text{Lnk}(M_{\text{WS}}^2, M_{\text{WL}}^1)} e^{ip^2\frac{2\pi nk}{(2k,n)^2} \int_{M^4} (b_{\text{WL}}^{\mathbb{Z}_n})^2} \times$$

$$\sum_{b_0^{\mathbb{Z}_n} \in H^2(M^4; \mathbb{Z}_n)} e^{ik\frac{2\pi}{n} \int_{M^4} b_0^{\mathbb{Z}_n} (\frac{2pn}{(2k,n)} b_{\text{WL}}^{\mathbb{Z}_n} + b_0^{\mathbb{Z}_n})}.$$

The above expression tells us the braiding statistics of point-like excitations and string-like excitations. Let assume  $H^2(M^4; \mathbb{Z}_n) = 0$ . In this case  $\int_{M^4} b_{\text{WL}}^{\mathbb{Z}_n} b_{\text{WL}}^{\mathbb{Z}_n}$  is an integer, and  $p^2\frac{2\pi nk}{(2k,n)^2}$  is also an integer. Thus  $e^{ip^2\frac{2\pi nk}{(2k,n)^2} \int_{M^4} b_{\text{WL}}^{\mathbb{Z}_n} b_{\text{WL}}^{\mathbb{Z}_n}}$  is always 1 when  $p^2\frac{2\pi nk}{(2k,n)^2} = \text{even}$  and  $e^{ip^2\frac{2\pi nk}{(2k,n)^2} \int_{M^4} b_{\text{WL}}^{\mathbb{Z}_n} b_{\text{WL}}^{\mathbb{Z}_n}}$  can be  $-1$  when  $p^2\frac{2\pi nk}{(2k,n)^2} = \text{odd}$ . This factor determines the statistics of the point-like excitations since  $b_{\text{WL}}^{\mathbb{Z}_n}$  is determined by the particle world-line  $M_{\text{WL}}^1$ . Comparing with the results obtained in the last section, we find that *when the factor  $e^{ip^2\frac{2\pi nk}{(2k,n)^2} \int_{M^4} b_{\text{WL}}^{\mathbb{Z}_n} b_{\text{WL}}^{\mathbb{Z}_n}}$  can be  $-1$  (depending on the braiding of the world-line  $M_{\text{WL}}^1$ ), then the correspond particle is a fermion*. This means that when  $p^2\frac{2\pi nk}{(2k,n)^2} = \text{odd}$ , the charge  $p$  particle is a fermion.

The factor  $e^{isp\frac{2\pi}{2k,n} \text{Lnk}(M_{\text{WS}}^2, M_{\text{WL}}^1)}$  determines the mutual statistics (i.e. the Aharonov-Bohm phase) between point-like excitations and string-like excitations. We see that it is the usual mutual statistics of  $Z_{(2k,n)}$ -gauge theory. We also see that there is no non-trivial braiding statistics between string-like excitations. This confirms our result in the last section that

the  $\mathbb{Z}_n$ -2-cocycle model produces a low energy effective  $Z_{(2k,n)}$ -gauge theory. It is a UT  $Z_{(2k,n)}$ -gauge theory if  $\frac{2\pi nk}{(2k,n)^2} = \text{even}$ , and a EF  $Z_{(2k,n)}$ -gauge theory if  $\frac{2\pi nk}{(2k,n)^2} = \text{odd}$ .

The term  $\sum_{b_0^{\mathbb{Z}_n} \in H^2(M^4; \mathbb{Z}_n)} e^{ik\frac{2\pi}{n} \int_{M^4} b_0^{\mathbb{Z}_n} (\frac{2pn}{(2k,n)} b_{\text{WL}}^{\mathbb{Z}_n} + b_0^{\mathbb{Z}_n})}$  tells us when the partition function will vanishes in the presence of emergent fermions, i.e. when  $\frac{2\pi nk}{(2k,n)^2} = \text{odd}$ . Let us assume there is no world-line and  $\frac{2\pi nk}{(2k,n)^2} = \text{odd}$ . In this case the above factor becomes  $\sum_{b_0^{\mathbb{Z}_n} \in H^2(M^4; \mathbb{Z}_n)} e^{ik\frac{2\pi}{n} \int_{M^4} (b_0^{\mathbb{Z}_n})^2}$ . We note that  $\frac{2\pi nk}{(2k,n)^2} = \text{odd}$  implies that  $k$  and  $\frac{n}{2}$  are both odd integers. Since  $n$  is even and  $\frac{n}{2}$  is odd, we have  $\mathbb{Z}_n = \mathbb{Z}_{n/2} \oplus \mathbb{Z}_2$ . Therefore,  $b_0^{\mathbb{Z}_n}$  can be expressed as

$$b_0^{\mathbb{Z}_n} = 2b_0^{\mathbb{Z}_{n/2}} + \frac{n}{2}b_0^{\mathbb{Z}_2}. \quad (94)$$

We obtain

$$\begin{aligned}
& \sum_{b_0^{\mathbb{Z}_n} \in H^2(M^4; \mathbb{Z}_n)} e^{ik \frac{2\pi}{n} \int_{M^4} (b_0^{\mathbb{Z}_n})^2} \\
&= \sum_{b_0^{\mathbb{Z}_n/2} \in H^2(M^4; \mathbb{Z}_{n/2})} e^{ik \frac{2\pi}{n} \int_{M^4} 4(b_0^{\mathbb{Z}_n/2})^2} \times \\
& \quad \sum_{b_0^{\mathbb{Z}_2} \in H^2(M^4; \mathbb{Z}_2)} e^{ik \frac{2\pi}{n} \int_{M^4} (\frac{n}{2})^2 (b_0^{\mathbb{Z}_2})^2} \\
&= \sum_{b_0^{\mathbb{Z}_n/2} \in H^2(M^4; \mathbb{Z}_{n/2})} e^{i2k \frac{2\pi}{n/2} \int_{M^4} (b_0^{\mathbb{Z}_n/2})^2} \times \\
& \quad \sum_{b_0^{\mathbb{Z}_2} \in H^2(M^4; \mathbb{Z}_2)} e^{i\pi \int_{M^4} (b_0^{\mathbb{Z}_2})^2} \quad (95)
\end{aligned}$$

The factor  $\sum_{b_0^{\mathbb{Z}_2} \in H^2(M^4; \mathbb{Z}_2)} e^{i\pi \int_{M^4} (b_0^{\mathbb{Z}_2})^2}$  can be rewritten as

$$\sum_{b_0^{\mathbb{Z}_2} \in H^2(M^4; \mathbb{Z}_2)} e^{i\pi \int_{M^4} (b_0^{\mathbb{Z}_2})^2} = \sum_{b_0^{\mathbb{Z}_2} \in H^2(M^4; \mathbb{Z}_2)} e^{i\pi \int_{M^4} w_2 b_0^{\mathbb{Z}_2}} \quad (96)$$

since  $M^4$  is orientable. Now we see that

$Z_{k;b^2\mathbb{Z}_n}(M^4) = 0$  when  $w_2 \neq 0$  (*i.e.* when the orientable  $M^4$  is not spin), if there is an emergence of fermions.

## F. 3+1D twisted $\mathbb{Z}_n a \mathbb{Z}_n b$ model

### 1. Model construction

In this section, we are going to construct a local bosonic model on space-time lattice  $M_{\text{Latt}}^4$ . Our model is a mixture of  $\mathbb{Z}_n$ -1-cocycle model and  $\mathbb{Z}_n$ -2-cocycle model. The local degrees of freedom of our model are  $\mathbb{Z}_n$  indices  $a_{ij}^{\mathbb{Z}_n}$  on the links and  $b_{ijk}^{\mathbb{Z}_n}$  on the triangles. We view  $a_{ij}^{\mathbb{Z}_n}$  as a 1-cochain in  $C^1(M_{\text{Latt}}^4; \mathbb{Z}_n)$  and  $b_{ijk}^{\mathbb{Z}_n}$  as a 2-cochain in  $C^2(M_{\text{Latt}}^4; \mathbb{Z}_n)$ .

Using the Bockstein homomorphism for  $\mathbb{Z}_n$ ,  $\mathcal{B}_n : H^m(M^d; \mathbb{Z}_n) \rightarrow H^{m+1}(M^d; \mathbb{Z}_n)$ , the partition function of our model is defined as

$$\begin{aligned}
& Z_{k_1 k_2; b\mathcal{B}a-bb\mathbb{Z}_n}(M_{\text{Latt}}^4) \quad (97) \\
&= \sum_{\substack{\{a_{ij}^{\mathbb{Z}_n}\} \\ d a^{\mathbb{Z}_n} = 0}} \sum_{\substack{\{b_{ijk}^{\mathbb{Z}_n}\} \\ d b^{\mathbb{Z}_n} = 0}} e^{i \frac{2\pi}{n} \int_{M_{\text{Latt}}^4} k_1 b^{\mathbb{Z}_n} \mathcal{B}_n a^{\mathbb{Z}_n} + k_2 b^{\mathbb{Z}_n} b^{\mathbb{Z}_n}}.
\end{aligned}$$

The volume-independent topological partition function is given by

$$\begin{aligned}
& Z_{k_1 k_2; b\mathcal{B}a-bb\mathbb{Z}_n}^{\text{top}}(M^4) \\
&= \sum_{\substack{a^{\mathbb{Z}_n} \in H^1(M^4; \mathbb{Z}_n) \\ b^{\mathbb{Z}_n} \in H^2(M^4; \mathbb{Z}_n)}} \frac{e^{i \frac{2\pi}{n} \int_{M^4} k_1 b^{\mathbb{Z}_n} \mathcal{B}_n a^{\mathbb{Z}_n} + k_2 b^{\mathbb{Z}_n} b^{\mathbb{Z}_n}}}{|H^1(M^4; \mathbb{Z}_n)|} \quad (98)
\end{aligned}$$

### 2. Topological partition functions

On  $M^4 = T^4$  or  $M^4 = S^2 \times T^2$ ,  $\mathcal{B}_n a^{\mathbb{Z}_n} = 0$ . Thus the partition function is a product of the partition function of the  $\mathbb{Z}_n a$  model in Section III B and the partition function of the  $\mathbb{Z}_n b$  model in Section III E. We find that (see Table III)

$$\begin{aligned}
& Z_{k_1 k_2; b\mathcal{B}a-bb\mathbb{Z}_n}^{\text{top}}(T^4) = n^3 \langle 2k_2, n \rangle^3, \\
& Z_{k_1 k_2; b\mathcal{B}a-bb\mathbb{Z}_n}^{\text{top}}(S^2 \times T^2) = n \langle 2k_2, n \rangle. \quad (99)
\end{aligned}$$

On  $M^4 = S^1 \times L^3(p)$ , for  $\langle n, p \rangle = 1$ , we find that  $\int_{M^4} b^{\mathbb{Z}_n} \mathcal{B}_n a^{\mathbb{Z}_n} = \int_{M^4} b^{\mathbb{Z}_n} b^{\mathbb{Z}_n} = 0$ , since  $H^2(S^1 \times L^3(p); \mathbb{Z}_n) = 0$ . So  $Z_{k_1 k_2; b\mathcal{B}a-bb\mathbb{Z}_n}^{\text{top}}(S^1 \times L^3(p)) = 1$ . For  $\langle n, p \rangle \neq 1$ , we can parametrize  $a^{\mathbb{Z}_n}$ ,  $b^{\mathbb{Z}_n}$  as

$$\begin{aligned}
& a^{\mathbb{Z}_n} = \alpha_1 a_1 + \alpha_2 a, \quad \alpha_1 \in \mathbb{Z}_n, \alpha_2 \in \mathbb{Z}_{\langle n, p \rangle}, \\
& b^{\mathbb{Z}_n} = \beta_1 a_1 a + \beta_2 b, \quad \beta_1, \beta_2 \in \mathbb{Z}_{\langle n, p \rangle}. \quad (100)
\end{aligned}$$

Using  $\mathcal{B}_n a = \frac{p}{\langle n, p \rangle} b$ ,  $\mathcal{B}_n a_1 = 0$ ,  $a_1 a b = \frac{n}{\langle n, p \rangle} c a_1$ , and  $b^2 = (a_1 a)^2 = 0$  (see eqn. (66), eqn. (67), and eqn. (68)), we find that

$$\begin{aligned}
& Z_{k_1 k_2; b\mathcal{B}a-bb\mathbb{Z}_n}^{\text{top}}(S^1 \times L^3(p)) \quad (101) \\
&= \sum_{\alpha_1 \in \mathbb{Z}_n; \alpha_2, \beta_1, \beta_2 \in \mathbb{Z}_{\langle n, p \rangle}} \frac{e^{i \frac{2\pi}{\langle n, p \rangle} (k_1 \frac{p}{\langle n, p \rangle} \alpha_2 \beta_1 + 2k_2 \beta_1 \beta_2)}}{n \langle n, p \rangle} \\
&= \sum_{\alpha_2, \beta_2 \in \mathbb{Z}_{\langle n, p \rangle}} \frac{\delta_{\langle n, p \rangle} (k_1 \frac{p}{\langle n, p \rangle} \alpha_2 + 2k_2 \beta_2)}{\langle n, p \rangle} \\
&= \langle n, p \rangle \langle 2k_2, k_1 \frac{p}{\langle n, p \rangle}, \langle n, p \rangle \rangle.
\end{aligned}$$

On  $M^4 = F^4$ , we note that the Bockstein homomorphism  $\mathcal{B}_n$  maps all 1-cocycles to 0. Thus the partition function is a product of the partition function of the  $\mathbb{Z}_n a$  model in Section III B and the partition function of the  $\mathbb{Z}_n b$  model in Section III E:

$$Z_{k_1 k_2; b\mathcal{B}a-bb\mathbb{Z}_n}^{\text{top}}(F^4) = \begin{cases} n \langle 2k_2, n \rangle, & \text{if } \frac{2k_2 n}{\langle 2k_2, n \rangle^2} = \text{even} \\ 0, & \text{if } \frac{2k_2 n}{\langle 2k_2, n \rangle^2} = \text{odd} \end{cases} \quad (102)$$

### 3. Point-like and string-like topological excitations

Here we are going to study more physical properties of the  $\mathbb{Z}_n a \mathbb{Z}_n b$  model. The GSD of our model on space  $M^3$  is given by  $\text{GSD}_{k_1 k_2; b\mathcal{B}a-bb\mathbb{Z}_n}(M^3) = Z_{k_1 k_2; b\mathcal{B}a-bb\mathbb{Z}_n}^{\text{top}}(S^1 \times M^3)$ . If we choose  $M^3 = S^1 \times S^2$ , we find that

$$\text{GSD}_{k_1 k_2; b\mathcal{B}a-bb\mathbb{Z}_n}(S^1 \times S^2) = n \langle 2k_2, n \rangle. \quad (103)$$

The GSD on  $S^1 \times S^2$  implies that there are  $n \langle 2k_2, n \rangle$  types of point-like and string-like excitations regardless the value of  $k_1$ . This result is unexpected, since one may

guess the number of types of point-like and string-like excitations are  $n^2$ . The reduction is due to confinement as will be explained below.

Again, we will view  $b^{\mathbb{Z}_n}$  as the field strength 2-form of a  $U(1)$  gauge theory

$$2\pi b^{\mathbb{Z}_n} = f, \quad (104)$$

We will also view  $\mathcal{B}_n a^{\mathbb{Z}_n}$  as the field strength 2-form of another  $U(1)$  gauge theory

$$2\pi \mathcal{B}_n a^{\mathbb{Z}_n} = f'. \quad (105)$$

So the twisted 3+1D  $b\mathcal{B}a-bb\mathbb{Z}_n$  model can be viewed as  $U(1) \times U'(1)$  gauge theory with some proper condensations. The  $U(1) \times U'(1)$  gauge theory has a form

$$\begin{aligned} & Z_{1,U^2(1)}(M^4) \\ &= \int D[a]D[a'] e^{i \frac{\Theta_1}{4\pi^2} \int_{M^4} f f' + i \frac{\Theta_2}{8\pi^2} \int_{M^4} f f + \dots}, \end{aligned} \quad (106)$$

with  $\Theta_1 = k_1 \frac{2\pi}{n}$  and  $\Theta_2 = k_2 \frac{4\pi}{n}$ .

Let us consider a more general  $U^\kappa(1)$  model

$$Z = \int \prod_I D[a_I] e^{i \frac{2\pi}{8\pi^2} \int_{M^4} f_I \Lambda_{IJ} f_J + \dots} \quad (107)$$

where  $\Lambda_{IJ}$  is a symmetric rational matrix. On the boundary, the action amplitude becomes

$$e^{i \frac{1}{4\pi} \int_{\partial M^4} \Lambda_{IJ} a_I da_J + \dots} \quad (108)$$

We see that  $2\pi$  flux of  $a_J$  carries  $a_I$ -charge  $Q_I = \Lambda_{IJ}$ .

Before the condensation, the point-like excitations are labeled by  $(q, m, q', m')$ . The magnetic charges for the two  $U(1)$  gauge fields are  $M = m$  and  $M' = m'$ . Using the above result with

$$\Lambda = \begin{pmatrix} 0 & \frac{k_1}{n} \\ \frac{k_1}{n} & \frac{2k_2}{n} \end{pmatrix}, \quad (109)$$

we see that the electric charges for the two  $U(1)$  gauge fields are  $Q = q + \frac{k_1}{n} m'$  and  $Q' = q' + \frac{k_1}{n} m + \frac{2k_2}{n} m'$ . The statistics of the  $(q, m, q', m')$ -excitation is  $e^{i\theta} = (-)^{qm+q'm'}$ .

Next, we condense  $(q, m, q', m') = (-k_1, 0, -2k_2, n)$  excitations that have  $Q = Q' = 0$ . Since  $(M, M') = (0, n)$  for such excitations, it breaks the second  $U'(1)$  to  $Z_n$  (in the dual picture). We also condense  $(q, m, q', m') = (n, 0, 0, 0)$  particles with  $(Q, Q', M, M') = (n, 0, 0, 0)$ . It breaks the first  $U(1)$  to  $Z_n$ . The unconfined particles must have  $M = Q' = 0$ , i.e.  $q' = m = 0$ . Thus the unconfined particles are generated by  $(q, m, q', m') = (1, 0, 0, 0)$  with  $(Q, M, Q', M') = (1, 0, 0, 0)$  and  $(q, m, q', m') = (0, 0, -\frac{2k_2}{\langle 2k_2, n \rangle}, \frac{n}{\langle 2k_2, n \rangle})$  with  $(Q, M, Q', M') = (\frac{k_1}{\langle 2k_2, n \rangle}, 0, 0, \frac{n}{\langle 2k_2, n \rangle})$ . We see that the point-like excitations are labeled by  $(p, p')$  (a bound state of  $p$  type- $(q, m, q', m') = (1, 0, 0, 0)$  and  $p'$  type- $(q, m, q', m') = (0, 0, -\frac{2k_2}{\langle 2k_2, n \rangle}, \frac{n}{\langle 2k_2, n \rangle})$  excitations). Two

particles that differ by a condensing particle are regarded as equivalent. Thus  $(p, p')$  labels have the following equivalent relation

$$(p + n, p') \sim (p, p') \sim (p - k_1, p' + \langle 2k_2, n \rangle). \quad (110)$$

So there are  $n\langle 2k_2, n \rangle$  distinct types of point-like excitations. The type- $(q, m, q', m') = (1, 0, 0, 0)$  excitation is a boson. The type- $(q, m, q', m') = (0, 0, -\frac{2k_2}{\langle 2k_2, n \rangle}, \frac{n}{\langle 2k_2, n \rangle})$  excitation has a statistics  $(-)^{\frac{2k_2 n}{\langle 2k_2, n \rangle^2}}$ .

We note that the point-like excitations are labeled by the integer points  $(p, p')$  in a 2-dimensional unit cell with basis vectors  $(n, 0)$  and  $(-k_1, \langle 2k_2, n \rangle)$ . We put the two basis vectors together to form a matrix  $\begin{pmatrix} n & 0 \\ -k_1 & \langle 2k_2, n \rangle \end{pmatrix}$ . The fusion of the point-like excitations is described by an abelian group  $G\left(\begin{smallmatrix} n & 0 \\ -k_1 & \langle 2k_2, n \rangle \end{smallmatrix}\right)$  characterized by the matrix. In general, the fusion rule of the point-like excitations is not given by  $Z_n \times Z_{\langle 2k_2, n \rangle}$ .

The string-like excitations are generated by the  $2\pi/n$  magnetic flux-line of the first  $U(1)$  and the  $1/n$ -unit electric flux-line of the second  $U'(1)$ . So the generic string-like excitations are labeled by  $(s, s')$ . Two strings that can join are regarded as equivalent<sup>31</sup>. Note we can attach a  $(q, m, q', m')$  excitation to change string  $(s, s')$  to an equivalent one, which generates the following equivalence relation

$$(s, s') \sim (s + nm, s' + nq' + k_1 m + 2k_2 m') \quad (111)$$

The above can be rewritten as

$$(s + n, s' + k_1) \sim (s, s') \sim (s, s' + \langle 2k_2, n \rangle). \quad (112)$$

We see that there are  $n\langle 2k_2, n \rangle$  distinct types of string-like excitations.

The fusion of the string-like excitations is described by an abelian group  $G\left(\begin{smallmatrix} n & k_1 \\ 0 & \langle 2k_2, n \rangle \end{smallmatrix}\right)$ . It turns out that the fusion of the point-like excitations and the fusion of the string-like excitations are described by the same abelian group

$$G\left(\begin{smallmatrix} n & 0 \\ -k_1 & \langle 2k_2, n \rangle \end{smallmatrix}\right) = G\left(\begin{smallmatrix} n & k_1 \\ 0 & \langle 2k_2, n \rangle \end{smallmatrix}\right). \quad (113)$$

In general, two integer matrices  $M_1$  and  $M_2$  describe the same abelian group if  $M_2 = WM_1U$  where  $U, W$  are invertible integer matrices. In this case, we say  $M_1 \sim M_2$ . Let  $\begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}$  be the Smith normal form of  $\begin{pmatrix} n & 0 \\ k_1 & \langle 2k_2, n \rangle \end{pmatrix}$ , i.e.

$$W\left(\begin{smallmatrix} n & 0 \\ k_1 & \langle 2k_2, n \rangle \end{smallmatrix}\right)U = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}. \quad (114)$$

This implies that

$$U_T\left(\begin{smallmatrix} n & k_1 \\ 0 & \langle 2k_2, n \rangle \end{smallmatrix}\right)W^T = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}. \quad (115)$$

We see that

$$\begin{pmatrix} n & k_1 \\ 0 & \langle 2k_2, n \rangle \end{pmatrix} \sim \begin{pmatrix} n & 0 \\ k_1 & \langle 2k_2, n \rangle \end{pmatrix} \sim \begin{pmatrix} n & 0 \\ -k_1 & \langle 2k_2, n \rangle \end{pmatrix}. \quad (116)$$

Via direct numerical calculation, we find that

$$G \begin{pmatrix} n & k_1 \\ 0 & \langle 2k_2, n \rangle \end{pmatrix} = Z_{\frac{n \langle n, 2k_2 \rangle}{\langle n, k_1, 2k_2 \rangle}} \times Z_{\langle n, k_1, 2k_2 \rangle} \quad (117)$$

The mutual braiding phase between a type- $(p, p')$  point-like excitation and a type- $(s, s')$  string-like excitation is given by

$$\theta = 2\pi \left( \frac{ps}{n} - \frac{p's'}{\langle 2k_2, n \rangle} + \frac{p'sk_1}{n \langle 2k_2, n \rangle} \right). \quad (118)$$

Since, both point-like excitations and string-like excitations have a fusion described by  $Z_{\frac{n \langle n, 2k_2 \rangle}{\langle n, k_1, 2k_2 \rangle}} \times Z_{\langle n, k_1, 2k_2 \rangle}$ , we will call the corresponding theory a  $Z_{\frac{n \langle n, 2k_2 \rangle}{\langle n, k_1, 2k_2 \rangle}} \times Z_{\langle n, k_1, 2k_2 \rangle}$  fusion theory. When  $\frac{2k_2 n}{\langle 2k_2, n \rangle^2} = \text{odd}$ , some point-like excitations are fermions.

#### 4. Including excitations in the path integral

In the  $\mathbb{Z}_n \text{a}\mathbb{Z}_n \text{b}$  model, there are two kinds of point-like excitations described by the world-lines  $M_{\text{WL}}^1$  and  $N_{\text{WL}}^1$ , which are  $\mathbb{Z}$ -valued 1-cycles. Let 3-coboundary  $C_{\text{WL}}^{\mathbb{Z}}$  be the Poincaré dual of  $M_{\text{WL}}^1$ . Then the point-like excitation that corresponds to  $M_{\text{WL}}^1$  is described the 2-cochain field  $b^{\mathbb{Z}_n}$  that satisfies

$$db^{\mathbb{Z}_n} \stackrel{n}{=} \tilde{p}_1 C_{\text{WL}}^{\mathbb{Z}}, \quad (119)$$

where  $\tilde{p}_1$  is the charge of the point-like excitation.

The  $\mathbb{Z}_n \text{a}\mathbb{Z}_n \text{b}$  model also contains two kind of string-like excitations described by the world-sheets  $M_{\text{WS}}^2$  and  $N_{\text{WS}}^2$  in space-time. The world-sheet  $N_{\text{WS}}^2$  can be viewed as a  $\mathbb{Z}$ -valued 2-cycle, which is dual to a  $\mathbb{Z}$ -valued 2-coboundary  $B_{\text{WS}}^{\mathbb{Z}}$ . Such a string-like excitation is described the 1-cochain field  $a^{\mathbb{Z}_n}$  that satisfies

$$da^{\mathbb{Z}_n} \stackrel{n}{=} s_2 B_{\text{WS}}^{\mathbb{Z}}, \quad (120)$$

where  $s_2$  is the charge of the string-like excitation. Therefore, in the presence of point-like topological excitations described by  $C_{\text{WL}}^{\mathbb{Z}}$ ,  $N_{\text{WL}}^1$  and string-like topological excitations described by  $M_{\text{WS}}^2$ ,  $B_{\text{WS}}^{\mathbb{Z}}$ , the partition function becomes

$$Z_{k_1 k_2; b\mathcal{B}a-bb\mathbb{Z}_n}(M^4, \tilde{p}_1 C_{\text{WL}}^{\mathbb{Z}_n}, p_2 N_{\text{WL}}^1, s_1 M_{\text{WS}}^2, s_2 B_{\text{WS}}^{\mathbb{Z}_n}) = \sum_{\{a_{ij}^{\mathbb{Z}_n}\}} \sum_{\{b_{ijk}^{\mathbb{Z}_n}\}} e^{i \frac{2\pi}{n} \int_{M^4} k_1 b^{\mathbb{Z}_n} \mathcal{B}_n a^{\mathbb{Z}_n} + k_2 (b^{\mathbb{Z}_n})^2} \times e^{i \frac{2\pi}{n} (p_2 \int_{N_{\text{WL}}^1} a^{\mathbb{Z}_n} + s_1 \int_{M_{\text{WS}}^2} b^{\mathbb{Z}_n})}. \quad (121)$$

where  $p_2$  is the charge of the point-like excitation, and  $s_1$  the charge of the string-like excitation. However, the

above partition function is not well defined. It is well defined only when  $b^{\mathbb{Z}_n}$  and  $a^{\mathbb{Z}_n}$  are cocycles. When  $b^{\mathbb{Z}_n}$  and  $a^{\mathbb{Z}_n}$  are not cocycles, the partition function is not invariant under the shift  $b^{\mathbb{Z}_n} \rightarrow b^{\mathbb{Z}_n} + n\tilde{b}^{\mathbb{Z}_n}$  and/or  $a^{\mathbb{Z}_n} \rightarrow a^{\mathbb{Z}_n} + n\tilde{a}^{\mathbb{Z}_n}$ .

So to remove such ambiguity, we write  $b^{\mathbb{Z}_n}$  and  $a^{\mathbb{Z}_n}$  as

$$\begin{aligned} b^{\mathbb{Z}_n} &= \tilde{p}_1 b_{\text{WL}}^{\mathbb{Z}} + b_0^{\mathbb{Z}_n} + d\tilde{a}^{\mathbb{Z}_n}, \\ a^{\mathbb{Z}_n} &= s_2 a_{\text{WS}}^{\mathbb{Z}} + a_0^{\mathbb{Z}_n} + dg^{\mathbb{Z}_n}. \end{aligned} \quad (122)$$

Here  $b_{\text{WL}}^{\mathbb{Z}}$  is a fixed  $\mathbb{Z}$ -valued 2-cochain field that satisfies

$$db_{\text{WL}}^{\mathbb{Z}} = C_{\text{WL}}^{\mathbb{Z}}, \quad (123)$$

and  $b_0^{\mathbb{Z}_n} \in H^2(M^4; \mathbb{Z}_n)$ . Also  $a_{\text{WS}}^{\mathbb{Z}}$  is a fixed  $\mathbb{Z}$ -valued 1-cochain fields that satisfies

$$da_{\text{WS}}^{\mathbb{Z}} = B_{\text{WS}}^{\mathbb{Z}}. \quad (124)$$

and  $a_0^{\mathbb{Z}_n} \in H^1(M^4; \mathbb{Z})$ .  $a_0^{\mathbb{Z}_n}$ ,  $g^{\mathbb{Z}_n}$ ,  $b_0^{\mathbb{Z}_n}$ ,  $\tilde{a}^{\mathbb{Z}_n}$  are  $\mathbb{Z}_n$ -valued. The partition function on orientable  $M^4$  is defined by summing over those  $\mathbb{Z}_n$ -valued fields:

$$\begin{aligned} Z_{k_1 k_2; b\mathcal{B}a-bb\mathbb{Z}_n}(M^4, \tilde{p}_1 C_{\text{WL}}^{\mathbb{Z}_n}, p_2 N_{\text{WL}}^1, s_1 M_{\text{WS}}^2, s_2 B_{\text{WS}}^{\mathbb{Z}_n}) &= \sum_{\{\tilde{a}_{ij}^{\mathbb{Z}_n}, g_i^{\mathbb{Z}_n}\}} \sum_{\substack{a_0^{\mathbb{Z}_n} \in H^1(M^4; \mathbb{Z}_n) \\ b_0^{\mathbb{Z}_n} \in H^2(M^4; \mathbb{Z}_n)}} e^{i \frac{2\pi}{n} \int_{M^4} k_2 (\tilde{p}_1 b_{\text{WL}}^{\mathbb{Z}} + b_0^{\mathbb{Z}_n} + d\tilde{a}^{\mathbb{Z}_n})^2} \times \\ &e^{i \frac{2\pi}{n} \int_{M^4} k_1 (\tilde{p}_1 b_{\text{WL}}^{\mathbb{Z}} + b_0^{\mathbb{Z}_n}) \mathcal{B}_n (s_2 a_{\text{WS}}^{\mathbb{Z}} + a_0^{\mathbb{Z}_n})} \times \\ &e^{i \frac{2\pi}{n} [p_2 \int_{N_{\text{WL}}^1} (s_2 a_{\text{WS}}^{\mathbb{Z}} + a_0^{\mathbb{Z}_n}) + s_1 \int_{M_{\text{WS}}^2} (\tilde{p}_1 b_{\text{WL}}^{\mathbb{Z}} + b_0^{\mathbb{Z}_n})]}. \end{aligned} \quad (125)$$

We note that

$$\begin{aligned} &e^{i \frac{2\pi}{n} \int_{M^4} k_2 (\tilde{p}_1 b_{\text{WL}}^{\mathbb{Z}} + b_0^{\mathbb{Z}_n} + d\tilde{a}^{\mathbb{Z}_n})^2} \\ &= e^{i \frac{2\pi}{n} \int_{M^4} k_2 \tilde{p}_1^2 (b_{\text{WL}}^{\mathbb{Z}})^2} e^{i \frac{2\pi}{n} \int_{M^4} k_2 (b_0^{\mathbb{Z}_n})^2} e^{i \frac{2\pi}{n} \int_{M^4} 2k_2 \tilde{p}_1 b_{\text{WL}}^{\mathbb{Z}} d\tilde{a}^{\mathbb{Z}_n}} \\ &= e^{i \frac{2\pi}{n} \int_{M^4} k_2 \tilde{p}_1^2 (b_{\text{WL}}^{\mathbb{Z}})^2} e^{i \frac{2\pi}{n} \int_{M^4} k_2 (b_0^{\mathbb{Z}_n})^2} e^{-i \frac{2\pi}{n} \int_{M^4} 2k_2 \tilde{p}_1 C_{\text{WL}}^{\mathbb{Z}} \tilde{a}^{\mathbb{Z}_n}} \\ &= e^{i \frac{2\pi}{n} \int_{M^4} k_2 \tilde{p}_1^2 (b_{\text{WL}}^{\mathbb{Z}})^2} e^{i \frac{2\pi}{n} \int_{M^4} k_2 (b_0^{\mathbb{Z}_n})^2} e^{-i \frac{2\pi}{n} \int_{M_{\text{WL}}^1} 2k_2 \tilde{p}_1 \tilde{a}^{\mathbb{Z}_n}} \end{aligned} \quad (126)$$

Also

$$\begin{aligned} &e^{i \frac{2\pi}{n} \int_{M^4} k_1 (\tilde{p}_1 b_{\text{WL}}^{\mathbb{Z}} + b_0^{\mathbb{Z}_n}) \mathcal{B}_n (s_2 a_{\text{WS}}^{\mathbb{Z}} + a_0^{\mathbb{Z}_n})} \\ &= e^{i \frac{2\pi}{n^2} \int_{M^4} k_1 \tilde{p}_1 s_2 b_{\text{WL}}^{\mathbb{Z}} B_{\text{WS}}^{\mathbb{Z}}} e^{i \frac{2\pi}{n^2} \int_{M^4} k_1 s_2 b_0^{\mathbb{Z}_n} B_{\text{WS}}^{\mathbb{Z}}} \times \\ &e^{i \frac{2\pi}{n^2} \int_{M^4} k_1 \tilde{p}_1 C_{\text{WL}}^{\mathbb{Z}} a_0^{\mathbb{Z}_n}} e^{i \frac{2\pi}{n^2} \int_{M^4} k_1 b_0^{\mathbb{Z}_n} \mathcal{B}_n a_0^{\mathbb{Z}_n}} \\ &= e^{i \frac{2\pi}{n^2} \int_{M^4} k_1 \tilde{p}_1 s_2 b_{\text{WL}}^{\mathbb{Z}} B_{\text{WS}}^{\mathbb{Z}}} e^{i \frac{2\pi}{n^2} \int_{N_{\text{WS}}^2} k_1 s_2 b_0^{\mathbb{Z}_n}} \times \\ &e^{i \frac{2\pi}{n^2} \int_{M_{\text{WL}}^1} k_1 \tilde{p}_1 a_0^{\mathbb{Z}_n}} e^{i \frac{2\pi}{n^2} \int_{M^4} k_1 b_0^{\mathbb{Z}_n} \mathcal{B}_n a_0^{\mathbb{Z}_n}} \end{aligned} \quad (127)$$

We can rewrite the partition function as

$$\begin{aligned}
& Z_{k_1 k_2; b\mathcal{B}a-bb\mathbb{Z}_n}(M^4, \tilde{p}_1 C_{\text{WL}}^{\mathbb{Z}_n}, p_2 N_{\text{WL}}^1, s_1 M_{\text{WS}}^2, s_2 B_{\text{WS}}^{\mathbb{Z}}) \\
& \propto e^{i \frac{2\pi}{n} \int_{M^4} k_2 \tilde{p}_1^2 (b_{\text{WL}}^{\mathbb{Z}})^2} e^{i \frac{2\pi}{n^2} \int_{M^4} k_1 \tilde{p}_1 s_2 b_{\text{WL}}^{\mathbb{Z}} B_{\text{WS}}^{\mathbb{Z}}} \times \\
& e^{i \frac{2\pi}{n} [p_2 \int_{N_{\text{WL}}^1} s_2 a_{\text{WS}}^{\mathbb{Z}} + s_1 \int_{M_{\text{WS}}^2} \tilde{p}_1 b_{\text{WL}}^{\mathbb{Z}}]} \times \\
& \sum_{\substack{a_0^{\mathbb{Z}_n} \in H^1(M^4; \mathbb{Z}_n) \\ b_0^{\mathbb{Z}_n} \in H^2(M^4; \mathbb{Z}_n)}} e^{i \frac{2\pi}{n} \int_{M^4} k_2 (b_0^{\mathbb{Z}})^2} e^{i \frac{2\pi}{n} \int_{M^4} k_1 b_0^{\mathbb{Z}_n} \mathcal{B}_n a_0^{\mathbb{Z}_n}} \times \\
& e^{i \frac{2\pi}{n^2} \int_{N_{\text{WS}}^2} k_1 s_2 b_0^{\mathbb{Z}_n}} e^{i \frac{2\pi}{n^2} \int_{M_{\text{WL}}^1} k_1 \tilde{p}_1 a_0^{\mathbb{Z}_n}} \times \\
& e^{i \frac{2\pi}{n} [p_2 \int_{N_{\text{WL}}^1} a_0^{\mathbb{Z}_n} + s_1 \int_{M_{\text{WS}}^2} b_0^{\mathbb{Z}_n}]} \sum_{\{\tilde{a}_{ij}^{\mathbb{Z}_n}, g_i^{\mathbb{Z}_n}\}} e^{-i \frac{2\pi}{n} \int_{M_{\text{WL}}^1} 2k_2 \tilde{p}_1 \tilde{a}^{\mathbb{Z}_n}}
\end{aligned} \tag{128}$$

Using the Poincaré duality, we can rewrite  $\int_{M^4} b_{\text{WL}}^{\mathbb{Z}_n} B_{\text{WS}}^{\mathbb{Z}_n}$  as  $\int_{N_{\text{WS}}^2} b_{\text{WL}}^{\mathbb{Z}_n}$ . Let  $D_{\text{WS}}^3$  be the extension of  $N_{\text{WS}}^2$ , i.e.  $\partial D_{\text{WS}}^3 = N_{\text{WS}}^2$ . Then we can rewrite  $\int_{N_{\text{WS}}^2} b_{\text{WL}}^{\mathbb{Z}_n} = \int_{D_{\text{WS}}^3} db_{\text{WL}}^{\mathbb{Z}_n} = \int_{D_{\text{WS}}^3} C_{\text{WL}}^{\mathbb{Z}_n}$ . In fact  $\int_{D_{\text{WS}}^3} C_{\text{WL}}^{\mathbb{Z}_n} = \text{Int}(D_{\text{WS}}^3, M_{\text{WL}}^1)$ , is the intersection number between  $D_{\text{WS}}^3$  and  $M_{\text{WL}}^1$  which is the linking number between  $N_{\text{WS}}^2$  and  $M_{\text{WL}}^1$ :  $\text{Lnk}(N_{\text{WS}}^2, M_{\text{WL}}^1)$ .

Also  $\sum_{\{\tilde{a}_{ij}^{\mathbb{Z}_n}\}} e^{i \frac{2\pi}{n} \int_{M_{\text{WL}}^1} 2k_2 \tilde{p}_1 \tilde{a}^{\mathbb{Z}_n}} \neq 0$  only when  $[2k_2 \tilde{p}_1]_n = 0$ , or when  $\tilde{p}_1$  is quantized as  $\tilde{p}_1 = p_1 \frac{n}{(2k_2, n)}$ ,  $p_1 \in \mathbb{Z}_{(2k_2, n)}$ . If  $\tilde{p}_1$  is not quantized as the above, the corresponding point-like excitation is confined.

Thus the above partition function for unconfined like excitations can be rewritten as

$$\begin{aligned}
& Z_{k_1 k_2; b\mathcal{B}a-bb\mathbb{Z}_n}(M^4, \frac{np_1 C_{\text{WL}}^{\mathbb{Z}_n}}{(2k_2, n)}, p_2 N_{\text{WL}}^1, s_1 M_{\text{WS}}^2, s_2 B_{\text{WS}}^{\mathbb{Z}_n}) \\
& = e^{i \pi \frac{2nk_2 p_1^2}{(2k_2, n)^2} \int_{M^4} b_{\text{WL}}^{\mathbb{Z}_n} b_{\text{WL}}^{\mathbb{Z}_n}} e^{i \frac{2\pi s_1 p_1}{(2k_2, n)} \text{Lnk}(M_{\text{WS}}^2, M_{\text{WL}}^1)} \\
& e^{i \frac{2\pi s_2 p_1 k_1}{n(2k_2, n)} \text{Lnk}(N_{\text{WS}}^2, M_{\text{WL}}^1)} e^{i \frac{2\pi s_2 p_2}{n} \text{Lnk}(N_{\text{WS}}^2, N_{\text{WL}}^1)} \times \\
& \sum_{\substack{a_0^{\mathbb{Z}_n} \in H^1(M^4; \mathbb{Z}_n) \\ b_0^{\mathbb{Z}_n} \in H^2(M^4; \mathbb{Z}_n)}} e^{i \frac{2\pi}{n} \int_{M^4} k_2 (b_0^{\mathbb{Z}})^2} e^{i \frac{2\pi}{n} \int_{M^4} k_1 b_0^{\mathbb{Z}_n} \mathcal{B}_n a_0^{\mathbb{Z}_n}} \times \\
& e^{i \frac{2\pi k_1 s_2}{n^2} \int_{N_{\text{WS}}^2} b_0^{\mathbb{Z}_n}} e^{i \frac{2\pi p_1 k_1}{n(2k_2, n)} \int_{M_{\text{WL}}^1} a_0^{\mathbb{Z}_n}} \times \\
& e^{i \frac{2\pi}{n} [p_2 \int_{N_{\text{WL}}^1} a_0^{\mathbb{Z}_n} + s_1 \int_{M_{\text{WS}}^2} b_0^{\mathbb{Z}_n}]}
\end{aligned} \tag{129}$$

The factors

$$\begin{aligned}
& e^{i \pi \frac{2nk_2 p_1^2}{(2k_2, n)^2} \int_{M^4} b_{\text{WL}}^{\mathbb{Z}_n} b_{\text{WL}}^{\mathbb{Z}_n}} e^{i \frac{2\pi s_1 p_1}{(2k_2, n)} \text{Lnk}(M_{\text{WS}}^2, M_{\text{WL}}^1)} \\
& e^{i \frac{2\pi s_2 p_1 k_1}{n(2k_2, n)} \text{Lnk}(N_{\text{WS}}^2, M_{\text{WL}}^1)} e^{i \frac{2\pi s_2 p_2}{n} \text{Lnk}(N_{\text{WS}}^2, N_{\text{WL}}^1)} \tag{130}
\end{aligned}$$

in the above expression determines the braiding statistics of point-like excitations and string-like excitations. We see that *there is no non-trivial braiding for string-like excitations. But there are non-trivial mutual statistics (i.e. the Aharonov-Bohm phase) between point-like excitations and string-like excitations. Also when  $\frac{2nk_2}{(2k_2, n)^2} = \text{odd}$ , the theory contain fermions.*

#### IV. COMPARISON BETWEEN THE 3+1D $\mathbb{Z}_n$ -2-COCYCLE MODEL AND 3+1D $\mathbb{Z}_n$ -1-COCYCLE MODEL

There is a well known duality between the 3+1D  $\mathbb{Z}_n$ -1-cocycle theory (with emergent  $\mathbb{Z}_n$ -gauge theory) and the above 3+1D  $\mathbb{Z}_n$ -2-cocycle theory with  $k = 0$ . In the following, we will compare the two theories in detail. We find that the two theories are equivalent, if they are viewed as pure topological theory without any symmetry. So both 3+1D  $\mathbb{Z}_n$ -1-cocycle theory and 3+1D  $\mathbb{Z}_n$ -2-cocycle theory realize the same topological order described by UT  $\mathbb{Z}_n$ -gauge theory. However, if we view the two theories as topological theory with time-reversal symmetry or parity symmetry, then the two theories are not equivalent. In other words, the two models realize the same topological orders, but different symmetry enriched topological orders (with time-reversal symmetry or parity symmetry).

##### A. Duality

To see the above mentioned duality, let us describe the lattice Hamiltonian of the two theories. We consider a 3D cubic lattice whose sites are labeled by  $i$ . To obtain a  $\mathbb{Z}_n$ -1-cocycle theory, we put a  $\mathbb{Z}_n$  degrees of freedom  $a_{ij}^{\mathbb{Z}_n} = 0, 1, \dots, n-1 = -a_{ji}^{\mathbb{Z}_n}$  on each nearest neighbor links  $(ij)$ . Let  $U_{ij} = e^{i \frac{2\pi}{n} a_{ij}^{\mathbb{Z}_n}}$  and  $V_{ij}$  is an operator that raise  $a_{ij}^{\mathbb{Z}_n}$  by one:  $V_{ij}|a_{ij}^{\mathbb{Z}_n} = m\rangle = |a_{ij}^{\mathbb{Z}_n} = m+1\rangle$ . Noting that the  $\mathbb{Z}_n$ -1-cocycle theory is a theory of closed  $\mathbb{Z}_n$ -loops at low energy, we find that the lattice Hamiltonian for the  $\mathbb{Z}_n$ -1-cocycle theory will be

$$\begin{aligned}
H_{\mathbb{Z}_n a} &= - \sum_i (Q_i + Q_i^\dagger) - \sum_{(ijkl)} (B_{ijkl} + B_{ijkl}^\dagger), \\
Q_i &= \prod_{j \text{ next to } i} U_{ij}, \\
B_{ijkl} &= V_{ij} V_{jk} V_{kl} V_{li},
\end{aligned} \tag{131}$$

where  $\sum_i$  sum over all sites and  $\sum_{(ijkl)}$  sum over all squares  $(ijkl)$ . The  $-(Q_i + Q_i^\dagger)$  terms enforce the closed-loop condition and the  $-(B_{ijkl} + B_{ijkl}^\dagger)$  terms are the loop hopping and/or loop creation/annihilation terms.

To obtain a  $\mathbb{Z}_n$ -2-cocycle theory, we put a  $\mathbb{Z}_n$  degrees of freedom  $b_{ijkl}^{\mathbb{Z}_n} = 0, 1, \dots, n-1 = -b_{lkji}^{\mathbb{Z}_n}$  on each square  $(ijkl)$ . But this is equivalent to put a  $\mathbb{Z}_n$  degrees of freedom  $a_{IJ}^{\mathbb{Z}_n} = 0, 1, \dots, n-1 = -a_{JI}^{\mathbb{Z}_n}$  on each link  $(IJ)$  of the dual lattice. The dual lattice of a cubic lattice is also a cubic lattice. The  $\mathbb{Z}_n$ -2-cocycle theory is a theory of closed  $\mathbb{Z}_n$ -membranes at low energy. Thus the lattice



Hamiltonian for the  $\mathbb{Z}_n$ -2-cocycle theory with  $k = 0$  is

$$H_{0;b^2\mathbb{Z}_n} = - \sum_I (Q_I + Q_I^\dagger) - \sum_{(IJKL)} (B_{IJKL} + B_{IJKL}^\dagger),$$

$$Q_I = \prod_{J \text{ next to } I} V_{IJ},$$

$$B_{IJKL} = U_{IJ} U_{JK} U_{KL} U_{LI}. \quad (132)$$

The  $-(B_{IJKL} + B_{IJKL}^\dagger)$  terms enforces the closed-membrane condition and the  $-(Q_I + Q_I^\dagger)$  are the membrane hopping and/or membrane creation/annihilation terms. The two Hamiltonians  $H_{\mathbb{Z}_n a}$  and  $H_{0;b^2\mathbb{Z}_n}$  are equivalent under a local unitary transformation that exchanges  $U$  and  $V$ . This implies that the two theories are really equivalent.

### B. Topological invariants for orientable space-time

To compare the two theory at Lagrangian level, we note that the volume-independent topological partition function for 3+1D  $\mathbb{Z}_n$ -1-cocycle theory is given by

$$Z_{\mathbb{Z}_n a}^{\text{top}}(M^4) = \frac{|H^1(M^4; \mathbb{Z}_n)|}{|H^0(M^4; \mathbb{Z}_n)|}. \quad (133)$$

while the volume-independent topological partition function for 3+1D  $\mathbb{Z}_n$ -2-cocycle theory (with  $k_1 = k_2 = 0$ ) is given by

$$Z_{00;b\mathcal{B}a-bb\mathbb{Z}_n}^{\text{top}}(M^4) = \frac{|H^0(M^4; \mathbb{Z}_n)| |H^2(M^4; \mathbb{Z}_n)|}{|H^1(M^4; \mathbb{Z}_n)|} \quad (134)$$

So their ratio is given by

$$\frac{Z_{00;b\mathcal{B}a-bb\mathbb{Z}_n}^{\text{top}}(M^4)}{Z_{\mathbb{Z}_n a}^{\text{top}}(M^4)} = \frac{|H^0(M^4; \mathbb{Z}_n)|^2 |H^2(M^4; \mathbb{Z}_n)|}{|H^1(M^4; \mathbb{Z}_n)|^2} \quad (135)$$

In Appendix C, we will show that for orientable close space-time  $M^4$ ,

$$\frac{Z_{00;b\mathcal{B}a-bb\mathbb{Z}_n}^{\text{top}}(M^4)}{Z_{\mathbb{Z}_n a}^{\text{top}}(M^4)} = n^{\chi(M^4)}, \quad (136)$$

where  $\chi(M^4)$  is the Euler number. The volume-independent topological partition functions of the two models are different, which may lead one to conclude that the  $\mathbb{Z}_n$ -1-cocycle model and the  $\mathbb{Z}_n$ -2-cocycle model realize different topological orders. However, in Ref. 31, it was conjectured that two 3+1D topological partition functions  $Z_1^{\text{top}}(M^4)$  and  $Z_2^{\text{top}}(M^4)$  describe the same L-type topological orders iff their ratio has a form

$$\frac{Z_1^{\text{top}}(M^4)}{Z_2^{\text{top}}(M^4)} = \rho^{\chi(M^4)} \lambda^{P_1(M^4)}. \quad (137)$$

where  $P_1(M^4)$  is the Pontryagin number of  $M^4$ . Therefore, the above result implies that the  $\mathbb{Z}_n$ -1-cocycle model and the  $\mathbb{Z}_n$ -2-cocycle model realize the same topological order.

### C. Ground state degeneracy for non-orientable spaces

Now we turn to study the ground state degeneracy of the two models. To calculate the GSD on closed space manifold  $M^3$ , we compute the volume-independent partition function on  $M^3 \times S^1$  space-time:

$$\text{GSD}(M^3) = Z^{\text{top}}(M^3 \times S^1). \quad (138)$$

We see that the ground state degeneracy of the two models are the same on orientable spaces  $M^3$  since their partition functions are the same on orientable space-times  $M^3 \times S^1$ .

However, for non orientable space  $M^3$ , the GSDs of the two models can be different. For example, let us assume the space to be  $M^3 = S^1 \times KB$ , where  $KB$  is the Klein bottle. We note that

$$H_2(KB; \mathbb{Z}) = 0, \quad H_1(KB; \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}_2, \quad H_0(KB; \mathbb{Z}) = \mathbb{Z} \quad (139)$$

and

$$H_2(S^1 \times KB; \mathbb{Z}) = H_2(KB; \mathbb{Z}) \oplus H_1(KB; \mathbb{Z}),$$

$$= \mathbb{Z} \oplus \mathbb{Z}_2;$$

$$H_1(S^1 \times KB; \mathbb{Z}) = H_1(KB; \mathbb{Z}) \oplus \mathbb{Z}.$$

$$= \mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}; \quad (140)$$

Then using the universal coefficient theorem eqn. (A8), we find that

$$H^2(S^1 \times KB; \mathbb{Z}_n) = \mathbb{Z}_n \oplus \mathbb{Z}_{\langle n, 2 \rangle}^{\oplus 2};$$

$$H^1(S^1 \times KB; \mathbb{Z}) = \mathbb{Z}_n^{\oplus 2} \mathbb{Z}_{\langle n, 2 \rangle}. \quad (141)$$

Thus

$$\text{GSD}_{0;b^2\mathbb{Z}_n}(S^1 \times KB) = n \langle n, 2 \rangle^2,$$

$$\text{GSD}_{\mathbb{Z}_n a}(S^1 \times KB) = n^2 \langle n, 2 \rangle. \quad (142)$$

When  $n > 2$ , the GSDs of the two theories are different. Since the difference only appear in non-orientable manifolds,

the  $\mathbb{Z}_n$ -2-cocycle model and the  $\mathbb{Z}_n$ -1-cocycle model realize two different time-reversal symmetry enriched topological orders.

This is consistent with the fact that the two theories realize the same topological order if we ignore the time-reversal symmetry.

Both topological orders has point-like excitations labeled by  $i \in \mathbb{Z}_n$  and string-like excitations labeled by  $s \in \mathbb{Z}_n$ . But they transform differently under time-reversal. For the  $\mathbb{Z}_n$ -1-cocycle theory  $(i, s) \rightarrow (i, -s)$  under time reversal. For the  $\mathbb{Z}_n$ -2-cocycle theory  $(i, s) \rightarrow (-i, s)$  under time reversal. Both the  $\mathbb{Z}_n$ -1-cocycle theory and the  $\mathbb{Z}_n$ -2-cocycle theory are described by the same Hamiltonian eqn. (131). But the time-reversal symmetry are

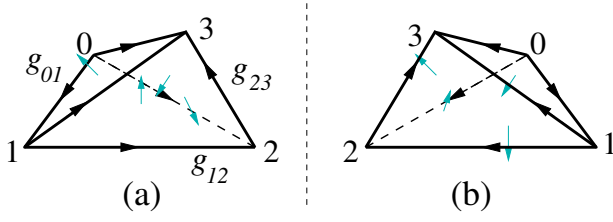


FIG. 5. (Color online) Two branched simplices with opposite orientations. (a) A branched simplex with positive orientation and (b) a branched simplex with negative orientation.

realized differently. In the  $\mathbb{Z}_n$ -1-cocycle theory, we assume  $|a_{ij}^{\mathbb{Z}_n}\rangle$ , the eigenstates of  $U_{ij}$ , are invariant under time-reversal. Thus  $(U_{ij}, V_{ij}) \rightarrow (U_{ij}^\dagger, V_{ij})$  under time reversal. In the  $\mathbb{Z}_n$ -2-cocycle theory, we assume that the eigenstates of  $V_{ij}$  are invariant under time-reversal. Thus  $(U_{ij}, V_{ij}) \rightarrow (U_{ij}, V_{ij}^\dagger)$  under time reversal.

## V. NON-ABELIAN COCYCLE MODELS

So far, we have constructed many local bosonic models – the cocycle models. But in those construction, the local degrees of freedom are always described by an abelian group, such as  $\mathbb{Z}_n$ . In this section, we will use group-cocycles in group cohomology theory (see Appendix G) to generalize the cocycle models so that the local degrees of freedom are described by a non-abelian group  $G$ . To use group-cocycles to construct the cocycle models, we need to map the group-cocycles in group cohomology theory to topological-cocycles in topological cohomology theory. To obtain such a map, we need to first introduce the branching structure in space-time lattice.

### A. The branching structure of space-time lattice

In order to define a generic lattice theory on the space-time complex  $M_{\text{latt}}^d$  using group cocycles, it is important to give the vertices of each simplex a local order. A nice local scheme to order the vertices is given by a branching structure.<sup>11,48,88</sup> A branching structure is a choice of orientation of each link in the  $d$ -dimensional complex so that there is no oriented loop on any triangle (see Fig. 5).

The branching structure induces a *local order* of the vertices on each simplex. The first vertex of a simplex is the vertex with no incoming links, and the second vertex is the vertex with only one incoming link, *etc*. So the simplex in Fig. 5a has the following vertex ordering: 0, 1, 2, 3.

The branching structure also gives the simplex (and its sub simplices) an canonical orientation. Fig. 5 illustrates two 3-simplices with opposite canonical orientations compare the 3-dimension space in which they are embedded.

The blue arrows indicate that canonical orientations of the 2-simplices. The black arrows indicate that canonical orientations of the 1-simplices.

### B. Group-vertex models that realize $G$ -SPT orders

Ref. 10, 11, and 48 have constructed exactly soluble local bosonic models using homogeneous group cocycles (see Appendix G) of group  $G$  to realize  $G$ -SPT orders. Those models are actually cocycle models on space-time lattice. In this section, we will review those results using the cocycle notation introduced above.

The local degrees of freedom of our model are now group elements living on the vertices of the orientable space-time lattice  $M_{\text{Latt}}^d$ :  $g_i \in G$ . Let  $\nu_n(g_0, \dots, g_n)$  be a homogeneous group  $n$ -cocycle:  $\nu_n(g_0, \dots, g_n) \in \mathcal{H}^n(G, \mathbb{R}/\mathbb{Z})$ . From  $\nu_n$ , we can construct a topological  $n$ -cocycle  $\tilde{\nu}_n$  on  $M_{\text{Latt}}^d$ :

$$\tilde{\nu}_n(i_0, i_1, \dots, i_n) = \nu_n(g_{i_0}, g_{i_1}, \dots, g_{i_n}) \quad (143)$$

where  $(i_0, i_1, \dots, i_n)$  is an  $n$ -simplex with the canonical orientation and the vertex ordering  $i_0 < i_1 < \dots < i_n$ . Below, we will drop the  $\sim$  and denote  $\tilde{\nu}_n(i_0, i_1, \dots, i_n)$  as  $\nu_n(g_{i_0}, g_{i_1}, \dots, g_{i_n})$ .

Using such mapping, we can construct a group-vertex model on orientable space-time  $M_{\text{Latt}}^d$ :

$$Z_{\nu_d}(M_{\text{Latt}}^d) = \sum_{\{g_i\}} e^{i2\pi \int_{M_{\text{Latt}}^d} \nu_d(\{g_i\})}. \quad (144)$$

Since  $\nu_d(\{fg_i\}) = \nu_d(\{g_i\})$ ,  $f \in G$ , the group-vertex model has a global on-site  $G$ -symmetry. Since  $e^{i2\pi \int_{M^d} \nu_d(\{g_i\})} = 1$  on any closed orientable manifold  $M^d$ . We find that the constructed model is gapped. We also see that

$$Z_{\nu_d}(M_{\text{Latt}}^d) = |G|^{N_v}. \quad (145)$$

So the volume-independent partition function  $Z_{\nu_d}^{\text{top}}(M^d) = 1$ , for all closed orientable manifolds  $M^d$ , which implies that the model does not have any topological order regardless the choice of the group cocycle  $\nu_d$ .  $Z_{\nu_d}^{\text{top}}(M^d) = 1$  also implies that the group-vortex model does not break the  $G$  symmetry (as one can see from the ground state degeneracy on closed orientable space manifold  $M_{\text{space}}^{d-1}$ :  $\text{GSD}_{\nu_d}^{\text{top}}(M_{\text{space}}^{d-1}) = Z_{\nu_d}^{\text{top}}(S^1 \times M_{\text{space}}^{d-1}) = 1$ ).

But  $Z_{\nu_d}^{\text{top}}(M^d) = 1$  also means that volume-independent partition function fails to detect SPT orders. In fact, we do not even know whether the lattice models with different  $\nu_d$ 's belong to different SPT phases, if we just look at  $Z_{\nu_d}^{\text{top}}(M^d)$ .

To detect SPT order via the partition function<sup>49,50,54,55</sup>, we need to add the symmetry twist<sup>53</sup> in space-time. A symmetry twist is described by  $a_{ij} \in G$  on each link (*i.e.* 1-simplex), that satisfy

$$a_{ij} = a_{ji}^{-1}, \quad a_{ij}a_{jk}a_{ki} = 1. \quad (146)$$

Such a  $a_{ij}$  configuration define a so called “flat  $G$ -connection” on space-time  $M^d$ . In the presence of symmetry twist, the partition function becomes

$$Z_{\nu_d}(M_{\text{Latt}}^d, a_{ij}) = \sum_{\{g_i\}} e^{i2\pi \int_{M_{\text{Latt}}^d} \nu_d^g(\{g_i\}, \{a_{ij}\})}, \quad (147)$$

where

$$\begin{aligned} \nu_d^g(\{g_i\}, \{a_{ij}\}) &\equiv \nu_d^g(g_{i_0}, g_{i_1}, \dots, g_{i_d}; a_{i_0 i_1}, a_{i_1 i_2}, \dots) \\ &\equiv \nu_d(g_{i_0}, a_{i_0 i_1} g_{i_1}, a_{i_0 i_1} a_{i_1 i_2} g_{i_2}, \dots). \end{aligned} \quad (148)$$

Clearly the partition function  $Z_{\nu_d}(M_{\text{Latt}}^d, a_{ij})$  is invariant under the gauge transformation

$$\begin{aligned} g_i &\rightarrow f_i g_i, \quad a_{ij} \rightarrow f_i a_{ij} f_j^{-1}; \\ \nu_d^g(\{f_i g_i\}, \{f_i a_{ij} f_j^{-1}\}) &= \nu_d^g(\{g_i\}, \{a_{ij}\}); \\ Z_{\nu_d}(M_{\text{Latt}}^d, a_{ij}) &= Z_{\nu_d}(M_{\text{Latt}}^d, f_i a_{ij} f_j^{-1}). \end{aligned} \quad (149)$$

So the partition function  $Z_{\nu_d}(M_{\text{Latt}}^d, a_{ij})$  only depend on the gauge equivalent class of the flat connection  $a_{ij}$ .

The volume-independent partition functions  $Z_{\nu_d}^{\text{top}}(M_{\text{Latt}}^d, a_{ij})$  are the so called SPT invariants that suppose to fully characterize the SPT order<sup>49–51,54,55</sup>. Using a gauge transformation to change  $g_i \rightarrow 1$ , we find the SPT invariant to be given by

$$\begin{aligned} Z_{\nu_d}^{\text{top}}(M_{\text{Latt}}^d, a_{ij}) &= e^{i2\pi \int_{M_{\text{Latt}}^d} \nu_d^g(\{g_i=1\}, \{a_{ij}\})} \\ &= e^{i2\pi \int_{M_{\text{Latt}}^d} \omega_d(\{a_{ij}\})} \end{aligned} \quad (150)$$

where  $\omega_d$  is the inhomogeneous group-cocycle that corresponds to the homogeneous group-cocycle  $\nu_d$  (see eqn. (G9)). The above expression allows us to compute the SPT invariant.

In the following, we will list some the SPT invariants for some simple SPT states:

1. The  $Z_n$ -SPT states in 2+1D are classified by  $\mathcal{H}^3(Z_n; \mathbb{R}/\mathbb{Z}) = \mathbb{Z}_n$ . For a  $Z_n$ -SPT state labeled by  $k \in \mathbb{Z}_n$ , its SPT invariant is

$$Z_k^{\text{top}}(M^3, a^{\mathbb{Z}_n}) = e^{ik \frac{2\pi}{n} \int_{M^3} a^{\mathbb{Z}_n} \mathcal{B}_n a^{\mathbb{Z}_n}}. \quad (151)$$

2. The  $Z_n \times \tilde{Z}_n$ -SPT states in 3+1D are classified by  $\mathcal{H}^4(Z_n \times \tilde{Z}_n; \mathbb{R}/\mathbb{Z}) = \mathbb{Z}_n^{\oplus 2}$ . For a  $Z_n \times \tilde{Z}_n$ -SPT state labeled by  $i(k_1, k_2) \in \mathbb{Z}_n^{\oplus 2}$ , its SPT invariant is

$$\begin{aligned} Z_{k_1, k_2}^{\text{top}}(M^4, a^{\mathbb{Z}_n}, \tilde{a}^{\mathbb{Z}_n}) \\ = e^{i \frac{2\pi}{n} \int_{M^4} k_1 a^{\mathbb{Z}_n} \tilde{a}^{\mathbb{Z}_n} \mathcal{B}_n \tilde{a}^{\mathbb{Z}_n} + k_2 \tilde{a}^{\mathbb{Z}_n} a^{\mathbb{Z}_n} \mathcal{B}_n a^{\mathbb{Z}_n}}. \end{aligned} \quad (152)$$

### C. Group-vertex models that realize $Z_2^T$ -SPT orders

To construct a local bosonic model that realize the time-reversal  $Z_2^T$  SPT order, we consider a  $\mathbb{Z}_2$ -group-vertex model:  $g_i \in \mathbb{Z}_2 = \{0, 1\}$ . The  $\mathbb{Z}_2$ -group-vertex

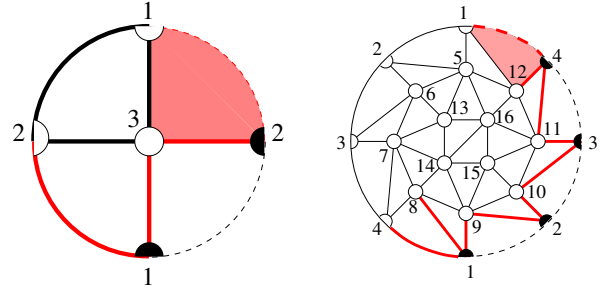


FIG. 6. (Color online) Two triangulations of  $\mathbb{RP}^2$  where the opposite points on the boundary are identified. One triangulation has 3 vertices and the other has 16 vertices. The open dots represent  $\tilde{g}_i = 0$  and the filled dots represent  $\tilde{g}_i = 1$  at the vertices.  $\tilde{g}_i$  is multivalued since it takes different values on the same vertex, such as vertex-1 and vertex-2. The black links represent  $a_{ij} = (d\tilde{g})_{ij} = \tilde{g}_i - \tilde{g}_j = (w_1)_{ij} = 0$  and the red links represent  $a_{ij} = (d\tilde{g})_{ij} = \tilde{g}_i - \tilde{g}_j = (w_1)_{ij} = 1 \bmod 2$ . The unshaded triangles represent  $(d\tilde{g}d\tilde{g})_{ijk} = (w_1^2)_{ijk} = \mathcal{B}_2 d\tilde{g} = \mathcal{B}_2 w_1 = 0$  and the shaded triangle represents  $(d\tilde{g}d\tilde{g})_{ijk} = (w_1^2)_{ijk} = \mathcal{B}_2 d\tilde{g} = \mathcal{B}_2 w_1 = 1$ . (By definition,  $(d\tilde{g}d\tilde{g})_{ijk} = [(g_i - \tilde{g}_j)(\tilde{g}_j - \tilde{g}_k)]_2$  and  $(\mathcal{B}_2 w_1)_{ijk} = \frac{(w_1)_{ij} + (w_1)_{jk} - (w_1)_{ik}}{2}$  where  $i, j, k$  are ordered as  $i < j < k$ .) We see that  $\int_{\mathbb{RP}^2} w_1^2 = \int_{\mathbb{RP}^2} d\tilde{g}d\tilde{g} = 1$ .

model on orientable space-time  $M_{\text{Latt}}^d$  is given by

$$Z_{\nu_d}(M_{\text{Latt}}^d) = \sum_{\{g_i\}} e^{i2\pi \int_{M_{\text{Latt}}^d} \nu_d(\{g_i\})}, \quad (153)$$

where the homogeneous  $\mathbb{Z}_2$ -group cocycle  $\nu_d(\{g_i\}) \in \mathcal{H}^d(\mathbb{Z}_2, (\mathbb{R}/\mathbb{Z})_{\mathbb{Z}_2})$  satisfies

$$\nu_d(\{g_i + 1\}) = -\nu_d(\{g_i\}) \bmod 1. \quad (154)$$

The extra “-” sign implies that the  $\mathbb{Z}_2$  group has a non-trivial action on  $\mathbb{R}/\mathbb{Z}$  which is indicated by the subscript  $\mathbb{Z}_2$  in  $(\mathbb{R}/\mathbb{Z})_{\mathbb{Z}_2}$ . For example, in 1+1D,

$$\nu_2(g_0, g_1, g_2) = \frac{1}{2} [g_1 - g_0]_2 [g_2 - g_1]_2 \quad (155)$$

Since the  $\mathbb{Z}_2$  action correspond to the time-reversal (or orientation reversal) transformation, to obtain partition function with the symmetry twist, we need to put the system on non-orientable space-time and to introduce a  $\mathbb{Z}_2$  valued 1-cocycle  $a_{ij}$  to describe orientation reversal:

$$Z_{\nu_d}(M_{\text{Latt}}^d) = \sum_{\{g_i\}} e^{i2\pi \int_{M_{\text{Latt}}^d} \nu_d^g(\{g_i\}, \{a_{ij}\})}. \quad (156)$$

where

$$\begin{aligned} \nu_d^g(\{g_i\}, \{a_{ij}\}) &\equiv \nu_d^g(g_{i_0}, g_{i_1}, \dots, g_{i_d}; a_{i_0 i_1}, a_{i_1 i_2}, \dots) \\ &\equiv \nu_d(g_{i_0}, a_{i_0 i_1} + g_{i_1}, a_{i_0 i_1} + a_{i_1 i_2} + g_{i_2}, \dots). \end{aligned} \quad (157)$$

Here  $a_{ij}$  is the  $\mathbb{Z}_2$  flat connection that describe the orientation of the manifold (see Fig. 6). In other words,

if the orientation does not change around a loop  $C$ , then  $\sum_{(ij) \in C} a_{ij} = \oint_C a = 0$ ; if the orientation changes around a loop  $C$ , then  $\sum_{(ij) \in C} a_{ij} = \oint_C a = 1$  (see Fig. 6). The above definition implies that  $a_{ij}$  is a  $\mathbb{Z}_2$  valued 1-cocycle  $a \in C^1(M^d; \mathbb{Z}_2)$ . In fact  $a = w_1$ .

We can use a multivalued  $\mathbb{Z}_2$  gauge transformation to make  $a_{ij} = 0$ , which changes the single-valued  $g_i$  to multivalued  $\tilde{g}_i$ . If the orientation changes around a loop  $C$ ,  $\tilde{g}$  will have to take different values on the same vertex somewhere on  $C$  (see Fig. 6). We see that to realize  $Z_2^T$ -SPT order, the local bosonic degrees of freedom must couple to space-time orientation. In other words,  $(-)^{\tilde{g}_i}$  is a pseudo scalar, which changes sign under time-reversal and parity transformations. In this paper, we will also refer  $\tilde{g}$  as a pseudo scalar field. Thus if we view  $\tilde{g}_i$  as a  $\mathbb{Z}_2$ -valued 0-cochain, we have (see Fig. 6)

$$a = w_1 = d\tilde{g}. \quad (158)$$

In terms of such multivalued  $\tilde{g}_i$ , the partition function can be written as

$$Z_{\nu_d}(M_{\text{Latt}}^d) = \sum_{\{\tilde{g}_i\}} e^{i2\pi \int_{M_{\text{Latt}}^d} \nu_d(\{\tilde{g}_i\})}. \quad (159)$$

The  $Z_2^T$  SPT invariant is given by the corresponding inhomogeneous cocycle  $\omega_d$ :

$$\begin{aligned} Z_{\nu_d}^{\text{top}}(M_{\text{Latt}}^d) &= e^{i2\pi \int_{M_{\text{Latt}}^d} \nu_d^g(\{g_i=1\}, \{a_{ij}\})} \\ &= e^{i2\pi \int_{M_{\text{Latt}}^d} \omega_d(\{a_{ij}\})}. \end{aligned} \quad (160)$$

We can express  $\omega_d(\{a_{ij}\})$  in terms of  $a_{ij}$  (see Fig. 6):

$$\omega_d(\{a_{ij}\}) = \begin{cases} \frac{1}{2}a^d & \text{if } d = \text{even}, \\ 0 & \text{if } d = \text{odd}. \end{cases} \quad (161)$$

Thus, the  $Z_2^T$  SPT invariant is given by

$$Z_{\nu_d}^{\text{top}}(M^d) = e^{i\pi \int_{M^d} w_1^d}. \quad (162)$$

From  $w_1^d = \text{Sq}^1(w_1^{d-1}) = (d-1)w_1^d$ , we see that  $w_1^d = 0 \bmod 2$  automatically, when  $d = \text{odd}$ . So the above expression for the  $Z_2^T$  SPT invariant is valid for both  $d = \text{even}$  and  $d = \text{odd}$ .

Last, we like to mention that, using multivalued  $\tilde{g}_i$ , we can also express the non-trivial homogeneous cocycle  $\nu_d(\{\tilde{g}_i\})$  as (see Fig. 6)

$$\nu_d(\{\tilde{g}_i\}) = \begin{cases} \frac{1}{2}(d\tilde{g})^d & \text{if } d = \text{even}, \\ 0 & \text{if } d = \text{odd}, \end{cases} \quad (163)$$

since  $a = d\tilde{g}$ . This allows us to rewrite eqn. (159) as (see Fig. 6)

$$Z_{\nu_d}(M_{\text{Latt}}^d) = \sum_{\{\tilde{g}_i\}} e^{i\pi \int_{M_{\text{Latt}}^d} (d\tilde{g})^d}, \quad (164)$$

for even  $d$ .

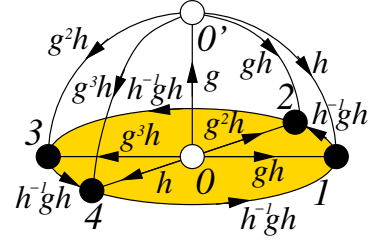


FIG. 7. (Color online) The lens space,  $L^3(p)$ , is obtain by identifying the bottom and the top disk after a  $2\pi/p$  rotation, eg link-(01) and link-(0'2) are identified, link-(02) and link-(0'3) are identified, link-(12) and link-(23) are identified, etc .

#### D. Group-link model and emergent Dijkgraaf-Witten gauge theory

Now let us construct local bosonic models – group-link models, whose topological orders are described by Dijkgraaf-Witten gauge theory. The local degrees of freedom of the group-link model are group elements living on the links of the space-time lattice  $M_{\text{Latt}}^d$ :  $a_{ij} \in G$  that satisfies  $a_{ij} = G_{ji}^{-1}$ . Then, using the inhomogeneous group-cocycle  $\omega_d(\{a_{ij}\})$ , we can construct a group-link model<sup>33,83,85,89</sup>

$$\begin{aligned} Z_{G, \omega_d}(M_{\text{Latt}}^d) & \\ = \sum_{\substack{\{a_{ij}\} \\ a_{ij} a_{jk} a_{ki} = 1}} e^{i2\pi \int_{M_{\text{Latt}}^d} \omega_d(\{a_{ij}\}) - U \sum_{(ijk)} |a_{ij} a_{jk} a_{ki} - 1|}, \end{aligned} \quad (165)$$

where  $\sum_{(ijk)}$  sums over all 3-simplices, and  $U \rightarrow +\infty$ .

Note that the above model is a local bosonic model, not the Dijkgraaf-Witten gauge theory. The Dijkgraaf-Witten gauge theory is defined by

$$\begin{aligned} Z_{G, \omega_d, \text{DW}}(M_{\text{Latt}}^d) & \\ = \sum_{\substack{[\{a_{ij}\}] \\ a_{ij} a_{jk} a_{ki} = 1}} e^{i2\pi \int_{M_{\text{Latt}}^d} \omega_d(\{a_{ij}\}) - U \sum_{(ijk)} |a_{ij} a_{jk} a_{ki} - 1|}, \end{aligned} \quad (166)$$

where the summation  $\sum_{[\{a_{ij}\}]}$  is over the gauge equivalent class,  $[\{a_{ij}\}]$ , of the configurations,  $\{a_{ij}\}$ . In contrast, the summation  $\sum_{\{a_{ij}\}}$  in the group-link model is over all the configurations,  $\{a_{ij}\}$  (without the gauge reduction). However, the volume-independent partition function of the two models are the same

$$Z_{G, \omega_d, \text{DW}}^{\text{top}}(M_{\text{Latt}}^d) = Z_{G, \omega_d}^{\text{top}}(M_{\text{Latt}}^d) \quad (167)$$

So the two models have the same emergent topological order.

As an example, let us compute the topological invariant for 2+1D lens space  $L^3(p)$ , using the explicit CW-

complex decomposition in Fig. 7:

$$Z_{G,\omega_3}^{\text{top}}(L^3(p)) = \frac{1}{|G|^2} \sum_{\substack{g,h \in G \\ g^p=1}} e^{i2\pi \sum_{m=0}^{p-1} \omega_3(g, g^m h, h^{-1}gh)}. \quad (168)$$

For  $G = \mathbb{Z}_n$ ,  $\omega_3 \in \mathcal{H}^3(\mathbb{Z}_n, \mathbb{R}/\mathbb{Z}) = \mathbb{Z}_n$  is labeled by  $k \in \mathbb{Z}_n$ :

$$\omega_3(g_1, g_2, g_3) = \frac{k}{n^2} g_1(g_2 + g_3 - [g_2 + g_3]_n). \quad (169)$$

We find that

$$Z_{\mathbb{Z}_n,k}^{\text{top}}(L^3(p)) = Z_{k;a\mathcal{B}a\mathbb{Z}_n}^{\text{top}}(L^3(p)). \quad (170)$$

In fact, the topological term in the  $\mathbb{Z}_n$  Dijkgraaf-Witten theory and the topological term in the  $\mathbb{Z}_n$ -1-cocycle model are directly related

$$2\pi \int_{M^3} \omega_3(\{a^{\mathbb{Z}_n}\}) = k \frac{2\pi}{n} \int_{M^3} a^{\mathbb{Z}_n} \mathcal{B}_n a^{\mathbb{Z}_n}, \quad (171)$$

as one can see from eqn. (169) and the explicit expression of  $a^{\mathbb{Z}_n} \mathcal{B}_n a^{\mathbb{Z}_n}$ :

$$\begin{aligned} \langle a^{\mathbb{Z}_n} \mathcal{B}_n a^{\mathbb{Z}_n}, (ijkl) \rangle &= a_{ij}^{\mathbb{Z}_n} \langle \mathcal{B}_n a^{\mathbb{Z}_n}, (jkl) \rangle; \\ \langle \mathcal{B}_n a^{\mathbb{Z}_n}, (jkl) \rangle &= \frac{1}{n} (a_{jk}^{\mathbb{Z}_n} + a_{kl}^{\mathbb{Z}_n} - a_{jl}^{\mathbb{Z}_n}) \\ &= \frac{1}{n} (a_{jk}^{\mathbb{Z}_n} + a_{kl}^{\mathbb{Z}_n} - [a_{jk}^{\mathbb{Z}_n} + a_{kl}^{\mathbb{Z}_n}]_n). \end{aligned} \quad (172)$$

Therefore, the  $\mathbb{Z}_n$ -1-cocycle model realizes the  $\mathbb{Z}_n$  Dijkgraaf-Witten theory.

### E. Symmetric topological orders described by gauge theories

We can also construct local bosonic models (called mixed group-vertex group-link models) that will produce topological orders described by a  $G_{\text{gauge}}$ -gauge theory that also have a symmetry  $G_{\text{symm}}$ . In the mixed model, the local degrees of freedom are group elements  $g_i \in G_{\text{symm}}$  living on the links group elements  $a_{ij} \in G_{\text{gauge}}$  living on the links of the space-time lattice  $M_{\text{Latt}}^d$ . Then, using the homogeneous group-cocycle  $\nu_n(\{g_i\}) \in \mathcal{H}^n(G_{\text{symm}}, \mathbb{R}/\mathbb{Z})$ , and the inhomogeneous group-cocycle  $\omega_{d-n}(\{a_{ij}\}) \in \mathcal{H}^{d-n}(G_{\text{gauge}}, \mathbb{R}/\mathbb{Z})$ , we can construct the mixed model

$$\begin{aligned} Z_{\nu_n \omega_{d-n}}(M_{\text{Latt}}^d) \\ = \sum_{\{g_i, a_{ij}\}, a_{ij} a_{jk} a_{ki} = 1} e^{i2\pi \int_{M_{\text{Latt}}^d} \nu_n(\{g_i\}) \omega_{d-n}(\{a_{ij}\})}, \end{aligned} \quad (173)$$

We can also construct a more general mixed model using inhomogeneous group-cocycle  $\omega_d \in \mathcal{H}^d(G_{\text{symm}} \times G_{\text{gauge}}, \mathbb{R}/\mathbb{Z})$ :

$$\begin{aligned} Z_{\nu_n \omega_{d-n}}(M_{\text{Latt}}^d) \\ = \sum_{\{g_i, a_{ij}\}, a_{ij} a_{jk} a_{ki} = 1} e^{i2\pi \int_{M_{\text{Latt}}^d} \omega_d(\{(g_i^{-1} g_j, a_{ij})\})}, \end{aligned} \quad (174)$$

where  $(g_i, a_{ij})$  is the group element of  $G_{\text{symm}} \times G_{\text{gauge}}$ .

We can construct an even more general mixed model using inhomogeneous group-cocycle  $\omega_d \in \mathcal{H}^d(G_{\text{PSG}}, \mathbb{R}/\mathbb{Z})$ <sup>85</sup>:

$$\begin{aligned} Z_{\omega_d; G_{\text{PSG}}}(M_{\text{Latt}}^d) \\ = \sum_{\{g_i, a_{ij}\}, a_{ij}^{\text{PSG}} a_{jk}^{\text{PSG}} a_{ki}^{\text{PSG}} = 1} e^{i2\pi \int_{M_{\text{Latt}}^d} \omega_d(\{(g_i^{-1} g_j, a_{ij})\})}, \end{aligned} \quad (175)$$

where  $G_{\text{PSG}}$  is a group that contains  $G_{\text{gauge}}$  as a normal subgroup such that  $G_{\text{PSG}}/G_{\text{gauge}} = G_{\text{symm}}$ , and  $a_{ij}^{\text{PSG}} = (g_i^{-1} g_j, a_{ij})$  is the group element of  $G_{\text{PSG}}$ <sup>69</sup>. In other words,  $G_{\text{PSG}}$  is an extension of  $G_{\text{symm}}$  by  $G_{\text{gauge}}$ , which is also described by the following short exact sequence

$$1 \rightarrow G_{\text{gauge}} \rightarrow G_{\text{PSG}} \rightarrow G_{\text{symm}} \rightarrow 1. \quad (176)$$

In this case, as discussed in Ref. 69, a gauge charge does not transform as a representation of  $G_{\text{gauge}}$ , but rather, transforms as a representation of  $G_{\text{PSG}}$ . Under the symmetry transformation, the gauge charge transforms according to  $G_{\text{PSG}}$  (which is called the projective symmetry group). In fact,  $G_{\text{PSG}}$  describes the so called ‘‘symmetry fractionalization’’.

If there is a symmetry twist described by  $a_{ij}^{\text{symm}} \in G_{\text{symm}}$  on the links, then the partition function will be

$$\begin{aligned} Z_{\omega_d; G_{\text{PSG}}}(M_{\text{Latt}}^d, a_{ij}^{\text{symm}}) \\ = \sum_{\{g_i, a_{ij}\}, a_{ij} a_{jk} a_{ki} = 1} e^{i2\pi \int_{M_{\text{Latt}}^d} \omega_d(\{(g_i^{-1} a_{ij}^{\text{symm}} g_j, a_{ij})\})}. \end{aligned} \quad (177)$$

The above construction also applies to the situation where  $G_{\text{symm}}$  contains time-reversal symmetry. In that case,  $a_{ij}^{\text{symm}}$  will contain contribution from the change of the orientations of the manifold, and  $\omega_d \in \mathcal{H}^d(G_{\text{PSG}}, (\mathbb{R}/\mathbb{Z})_T)$  where time-reversal  $T \in G_{\text{PSG}}$  will have a sign-changing action on  $\mathbb{R}/\mathbb{Z}$ .

If we include  $\mathbb{Z}_n$ -2-cochain field  $b^{\mathbb{Z}_n}$ , we can construct new general local boson models with emergent symmetric topological order, such as<sup>45</sup>

$$\begin{aligned} Z_{b^{\mathbb{Z}_n} \omega_{d-2}; G_{\text{PSG}}}(M_{\text{Latt}}^d, a_{ij}^{\text{symm}}) = \\ \sum_{\substack{\{g_i, a_{ij}, b_{ijk}^{\mathbb{Z}_n}\}, \text{db}^{\mathbb{Z}_n} = 0 \\ a_{ij} a_{jk} a_{ki} = 1}} e^{i2\pi \int_{M_{\text{Latt}}^d} b^{\mathbb{Z}_n} \omega_{d-2}(\{(g_i^{-1} a_{ij}^{\text{symm}} g_j, a_{ij})\})}, \end{aligned} \quad (178)$$

where we have assumed that  $n\omega_{d-2} = 0$ . This model has an emergent  $\mathbb{Z}_n \times G_{\text{gauge}}$ -gauge theory with  $G_{\text{symm}}$  symmetry. When,  $G_{\text{gauge}} = 1$ , the  $\mathbb{Z}_n$  charge may carry a projective representation of  $G_{\text{symm}}$ . When,  $G_{\text{symm}} = 1$ , the  $\mathbb{Z}_n$  charge may carry a projective representation of  $G_{\text{gauge}}$ . In general, the  $\mathbb{Z}_n$  charge may carry projective representation of  $G_{\text{PSG}}$  (i.e. with mixed fractionalized symmetry  $G_{\text{symm}}$  charge and gauge  $G_{\text{gauge}}$  charge).



## VI. TIME-REVERSAL SYMMETRIC TOPOLOGICAL ORDERS

In this section, we are going to construct exactly soluble local bosonic models that have time-reversal symmetry and emergent time-reversal symmetric topological orders. The time-reversal symmetry  $T$  is described by the symmetry group  $Z_2^T$ , which means  $T^2 = 1$ . We will first construct 2+1D models and then 3+1D models. All the 3+1D models realize time-reversal symmetric  $Z_2$ -gauge theories at low energies.

### A. 2+1D time-reversal symmetric $Z_2$ -1-cocycle models

#### 1. Model construction

We start with the  $Z_2$ -1-cocycle models which produce time-reversal-symmetry enriched  $Z_2$  topological orders and double-semion topological orders in 2+1D. The partition function has a form

$$Z_{Z_2^T T}(M_{\text{Latt}}^3) = \sum_{\{a_{ij}^{Z_2}\}, da^{Z_2}=0} e^{i\pi \int_{M_{\text{Latt}}^3} W(a^{Z_2}, w_m)} \quad (179)$$

The possible topological terms  $W(a^{Z_2}, w_m)$  are mixture of 1-cocycle  $a^{Z_2}$  and Stiefel-Whitney classes  $w_m$ . Here  $W(a^{Z_2}, w_m)$  has its value in  $\mathbb{Z}_2$ . Thus  $e^{i\pi \int_{M_{\text{Latt}}^3} W(a^{Z_2}, w_m)} = \pm 1$  and there is time-reversal symmetry in our model. Also since  $W(a^{Z_2}, w_m) \in C^3(M_{\text{Latt}}^3; \mathbb{Z}_2)$ ,  $e^{i\pi \int_{M_{\text{Latt}}^3} W(a^{Z_2}, w_m)}$  is well defined even for non-orientable manifold  $M_{\text{non}}^3$  where  $H_3(M_{\text{non}}^3; \mathbb{Z}) = 0$  but  $H_3(M_{\text{non}}^3; \mathbb{Z}_2) = \mathbb{Z}_2$ . We also note that for non-orientable manifold,  $M_{\text{non}}^3$  itself is a chain with boundary (*i.e.*  $M_{\text{non}}^3$  is not a cycle). Therefore  $\int_{M_{\text{non}}^3} db \neq 0$ , for a 2-cochain  $b$ .

The possible topological terms are given by the combinations of the following six 3-cocycles:

$$\begin{array}{lll} w_1^3, & w_1 w_2, & w_3, \\ (a^{Z_2})^3, & w_1 (a^{Z_2})^2, & w_1^2 a^{Z_2}. \end{array} \quad (180)$$

From Appendix D3, we find many relations between Stiefel-Whitney and the  $Z_2$ -1-cocycle:

$$\begin{aligned} w_1^2 &= w_2, \quad w_1 w_2 = w_3 = 0, \\ w_1 (a^{Z_2})^2 &= \text{Sq}^1((a^{Z_2})^2) = 2(a^{Z_2})^3 = 0. \end{aligned} \quad (181)$$

So the most general time-reversal symmetric  $Z_2$ -1-cocycle model that couples to Stiefel-Whitney classes is given by

$$\begin{aligned} & Z_{k_1 k_2; tZ_2^T T}(M_{\text{Latt}}^3) \\ &= \sum_{\{a_{ij}^{Z_2}\}, da^{Z_2}=0} e^{i\pi \int_{M_{\text{Latt}}^3} k_1 a^{Z_2} \mathcal{B}_2 a^{Z_2} + k_2 w_1^2 a^{Z_2}} \end{aligned} \quad (182)$$

where  $k_1, k_2 \in \mathbb{Z}_2$ , and we have used  $(a^{Z_2})^3 = a^{Z_2} \mathcal{B}_2 a^{Z_2}$ .

We like to remark that the Stiefel-Whitney class  $w_1$  in the above path integral can be induced by a local degrees of freedom – a pseudo-scalar  $\tilde{g}_i$  introduced in Section VC. Using  $w_1 = d\tilde{g}_i - dg_i$ , where  $g_i$  is  $\mathbb{Z}_2$ -single-valued 0-cochain, we can rewrite the above path integral as (the  $g_i$  dependence disappears)

$$\begin{aligned} & Z_{k_1 k_2; tZ_2^T T}(M_{\text{Latt}}^3) \\ &= \sum_{\{\tilde{g}_i, a_{ij}^{Z_2}\}, da^{Z_2}=0} e^{i\pi \int_{M_{\text{Latt}}^3} k_1 a^{Z_2} \mathcal{B}_2 a^{Z_2} + k_2 \mathcal{B}_2 d\tilde{g} a^{Z_2}} \end{aligned} \quad (183)$$

which is a pure local bosonic model.

The above four local bosonic models with different values of  $k_1, k_2$  give rise to four different time-reversal-symmetry enriched topological orders. If we break the time-reversal symmetry, the above local bosonic model will only give rise to two different topological orders labeled by  $k_1$ : the  $Z_2$  topological order (*i.e.* the  $Z_2$  gauge theory) for  $k_1 = 0$  and the double-semion topological order for  $k_1 = 1$ .

#### 2. Topological partition functions

Next, we will compute the volume independent partition function, which is given by

$$\begin{aligned} & Z_{k_1 k_2; tZ_2^T T}^{\text{top}}(M^3) \\ &= \frac{1}{2} \sum_{a^{Z_2} \in H^1(M^3; \mathbb{Z}_2)} e^{i\pi \int_{M^3} k_1 a^{Z_2} \mathcal{B}_2 a^{Z_2} + k_2 w_1^2 a^{Z_2}} \end{aligned} \quad (184)$$

On  $M^3 = S^1 \times \Sigma_g$ ,  $\int_{M^3} k_1 a^{Z_2} \mathcal{B}_2 a^{Z_2} + k_2 w_1^2 a^{Z_2} = 0$ . Thus

$$Z_{k_1 k_2; tZ_2^T T}^{\text{top}}(S^1 \times \Sigma_g) = 2^{2g} \quad (185)$$

On  $M^3 = S^1 \times \Sigma_g^{\text{non}}$ , we note that the cohomology ring  $H^*(S^1 \times \Sigma_g^{\text{non}}; \mathbb{Z}_2)$  has a basis

$$H^*(S^1 \times \Sigma_g^{\text{non}}; \mathbb{Z}_2) = \{a_0, a_i |_{i=1, \dots, g}, a_0 a_i, b, a_0 b\} \quad (186)$$

with  $a_0, a_i \in H^1(S^1 \times \Sigma_g^{\text{non}}; \mathbb{Z}_2)$  and  $b \in H^2(S^1 \times \Sigma_g^{\text{non}}; \mathbb{Z}_2)$ , which have the following cup product:

$$a_i^2 = b, \quad a_0^2 = a_i b = 0. \quad (187)$$

The Stiefel-Whitney classes are given by

$$w_1 = \sum_{i=1}^g a_i, \quad w_2 = w_1^2 = [g]_2 b, \quad (188)$$

and the Bockstein homomorphism is given by

$$\mathcal{B}_2 a_i = (a_i)^2 = b, \quad \mathcal{B}_2 a_0 = 0. \quad (189)$$

Expand

$$a^{Z_2} = \sum_{\mu=0}^g \alpha_{\mu} a_{\mu}, \quad (190)$$

we find that

$$\begin{aligned}
& Z_{k_1 k_2; \mathbb{Z}_2 a T}^{\text{top}}(S^1 \times \Sigma_g^{\text{non}}) \\
&= \frac{1}{2} \sum_{a_\mu=0,1} e^{i\pi(k_1 \alpha_0 \sum_{i=1}^g \alpha_i + k_2 g \alpha_0)} \\
&= \sum_{a_i=0,1} \delta_2(k_1 \sum_{i=1}^g \alpha_i + k_2 g) \\
&= (1 - k_1)[k_2 g + 1]_2 2^g + k_1 2^{g-1} \quad (191)
\end{aligned}$$

The results are summarized in Table I.

We like to remark that  $Z_2 \times Z_2^T$ -SPT states are classified by  $\mathcal{H}^3(Z_2 \times Z_2^T; (\mathbb{R}/\mathbb{Z})_T) = \mathbb{Z}_2^{\oplus 2}$ . For a  $Z_2 \times Z_2^T$ -SPT state labeled by  $(k_1, k_2) \in \mathbb{Z}_2^{\oplus 2}$ , its SPT invariant is given by  $Z^{\text{top}}(M^3, a^{\mathbb{Z}_2}) = e^{i\pi \int_{M^3} k_1 a^{\mathbb{Z}_2} \mathcal{B}_2 a^{\mathbb{Z}_2} + k_2 w_1^2 a^{\mathbb{Z}_2}}$ , where  $a^{\mathbb{Z}_2}$  describes the  $Z_2$ -symmetry twist on  $M^3$ . Such SPT invariant happen to be the phase factor in eqn. (184), and the in summation in eqn. (184) happen to be the summation of all possible  $Z_2$ -symmetry twists. The implies that the topological orders produced by the 2+1D  $\mathbb{Z}_2$ -1-cocycle model can be regarded as the  $Z_2$ -gauged  $Z_2 \times Z_2^T$ -SPT states.

### 3. Properties of excitations

When  $k_1 = 0$ , the 2+1D  $\mathbb{Z}_2$ -1-cocycle model has an emergent  $Z_2$  topological order described by a low energy  $Z_2$ -gauge theory. It has four types of point-like excitations: 1,  $e$ ,  $m$ ,  $\varepsilon = em$ , where  $\varepsilon$  is a fermion and others are bosons. When  $k_1 = 1$ , the cocycle model has an emergent double-semion topological order. It has four types of point-like excitations: 1,  $e$ ,  $m$ ,  $\varepsilon$ , where  $e$  is an semion with spin  $\frac{1}{4}$ , and  $\varepsilon$  an semion with spin  $-\frac{1}{4}$ . 1 and  $e$  are bosons, and they carry  $Z_2$ -charge 0 and 1 respectively.

To obtain more properties of the excitations in those  $T$ -symmetric topological orders, let us consider dimension reduction. In general, when we reduce a stable phase  $\mathcal{C}^d$  in  $d$ -dimension to lower dimension  $d'$  via a compactification  $M^d \rightarrow M^{d'} \times N^{d-d'}$ , the resulting lower dimensional phase on  $M^{d'}$  may correspond to several stable phases  $\mathcal{C}_i^{d'}$  with accidental degenerate energy.<sup>90</sup> We denote such dimension reduction as

$$\mathcal{C}^d = \bigoplus_i \mathcal{C}_i^{d'}, \quad (192)$$

and refer  $\mathcal{C}_i^{d'}$ 's as different sectors. The different sector arise from different field configurations on  $N^{d-d'}$ . We like to ask: what are effective theories for those  $d'$ -dimensional systems in each sector?

To apply the above general picture to our case, let us assume the space-time to be  $M^3 = M^2 \times S^1$  and  $S^1$  is a small circle. We can view the 2+1D  $\mathbb{Z}_2$ -1-cocycle models as a 1+1D local bosonic systems. Then what is the effective theory for such 1+1D systems?

To answer the above question, we can write  $a^{\mathbb{Z}_2}$  as  $a^{\mathbb{Z}_2} = a_{M^2}^{\mathbb{Z}_2} + a_{S^1}^{\mathbb{Z}_2}$ , where  $a_{M^2}^{\mathbb{Z}_2}$  are low energy degrees



FIG. 8. (Color online) In a dimension reduction from 2D space to 1D space (a cylinder), a hole in the 2D space becomes an end of the 1D space. The  $Z_2$ -vortex with  $\int_C a^{\mathbb{Z}_2} = 1$  in 2D space becomes the  $\int_C a^{\mathbb{Z}_2} = 1$  sector in the 1D space.

of freedom only live on  $M^2$  (*i.e.* constant in the  $S^1$  direction), and  $a_{S^1}^{\mathbb{Z}_2}$  are high energy degrees of freedom only live on  $S^1$  (*i.e.* constant in the  $M^2$  directions). The different field configurations on  $S^1$  are labeled by  $\alpha = \int_S a_{S^1}^{\mathbb{Z}_2} \in \mathbb{Z}_2$ . So the different sectors are also labeled by  $\alpha = 0, 1$ . The partition function on  $M^2 \times S^1$  becomes

$$\begin{aligned}
& Z_{k_1 k_2; \mathbb{Z}_2 a T}(M^2 \times S^1) \\
&= \sum_{\{a_{ij}^{\mathbb{Z}_2}\}, \text{da}^{\mathbb{Z}_2}=0} e^{i\pi \int_{M^2 \times S^1} k_1 a^{\mathbb{Z}_2} \mathcal{B}_2 a^{\mathbb{Z}_2} + k_2 \mathcal{B}_2 d\tilde{g} a^{\mathbb{Z}_2}} \\
&= \sum_{\{a_{ij}^{\mathbb{Z}_2}\}, \text{da}_{M^2}^{\mathbb{Z}_2}=0} e^{i\pi \alpha \int_{M^2} k_1 \mathcal{B}_2 a_{M^2}^{\mathbb{Z}_2} + k_2 \mathcal{B}_2 d\tilde{g}} \quad (193)
\end{aligned}$$

We see that in the sector  $\alpha = 0$ , the resulting 1+1D  $Z_2^T$  SPT order is trivial. In contrast, in the sector  $\alpha = 1$ , the resulting 1+1D  $Z_2^T$ -SPT order is non-trivial. Usually, in 1+1D, the gauge field  $a_{M^2}^{\mathbb{Z}_2}$  fluctuate strongly. Here, we want to treat the 1+1D system as reduced from the 2+1D system as shown in Fig. 8. In this case, we can assume the gauge field  $a_{M^2}^{\mathbb{Z}_2}$  to fluctuate weakly, and treat  $a_{M^2}^{\mathbb{Z}_2}$  as a background probe field. Therefore, we can view the 1+1D system as a system with  $Z_2 \times Z_2^T$  symmetry. Then form the 1+1D effective theory eqn. (193) which can be viewed as an SPT invariant<sup>55</sup>, we see that in the sector  $\alpha = 1$  is described by a  $Z_2 \times Z_2^T$ -SPT state labeled by  $(k_1, k_2)$ , which agrees with the group cohomology result  $\mathcal{H}^2(Z_2 \times Z_2^T, \mathbb{R}/\mathbb{Z}_T) = \mathbb{Z}_2^{\oplus 2}$ .

If  $(k_1, k_2) = (0, 1)$ , the 1+1D SPT state is a pure  $Z_2^T$ -SPT state as indicated by the term  $e^{i\pi \int_{M^2} k_2 \mathcal{B}_2 d\tilde{g}}$ . Such SPT state has Kramer doublet at the chain end. In fact, the chain end has to sector with  $Z_2$ -charge 0 and with  $Z_2$ -charge 1. Both sectors are Kramer doublets. We may view the 1+1D system with a chain end as a 2+1D system with a hole as described in Fig. 8. Thus the  $\alpha = 1$  sector, correspond to a  $\pi$ -flux in 2+1D. We see that a  $\pi$ -flux carries a Kramer doublet regardless if it carries addition  $Z_2$ -charge or not. Similarly, the  $\alpha = 0$  sector gives rise to trivial 1+1D SPT state, and thus a  $\pi$ -flux carries a time-reversal singlet regardless if it carries addition  $Z_2$ -charge or not. To summarize,

the 2+1D  $\mathbb{Z}_2$ -1-cocycle model labeled by  $(k_1, k_2) = (0, 1)$  has four types of point-like excitations  $1, e, m, \varepsilon = em$ . The excitations  $m, \varepsilon$  carry  $\pi$ -flux, while the excitations  $e, \varepsilon$  carry a  $Z_2$ -gauge 1. The excitations  $m, \varepsilon$  are Kramer doublets and the excitation  $\varepsilon$  is a fermion (see Table I).

The time-reversal singlet has a quantum dimension  $d = 1$  and the Kramer doublet has a quantum dimension  $d = 2$ . (Quantum dimension is the dimension of the Hilbert space for the internal degrees of freedom carried by a particle.) Thus the four types of particles have the following quantum dimensions  $(d_1, d_e, d_m, d_f) = (1, 1, 2_-, 2_-)$ , where the subscript  $-$  indicates the Kramer doublet. A particle can also carry spin  $s$ , which is defined mod 1. A boson has spin 0 mod 1 and a fermion has spin  $\frac{1}{2}$  mod 1. Thus, the four types of particles have the following spins  $(s_1, s_e, s_m, s_f) = (0, 0, 0, \frac{1}{2})$  (see Table I).

If  $(k_1, k_2) = (1, 0)$ , the cocycle model has four excitations:  $1, e, m, \varepsilon$ .  $1$  and  $e$  transform as time-reversal singlet.  $m$  and  $\varepsilon$  transform into each other and form a time-reversal doublet. Since  $m$  and  $\varepsilon$  are always degenerate with time-reversal symmetry, we view them as a single type of excitation with quantum dimension 2. Thus

the 2+1D  $\mathbb{Z}_2$ -1-cocycle model labeled by  $(k_1, k_2) = (1, 0)$  has three types of point-like excitations with quantum dimensions  $(d_i) = (1, 1, 2)$  and spins  $(s_i) = (0, 0, [\frac{1}{4}, \frac{3}{4}])$ .

Under the dimension reduction, the 1+1D state in  $\alpha = 1$  sector is a  $Z_2 \times Z_2^T$ -SPT state described by the SPT invariant  $e^{i\pi \int_{M^2} \mathcal{B}_2 a_{M^2}^{\mathbb{Z}_2}}$ . The chain end for such a  $Z_2 \times Z_2^T$ -SPT is a doublet with fraction  $Z_2$ -charge  $\pm \frac{1}{2}$ . Under the time reversal, the  $+\frac{1}{2}$  and  $-\frac{1}{2}$   $Z_2$ -charge states get exchanged and  $T^2 = 1$ . Thus, the  $\pi$ -flux in 2+1D ground state will carries a doublet of  $\pm \frac{1}{2}$   $Z_2$ -charges. There are 2 types of 0-flux excitations with 0 and 1  $Z_2$ -charges. Those two types of excitations are time-reversal singlet. Thus we denote that quantum dimensions for those excitations as  $(d_i) = (1, 1, 2_+)$ , where subscript  $+$  indicates  $T^2 = 1$  (see Table I).

If  $(k_1, k_2) = (1, 1)$ , under the dimension reduction, the 1+1D state in  $\alpha = 1$  sector is a  $Z_2 \times Z_2^T$ -SPT state described by the SPT invariant  $e^{i\pi \int_{M^2} \mathcal{B}_2 a_{M^2}^{\mathbb{Z}_2} + \mathcal{B}_2 d\tilde{g}}$ . The chain end for such a  $Z_2 \times Z_2^T$ -SPT states may contain four degenerate states formed by a doublet with fraction  $Z_2$ -charge  $\pm \frac{1}{2}$  and a Kramer doublet. The time-reversal transformation is described by

$$T = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} K = \sigma^1 \otimes i\sigma^2 K, \quad T^2 = -1, \quad (194)$$

where  $K$  is the anti-unitary transformation, and  $\sigma^{1,2,3}$  are the Pauli matrices. The  $Z_2$  symmetry is generated

by

$$Q = \begin{pmatrix} i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \end{pmatrix} = i\sigma^3 \otimes \sigma^0, \quad Q^2 = -1. \quad (195)$$

However, the four states can be split by a time-reversal and  $Z_2$  symmetric perturbation

$$\delta H = \Delta \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \Delta \sigma^3 \otimes \sigma^3. \quad (196)$$

Thus the chain end in general has a doublet with fractional  $Z_2$ -charge  $\pm \frac{1}{2}$  which is also a  $T^2 = -1$  Kramer doublet at the same time. As a result, the  $\pi$ -flux in 2+1D ground state will carries a Kramer doublet with fractional  $Z_2$ -charge  $\pm \frac{1}{2}$ . We stress that there is no time-reversal symmetric perturbation that can give rise to  $T^2 = 1$  doublet. To summarize,

the 2+1D  $\mathbb{Z}_2$ -1-cocycle model labeled by  $(k_1, k_2) = (1, 1)$  has three types of point-like excitations with quantum dimensions  $(d_1, d_e, d_s) = (1, 1, 2_-)$  and spins  $(s_1, s_e, s_s) = (0, 0, [\frac{1}{4}, \frac{3}{4}])$ , where subscript “ $-$ ” indicates  $T^2 = -1$  (see Table I).

## B. 2+1D time-reversal symmetric $Z_4^T$ -group cohomology models

### 1. Model construction

Using the group cocycles, we can construct more local bosonic models that can produce time-reversal symmetric 2+1D (twisted)  $Z_2$ -gauge theories at low energy (see eqn. (177)). In this section, we will discuss those models.

We put  $\mathbb{Z}_2$  degrees of freedom on both vertices and links:  $\tilde{g}_i \in \mathbb{Z}_2$  and  $a_{ij}^{\mathbb{Z}_2} \in \mathbb{Z}_2$ . Note that  $\tilde{g}_i$  is a pseudo scalar as discussed in Section V C (see Fig. 6). Using

$$1 \rightarrow \mathbb{Z}_2 \rightarrow \mathbb{Z}_4 \rightarrow \mathbb{Z}_2 \rightarrow 1, \quad (197)$$

we can construct a  $\mathbb{Z}_4$ -1-cocycle field

$$a_{ij}^{\mathbb{Z}_4} = 2a_{ij}^{\mathbb{Z}_2} + (d\tilde{g})_{ij}. \quad (198)$$

Notice that  $\mathcal{H}^3(\mathbb{Z}_4, (\mathbb{R}/\mathbb{Z})_{\mathbb{Z}_4}) = 0$ . Thus there is no group cocycle term in the action amplitude. We obtain the following time-reversal symmetric model

$$Z_{Z_4^T}(M^3) = \sum_{\substack{\{a_{ij}^{\mathbb{Z}_2}, \tilde{g}_i\}, d a^{\mathbb{Z}_4} = 0 \\ a^{\mathbb{Z}_4} = 2a^{\mathbb{Z}_2} + d\tilde{g}}} 1 \quad (199)$$

The condition  $da^{\mathbb{Z}_4} = 0$  becomes (when we view the cochains as  $\mathbb{Z}$  valued)

$$\begin{aligned} da^{\mathbb{Z}_4} &= 2da^{\mathbb{Z}_2} + d(d\tilde{g}) \stackrel{4}{=} 0 \\ \rightarrow da^{\mathbb{Z}_2} + \mathcal{B}_2 d\tilde{g} &\stackrel{2}{=} 0. \end{aligned} \quad (200)$$

We can rewrite the above partition function as

$$Z_{Z_4^T}(M^3) = \sum_{\{a_{ij}^{\mathbb{Z}_2}, \tilde{g}_i\}, da^{\mathbb{Z}_2} \stackrel{2}{=} \mathcal{B}_2 d\tilde{g}} 1. \quad (201)$$

We see that such a model is different from the model eqn. (183) with  $k_{1,2} = 0$ . The condition  $da^{\mathbb{Z}_2} = \mathcal{B}_2 d\tilde{g}$  encode the non-trivial group extension (197).

Due to the relation  $\mathcal{B}_2 d\tilde{g} = \mathcal{B}_2 w_1 \stackrel{2}{=} w_1^2$ ,  $Z_{Z_4^T}(M^3) \neq 0$  only when  $w_1^2 = 0$  as  $\mathbb{Z}_2$ -valued cohomology class. Thus we introduce

$$\bar{\delta}_m(c) = \begin{cases} 0 & \text{if } c \neq db \text{ mod } m, \\ 1 & \text{if } c = db \text{ mod } m. \end{cases} \quad (202)$$

So  $Z_{Z_4^T}(M^3)$  contains a factor  $\bar{\delta}_2(\mathcal{B}_2 w_1)$ . Furthermore, on space-time  $M^3$  with  $\mathcal{B}_2 w_1 = 0$ , we have  $da^{\mathbb{Z}_2} \stackrel{2}{=} 0$ . In this case, we can combine the  $Z_2$ -1-cocycle model and the  $Z_4^T$  group cohomology model together:

$$\begin{aligned} &Z_{k_0 k_1 k_2; tZ_2 aT}(M^3) \\ &= \sum_{\{\tilde{g}_i, a_{ij}^{\mathbb{Z}_2}\}, da^{\mathbb{Z}_2} \stackrel{2}{=} k_0 \mathcal{B}_2 d\tilde{g}} e^{i\pi \int_{M^3} k_1 (a^{\mathbb{Z}_2})^3 + k_2 a^{\mathbb{Z}_2} \mathcal{B}_2 d\tilde{g}} \\ &= \sum_{\{\tilde{g}_i, a_{ij}^{\mathbb{Z}_2}, \tilde{a}_{ij}^{\mathbb{Z}_2}\}} e^{i\pi \int_{M^3} k_1 (a^{\mathbb{Z}_2})^3 + \tilde{a}^{\mathbb{Z}_2} da^{\mathbb{Z}_2} + k_0 \tilde{a}^{\mathbb{Z}_2} \mathcal{B}_2 d\tilde{g} + k_2 a^{\mathbb{Z}_2} \mathcal{B}_2 d\tilde{g}}. \end{aligned} \quad (203)$$

When  $k_0 = 0$ , the above model reduces to the  $Z_2$ -1-cocycle model eqn. (183). When  $k_0 = 1$  and  $k_1 = k_2 = 0$ , the above becomes the  $Z_4^T$  group cohomology model. The volume-independent partition function is given by

$$\begin{aligned} Z_{k_0 k_1 k_2; tZ_2 aT}^{\text{top}}(M^3) &= \frac{\bar{\delta}_2(k_0 \mathcal{B}_2 w_1)}{|H^0(M^3; \mathbb{Z}_2)|} \times \\ &\sum_{a^{\mathbb{Z}_2} \in H^1(M^3; \mathbb{Z}_2)} e^{i\pi \int_{M^3} k_1 (a^{\mathbb{Z}_2})^3 + k_2 w_1^2 a^{\mathbb{Z}_2}}. \end{aligned} \quad (204)$$

In the above, we have assumed that when  $k_0 \mathcal{B}_2 w_1$  is a coboundary, we will choose such a coboundary to be zero. We note that  $Z_{k_0 k_1 k_2; tZ_2 aT}^{\text{top}}(M^3)$  is simply given by  $Z_{k_1 k_2; tZ_2 aT}^{\text{top}}(M^3)$  (see Section VIA 2) with an extra  $\bar{\delta}_2(k_0 \mathcal{B}_2 w_1)$  term.

When  $k_0 = 0$ , the above model becomes the one studied in Section VIA, and the topological order that it produces can be viewed as a gauged  $Z_2 \times Z_2^T$ -SPT state.

## 2. Properties of excitations

When  $k_0 = 1$ , the non-trivial group extension makes the time-reversal transformation  $T$  to have a property

that  $T^2$  is a  $Z_2$ -gauge transformation. So  $T^2 = -1$  for a non-trivial  $Z_2$ -charge. In other words, the  $e$  particle with  $Z_2$ -charge 1 carries a Kramer doublet.  $e$  is also a boson, since if we break the time-reversal symmetry, the above model give rise to the  $Z_2$  or double-semion topological orders, where in both cases, the  $Z_2$ -charge is a boson. We also note that when  $k_0 = 1$ ,  $k_2 = 0, 1$  gives rise to the same model.

When  $(k_0, k_1, k_2) = (1, 0, *)$ , the dimension reduction  $M^3 \rightarrow M^2 \times S^1$  does not produce non-trivial  $Z_2 \times Z_2^T$  SPT state in 1+1D, thus the  $Z_2$  vortex  $m$  in 2+1D is a time-reversal singlet and is a boson. The bound state of a  $Z_2$  charge and a  $Z_2$  vortex is a fermion that carries a Kramer doublet. The results are summarized in Table I.

When  $(k_0, k_1, k_2) = (1, 1, 0)$ , the dimension reduction  $M^3 \rightarrow M^2 \times S^1$  produce a non-trivial  $Z_2 \times Z_2^T$  SPT state in 1+1D, thus the  $Z_2$  vortex  $m$ . In fact the  $Z_2$  vortex  $m$  is a  $T^2 = 1$  time-reversal doublet that carries  $Z_2$ -gauge-charge  $\pm 1/2$  (the same as discussed in Section VIA 3 for the  $(k_0, k_1, k_2) = (0, 1, 0)$  case). The  $Z_2$ -gauge-charge  $\pm 1/2$  doublet is formed by a semion with spin  $s = 1/4$  and a conjugate semion with spin  $s = 3/4$ . The bound state of a  $Z_2$  charge and a  $Z_2$  vortex is  $\varepsilon$  which also form a time-reversal doublet. But  $\varepsilon$  is a  $T^2 = -1$  Kramer doublet that carries  $Z_2$ -gauge-charge  $\pm 1/2$ . To summarize,

the 2+1D  $\mathbb{Z}_4^T$  group-cocycle model labeled by  $(k_0, k_1, k_2) = (1, 1, 0)$  has four types of point-like excitations with quantum dimensions  $(d_1, d_e, d_m, d_\varepsilon) = (1, 2_-, 2_+, 2_-)$  and spins  $(s_1, s_e, s_m, s_\varepsilon) = (0, 0, [\frac{1}{4}, \frac{3}{4}], [\frac{1}{4}, \frac{3}{4}])$  (see Table I).

For  $(k_0, k_1, k_2) = (1, 1, 1)$  the results is the same as that for  $(k_0, k_1, k_2) = (1, 1, 0)$ , except that the properties of  $m$  and  $\varepsilon$  are exchanged. This is why  $(k_0, k_1, k_2) = (1, 1, 0)$  and  $(k_0, k_1, k_2) = (1, 1, 1)$  correspond to the same time-reversal SET order.

## 3. Including excitations in the path integral

Now, let us include the excitations in the partition function eqn. (203). Let  $M_e^1$  be the  $\mathbb{Z}_2$ -valued 1-cycle that correspond to the world-line of the  $Z_2$ -charge  $e$ :  $M_e^1 \in Z_1(M^4; \mathbb{Z}_2)$ . Let  $M_m^1$  be the  $\mathbb{Z}_2$ -valued 1-cycle that correspond to the world-line of the  $Z_2$ -vortex  $m$ :  $M_m^1 \in Z_1(M^4; \mathbb{Z}_2)$ . The Poincaré dual of  $M_e^1$  is a  $\mathbb{Z}_2$ -valued 2-cocycle  $B_e^{\mathbb{Z}_2}$  and the Poincaré dual of  $M_m^1$  is a  $\mathbb{Z}_2$ -valued 2-cocycle  $B_m^{\mathbb{Z}_2}$ :  $B_e^{\mathbb{Z}_2} \in Z^2(M^4; \mathbb{Z}_2)$  and  $B_m^{\mathbb{Z}_2} \in Z^2(M^4; \mathbb{Z}_2)$ . The partition function with excita-

tions is given by

$$\begin{aligned}
& Z_{k_0 k_1 k_2; t\mathbb{Z}_2 aT}(M^3) \\
&= \sum_{\{\tilde{g}_i, a_{ij}^{\mathbb{Z}_2}\}, da^{\mathbb{Z}_2} \equiv k_0 \mathcal{B}_2 d\tilde{g} + B_m^{\mathbb{Z}_2}} e^{i\pi \int_{M^3} k_1 (a^{\mathbb{Z}_2})^3 + k_2 a^{\mathbb{Z}_2} \mathcal{B}_2 d\tilde{g}} e^{i\pi \int_{M_e^1} a^{\mathbb{Z}_2}} \\
&= \sum_{\{\tilde{g}_i, a_{ij}^{\mathbb{Z}_2}, \tilde{a}_{ij}^{\mathbb{Z}_2}\}} e^{i\pi \int_{M^3} \tilde{a}^{\mathbb{Z}_2} (da^{\mathbb{Z}_2} + k_0 \mathcal{B}_2 d\tilde{g} + B_m^{\mathbb{Z}_2})} \\
&\quad e^{i\pi \int_{M^3} k_1 (a^{\mathbb{Z}_2})^3 + k_2 a^{\mathbb{Z}_2} \mathcal{B}_2 d\tilde{g}} e^{i\pi \int_{M_e^1} a^{\mathbb{Z}_2}} \\
&= \sum_{\{\tilde{g}_i, a_{ij}^{\mathbb{Z}_2}, \tilde{a}_{ij}^{\mathbb{Z}_2}\}} e^{i\pi \int_{M^3} \tilde{a}^{\mathbb{Z}_2} da^{\mathbb{Z}_2}} e^{i\pi \int_{M^3} k_1 (a^{\mathbb{Z}_2})^3} \\
&\quad e^{i\pi \int_{M^3} \tilde{a}^{\mathbb{Z}_2} (k_0 \mathcal{B}_2 d\tilde{g} + B_m^{\mathbb{Z}_2}) + a^{\mathbb{Z}_2} (k_2 \mathcal{B}_2 d\tilde{g} + B_e^{\mathbb{Z}_2})}.
\end{aligned} \tag{205}$$

Let us change the variables to

$$a^{\mathbb{Z}_2} \stackrel{2}{=} a_m^{\mathbb{Z}_2} + a_0^{\mathbb{Z}_2}, \quad \tilde{a}^{\mathbb{Z}_2} \stackrel{2}{=} a_e^{\mathbb{Z}_2} + \tilde{a}_0^{\mathbb{Z}_2}, \tag{206}$$

where  $a_0^{\mathbb{Z}_2}, \tilde{a}_0^{\mathbb{Z}_2} \in C^1(M^2; \mathbb{Z}_2)$ , and  $a_m^{\mathbb{Z}_2}, a_e^{\mathbb{Z}_2}$  are fixed  $\mathbb{Z}_2$ -valued 1-cochains satisfying

$$da_m^{\mathbb{Z}_2} \stackrel{2}{=} B_m^{\mathbb{Z}_2} + k_0 \mathcal{B}_2 w_1, \quad da_e^{\mathbb{Z}_2} \stackrel{2}{=} B_e^{\mathbb{Z}_2} + k_2 \mathcal{B}_2 w_1. \tag{207}$$

(Here we have assumed that  $B_m^{\mathbb{Z}_2} + k_0 \mathcal{B}_2 w_1$  and  $B_e^{\mathbb{Z}_2} + k_2 \mathcal{B}_2 w_1$  are coboundaries.) Now we can rewrite the partition function as

$$\begin{aligned}
& Z_{k_0 k_1 k_2; t\mathbb{Z}_2 aT}(M^3) \\
&= \sum_{\tilde{g}, a_0^{\mathbb{Z}_2}, \tilde{a}_0^{\mathbb{Z}_2}} e^{i\pi \int_{M^3} a_e^{\mathbb{Z}_2} da_m^{\mathbb{Z}_2} + \tilde{a}_0^{\mathbb{Z}_2} da_0^{\mathbb{Z}_2}} e^{i\pi \int_{M^3} k_1 (a_m^{\mathbb{Z}_2} + a_0^{\mathbb{Z}_2})^3} \\
&\quad e^{i\pi \int_{M^3} a_e^{\mathbb{Z}_2} (k_0 \mathcal{B}_2 d\tilde{g} + B_m^{\mathbb{Z}_2}) + a_m^{\mathbb{Z}_2} (k_2 \mathcal{B}_2 d\tilde{g} + B_e^{\mathbb{Z}_2})} \\
&= \sum_{\tilde{g}, da_0^{\mathbb{Z}_2}=0} e^{i\pi \int_{M^3} a_e^{\mathbb{Z}_2} da_m^{\mathbb{Z}_2}} e^{i\pi \int_{M^3} k_1 (a_m^{\mathbb{Z}_2} + a_0^{\mathbb{Z}_2})^3} \\
&\quad e^{i\pi \int_{M^3} a_e^{\mathbb{Z}_2} (k_0 \mathcal{B}_2 d\tilde{g} + B_m^{\mathbb{Z}_2}) + a_m^{\mathbb{Z}_2} (k_2 \mathcal{B}_2 d\tilde{g} + B_e^{\mathbb{Z}_2})}.
\end{aligned} \tag{208}$$

Since  $a_0^{\mathbb{Z}_2}$  becomes a cocycle, we can further simplify the factor  $e^{i\pi \int_{M^3} k_1 (a_m^{\mathbb{Z}_2} + a_0^{\mathbb{Z}_2})^3}$  using eqn. (21)

$$\begin{aligned}
& e^{i\pi \int_{M^3} k_1 (a_m^{\mathbb{Z}_2} + a_0^{\mathbb{Z}_2})^3} \\
&= e^{i\pi \int_{M^3} k_1 [(a_m^{\mathbb{Z}_2})^3 + (a_0^{\mathbb{Z}_2})^3 + (a_m^{\mathbb{Z}_2})^2 a_0^{\mathbb{Z}_2} + a_m^{\mathbb{Z}_2} (a_0^{\mathbb{Z}_2})^2]}.
\end{aligned} \tag{209}$$

The partition function now becomes

$$\begin{aligned}
& Z_{k_0 k_1 k_2; t\mathbb{Z}_2 aT}(M^3) \\
&= e^{i\pi \int_{M^3} a_m^{\mathbb{Z}_2} B_e^{\mathbb{Z}_2}} \sum_{\tilde{g}, da_0^{\mathbb{Z}_2}=0} e^{i\pi \int_{M^3} k_0 a_e^{\mathbb{Z}_2} \mathcal{B}_2 d\tilde{g} + k_2 a_m^{\mathbb{Z}_2} \mathcal{B}_2 d\tilde{g}} \times \\
&\quad e^{i\pi \int_{M^3} k_1 [(a_m^{\mathbb{Z}_2})^3 + (a_0^{\mathbb{Z}_2})^3 + (a_m^{\mathbb{Z}_2})^2 a_0^{\mathbb{Z}_2} + a_m^{\mathbb{Z}_2} (a_0^{\mathbb{Z}_2})^2]}.
\end{aligned} \tag{210}$$

The above partition function can be expressed in terms of linking numbers. Consider  $\int_{M^3} B_e^{\mathbb{Z}_2} a_m^{\mathbb{Z}_2} = \int_{M_e^1} a_m^{\mathbb{Z}_2}$ . If

$M_e^1$  is a boundary  $M_e^1 = \partial D_e^2$ , then we can relate the above to the intersection number and the linking number:

$$\begin{aligned}
& \int_{M_e^1} a_m^{\mathbb{Z}_2} = \int_{D_e^2} da_m^{\mathbb{Z}_2} = \int_{D_e^2} B_m^{\mathbb{Z}_2} + k_0 w_1^2 \\
&= \text{Int}(D_e^2, M_m^1 + k_0 M_w^1) = \text{Lnk}(M_e^1, M_m^1 + k_0 M_w^1).
\end{aligned} \tag{211}$$

where  $M_w^1$  is the  $\mathbb{Z}_2$ -valued 1-cycle which is the Poincaré dual of  $\mathcal{B}_2 w_1$ . Here  $\text{Int}(D_e^2, M_m^1)$  is the intersection number between  $D_e^2$  and  $M_m^1$ , and  $\text{Lnk}(M_e^1, M_m^1)$  the linking number between  $M_e^1$  and  $M_m^1$ . The linking number satisfies

$$\text{Lnk}(M_e^1, M_m^1) = \text{Lnk}(M_m^1, M_e^1). \tag{212}$$

Using the linking number, we can rewrite the partition function as

$$\begin{aligned}
& Z_{k_0 k_1 k_2; t\mathbb{Z}_2 aT}(M^3) \\
&\propto e^{i\pi \int_{M^3} k_1 (a_m^{\mathbb{Z}_2})^3} e^{i\pi \text{Lnk}(k_2 M_w^1 + M_e^1, M_m^1)} \times \\
&\quad \sum_{\tilde{g}, da_0^{\mathbb{Z}_2}=0} e^{i\pi \int_{M^3} k_0 a_e^{\mathbb{Z}_2} \mathcal{B}_2 d\tilde{g} + k_2 a_m^{\mathbb{Z}_2} \mathcal{B}_2 d\tilde{g}} \times \\
&\quad e^{i\pi \int_{M^3} k_1 [(a_0^{\mathbb{Z}_2})^3 + (a_m^{\mathbb{Z}_2})^2 a_0^{\mathbb{Z}_2} + a_m^{\mathbb{Z}_2} (a_0^{\mathbb{Z}_2})^2]}.
\end{aligned} \tag{213}$$

We like to stress that the above path integral has a time-reversal symmetry: it is invariant under a combined transformation  $\tilde{g}_i \rightarrow [\tilde{g}_i + 1]_2$ ,  $a_0^{\mathbb{Z}_2} \rightarrow a_{0,ij}^{\mathbb{Z}_2}$ , and complex conjugation.

The physical properties of excitations can be obtained from the above effective theory. Let us first assume  $k_1 = 0$ , and rewrite the partition function as

$$\begin{aligned}
& Z_{k_0 k_1 k_2; t\mathbb{Z}_2 aT}(M^3) \\
&\propto \bar{\delta}_2(M_m^1 + k_0 M_w^1) \bar{\delta}_2(M_e^1 + k_2 M_w^1) e^{i\pi \text{Lnk}(k_2 M_w^1 + M_e^1, M_m^1)} \\
&\quad \sum_{\tilde{g}, da_0^{\mathbb{Z}_2}=0} e^{i\pi \int_{M^3} k_0 a_e^{\mathbb{Z}_2} \mathcal{B}_2 d\tilde{g} + k_2 a_m^{\mathbb{Z}_2} \mathcal{B}_2 d\tilde{g}},
\end{aligned} \tag{214}$$

where we have restored the two  $\delta$ -functions. For simplicity, we will also assume  $w_1^2 = 0$ , and choose  $a_e^{\mathbb{Z}_2}$  be the Poincaré dual of  $D_e^2$  and  $a_m^{\mathbb{Z}_2}$  be the Poincaré dual of  $D_m^2$ . Here  $D_e^2$  and  $D_m^2$  are the disks bonded by the world-lines  $M_e^1$  and  $M_m^1$ . The dynamical part of the partition function can be written as

$$\begin{aligned}
& \sum_{\tilde{g}} e^{i\pi \int_{M^3} k_0 a_e^{\mathbb{Z}_2} \mathcal{B}_2 d\tilde{g} + k_2 a_m^{\mathbb{Z}_2} \mathcal{B}_2 d\tilde{g}} \\
&= \sum_{\tilde{g}} e^{i\pi \int_{D_e^2} k_0 \mathcal{B}_2 d\tilde{g}} e^{i\pi \int_{D_m^2} k_2 \mathcal{B}_2 d\tilde{g}} \\
&\propto e^{i\pi \int_{D_e^2} k_0 \mathcal{B}_2 w_1} e^{i\pi \int_{D_m^2} k_2 \mathcal{B}_2 w_1}.
\end{aligned}$$

From the above, we see that, when  $k_0 = 1$ , there is  $\mathbb{Z}_2^T$ -SPT state described by the SPT invariant  $e^{i\pi \int_{D_e^2} \mathcal{B}_2 w_1}$  on  $D_e^2$ . In this case, the boundary of  $D_e^2$ , *i.e.* the  $e$  particle described by the world-line  $M_e^1 = \partial D_e^2$ , will carry a Kramer doublet. This agrees with the result in Section



**VIB 2.** Similarly, when  $k_2 = 1$ , there is  $Z_2^T$ -SPT state described by the SPT invariant  $e^{i\pi \int_{D_m^2} \mathcal{B}_2 w_1}$  on  $D_m^2$ , and the  $m$  particle will carry a Kramer doublet.

The term  $e^{i\pi \text{Lnk}(M_e^1, M_m^1)}$  tell us that the  $e$  and  $m$  have a mutual  $\pi$  statistics between them. The absence of self-linking terms,  $e^{i\theta \text{Lnk}(M_e^1, M_e^1)}$  and  $e^{i\theta \text{Lnk}(M_m^1, M_m^1)}$ , implies that the  $e$  and  $m$  are bosons. We also see that the emergence of Kramer-doublet bosons cause the partition function to vanish on the space-time with  $w_1^2 \neq 0$ . From the form of  $\bar{\delta}_2(M_m^1 + k_0 M_w^1) \bar{\delta}_2(M_e^1 + k_2 M_w^1)$ , we see that space-time with  $w_1^2 \neq 0$  will nuclear a  $m$  particle (or more precisely, a non-contractible world-line of the  $m$ ) if the bosonic  $e$  particle is a Kramer doublet. Similarly, space-time with  $w_1^2 \neq 0$  will nuclear a  $e$  particle if the bosonic  $m$  particle is a Kramer doublet. In other words,

if there is an emergent bosonic Kramer doublet, then a space-time with  $w_1^2 \neq 0$  will create a world-line of a particle that has a mutual  $\pi$  statistics with the bosonic Kramer doublet. The world-line is equal to the Poincaré dual of  $w_1^2$ .

Those results are summarized by the top three rows in Table I.

Next, we consider the case of  $k_1 = 1$ . The partition function now reads

$$\begin{aligned} & Z_{k_0 k_1 k_2; t Z_2 a T}(M^3) \\ & \propto \bar{\delta}_2(M_m^1 + k_0 M_w^1) e^{i\pi \int_{M^3} k_1 (a_m^{Z_2})^3} e^{i\pi \text{Lnk}(k_2 M_w^1 + M_e^1, M_m^1)} \times \\ & \sum_{\tilde{g}, da_0^{Z_2}=0} e^{i\pi \int_{M^3} k_0 a_e^{Z_2} \mathcal{B}_2 d\tilde{g} + k_2 a_m^{Z_2} \mathcal{B}_2 d\tilde{g}} \times \\ & e^{i\pi \int_{M^3} (a_0^{Z_2})^3 + (a_m^{Z_2})^2 a_0^{Z_2} + a_m^{Z_2} (a_0^{Z_2})^2}. \end{aligned} \quad (215)$$

Note that we only have one  $\delta$ -function in this case. The above result for the  $e$  particle is not changed: the  $e$  is still a boson, which carries Kramer doublet if  $k_0 = 1$  and time-reversal singlet if  $k_0 = 0$ .

But the result for the  $m$  particle is changed. The effective theory on  $D_m^2$  now becomes

$$\sum_{\tilde{g}, a_0^{Z_2}} e^{i\pi \int_{D_m^2} k_2 \mathcal{B}_2 d\tilde{g} + (a_0^{Z_2})^2}. \quad (216)$$

If we treat the emergent  $Z_2$  gauge symmetry as a  $Z_2$  symmetry, then the above can be viewed as a  $Z_2 \times Z_2^T$ -SPT state on  $D_m^2$ . The SPT state is characterized by SPT invariant  $e^{i\pi \int_{D_m^2} k_2 \mathcal{B}_2 w_1 + (a^{Z_2})^2}$  where  $a^{Z_2}$  is the symmetry twist of  $Z_2$ . As discussed in Section VIA 3, when  $k_2 = 0$ , the  $m$ -particle will carry  $\pm 1/2$   $Z_2$ -gauge-charge, which form a  $T^2 = 1$  time-reversal doublet (labeled by  $2_+$ ). When  $k_2 = 1$ , the  $m$ -particle will carry  $\pm 1/2$   $Z_2$ -gauge-charge, which form a  $T^2 = -1$  Kramer doublet (labeled by  $2_-$ ). The above applies for both  $k_0 = 0, 1$  cases.

For the bond state of  $e$  and  $m$ , the  $\varepsilon$  particle, the  $Z_2 \times$

$Z_2^T$ -SPT state on the corresponding  $D_\varepsilon^2$  is described by

$$\sum_{\tilde{g}, a_0^{Z_2}} e^{i\pi \int_{D_m^2} (k_0 + k_2) \mathcal{B}_2 d\tilde{g} + k_1 (a_0^{Z_2})^2}. \quad (217)$$

We see that the  $\varepsilon$  is always a  $\pm 1/2$   $Z_2$ -gauge-charge doublet. It is a  $T^2 = -1$  Kramer doublet ( $2_-$ ) if  $(k_0 + k_2) = 1$  and a  $T^2 = 1$  time-reversal doublet ( $2_+$ ) if  $(k_0 + k_2) = 0$ .

The statistics of the  $m$  particle is no longer bosonic due the self-braiding term  $e^{i\pi \int_{M^3} k_1 (a_m^{Z_2})^3}$  (which can be viewed as the triple self-intersection of  $D_m^2$ ). We note that  $e^{i\pi \int_{M^3} k_1 (a_m^{Z_2})^3} = \pm 1$  respects the time-reversal symmetry. But one expect  $m$  to a semion described by the self-linking term  $e^{i\frac{\pi}{2} \text{Lnk}(M_m^1, M_m^1)}$ . In fact, the above self-linking term breaks the time-reversal symmetry, and does not describe the statics of  $m$  which in our case is a particle the respect the time-reversal symmetry. In other words, due to the time reversal symmetry,  $m$  is not a semion.

In fact,  $m$  is a  $T^2 = -1$  Kramer doublet or a  $T^2 = 1$  time-reversal doublet formed by a semion (with spin  $s = 1/4$ ) and a conjugate semion (with spin  $s = 3/4$ ). The statistics of such a time-reversal symmetric doublet is not described by the self-linking term  $e^{i\frac{\pi}{2} \text{Lnk}(M_m^1, M_m^1)}$  or the self-linking term  $e^{-i\frac{\pi}{2} \text{Lnk}(M_m^1, M_m^1)}$ . Our calculation suggest that the statistics of the time-reversal symmetric doublet is described by  $e^{i\pi \int_{M^3} k_1 (a_m^{Z_2})^3}$  – the triple self-intersection of  $D_m^2$ . Those results are summarized by the bottom three rows in Table I.

### C. 3+1D time-reversal symmetric model

#### 1. Model construction

In this section, we are going to study a class of 3+1D time-reversal symmetric local bosonic models, that can produce the simplest time-reversal symmetric topological orders. The 3+1D time-reversal symmetric local bosonic models contain  $\mathbb{Z}_2$ -multi-valued 0-cochain field  $\tilde{g}_i$ ,  $\mathbb{Z}_2$ -valued 1-cochain field  $a_{ij}^{Z_2}$ , and  $\mathbb{Z}_2$ -valued 2-cochain field  $b_{ijk}^{Z_2}$ . Its path integral is given by

$$\begin{aligned} & Z_{k_1 k_2 k_3 k_4 k_5 k_6}(M^4) \\ & = \sum_{\{\tilde{g}_i^{Z_2}, a_{ij}^{Z_2}, b_{ijk}^{Z_2}\}} e^{i\pi \int_{M^4} b^{Z_2} da^{Z_2}} e^{i\pi \int_{M^4} (k_3 + k_4) b^{Z_2} \mathcal{B}_2 d\tilde{g} + k_4 (b^{Z_2})^2} \times \\ & e^{i\pi \int_{M^4} k_1 (a^{Z_2})^4 + (k_2 + k_1) a^{Z_2} (d\tilde{g})^3} e^{i\pi \int_{M^4} k_5 (d\tilde{g})^4 + k_6 (w_2)^2}. \end{aligned} \quad (218)$$

The 0-cocycle field  $\tilde{g}_i$  is a pseudo scalar as introduced in Section VC. It satisfies  $d\tilde{g}_i = w_1 + dg$ , where  $g_i$  is a  $\mathbb{Z}_2$ -single-valued 0-cochain field. Thus  $\mathcal{B}_2 d\tilde{g}_i = \mathcal{B}_2 w_1$ . The above path integral defines the system for both closed and open space-time manifold  $M^4$ . But in the following, we will assume  $M^4$  to be closed. The index  $k_I = 0, 1$ . So there are  $2^6 = 64$  different models.

We note that the above path integral has the time-reversal symmetry  $Z_2^T$ , *i.e.* invariant under the combined transformation of  $\tilde{g}_i \rightarrow [\tilde{g}_i + 1]_2$  and complex conjugation. (Under the transformation  $\tilde{g}_i \rightarrow \tilde{g}'_i = [\tilde{g}_i + 1]_2$ ,  $d\tilde{g}_i = -d\tilde{g}'_i$ .) This is a designed property. However, the path integral also has an extra  $Z'_2$  symmetry:  $\tilde{g}_i \rightarrow [\tilde{g}_i + 1]_2$  (without the complex conjugation).

Let us also include the excitations in the path integral. We know that the point-like excitations are described by the world-lines in space-time. A world line  $M_{\text{WL}}^1$  can be viewed as a  $\mathbb{Z}_2$ -valued 1-cycle, which is Poincaré dual to a  $\mathbb{Z}_2$ -valued 3-cochain  $C_{\text{WL}}^3$ . The string-like excitations are described by the world-sheet in space-time, which can be viewed as  $\mathbb{Z}_2$ -valued 2-cycles  $M_{\text{WS}}^2$  in the space-time lattice, whose Poincaré dual is a  $\mathbb{Z}_2$ -valued 2-cocycle  $B_{\text{WS}}^2$ .

Just like the  $Z_2$ -gauge theory, we can include those excitations in path integral eqn. (218), by adding the  $Z_2$ -charge coupling term  $e^{i\pi \int_{M_{\text{WL}}^1} a^{\mathbb{Z}_2}}$  and the  $Z_2$ -flux coupling term  $e^{i\pi \int_{M_{\text{WS}}^2} b^{\mathbb{Z}_2}}$ . Due to the Poincaré duality,

$$\begin{aligned} e^{i\pi \int_{M_{\text{WL}}^1} a^{\mathbb{Z}_2}} &= e^{i\pi \int_{M^4} C_{\text{WL}}^3 a^{\mathbb{Z}_2}}, \\ e^{i\pi \int_{M_{\text{WS}}^2} b^{\mathbb{Z}_2}} &= e^{i\pi \int_{M^4} B_{\text{WS}}^2 b^{\mathbb{Z}_2}}. \end{aligned} \quad (219)$$

Thus, in the presence of point-like topological excitations described by  $C_{\text{WL}}^3$  and string-like topological excitations described by  $B_{\text{WS}}^2$ , the partition function eqn. (218) becomes

$$\begin{aligned} &Z_{k_1 k_2 k_3 k_4 k_5 k_6}(M^4) \\ &= \sum_{\tilde{g}, a^{\mathbb{Z}_2}, b^{\mathbb{Z}_2}} e^{i\pi \int_{M^4} b^{\mathbb{Z}_2} (da^{\mathbb{Z}_2} + k_4 b^{\mathbb{Z}_2} + (k_3 + k_4) \mathcal{B}_2 d\tilde{g} + B_{\text{WS}}^2)} \times \\ &e^{i\pi \int_{M^4} k_1 (a^{\mathbb{Z}_2})^4 + [(k_2 + k_1)(d\tilde{g})^3 + C_{\text{WL}}^3] a^{\mathbb{Z}_2}} e^{i\pi \int_{M^4} k_5 (d\tilde{g})^4 + k_6 w_2^2}. \end{aligned} \quad (220)$$

## 2. Partition function

To understand the physical properties of those 64 models, we like to compute the corresponding partition functions on closed space-time  $M^4$ . However, unlike other models constructed in this paper, the above models are not exactly soluble. They are exactly soluble only in the cases  $k_1 = 0$  or  $k_4 = 0$ . So we will calculate the partition functions for those two cases.

When  $k_4 = 0$ , the action is linear in  $b^{\mathbb{Z}_2}$ , and we can integrate out  $b^{\mathbb{Z}_2}$  first, which lead to a constraint

$$da^{\mathbb{Z}_2} \stackrel{2}{=} k_3 \mathcal{B}_2 d\tilde{g} + B_{\text{WS}}^2 = k_3 \mathcal{B}_2 d\tilde{g}_0 + B_{\text{WS}}^2. \quad (221)$$

where  $\tilde{g}_0$  is a fixed  $\mathbb{Z}_2$ -multi-valued 0-cochain such that

$$\tilde{g} - \tilde{g}_0 \stackrel{2}{=} g \quad (222)$$

is a  $\mathbb{Z}_2$ -single-valued 0-cochain  $g$ . We see that the partition function is zero when  $k_3 \mathcal{B}_2 d\tilde{g}_0 + B_{\text{WS}}^2$  is not a

coboundary. Thus the partition function contain a factor  $\bar{\delta}_2(k_3 \mathcal{B}_2 d\tilde{g}_0 + B_{\text{WS}}^2)$ . We may solve the  $da^{\mathbb{Z}_2} \stackrel{2}{=} k_3 \mathcal{B}_2 d\tilde{g} + B_{\text{WS}}^2$  constraint via the following ansatz

$$a^{\mathbb{Z}_2} \stackrel{2}{=} a_{\text{WS}}^{\mathbb{Z}_2} + a_0^{\mathbb{Z}_2} \quad (223)$$

where  $a_{\text{WS}}^{\mathbb{Z}_2}$  is a  $\mathbb{Z}_2$ -valued 1-cochain that satisfies

$$da_{\text{WS}}^{\mathbb{Z}_2} \stackrel{2}{=} k_3 \mathcal{B}_2 d\tilde{g} + B_{\text{WS}}^2, \quad (224)$$

and  $a_0^{\mathbb{Z}_2}$  is a  $\mathbb{Z}_2$ -valued 1-cocycle field  $a_0^{\mathbb{Z}_2} \in Z^1(M^4; \mathbb{Z}_2)$ . The partition function now becomes

$$\begin{aligned} &Z_{k_1 k_2 k_3 0 k_5 k_6}(M^4) = \bar{\delta}_2(k_3 \mathcal{B}_2 d\tilde{g} + B_{\text{WS}}^2) \\ &\sum_{\tilde{g}, da_0^{\mathbb{Z}_2} \stackrel{2}{=} 0} e^{i\pi \int_{M^4} k_1 (a_{\text{WS}}^{\mathbb{Z}_2})^4 + [(k_2 + k_1)(d\tilde{g})^3 + C_{\text{WL}}^3] a_{\text{WS}}^{\mathbb{Z}_2}} \times \\ &e^{i\pi \int_{M^4} k_1 (a_0^{\mathbb{Z}_2})^4 + [(k_2 + k_1)(d\tilde{g})^3 + C_{\text{WL}}^3] a_0^{\mathbb{Z}_2}} e^{i\pi \int_{M^4} k_5 (d\tilde{g})^4 + k_6 w_2^2}. \end{aligned} \quad (225)$$

Since  $a_0^{\mathbb{Z}_2}$  is a cocycle, we can replace  $d\tilde{g}$  by  $d\tilde{g}_0$  in the last line above, and use many relations between  $a_0^{\mathbb{Z}_2}$  and Stiefel-Whitney classes, such as (see Appendix D 4 where  $w_1$  is replaced by  $d\tilde{g}_0$ )

$$\begin{aligned} (d\tilde{g}_0)^2 (a_0^{\mathbb{Z}_2})^2 &\stackrel{2, d}{=} (d\tilde{g}_0)^3 a_0^{\mathbb{Z}_2}, \quad w_2 (a_0^{\mathbb{Z}_2})^2 \stackrel{2, d}{=} w_3 a_0^{\mathbb{Z}_2}, \\ (a_0^{\mathbb{Z}_2})^4 &\stackrel{2, d}{=} d\tilde{g}_0 (a_0^{\mathbb{Z}_2})^3, \quad [a_0^{\mathbb{Z}_2}]^2 + (d\tilde{g}_0)^2 + w_2 (a_0^{\mathbb{Z}_2})^2 \stackrel{2, d}{=} 0, \end{aligned} \quad (226)$$

to simplify the last line. Note that those relations are valid only when  $a_0^{\mathbb{Z}_2}$  is a cocycle and when  $M^4$  is closed. Therefore, we can rewrite the above partition function on closed  $M^4$  as

$$\begin{aligned} &Z_{k_1 k_2 k_3 0 k_5 k_6}(M^4) = \bar{\delta}_2(k_3 \mathcal{B}_2 d\tilde{g}_0 + B_{\text{WS}}^2) \\ &\sum_{\tilde{g}, da_0^{\mathbb{Z}_2} \stackrel{2}{=} 0} e^{i\pi \int_{M^4} k_1 (a_{\text{WS}}^{\mathbb{Z}_2})^4 + [(k_2 + k_1)(d\tilde{g})^3 + C_{\text{WL}}^3] a_{\text{WS}}^{\mathbb{Z}_2}} \times \\ &e^{i\pi \int_{M^4} (k_1 w_3 + k_2 (d\tilde{g}_0)^3 + C_{\text{WL}}^3) a_0^{\mathbb{Z}_2}} e^{i\pi \int_{M^4} k_5 (d\tilde{g})^4 + k_6 w_2^2} \\ &= \bar{\delta}_2(k_3 \mathcal{B}_2 d\tilde{g}_0 + B_{\text{WS}}^2) \bar{\delta}_2(k_1 w_3 + k_2 (d\tilde{g}_0)^3 + C_{\text{WL}}^3) \\ &e^{i\pi \int_{M^4} k_5 (d\tilde{g}_0)^4 + k_6 w_2^2} \times \\ &\sum_{\tilde{g}} e^{i\pi \int_{M^4} k_1 (a_{\text{WS}}^{\mathbb{Z}_2})^4 + [(k_2 + k_1)(d\tilde{g})^3 + C_{\text{WL}}^3] a_{\text{WS}}^{\mathbb{Z}_2}}. \end{aligned} \quad (227)$$

We note that  $x^2 \stackrel{2}{=} \mathcal{B}_2 x$  for any  $\mathbb{Z}_2$ -valued 1-cocycle  $x$ , and  $d\tilde{g} = d\tilde{g}_0 + dg$ . Thus

$$\begin{aligned} &e^{i\pi \int_{M^4} (k_2 + k_1)(d\tilde{g})^3} a_{\text{WS}}^{\mathbb{Z}_2} = e^{i\pi \int_{M^4} (k_2 + k_1) d\tilde{g} \mathcal{B}_2 d\tilde{g}_0 a_{\text{WS}}^{\mathbb{Z}_2}} \\ &= e^{i\pi \int_{M^4} (k_2 + k_1) d\tilde{g}_0 \mathcal{B}_2 d\tilde{g}_0 a_{\text{WS}}^{\mathbb{Z}_2}} e^{i\pi \int_{M^4} (k_2 + k_1) dg \mathcal{B}_2 d\tilde{g}_0 a_{\text{WS}}^{\mathbb{Z}_2}} \\ &= e^{i\pi \int_{M^4} (k_2 + k_1)(d\tilde{g}_0)^3} a_{\text{WS}}^{\mathbb{Z}_2} e^{i\pi \int_{M^4} (k_2 + k_1) g \mathcal{B}_2 d\tilde{g}_0 (k_3 \mathcal{B}_2 d\tilde{g}_0 + B_{\text{WS}}^2)} \end{aligned} \quad (228)$$

Therefore, the volume independent partition function is

given by

$$Z_{k_1 k_2 k_3 0 k_5 k_6}^{\text{top}}(M^4) = \frac{|H^0(M^4; \mathbb{Z}_2)| |H^2(M^4; \mathbb{Z}_2)|}{|H^1(M^4; \mathbb{Z}_2)|} \times \\ e^{i\pi \int_{M^4} k_5 (d\tilde{g}_0)^4 + k_6 w_2^2 + k_1 (a_{\text{WS}}^{\mathbb{Z}_2})^4 + [(k_2 + k_1)(d\tilde{g}_0)^3 + C_{\text{WL}}^{\mathbb{Z}_2}] a_{\text{WS}}^{\mathbb{Z}_2}} \times \\ \bar{\delta}_2(k_3 \mathcal{B}_2 d\tilde{g}_0 + B_{\text{WS}}^{\mathbb{Z}_2}) \bar{\delta}_2(k_1 w_3 + k_2 (d\tilde{g}_0)^3 + C_{\text{WL}}^{\mathbb{Z}_2}) \times \\ \delta_2[(k_2 + k_1) \mathcal{B}_2 d\tilde{g}_0 (k_3 \mathcal{B}_2 d\tilde{g}_0 + B_{\text{WS}}^{\mathbb{Z}_2})].$$

We note that  $\tilde{g}_0$  is multivalued only on  $\partial M^4$  (which is a non-zero even cycle when  $M^4$  is not orientable). So  $\mathcal{B}_2 d\tilde{g}_0$  is non-zero only near the “boundary”  $\partial M^4$  (see Fig. 6). Therefore,  $\delta_2[(k_2 + k_1) \mathcal{B}_2 d\tilde{g}_0 (k_3 \mathcal{B}_2 d\tilde{g}_0 + B_{\text{WS}}^{\mathbb{Z}_2})]$  is a boundary term.

When  $k_1 = 0$ , the action is linear in  $a^{\mathbb{Z}_2}$ . In this case, we can integrate out  $a^{\mathbb{Z}_2}$  first, which lead to a constraint

$$db^{\mathbb{Z}_2} = k_2 (d\tilde{g})^3 + C_{\text{WL}}^{\mathbb{Z}_2}. \quad (229)$$

So the partition function is zero when  $k_2 (d\tilde{g}_0)^3 + C_{\text{WL}}^{\mathbb{Z}_2}$  is not a coboundary. Thus the partition function contains a factor  $\bar{\delta}_2(k_2 (d\tilde{g}_0)^3 + C_{\text{WL}}^{\mathbb{Z}_2})$ . We may solve the  $db^{\mathbb{Z}_2} \stackrel{=}{=} k_2 (d\tilde{g})^3 + C_{\text{WL}}^{\mathbb{Z}_2}$  constraint via the following ansatz

$$b^{\mathbb{Z}_2} \stackrel{=}{=} b_{\text{WL}}^{\mathbb{Z}_2} + b_0^{\mathbb{Z}_2} \quad (230)$$

where  $b_{\text{WL}}^{\mathbb{Z}_2}$  is a  $\mathbb{Z}_2$ -valued 2-cochain that satisfies

$$db_{\text{WL}}^{\mathbb{Z}_2} \stackrel{=}{=} k_2 (d\tilde{g})^3 + C_{\text{WL}}^{\mathbb{Z}_2}, \quad (231)$$

and  $b_0^{\mathbb{Z}_2}$  is a  $\mathbb{Z}_2$ -valued 2-cocycle field  $b_0^{\mathbb{Z}_2} \in Z^2(M^4; \mathbb{Z}_2)$ . The partition function now becomes

$$Z_{0k_2 k_3 k_4 k_5 k_6}(M^4) = \bar{\delta}_2[k_2 (d\tilde{g}_0)^3 + C_{\text{WL}}^{\mathbb{Z}_2}] \times \\ \sum_{\tilde{g}, db_0^{\mathbb{Z}_2} \stackrel{=}{=} 0} e^{i\pi \int_{M^4} b_{\text{WL}}^{\mathbb{Z}_2} [k_4 b_{\text{WL}}^{\mathbb{Z}_2} + (k_3 + k_4) \mathcal{B}_2 d\tilde{g} + B_{\text{WS}}^{\mathbb{Z}_2}]} \times \\ e^{i\pi \int_{M^4} b_0^{\mathbb{Z}_2} [k_4 b_0^{\mathbb{Z}_2} + (k_3 + k_4) \mathcal{B}_2 d\tilde{g} + B_{\text{WS}}^{\mathbb{Z}_2}]} e^{i\pi \int_{M^4} k_5 (d\tilde{g}_0)^4 + k_6 w_2^2}. \quad (232)$$

Since  $b_0^{\mathbb{Z}_2}$  is a cocycle, we can replace  $d\tilde{g}$  by  $d\tilde{g}_0$  in the last line above, and use many relations between  $b_0^{\mathbb{Z}_2}$  and Stiefel-Whitney classes, such as (see Appendix D 5)

$$d\tilde{g}_0 \mathcal{B}_2 b^{\mathbb{Z}_2} = 0, \quad (b^{\mathbb{Z}_2})^2 + [(d\tilde{g}_0)^2 + w_2] b^{\mathbb{Z}_2} = 0, \quad (233)$$

to simplify the last line. Therefore, we can rewrite the above partition function on closed  $M^4$  as

$$Z_{0k_2 k_3 k_4 k_5 k_6}(M^4) = \bar{\delta}_2(k_2 (d\tilde{g}_0)^3 + C_{\text{WL}}^{\mathbb{Z}_2}) \times \\ \sum_{\tilde{g}, db_0^{\mathbb{Z}_2} \stackrel{=}{=} 0} e^{i\pi \int_{M^4} b_{\text{WL}}^{\mathbb{Z}_2} [k_4 b_{\text{WL}}^{\mathbb{Z}_2} + (k_3 + k_4) \mathcal{B}_2 d\tilde{g} + B_{\text{WS}}^{\mathbb{Z}_2}]} \times \\ e^{i\pi \int_{M^4} b_0^{\mathbb{Z}_2} [k_4 w_2 + k_3 \mathcal{B}_2 d\tilde{g}_0 + B_{\text{WS}}^{\mathbb{Z}_2}]} e^{i\pi \int_{M^4} k_5 (d\tilde{g}_0)^4 + k_6 w_2^2} \\ = \bar{\delta}_2(k_2 (d\tilde{g}_0)^3 + C_{\text{WL}}^{\mathbb{Z}_2}) \bar{\delta}_2(k_4 w_2 + k_3 \mathcal{B}_2 d\tilde{g}_0 + B_{\text{WS}}^{\mathbb{Z}_2}) \times \\ e^{i\pi \int_{M^4} k_5 (d\tilde{g}_0)^4 + k_6 w_2^2} \times \\ \sum_{\tilde{g}} e^{i\pi \int_{M^4} b_{\text{WL}}^{\mathbb{Z}_2} [k_4 b_{\text{WL}}^{\mathbb{Z}_2} + (k_3 + k_4) \mathcal{B}_2 d\tilde{g} + B_{\text{WS}}^{\mathbb{Z}_2}]}.$$

The above partition function can be simplified further. Let  $\bar{b}_{\text{WL}}^{\mathbb{Z}_2}$  be a fixed 2-cocycle that satisfies

$$d\bar{b}_{\text{WL}}^{\mathbb{Z}_2} \stackrel{=}{=} k_2 (d\tilde{g}_0)^3 + C_{\text{WL}}^{\mathbb{Z}_2}. \quad (235)$$

and let

$$b_{\text{WL}}^{\mathbb{Z}_2} \stackrel{=}{=} \bar{b}_{\text{WL}}^{\mathbb{Z}_2} + b_1^{\mathbb{Z}_2} \quad (236)$$

In this case,  $b_1^{\mathbb{Z}_2}$  satisfies

$$db_1^{\mathbb{Z}_2} \stackrel{=}{=} k_2 [(d\tilde{g})^3 - (d\tilde{g}_0)^3] \stackrel{=}{=} k_2 [d\tilde{g} \mathcal{B}_2 d\tilde{g} - d\tilde{g}_0 \mathcal{B}_2 d\tilde{g}_0] \\ = k_2 d(g \mathcal{B}_2 d\tilde{g}_0) \quad (237)$$

where we have used  $d\tilde{g} = d\tilde{g}_0 + dg$  and  $x^2 \stackrel{=}{=} \mathcal{B}_2 x$  for any  $\mathbb{Z}_2$ -valued 1-cocycle  $x$ . So  $b_{\text{WL}}^{\mathbb{Z}_2}$  is given by

$$b_{\text{WL}}^{\mathbb{Z}_2} \stackrel{=}{=} k_2 g \mathcal{B}_2 d\tilde{g}_0 + \bar{b}_{\text{WL}}^{\mathbb{Z}_2}. \quad (238)$$

Now, we can rewrite the partition function eqn. (232) in the following form (using the relations obtained in Appendix D 5)

$$Z_{0k_2 k_3 k_4 k_5 k_6}(M^4) \\ = \bar{\delta}_2[k_2 (d\tilde{g}_0)^3 + C_{\text{WL}}^{\mathbb{Z}_2}] \bar{\delta}_2(k_4 w_2 + k_3 \mathcal{B}_2 d\tilde{g}_0 + B_{\text{WS}}^{\mathbb{Z}_2}) \times \\ e^{i\pi \int_{M^4} k_5 (d\tilde{g}_0)^4 + k_6 w_2^2} e^{i\pi \int_{M^4} b_{\text{WL}}^{\mathbb{Z}_2} [k_4 \bar{b}_{\text{WL}}^{\mathbb{Z}_2} + (k_3 + k_4) \mathcal{B}_2 d\tilde{g}_0 + B_{\text{WS}}^{\mathbb{Z}_2}]} \times \\ \sum_g e^{i\pi \int_{M^4} k_2 g \mathcal{B}_2 d\tilde{g}_0 [k_4 g \mathcal{B}_2 d\tilde{g}_0 + (k_3 + k_4) \mathcal{B}_2 d\tilde{g}_0 + B_{\text{WS}}^{\mathbb{Z}_2}]}.$$

Using the fact (see eqn. (21))  $g \mathcal{B}_2 d\tilde{g}_0 g \stackrel{2, d}{=} g^2 \mathcal{B}_2 d\tilde{g}_0 = g \mathcal{B}_2 d\tilde{g}_0$ , we can simplify

$$\sum_g e^{i\pi \int_{M^4} k_2 g \mathcal{B}_2 d\tilde{g}_0 [k_4 g \mathcal{B}_2 d\tilde{g}_0 + (k_3 + k_4) \mathcal{B}_2 d\tilde{g}_0 + B_{\text{WS}}^{\mathbb{Z}_2}]} \\ = \sum_g e^{i\pi \int_{M^4} k_2 g \mathcal{B}_2 d\tilde{g}_0 [k_4 \mathcal{B}_2 d\tilde{g}_0 + (k_3 + k_4) \mathcal{B}_2 d\tilde{g}_0 + B_{\text{WS}}^{\mathbb{Z}_2}]} \\ = \delta_2[k_2 \mathcal{B}_2 d\tilde{g}_0 (k_3 \mathcal{B}_2 d\tilde{g}_0 + B_{\text{WS}}^{\mathbb{Z}_2})] \quad (240)$$

Thus, the volume independent partition function is given by

$$Z_{0k_2 k_3 k_4 k_5 k_6}^{\text{top}}(M^4) = \frac{|H^0(M^4; \mathbb{Z}_2)| |H^2(M^4; \mathbb{Z}_2)|}{|H^1(M^4; \mathbb{Z}_2)|} \times \\ e^{i\pi \int_{M^4} k_5 (d\tilde{g}_0)^4 + k_6 w_2^2 + \bar{b}_{\text{WL}}^{\mathbb{Z}_2} [k_4 \bar{b}_{\text{WL}}^{\mathbb{Z}_2} + (k_3 + k_4) \mathcal{B}_2 d\tilde{g}_0 + B_{\text{WS}}^{\mathbb{Z}_2}]} \times \\ \bar{\delta}_2[k_2 (d\tilde{g}_0)^3 + C_{\text{WL}}^{\mathbb{Z}_2}] \bar{\delta}_2(k_4 w_2 + k_3 \mathcal{B}_2 d\tilde{g}_0 + B_{\text{WS}}^{\mathbb{Z}_2}) \times \\ \delta_2[k_2 \mathcal{B}_2 d\tilde{g}_0 (k_3 \mathcal{B}_2 d\tilde{g}_0 + B_{\text{WS}}^{\mathbb{Z}_2})].$$

### 3. Physical properties of ground states

Using the above partition functions, we can obtain many physical properties of ground states by setting  $B_{\text{WS}}^{\mathbb{Z}_2} = C_{\text{WL}}^{\mathbb{Z}_2} = 0$ . For simplicity, we will also assume that  $w_i = 0$  on  $M^4$ , so that we can choose  $a_{\text{WS}}^{\mathbb{Z}_2} = \bar{b}_{\text{WL}}^{\mathbb{Z}_2} = 0$ .

First, we see that the partition functions for different  $k_I$ 's do not depend the shape or the metrics of space-time manifold  $M^4$ . So the ground states of the 48 models with  $k_1 k_4 = 0$  are all gapped. The partition functions also do not depend on the triangulation of the space-time. So the ground states are all gapped liquids.<sup>16,17</sup> If we choose space-time to be  $M^4 = S^1 \times S^3$  where  $w_1 = w_2 = w_3 = 0$ , we find the volume-independent partition functions to be  $Z_{k_1 k_2 k_3 k_4 k_5 k_6}^{\text{top}}(S^1 \times S^3) = 1$ . This means that the ground state degeneracies on  $S^3$  for the 48 models (with  $k_1 k_4 = 0$ ) are all equal to 1, and there is no spontaneous symmetry breaking of  $Z_2^T$  or  $Z_2'$ .

The volume-independent partition functions are not equal to 1 for other closed space-times with vanishing Euler number and Pontryagin number. For example on  $M^4 = T^2 \times S^2$  where  $w_1 = w_2 = w_3 = 0$ ,  $Z_{k_1 k_2 k_3 k_4 k_5 k_6}^{\text{top}}(T^2 \times S^2) = 2$ . Thus, those 48 models all realize non-trivial 3+1D topological orders in their ground states. The 2-fold ground state degeneracy on space  $S^1 \times S^2$  tells us that the topological orders are simple since they all have only one non-trivial point-like topological excitation and one non-trivial string-like topological excitation. In fact the emergent topological orders are  $Z_2$  topological orders described by UT or EF  $Z_2$  gauge theories with  $a^{Z_2}$  as the  $Z_2$ -gauge field. Because the ground states also have symmetries, we may view those topological orders as  $Z_2^T$ -SET orders or as  $Z_2' \times Z_2^T$ -SET orders.

We remark that the action amplitude  $e^{i\pi \int_{M^4} k_5 (d\tilde{g})^4 + k_6 w_2^2} = e^{i\pi \int_{M^4} k_5 w_1^4 + k_6 w_2^2}$  is the SPT invariant for the  $Z_2^T$  SPT states. So different  $k_5, k_6$  correspond to stacking with different  $Z_2^T$ -SPT states.

#### 4. Properties of point-like excitations

First, if we break the time-reversal symmetry (*i.e.* only put the system on orientable space-time with  $w_1 = 0$ ), then our models with  $k_1 = 0$  reduce to the  $\mathbb{Z}_n$ -2-cocycle model eqn. (74) with  $n = 2$  and  $k = k_4$ . So when  $k_1 = 0$ , the point-like topological excitation in our model is a fermion if  $k_4 = 1$ , and a boson if  $k_4 = 0$  (see Table II where a fermion is indicated by spin  $s_2 = \frac{1}{2}$  and a boson by spin  $s_2 = 0$ ).

When  $k_4 = 0$  (and without time-reversal symmetry), our model reduce to the UT  $Z_2$ -gauge theory (note that  $(a_0^{Z_2})^4 = (a_0^{Z_2})^3 w_1$ , and  $(a_0^{Z_2})^4 = 0$  when  $w_1 = 0$ ). So the point-like topological excitation in our model is a boson if  $k_4 = 0$ , even when  $k_1 \neq 0$ .

In the presence of time-reversal symmetry  $Z_2^T$  with  $T^2 = 1$ , the point-like topological excitation may carry fractionalized time-reversal symmetry with  $T^2 = -1$ , *i.e.* it may carry Kramer doublet. In fact, in this section, we will consider both time-reversal symmetry and the extra  $Z_2'$  symmetry  $\tilde{g}_i \rightarrow [\tilde{g}_i + 1]_2$  of our models. So the total symmetry group is  $Z_2' \times Z_2^T$ . In this case, the multivalueness of  $\tilde{g}_i$  is not only due the orientation twist around a loop, it is also due to the  $Z_2'$ -symmetry twist

around a loop. Thus

$$d\tilde{g} = w_1 + a'^{Z_2} \quad (241)$$

where  $a'^{Z_2}$  is the 1-cocycle that describes the  $Z_2'$ -symmetry twist in space-time.<sup>53-56</sup>

To see the time-reversal and  $Z_2'$  symmetry properties of the point-like topological excitation, we first consider the  $k_4 = 0$  case and start with the path integral eqn. (225). We like to stress that in our calculation to obtain eqn. (225), we did not use the relation  $d\tilde{g} = w_1$  which is not valid in the presence of  $Z_2'$ -symmetry twist, which is necessary to consider  $Z_2'$  symmetry. The only term that involves the world-line of the point-like topological excitation is  $e^{i\pi \int_{M^4} C_{\text{WL}}^{Z_2}(a_{\text{WS}}^{Z_2} + a_0^{Z_2})}$ , which can be expanded as

$$\begin{aligned} e^{i\pi \int_{M^4} C_{\text{WL}}^{Z_2}(a_{\text{WS}}^{Z_2} + a_0^{Z_2})} &= e^{i\pi \int_{M_{\text{WL}}^1} a_{\text{WS}}^{Z_2} + a_0^{Z_2}} = e^{i\pi \int_{D_{\text{WL}}^2} da_{\text{WS}}^{Z_2}} \\ &= e^{i\pi \int_{D_{\text{WL}}^2} k_3 \mathcal{B}_2 d\tilde{g} + B_{\text{WS}}^{Z_2}}, \end{aligned} \quad (242)$$

where  $D_{\text{WL}}^2$  is the 2-dimensional submanifold whose boundary is the world-line  $\partial D_{\text{WL}}^2 = M_{\text{WL}}^1$ . The term  $e^{i\pi \int_{M_{\text{WL}}^1} a_0^{Z_2}}$  indicates that the point-like excitation carries an unit of  $Z_2$ -gauge charge.

The term  $e^{i\pi \int_{D_{\text{WL}}^2} B_{\text{WS}}^{Z_2}} = e^{i\pi \text{Lnk}(M_{\text{WL}}^1, M_{\text{WS}}^2)}$  is determined by the linking number  $\text{Lnk}(M_{\text{WL}}^1, M_{\text{WS}}^2)$  between the world-line  $M_{\text{WL}}^1$  of the point-like excitation and the world-sheet  $M_{\text{WS}}^2$  of the string-like excitation. It describes the  $\pi$  phase change as a point-like excitation goes around the string-like excitation.

The term  $e^{i\pi \int_{D_{\text{WL}}^2} k_3 \mathcal{B}_2 d\tilde{g}}$  give rise to a  $Z_2' \times Z_2^T$ -SPT invariant

$$e^{i\pi \int_{D_{\text{WL}}^2} k_3 \mathcal{B}_2 d\tilde{g}} \rightarrow e^{i\pi \int_{D_{\text{WL}}^2} k_3 (\mathcal{B}_2 w_1 + \mathcal{B}_2 a'^{Z_2})} \quad (243)$$

which describes a 1+1D  $Z_2' \times Z_2^T$ -SPT state on  $D_{\text{WL}}^2$  when  $k_3 = 1$ . Due to the term  $\mathcal{B}_2 w_1$  in the SPT invariant, the boundary of the 1+1D  $Z_2^T$ -SPT state carries a Kramer doublet. Thus

the world-line, *i.e.* the point-like excitation, carries a Kramer doublet if  $k_3 = 1$  and carries a time-reversal singlet if  $k_3 = 0$

(see Table II where a Kramer doublet is indicated by quantum dimension  $d_2 = 2_-$  and a time-reversal singlet by quantum dimension  $d_2 = 1$ ). Due to the term  $\mathcal{B}_2 a'^{Z_2}$ , the Kramer doublet on the point-like excitation is formed by  $Z_2'$ -charge  $\pm \frac{1}{2}$  states. So

the Kramer doublet also carries fractional  $Z_2'$ -charge  $\pm \frac{1}{2}$ .

We next consider the  $k_1 = 0$  case and start with the path integral eqn. (232). Again, in our calculation to obtain eqn. (232), we did not use the relation  $d\tilde{g} = w_1$ .

The only term that involves the world line of the point-like topological excitation is

$$e^{i\pi \int_{M^4} b_{\text{WL}}^{\mathbb{Z}_2} [k_4 b_{\text{WL}}^{\mathbb{Z}_2} + (k_3 + k_4) \mathcal{B}_2 d\tilde{g} + B_{\text{WS}}^{\mathbb{Z}_2}]} \quad (244)$$

Let us consider a particular world-line which is a boundary:  $M_{\text{WL}}^1 = \partial D_{\text{WL}}^2$ , and write  $b_{\text{WL}}^{\mathbb{Z}_2}$  as

$$b_{\text{WL}}^{\mathbb{Z}_2} \stackrel{2}{=} \bar{b}_{\text{WL}}^{\mathbb{Z}_2} + b'_{\text{WL}}{}^{\mathbb{Z}_2} \quad (245)$$

where

$$db'_{\text{WL}}{}^{\mathbb{Z}_2} = C_{\text{WL}}^{\mathbb{Z}_2} \quad (246)$$

comes from the world-line and  $\bar{b}_{\text{WL}}^{\mathbb{Z}_2}$  from the background Stiefel-Whitney class and other world-lines. We obtain

$$\begin{aligned} & e^{i\pi \int_{M^4} b_{\text{WL}}^{\mathbb{Z}_2} [k_4 b_{\text{WL}}^{\mathbb{Z}_2} + (k_3 + k_4) \mathcal{B}_2 d\tilde{g} + B_{\text{WS}}^{\mathbb{Z}_2}]} \\ &= e^{i\pi \int_{M^4} \bar{b}_{\text{WL}}^{\mathbb{Z}_2} [k_4 \bar{b}_{\text{WL}}^{\mathbb{Z}_2} + (k_3 + k_4) \mathcal{B}_2 d\tilde{g} + B_{\text{WS}}^{\mathbb{Z}_2}]} \times \\ & e^{i\pi \int_{M^4} b'_{\text{WL}}{}^{\mathbb{Z}_2} [k_4 b'_{\text{WL}}{}^{\mathbb{Z}_2} + (k_3 + k_4) \mathcal{B}_2 d\tilde{g} + B_{\text{WS}}^{\mathbb{Z}_2}]} \end{aligned} \quad (247)$$

which can be viewed as the effective action amplitude on the world line.

Compare with our previous result, we see that

the term  $e^{i\pi \int_{M^4} k_4 (b'_{\text{WL}}{}^{\mathbb{Z}_2})^2}$  should describe the Fermi statistics of the point-like excitation when  $k_4 = 1$ .

Using the fact that Poincaré dual of  $b'_{\text{WL}}{}^{\mathbb{Z}_2}$  is  $D_{\text{WL}}^2$ , we can express  $e^{i\pi \int_{M^4} k_4 (b'_{\text{WL}}{}^{\mathbb{Z}_2})^2}$  in terms of self-intersection number of  $D_{\text{WL}}^2$ :

$$e^{i\pi \int_{M^4} k_4 (b'_{\text{WL}}{}^{\mathbb{Z}_2})^2} = e^{i\pi \text{Int}(D_{\text{WL}}^2, D_{\text{WL}}^2)}. \quad (248)$$

We see that

the Fermi statistics in 3+1D is described by the self-intersection number of the disk whose boundary is the world-line of the fermion.

The term  $e^{i\pi \int_{M^4} b_{\text{WL}}^{\mathbb{Z}_2} B_{\text{WS}}^{\mathbb{Z}_2}}$  describes the  $\pi$  phase change as a point-like excitation goes around the string-like excitation.

Now, let us concentrate on

$$e^{i\pi \int_{M^4} (k_3 + k_4) b_{\text{WL}}^{\mathbb{Z}_2} \mathcal{B}_2 d\tilde{g}} = e^{i\pi \int_{D_{\text{WL}}^2} (k_3 + k_4) \mathcal{B}_2 d\tilde{g}} \quad (249)$$

where we have used the fact that Poincaré dual of  $b_{\text{WL}}^{\mathbb{Z}_2}$  is  $D_{\text{WL}}^2$ . As discussed before, due to such a term will make

the point-like excitation carry a Kramer doublet formed by fractional  $Z'_2$ -charge  $\pm \frac{1}{2}$ , if  $k_3 + k_4 \stackrel{2}{=} 1$  and carry a time-reversal singlet with integer  $Z'_2$ -charge, if  $k_3 + k_4 \stackrel{2}{=} 0$ .

## 5. Properties of string-like excitations

To obtain physical properties of string excitations, let us consider a dimension reduction  $M^4 = M^3 \times S^1$  (for details see Section VIA 3).

Let us first consider the case for  $k_4 = 0$  and start from eqn. (225). We can choose  $a_{\text{WS}}^{\mathbb{Z}_2}$  to make  $\int_{S^1} a_{\text{WS}}^{\mathbb{Z}_2} = 0$ . The two sectors after the reduction are labeled by  $\alpha = \int_{S^1} a^{\mathbb{Z}_2} = \int_{S^1} a_0^{\mathbb{Z}_2}$ . The effective theory on  $M^3$  after the dimension reduction is given by

$$\begin{aligned} & Z_{k_1 k_2 k_3 k_4 k_5 k_6}(M^4) = \bar{\delta}_2(k_3 \mathcal{B}_2 w_1 + B_{\text{WS}}^{\mathbb{Z}_2}) \times \\ & \sum_{\tilde{g}, da_0^{\mathbb{Z}_2} \stackrel{2}{=} 0} e^{i\pi \int_{M^3} B_{\text{WL}}^{\mathbb{Z}_2} a_{\text{WS}}^{\mathbb{Z}_2}} e^{i\pi \int_{M^3} B_{\text{WL}}^{\mathbb{Z}_2} a_0^{\mathbb{Z}_2} + \alpha(k_2 + k_1)(d\tilde{g})^3}, \end{aligned}$$

where  $a_0^{\mathbb{Z}_2}$  now lives on  $M^3$  and  $B_{\text{WL}}^{\mathbb{Z}_2}$  is the Poincaré dual of the world-line in  $M^3$ .

For simplicity, let us choose the world-line to make  $B_{\text{WL}}^{\mathbb{Z}_2} = 0$ . The effective theory on  $M^3$  now becomes (only the dynamical part)

$$Z_{k_1 k_2 k_3 k_4 k_5 k_6}(M^4) = \sum_{\tilde{g}, da_0^{\mathbb{Z}_2} \stackrel{2}{=} 0} e^{i\pi \int_{M^3} \alpha(k_2 + k_1)(d\tilde{g})^3}. \quad (250)$$

If we view the above effective theory as a 2+1D theory with time-reversal  $Z_2^T$  symmetry that acts on  $\tilde{g}_i$ , then the above effective theory describe trivial  $Z_2^T$ -SPT states since the SPT invariant

$$\begin{aligned} & e^{i\pi \int_{M^3} \alpha(k_2 + k_1)(d\tilde{g})^3} \\ &= e^{i\pi \int_{M^3} \alpha(k_2 + k_1) w_1^3} = 1 \end{aligned} \quad (251)$$

becomes trivial in 2+1D (see Appendix D 3). The 1+1D boundary of the 2+1D theory in the  $\alpha = 1$  sector corresponds to the  $Z_2$ -vortex line. So the above result implies that the  $Z_2$ -vortex line of our model just behave like the  $Z_2$ -vortex line of UT  $Z_2$ -gauge theory regardless the values of  $k_I$ .

Our model actually has a  $Z'_2 \times Z_2^T$  symmetry. So the 2+1D effective theory can be viewed as a model with  $Z'_2$  symmetry. In this case, the model describes a non-trivial  $Z'_2$ -SPT state, when  $\alpha(k_2 + k_1) \neq 0$ . To see this, we note that the  $Z'_2$  acts like  $\tilde{g}_i \rightarrow [\tilde{g}_i + 1]_2$ . So to obtain the  $Z'_2$ -SPT invariant, we need to gauge the  $Z'_2$  symmetry (see Section VB) by replacing  $d\tilde{g}$  by  $a'^{\mathbb{Z}_2}$ :

$$e^{i\pi \int_{M^3} \alpha(k_2 + k_1)(d\tilde{g})^3} = e^{i\pi \int_{M^3} \alpha(k_2 + k_1)(a'^{\mathbb{Z}_2})^3}. \quad (252)$$

The above SPT invariant allows us to show our 2+1D effective theory leads to a non-trivial  $Z'_2$ -SPT state, which was first studied in Ref. 10. Since the 1+1D boundary of the 2+1D theory in the  $\alpha = 1$  sector corresponds to the  $Z_2$ -vortex line, so the above result implies that

the  $Z_2$ -vortex line of our model carry non-trivial edge excitations of  $Z'_2$ -SPT state described by SPT invariant  $e^{i\pi \int_{M^3} (k_2 + k_1)(a'^{\mathbb{Z}_2})^3}$ .



The above results about the  $Z_2$ -vortex line can be obtained by directly calculating the effective theory on the  $Z_2$ -vortex line. We start from the theory with excitations eqn. (225). Let the world-sheet of the string (*i.e.* the  $Z_2$ -vortex line),  $M_{\text{WS}}^2$ , be the boundary of  $D_{\text{WS}}^3$ . For simplicity, let us assume that  $w_2 = w_1 = 0$  and  $a'^{\mathbb{Z}_2} = 0$  (*i.e.* no  $Z_2'$ -symmetry twist) on  $M^4$ . In this case,  $a_{\text{WS}}^{\mathbb{Z}}$  can be chosen to be the Poincaré dual of  $D_{\text{WS}}^3$ .

The effective theory on the string comes from the factor  $e^{i\pi \int_{M^4} (k_2 + k_1) a_{\text{WS}}^{\mathbb{Z}_2} (d\tilde{g})^3}$  in eqn. (225), which leads to the following effective theory

$$Z = \sum_{\tilde{g}_i} e^{i\pi \int_{D_{\text{WS}}^3} (k_2 + k_1) (d\tilde{g})^3}. \quad (253)$$

If we identify  $(-)^{\tilde{g}_i}$  as  $\sigma_i^z$  of spin-1/2, then the above action amplitude describes a 2+1D spin-1/2 model with  $Z_2' \times Z_2^T$  symmetry acting on  $\tilde{g}_i$ 's:

$$Z_2' : \prod_i \sigma_i^x, \quad Z_2^T : K \prod_i \sigma_i^x, \quad (254)$$

where  $K$  is the complex conjugation. The effective theory actually describes a non-trivial  $Z_2' \times Z_2^T$ -SPT state on  $D_{\text{WS}}^3$ . So the effective theory on the world-sheet  $M_{\text{WS}}^2$  is the effective boundary theory of the  $Z_2' \times Z_2^T$ -SPT state. In other words, the string will carry non-trivial boundary excitations of the 2+1D  $Z_2' \times Z_2^T$ -SPT state. The non-trivialness of the excitations on the string is protected by the anomalous symmetry on the boundary.<sup>65</sup> This can be viewed as the symmetry fractionalization (or quantum number fractionalization) on strings. We have seen that on point-like excitation, the  $T^2 = 1$   $Z_2^T$  time-reversal symmetry can be fractionalized into  $T^2 = -1$  Kramer doublet. In contrast, on strings, the symmetry fractionalization is realized as the anomalous (*i.e.* non-on-site) symmetry that constrains the effective theory for degrees of freedom on the strings.

So the key to calculate the symmetry fractionalization is to calculate the non-on-site (*i.e.* anomalous) symmetry on the strings. Let us do the calculation for the case  $k_2 + k_1 = 1$ , which leads to the following effective theory

$$Z = \sum_{\tilde{g}_i} e^{i\pi \int_{D_{\text{WS}}^3} (d\tilde{g})^3}. \quad (255)$$

which describes a  $Z_2' \times Z_2^T$ -SPT state. The group-cocycle that describes the  $Z_2' \times Z_2^T$ -SPT phase is in fact the topological term  $\int_{D_{\text{WS}}^3} (d\tilde{g})^3$ :

$$\nu_3(\tilde{g}_0, \tilde{g}_1, \tilde{g}_2, \tilde{g}_3) = -(\tilde{g}_0 - \tilde{g}_1)(\tilde{g}_1 - \tilde{g}_2)(\tilde{g}_2 - \tilde{g}_3) \quad (256)$$

The  $Z_2 \times Z_2^T$  symmetry on the string are twisted by the group-cocycle and becomes non-on-site

$$\begin{aligned} Z_2' : & \prod_I \sigma_I^x e^{i\pi \nu_3(1,0,\tilde{g}_I,\tilde{g}_{I+1})}, \\ Z_2^T : & K \prod_I \sigma_I^x e^{i\pi \nu_3(1,0,\tilde{g}_I,\tilde{g}_{I+1})}, \end{aligned} \quad (257)$$

where

$$\begin{aligned} e^{i\pi \nu_3(1,0,\tilde{g}_I,\tilde{g}_J)} &= (-)^{\tilde{g}_I(\tilde{g}_I - \tilde{g}_J)} = (-)^{\tilde{g}_I} (-)^{\tilde{g}_I \tilde{g}_J} \\ &= \sigma_I^z \frac{1 + \sigma_I^z + \sigma_J^z - \sigma_I^z \sigma_J^z}{2} \end{aligned} \quad (258)$$

The effective Hamiltonian on the string respects the anomalous  $Z_2' \times Z_2^T$  symmetry, which may take a form

$$H_{\text{str}} = \sum_I J_I^z \sigma_I^z \sigma_{I+1}^z + \sum_I K_I^x (\sigma_{I-1}^z \sigma_I^x \sigma_{I+1}^z - \sigma_I^x) \quad (259)$$

The ground state of such Hamiltonian is gapless or spontaneously breaks the  $Z_2'$  symmetry. So when  $k_1 + k_2 = 1$  and  $k_4 = 0$ , the strings carry non-trivial excitations described by the above Hamiltonian with an anomalous  $Z_2' \times Z_2^T$  symmetry:  $U' = \prod_I \sigma_I^x \prod_I \sigma_I^z \frac{1 + \sigma_I^z + \sigma_{I+1}^z - \sigma_I^z \sigma_{I+1}^z}{2}$  and  $U_T = K \prod_I \sigma_I^x \prod_I \sigma_I^z \frac{1 + \sigma_I^z + \sigma_{I+1}^z - \sigma_I^z \sigma_{I+1}^z}{2}$ .

Next, let us consider the case for  $k_1 = 0$  and start from eqn. (234). The only term that involve the word sheet is  $e^{i\pi \int_{M^4} b_{\text{WL}}^{\mathbb{Z}_2} B_{\text{WS}}^{\mathbb{Z}_2}}$ , which can be rewritten as

$$\begin{aligned} e^{i\pi \int_{M^4} b_{\text{WL}}^{\mathbb{Z}_2} B_{\text{WS}}^{\mathbb{Z}_2}} &= e^{i\pi \int_{M_{\text{WS}}^2} b_{\text{WL}}^{\mathbb{Z}_2}} = e^{i\pi \int_{D_{\text{WS}}^3} db_{\text{WL}}^{\mathbb{Z}_2}} \\ &= e^{i\pi \int_{D_{\text{WS}}^3} k_2 (d\tilde{g})^3 + C_{\text{WL}}^{\mathbb{Z}_2}} \end{aligned} \quad (260)$$

Repeat the above calculation, we see that

when  $k_2 = 1$  and  $k_1 = 0$ , the strings carry non-trivial excitations with an anomalous  $Z_2' \times Z_2^T$  symmetry:  $U' = \prod_I \sigma_I^x \prod_I \sigma_I^z \frac{1 + \sigma_I^z + \sigma_{I+1}^z - \sigma_I^z \sigma_{I+1}^z}{2}$  and  $U_T = K \prod_I \sigma_I^x \prod_I \sigma_I^z \frac{1 + \sigma_I^z + \sigma_{I+1}^z - \sigma_I^z \sigma_{I+1}^z}{2}$ .

We like remark that potentially, the strings may carry an anomalous  $Z_2 \times Z_2' \times Z_2^T$  symmetry, where  $Z_2$  is associated with  $a^{\mathbb{Z}_2}$ . From the above calculation, we see that the anomalous symmetry only come from the  $Z_2'$  symmetry. There is no anomalous symmetry from  $Z_2$ .

## VII. ACKNOWLEDGEMENT

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## Appendix A: The Künneth formula

The Künneth formula is a very helpful formula that allows us to calculate the cohomology of chain complex  $X \times X'$  in terms of the cohomology of chain complex  $X$  and chain complex  $X'$ . The Künneth formula is expressed in terms of the tensor-product operation  $\otimes_R$  and the torsion-product operation  $\text{Tor}_1^R$  that act on  $R$ -modules  $\mathbb{M}, \mathbb{M}', \mathbb{M}''$ . Here  $R$  is a ring and a  $R$ -module is

like a vector space over  $R$  (*i.e.* we can “multiply” a “vector” in  $\mathbb{M}$  by an element of  $R$ , and two “vectors” in  $\mathbb{M}$  can add.) The tensor-product operation  $\otimes_R$  has the following properties:

$$\begin{aligned} \mathbb{M} \otimes_{\mathbb{Z}} \mathbb{M}' &\simeq \mathbb{M}' \otimes_{\mathbb{Z}} \mathbb{M}, \\ (\mathbb{M}' \oplus \mathbb{M}'') \otimes_R \mathbb{M} &= (\mathbb{M}' \otimes_R \mathbb{M}) \oplus (\mathbb{M}'' \otimes_R \mathbb{M}), \\ \mathbb{M} \otimes_R (\mathbb{M}' \oplus \mathbb{M}'') &= (\mathbb{M} \otimes_R \mathbb{M}') \oplus (\mathbb{M} \otimes_R \mathbb{M}''); \\ \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{M} &\simeq \mathbb{M} \otimes_{\mathbb{Z}} \mathbb{Z} = \mathbb{M}, \\ \mathbb{Z}_n \otimes_{\mathbb{Z}} \mathbb{M} &\simeq \mathbb{M} \otimes_{\mathbb{Z}} \mathbb{Z}_n = \mathbb{M}/n\mathbb{M}, \\ \mathbb{Z}_m \otimes_{\mathbb{Z}} \mathbb{Z}_n &= \mathbb{Z}_{\langle m, n \rangle}, \end{aligned} \quad (\text{A1})$$

The torsion-product operation  $\text{Tor}_1^R$  has the following properties:

$$\begin{aligned} \text{Tor}_R^1(\mathbb{M}, \mathbb{M}') &\simeq \text{Tor}_R^1(\mathbb{M}', \mathbb{M}), \\ \text{Tor}_R^1(\mathbb{M}' \oplus \mathbb{M}'', \mathbb{M}) &= \text{Tor}_R^1(\mathbb{M}', \mathbb{M}) \oplus \text{Tor}_R^1(\mathbb{M}'', \mathbb{M}), \\ \text{Tor}_R^1(\mathbb{M}, \mathbb{M}' \oplus \mathbb{M}'') &= \text{Tor}_R^1(\mathbb{M}, \mathbb{M}') \oplus \text{Tor}_R^1(\mathbb{M}, \mathbb{M}'') \\ \text{Tor}_{\mathbb{Z}}^1(\mathbb{Z}, \mathbb{M}) &= \text{Tor}_{\mathbb{Z}}^1(\mathbb{M}, \mathbb{Z}) = 0, \\ \text{Tor}_{\mathbb{Z}}^1(\mathbb{Z}_n, \mathbb{M}) &= \{m \in \mathbb{M} | nm = 0\}, \\ \text{Tor}_{\mathbb{Z}}^1(\mathbb{Z}_m, \mathbb{Z}_n) &= \mathbb{Z}_{\langle m, n \rangle}, \end{aligned} \quad (\text{A2})$$

where  $\langle m, n \rangle$  is the greatest common divisor of  $m$  and  $n$ . These expressions allow us to compute the tensor-product  $\otimes_R$  and the torsion-product  $\text{Tor}_R^1$ . We will use abbreviated  $\text{Tor}$  to denote  $\text{Tor}_{\mathbb{Z}}^1$ .

The Künneth formula itself is given by (see Ref. 91 page 247)

$$\begin{aligned} H^d(X \times X', \mathbb{M} \otimes_R \mathbb{M}') &\simeq \left[ \oplus_{k=0}^d H^k(X, \mathbb{M}) \otimes_R H^{d-k}(X', \mathbb{M}') \right] \oplus \\ &\left[ \oplus_{k=0}^{d+1} \text{Tor}_R^1(H^k(X, \mathbb{M}), H^{d-k+1}(X', \mathbb{M}')) \right]. \end{aligned} \quad (\text{A3})$$

Here  $R$  is a principle ideal domain and  $\mathbb{M}, \mathbb{M}'$  are  $R$ -modules such that  $\text{Tor}_R^1(\mathbb{M}, \mathbb{M}') = 0$ . We also require either

- (1)  $H_d(X; \mathbb{Z})$  and  $H_d(X'; \mathbb{Z})$  are finitely generated, or
- (2)  $\mathbb{M}'$  and  $H_d(X'; \mathbb{Z})$  are finitely generated.

For more details on principal ideal domain and  $R$ -module, see the corresponding Wiki articles. Note that  $\mathbb{Z}$  and  $\mathbb{R}$  are principal ideal domains, while  $\mathbb{R}/\mathbb{Z}$  is not. Also,  $\mathbb{R}$  and  $\mathbb{R}/\mathbb{Z}$  are not finitely generate  $R$ -modules if  $R = \mathbb{Z}$ . The Künneth formula also works for topological cohomology where  $X$  and  $X'$  are treated as topological spaces.

For homology, there is a similar Künneth formula

$$\begin{aligned} H_d(X \times X'; \mathbb{Z}) &\simeq \left[ \oplus_{k=0}^d H_k(X; \mathbb{Z}) \otimes H_{d-k}(X'; \mathbb{Z}) \right] \oplus \\ &\left[ \oplus_{k=0}^{d-1} \text{Tor}(H_k(X; \mathbb{Z}), H_{d-k-1}(X'; \mathbb{Z})) \right]. \end{aligned} \quad (\text{A4})$$

As the first application of Künneth formula, we like to use it to calculate  $H^*(X', \mathbb{M})$  from  $H^*(X'; \mathbb{Z})$ , by

choosing  $R = \mathbb{M}' = \mathbb{Z}$ . In this case, the condition  $\text{Tor}_R^1(\mathbb{M}, \mathbb{M}') = \text{Tor}_{\mathbb{Z}}^1(\mathbb{M}, \mathbb{Z}) = 0$  is always satisfied.  $\mathbb{M}$  can be  $\mathbb{R}/\mathbb{Z}$ ,  $\mathbb{Z}$ ,  $\mathbb{Z}_n$  etc. So we have

$$\begin{aligned} H^d(X \times X', \mathbb{M}) &\simeq \left[ \oplus_{k=0}^d H^k(X, \mathbb{M}) \otimes_{\mathbb{Z}} H^{d-k}(X'; \mathbb{Z}) \right] \oplus \\ &\left[ \oplus_{k=0}^{d+1} \text{Tor}(H^k(X, \mathbb{M}), H^{d-k+1}(X'; \mathbb{Z})) \right]. \end{aligned} \quad (\text{A5})$$

The above is also valid for topological cohomology.

We can further choose  $X$  to be the space of one point in eqn. (A5), and use

$$H^d(X, \mathbb{M}) = \begin{cases} \mathbb{M}, & \text{if } d = 0, \\ 0, & \text{if } d > 0, \end{cases} \quad (\text{A6})$$

to reduce eqn. (A5) to

$$H^d(X, \mathbb{M}) \simeq H^d(X; \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{M} \oplus \text{Tor}(H^{d+1}(X; \mathbb{Z}), \mathbb{M}). \quad (\text{A7})$$

where  $X'$  is renamed as  $X$ . The above is a form of the universal coefficient theorem which can be used to calculate  $H^*(X, \mathbb{M})$  from  $H^*(X; \mathbb{Z})$  and the module  $\mathbb{M}$ . The universal coefficient theorem works for topological cohomology where  $X$  is a topological space. In fact, we also have a similar universal coefficient theorem for homology

$$H_d(X, \mathbb{M}) \simeq H_d(X; \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{M} \oplus \text{Tor}(H_{d-1}(X; \mathbb{Z}), \mathbb{M}). \quad (\text{A8})$$

Using the universal coefficient theorem, we can rewrite eqn. (A5) as

$$H^d(X \times X', \mathbb{M}) \simeq \oplus_{k=0}^d H^k[X, H^{d-k}(X', \mathbb{M})]. \quad (\text{A9})$$

The above is also valid for topological cohomology. We note that

$$H^0(X, \mathbb{M}) = \mathbb{M}. \quad (\text{A10})$$

There is also a universal coefficient theorem between homology and cohomology

$$H^d(X, \mathbb{M}) \simeq \text{Hom}(H_d(X; \mathbb{Z}), \mathbb{M}) \oplus \text{Ext}(H_{d-1}(X; \mathbb{Z}), \mathbb{M}). \quad (\text{A11})$$

Here  $\text{Ext}$  operation on modules is given by

$$\begin{aligned} \text{Ext}_R^1(\mathbb{M}' \oplus \mathbb{M}'', \mathbb{M}) &= \text{Ext}_R^1(\mathbb{M}', \mathbb{M}) \oplus \text{Ext}_R^1(\mathbb{M}'', \mathbb{M}), \\ \text{Ext}_R^1(\mathbb{M}, \mathbb{M}' \oplus \mathbb{M}'') &= \text{Ext}_R^1(\mathbb{M}, \mathbb{M}') \oplus \text{Ext}_R^1(\mathbb{M}, \mathbb{M}'') \\ \text{Ext}_{\mathbb{Z}}^1(\mathbb{Z}, \mathbb{M}) &= 0, \\ \text{Ext}_{\mathbb{Z}}^1(\mathbb{Z}_n, \mathbb{M}) &= \mathbb{M}/n\mathbb{M}, \\ \text{Ext}_{\mathbb{Z}}^1(\mathbb{Z}_n, \mathbb{Z}) &= \mathbb{Z}_n, \\ \text{Ext}_{\mathbb{Z}}^1(\mathbb{Z}_m, \mathbb{Z}_n) &= \mathbb{Z}_{\langle m, n \rangle}. \end{aligned} \quad (\text{A12})$$

The Hom operation on modules is given by

$$\begin{aligned}
\text{Hom}_R(\mathbb{M}' \oplus \mathbb{M}'', \mathbb{M}) &= \text{Hom}_R(\mathbb{M}', \mathbb{M}) \oplus \text{Hom}_R(\mathbb{M}'', \mathbb{M}), \\
\text{Hom}_R(\mathbb{M}, \mathbb{M}' \oplus \mathbb{M}'') &= \text{Hom}_R(\mathbb{M}, \mathbb{M}') \oplus \text{Hom}_R(\mathbb{M}, \mathbb{M}'') \\
\text{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{M}) &= \mathbb{M}, \\
\text{Hom}_{\mathbb{Z}}(\mathbb{Z}_n, \mathbb{M}) &= \{m \in \mathbb{M} | nm = 0\}, \\
\text{Hom}_{\mathbb{Z}}(\mathbb{Z}_n, \mathbb{Z}) &= 0, \\
\text{Hom}_{\mathbb{Z}}(\mathbb{Z}_m, \mathbb{Z}_n) &= \mathbb{Z}_{\langle m, n \rangle}, \quad (A13)
\end{aligned}$$

We will use abbreviated Ext and Hom to denote  $\text{Ext}_{\mathbb{Z}}^1$  and  $\text{Hom}_{\mathbb{Z}}$ .

### Appendix B: Poincaré Duality

Poincaré Duality relates  $H^k(M^d, R)$  and  $H_{d-k}(M^d, R)$ . We note that for a closed connected  $d$ -dimensional space  $M^d$ ,  $H_0(M^d; \mathbb{Z}) = \mathbb{Z}$ ,  $H_d(M^d; \mathbb{Z}) = \mathbb{Z}$  if  $M^d$  is orientable, and  $H_d(M^d; \mathbb{Z}) = 0$  if  $M^d$  is non-orientable. Similarly,  $H^0(M^d; \mathbb{Z}) = \mathbb{Z}$ ,  $H^d(M^d; \mathbb{Z}) = \mathbb{Z}$  if  $M^d$  is orientable, and  $H^d(M^d; \mathbb{Z}) = \mathbb{Z}_2$  if  $M^d$  is non-orientable.

**Poincaré Duality:** If  $M$  is a closed  $R$ -orientable  $n$ -dimensional manifold with fundamental class  $[M] \in H^n(M, R)$  (here  $R$  is a ring), then the map  $D : H^k(M; R) \rightarrow H_{n-k}(M; R)$  defined by  $D(\alpha) = [M] \cap \alpha$  is an isomorphism for all  $k$ .

The cup product pairing between  $H^k(M^d, R)$  and  $H^{d-k}(M^d, R)$  is non singular for closed  $R$ -orientable manifolds when  $R$  is a field, or when  $R = \mathbb{Z}$  and torsion in  $H^*(M; \mathbb{Z})$  is factored out. This implies that the free part of  $H^k(M^d; \mathbb{Z})$  and  $H^{d-k}(M^d; \mathbb{Z})$  has the same dimension.

### Appendix C: The factor $\frac{|H^0(M^4; \mathbb{Z}_n)|^2 |H^2(M^4; \mathbb{Z}_n)|}{|H^1(M^4; \mathbb{Z}_n)|^2}$

To calculate the factor  $\frac{|H^0(M^4; \mathbb{Z}_n)|^2 |H^2(M^4; \mathbb{Z}_n)|}{|H^1(M^4; \mathbb{Z}_n)|^2}$  we first use eqn. (A11) to show

$$\begin{aligned}
H^1(M^4; \mathbb{Z}_n) & \quad (C1) \\
&= \text{Hom}(H_1(M^4; \mathbb{Z}); \mathbb{Z}_n) \oplus \text{Ext}(H_0(M^4; \mathbb{Z}), \mathbb{Z}_n) \\
&= \text{Hom}(\text{f}H_1(M^4; \mathbb{Z}); \mathbb{Z}_n) \oplus \text{Hom}(\text{t}H_1(M^4; \mathbb{Z}); \mathbb{Z}_n) \\
&= \mathbb{Z}_n^{\oplus b_1} \oplus \text{Hom}(\text{t}H_1(M^4; \mathbb{Z}); \mathbb{Z}_n), \quad (C2)
\end{aligned}$$

and

$$\begin{aligned}
H^2(M^4; \mathbb{Z}_n) & \quad (C3) \\
&= \text{Hom}(H_2(M^4; \mathbb{Z}); \mathbb{Z}_n) \oplus \text{Ext}(H_1(M^4; \mathbb{Z}), \mathbb{Z}_n) \\
&= \text{Hom}(\text{f}H_2(M^4; \mathbb{Z}); \mathbb{Z}_n) \oplus \text{Hom}(\text{t}H_2(M^4; \mathbb{Z}); \mathbb{Z}_n) \oplus \\
&\quad \text{Ext}(\text{t}H_1(M^4; \mathbb{Z}), \mathbb{Z}_n), \\
&= \mathbb{Z}_n^{\oplus b_2} \oplus \text{Hom}(\text{t}H_2(M^4; \mathbb{Z}); \mathbb{Z}_n) \oplus \text{Hom}(\text{t}H_1(M^4; \mathbb{Z}), \mathbb{Z}_n).
\end{aligned}$$

where “f” and “t” indicate the free and torsion parts of a discrete abelian group and  $b_n$  is the dimension of  $\text{f}H_n(M^4; \mathbb{Z})$  (i.e. the  $n^{\text{th}}$  Betti number). Using  $H^0(M^4; \mathbb{Z}_n) = \mathbb{Z}_n^{\oplus b_0}$ , we find that

$$\begin{aligned}
&\frac{|H^0(M^4; \mathbb{Z}_n)|^2 |H^2(M^4; \mathbb{Z}_n)|}{|H^1(M^4; \mathbb{Z}_n)|^2} \\
&= n^{2b_0 + b_2 - 2b_1} \frac{|\text{Hom}(\text{t}H_2(M^4; \mathbb{Z}); \mathbb{Z}_n)|}{|\text{Hom}(\text{t}H_1(M^4; \mathbb{Z}); \mathbb{Z}_n)|}. \quad (C4)
\end{aligned}$$

We note that, according to eqn. (A11)

$$\begin{aligned}
H^2(M^d; \mathbb{Z}) & \\
&= \text{Hom}(H_2(M^d; \mathbb{Z}), \mathbb{Z}) \oplus \text{Ext}(H_1(M^d; \mathbb{Z}), \mathbb{Z}). \quad (C5)
\end{aligned}$$

Since  $\text{Hom}(\mathbb{Z}_n, \mathbb{Z}) = 0$ ,  $\text{Ext}(\mathbb{Z}_n, \mathbb{Z}) = \mathbb{Z}_n$ , and  $\text{Ext}(\mathbb{Z}, \mathbb{Z}) = 0$ , we see that  $\text{t}H^2(M^d; \mathbb{Z}) = \text{t}H_1(M^d; \mathbb{Z})$ . We get

$$\begin{aligned}
&\frac{|H^0(M^4; \mathbb{Z}_n)|^2 |H^2(M^4; \mathbb{Z}_n)|}{|H^1(M^4; \mathbb{Z}_n)|^2} \\
&= n^{2b_0 + b_2 - 2b_1} \frac{|\text{Hom}(\text{t}H_2(M^4; \mathbb{Z}); \mathbb{Z}_n)|}{|\text{Hom}(\text{t}H^2(M^4; \mathbb{Z}); \mathbb{Z}_n)|}. \quad (C6)
\end{aligned}$$

For 4-dimensional closed orientable manifolds  $b_1 = b_3$ ,  $b_0 = b_4$  and  $\chi(M^4) = \sum_{n=0}^4 (-1)^n b_n$  is the Euler number of  $M^4$ . Using Poincaré duality  $H^2(M^4; \mathbb{Z}) = H_2(M^4; \mathbb{Z})$ , we can show that

$$\frac{|H^0(M^4; \mathbb{Z}_n)|^2 |H^2(M^4; \mathbb{Z}_n)|}{|H^1(M^4; \mathbb{Z}_n)|^2} = n^{\chi(M^4)}. \quad (C7)$$

When  $n = 2$  we have a Poincaré duality  $H^k(M^d; \mathbb{Z}_2) = H^{d-k}(M^d; \mathbb{Z}_2)$  for any closed manifold  $M^d$  regardless if  $M^d$  is orientable or not (since  $M^d$  is always  $\mathbb{Z}_2$ -orientable). Thus

$$\begin{aligned}
&\frac{|H^0(M^4; \mathbb{Z}_2)|^2 |H^2(M^4; \mathbb{Z}_2)|}{|H^1(M^4; \mathbb{Z}_2)|^2} \\
&= \frac{|H^0(M^4; \mathbb{Z}_2)| |H^2(M^4; \mathbb{Z}_2)| |H^4(M^4; \mathbb{Z}_2)|}{|H^1(M^4; \mathbb{Z}_2)| |H^3(M^4; \mathbb{Z}_2)|}. \quad (C8)
\end{aligned}$$

According to eqn. (A11)

$$\begin{aligned}
H^k(M^d; \mathbb{Z}_2) & \\
&= \text{Hom}(H_k(M^d; \mathbb{Z}), \mathbb{Z}_2) \oplus \text{Ext}(H_{k-1}(M^d; \mathbb{Z}), \mathbb{Z}_2) \\
&= \mathbb{Z}_2^{\oplus b_k} \oplus \text{Hom}(\text{t}H_k(M^d; \mathbb{Z}), \mathbb{Z}_2) \oplus \\
&\quad \text{Hom}(\text{t}H_{k-1}(M^d; \mathbb{Z}), \mathbb{Z}_2). \quad (C9)
\end{aligned}$$

This allows us to show

$$\frac{|H^0(M^4; \mathbb{Z}_2)|^2 |H^2(M^4; \mathbb{Z}_2)|}{|H^1(M^4; \mathbb{Z}_2)|^2} = 2^{\chi(M^4)}, \quad (C10)$$

where we have used the fact that  $\text{t}H_4(M^4; \mathbb{Z}) = 0$  for both orientable and non-orientable closed manifolds. On the other hand, the factor  $\frac{|H^0(M^4; \mathbb{Z}_n)|^2 |H^2(M^4; \mathbb{Z}_n)|}{|H^1(M^4; \mathbb{Z}_n)|^2}$  is in general not of the form  $\rho^{\chi(M^4)}$  for non-orientable manifolds when  $n > 2$ .

## Appendix D: Relations between cocycles and Stiefel-Whitney classes on a closed manifold

The cocycles and the Stiefel-Whitney classes on a closed manifold many satisfy many relations. In this section, we will show how to generate those relations.

### 1. Introduction to Stiefel-Whitney classes

The Stiefel-Whitney classes  $w_i \in H^i(M^d; \mathbb{Z}_2)$  is defined for an  $O(n)$  vector bundle on a  $d$ -dimensional space with  $n \rightarrow \infty$ . If the  $O(\infty)$  vector bundle on  $d$ -dimensional space,  $M^d$ , happen to be the tangent bundle of  $M^d$  direct summed with a trivial  $\infty$ -dimensional vector bundle, then the corresponding Stiefel-Whitney classes are referred as the Stiefel-Whitney classes of the manifold  $M^d$ .

The Stiefel-Whitney classes of manifold behave well under the connected sum of manifolds. Let  $w(M)$  be the total Stiefel-Whitney class of a manifold  $M$ . If we know  $w(M)$  and  $w(N)$ , then we can obtain  $w(M \# N)$ :

$$w(M \# N) = w(M) + w(N) - 1. \quad (D1)$$

Under the product of manifolds, we have

$$w(M \times N) = w(M)w(N). \quad (D2)$$

The Stiefel-Whitney numbers are non-oriented cobordism invariant. All the Stiefel-Whitney numbers of a smooth compact manifold vanish iff the manifold is the boundary of some smooth compact manifold. Here the manifold can be non-orientable.

The Stiefel-Whitney numbers and Pontryagin numbers are oriented cobordism invariant. All the Stiefel-Whitney numbers and Pontryagin numbers of a smooth compact orientable manifold vanish iff the manifold is the boundary of some smooth compact orientable manifold.

### 2. Relations between Stiefel-Whitney classes of the tangent bundle

For generic  $O(\infty)$  vector bundle, the Stiefel-Whitney classes are all independent. However, the Stiefel-Whitney classes for a manifold (*i.e.* for the tangent bundle) are not independent and satisfy many relations.

To obtain those relations, we note that, for any  $O(\infty)$  vector bundle, the total Stiefel-Whitney class  $w = 1 + w_1 + w_2 + \dots$  is related to the total Wu class  $u = 1 + u_1 + u_2 + \dots$  through the total Steenrod square<sup>92</sup>:

$$w = Sq(u), \quad Sq = 1 + Sq^1 + Sq^2 + \dots \quad (D3)$$

Therefore,  $w_n = \sum_{i=0}^n Sq^i(u_{n-i})$ . The Steenrod squares have the following properties:

$$Sq^i(x_j) = 0, \quad i > j, \quad Sq^j(x_j) = x_j x_j, \quad Sq^0 = 1, \quad (D4)$$

for any  $x_j \in H^j(M^d; \mathbb{Z}_2)$ . Thus

$$u_n = w_n + \sum_{i=1, 2i \leq n} Sq^i(u_{n-i}). \quad (D5)$$

This allows us to compute  $u_n$  iteratively, using Wu formula

$$Sq^i(w_j) = 0, \quad i > j, \quad Sq^i(w_i) = w_i w_i, \quad (D6)$$

$$Sq^i(w_j) = w_i w_j + \sum_{k=1}^i \frac{(j-i-1+k)!}{(j-i-1)!k!} w_{i-k} w_{j+k}, \quad i < j,$$

$$Sq^1(w_j) = w_1 w_j + (j-1)w_{j+1},$$

and the Steenrod relation

$$Sq^n(xy) = \sum_{i=0}^n Sq^i(x) Sq^{n-i}(y). \quad (D7)$$

We find

$$\begin{aligned} u_0 &= 1, & u_1 &= w_1, & u_2 &= w_1^2 + w_2, \\ u_3 &= w_1 w_2, & u_4 &= w_1^4 + w_2^2 + w_1 w_3 + w_4, \\ u_5 &= w_1^3 w_2 + w_1 w_2^2 + w_1^2 w_3 + w_1 w_4, \\ u_6 &= w_1^2 w_2^2 + w_1^3 w_3 + w_1 w_2 w_3 + w_2^3 + w_1^2 w_4 + w_2 w_4, \\ u_7 &= w_1^2 w_2 w_3 + w_1 w_2^2 + w_1 w_2 w_4, \\ u_8 &= w_1^8 + w_2^4 + w_1^2 w_3^2 + w_1^2 w_2 w_4 + w_1 w_3 w_4 + w_4^2 \\ &\quad + w_1^3 w_5 + w_3 w_5 + w_1^2 w_6 + w_2 w_6 + w_1 w_7 + w_8. \end{aligned} \quad (D8)$$

If the  $O(\infty)$  vector bundle on  $d$ -dimensional space,  $M^d$ , happen to be the tangent bundle of  $M^d$ , then the corresponding Wu class and the Steenrod square satisfy

$$Sq^{d-j}(x_j) = u_{d-j} x_j, \quad \text{for any } x_j \in H^j(M^d; \mathbb{Z}_2). \quad (D9)$$

We can generate many relations for cocycles and Stiefel-Whitney classes on a manifold using the above result:

1. If we choose  $x_j$  to be a combination of Stiefel-Whitney classes, the above will generate many relations between Stiefel-Whitney classes.
2. If we choose  $x_j$  to be a combination of Stiefel-Whitney classes and cocycles, the above will generate many relations between Stiefel-Whitney classes and cocycles.
3. Since  $Sq^i(x_j) = 0$  if  $i > j$ , therefore  $u_i x_{d-i} = 0$  for any  $x_{d-i} \in H^{d-i}(M^d; \mathbb{Z}_2)$  if  $i > d-i$ . Since  $\mathbb{Z}_2$  is a field and according to the Poincaré duality, this implies that  $u_i = 0$  for  $2i > d$ .
4.  $Sq^n \dots Sq^m(u_i) = 0$  if  $2i > d$ . This also gives us relations among Stiefel-Whitney classes.

### 3. Relations between Stiefel-Whitney classes and a $\mathbb{Z}_2$ -valued 1-cocycle in 3-dimensions

On a 3-dimensional manifold, we can find many relations between Stiefel-Whitney classes:

- (1)  $u_2 = w_1^2 + w_2 = 0$ .
  - (2)  $u_3 = w_1 w_2 = 0$ .
  - (3)  $Sq^1(u_2) = 0$ . Using  $Sq^1(w_i) = w_1 w_i + (i+1)w_{i+1}$ , we find that  $Sq^1(w_1^2 + w_2) = Sq^1(w_1)w_1 + w_1 Sq^1(w_1) + Sq^1(w_2) = w_1 w_2 + w_3 = 0$ .
- This gives us three relations

$$w_1^2 = w_2, \quad w_1 w_2 = w_3 = 0. \quad (D10)$$

Let  $a^{\mathbb{Z}_2}$  be a  $\mathbb{Z}_2$ -valued 1-cocycle. We can also find a relation between the Stiefel-Whitney classes and  $a^{\mathbb{Z}_2}$ :

$$w_1(a^{\mathbb{Z}_2})^2 = Sq^1((a^{\mathbb{Z}_2})^2) = 2(a^{\mathbb{Z}_2})^3 = 0. \quad (D11)$$

There are six possible 3-cocycles that can be constructed from the Stiefel-Whitney classes and the 1-cocycle  $a^{\mathbb{Z}_2}$ :

$$\begin{array}{ccc} (w_1)^3, & w_1 w_2, & w_3, \\ (a^{\mathbb{Z}_2})^3, & w_1(a^{\mathbb{Z}_2})^2, & w_1^2 a^{\mathbb{Z}_2}. \end{array} \quad (D12)$$

From the above relations, we see that only two of them are non-zero:

$$(a^{\mathbb{Z}_2})^3, \quad w_1^2 a^{\mathbb{Z}_2}. \quad (D13)$$

### 4. Relations between Stiefel-Whitney classes and a $\mathbb{Z}_2$ -valued 1-cocycle in 4-dimensions

The relations between the Stiefel-Whitney classes for 4-dimensional manifold can be listed:

- (1)  $u_3 = w_1 w_2 = 0$ .
- (2)  $u_4 = w_1^4 + w_2^2 + w_1 w_3 + w_4 = 0$ .
- (3)  $Sq^1(u_3) = 0$ , which implies  $Sq^1(w_1 w_2) = Sq^1(w_1)w_2 + w_1 Sq^1(w_2) = w_1^2 w_2 + w_1^2 w_2 + w_1 w_3 = w_1 w_3 = 0$ , which can be summarized as

$$w_1 w_2 = 0, \quad w_1 w_3 = 0, \quad w_1^4 + w_2^2 + w_4 = 0. \quad (D14)$$

We also have many relations between the Stiefel-Whitney classes and  $a^{\mathbb{Z}_2}$ :

- (1)  $Sq^1((a^{\mathbb{Z}_2})^3) = (a^{\mathbb{Z}_2})^4 = w_1(a^{\mathbb{Z}_2})^3$ .
- (2)  $Sq^1(w_1^2 a^{\mathbb{Z}_2}) = w_1^2(a^{\mathbb{Z}_2})^2 = w_1^3 a^{\mathbb{Z}_2}$ .
- (3)  $Sq^1(w_2 a^{\mathbb{Z}_2}) = (w_1 w_2 + w_3)a^{\mathbb{Z}_2} + w_2(a^{\mathbb{Z}_2})^2 = w_1 w_2 a^{\mathbb{Z}_2}$ , which implies that  $w_3 a^{\mathbb{Z}_2} = w_2(a^{\mathbb{Z}_2})^2$ .
- (4)  $Sq^2((a^{\mathbb{Z}_2})^2) = (a^{\mathbb{Z}_2})^4 = (w_1^2 + w_2)(a^{\mathbb{Z}_2})^2$ .
- (5)  $Sq^2(w_1 a^{\mathbb{Z}_2}) = w_1^2(a^{\mathbb{Z}_2})^2 = (w_1^2 + w_2)w_1 a^{\mathbb{Z}_2} = w_1^3 a^{\mathbb{Z}_2}$ , which is the same as (2).

To summarize

$$\begin{aligned} w_1^2(a^{\mathbb{Z}_2})^2 &= w_1^3 a^{\mathbb{Z}_2}, & (a^{\mathbb{Z}_2})^4 &= w_1(a^{\mathbb{Z}_2})^3, \\ w_2(a^{\mathbb{Z}_2})^2 &= w_3 a^{\mathbb{Z}_2}, & (a^{\mathbb{Z}_2})^4 + w_1^2(a^{\mathbb{Z}_2})^2 + w_2(a^{\mathbb{Z}_2})^2 &= 0. \end{aligned} \quad (D15)$$

There are nine 4-cocycles that can be constructed from Stiefel-Whitney classes and a 1-cocycle  $a^{\mathbb{Z}_2}$ :

$$\begin{array}{ccc} (a^{\mathbb{Z}_2})^4, & w_1(a^{\mathbb{Z}_2})^3, & w_1^2(a^{\mathbb{Z}_2})^2, \\ w_2(a^{\mathbb{Z}_2})^2, & w_1^3 a^{\mathbb{Z}_2}, & w_3 a^{\mathbb{Z}_2}, \\ w_1^4, & w_2^2, & w_4. \end{array} \quad (D16)$$

Only four of them are independent

$$w_1^4, \quad w_2^2, \quad w_3 a^{\mathbb{Z}_2}, \quad w_1^3 a^{\mathbb{Z}_2}. \quad (D17)$$

### 5. Relations between Stiefel-Whitney classes and a $\mathbb{Z}_2$ -valued 2-cocycle in 4-dimensions

There are two relations between the Stiefel-Whitney classes and a  $\mathbb{Z}_2$ -valued 2-cocycle  $b^{\mathbb{Z}_2}$ :

- (1)  $Sq^1(w_1 b^{\mathbb{Z}_2}) = w_1^2 b^{\mathbb{Z}_2} + w_1 \mathcal{B}_2 b^{\mathbb{Z}_2} = w_1^2 b^{\mathbb{Z}_2}$ , which implies  $w_1 \mathcal{B}_2 b^{\mathbb{Z}_2} = 0$ .
- (2)  $Sq^2(b^{\mathbb{Z}_2}) = (b^{\mathbb{Z}_2})^2 = (w_1^2 + w_2)b^{\mathbb{Z}_2}$ .

There are seven 4-cocycles that can be constructed from Stiefel-Whitney classes and a  $\mathbb{Z}_2$ -valued 2-cocycle  $b^{\mathbb{Z}_2}$ :

$$\begin{array}{ccc} (b^{\mathbb{Z}_2})^2, & w_1 \mathcal{B}_2 b^{\mathbb{Z}_2}, & w_1^2 b^{\mathbb{Z}_2}, \quad w_2 b^{\mathbb{Z}_2}, \\ w_1^4, & w_2^2, & w_4. \end{array} \quad (D18)$$

So the following four 4-cocycles are independent

$$w_1^4, \quad w_2^2, \quad w_2 b^{\mathbb{Z}_2}, \quad w_1^2 b^{\mathbb{Z}_2}. \quad (D19)$$

## Appendix E: Spin and Pin structures

Stiefel-Whitney classes can determine when a manifold can have a spin structure. The spin structure is defined only for orientable manifolds. The tangent bundle for an orientable manifold  $M^d$  is a  $SO(d)$  bundle. The group  $SO(d)$  has a central extension to the group  $\text{Spin}(d)$ . Note that  $\pi_1(SO(d)) = \mathbb{Z}_2$ . The group  $\text{Spin}(d)$  is the double covering of the group  $SO(d)$ . A spin structure on  $M^d$  is a  $\text{Spin}(d)$  bundle, such that under the group reduction  $\text{Spin}(d) \rightarrow SO(d)$ , the  $\text{Spin}(d)$  bundle reduces to the  $SO(d)$  bundle. Some manifolds cannot have such a lifting from  $SO(d)$  tangent bundle to the  $\text{Spin}(d)$  spinor bundle. The manifolds that have such a lifting is called spin manifold. A manifold is a spin manifold iff its first and second Stiefel-Whitney class vanishes  $w_1 = w_2 = 0$ .

For a non-orientable manifold  $N^d$ , the tangent bundle is a  $O(d)$  bundle. The non-connected group  $O(d)$  has two nontrivial central extensions (double covers) by  $\mathbb{Z}_2$  with different group structures, denoted by  $\text{Pin}^+(d)$  and  $\text{Pin}^-(d)$ . So the  $O(d)$  tangent bundle has two types of lifting to a  $\text{Pin}^+$  bundle and a  $\text{Pin}^-$  bundle, which are called  $\text{Pin}^+$  structure and  $\text{Pin}^-$  structure respectively. The manifolds with such liftings are called  $\text{Pin}^+$  manifolds or  $\text{Pin}^-$  manifolds. We see that the concept of  $\text{Pin}^\pm$  structure applies to both orientable and non-orientable manifolds. A manifold is a  $\text{Pin}^+$  manifold iff  $w_2 = 0$ . A



manifold is a  $\text{Pin}^-$  manifold iff  $w_2 + w_1^2 = 0$ . If a manifold  $N^d$  does admit  $\text{Pin}^+$  or  $\text{Pin}^-$  structures, then the set of isomorphism classes of  $\text{Pin}^+$ -structures (or  $\text{Pin}^-$ -structures) can be labeled by elements in  $H^1(N^d; \mathbb{Z}_2)$ . For example  $\mathbb{R}P^4$  admits two  $\text{Pin}^+$ -structures and no  $\text{Pin}^-$ -structures since  $w_2(\mathbb{R}P^4) = 0$  and  $w_2(\mathbb{R}P^4) + w_1^2(\mathbb{R}P^4) \neq 0$ .

From eqn. (D1), we see that  $M \# N$  is  $\text{pin}^+$  iff both  $M$  and  $N$  are  $\text{pin}^+$ . Similarly,  $M \# N$  is  $\text{pin}^-$  iff both  $M$  and  $N$  are  $\text{pin}^-$ .

## Appendix F: Cohomology rings

In this section, we list some cohomology rings  $H^*(M^4; \mathbb{Z}_n)$ , that are used in the main text of the paper. First, let us list a few theorems:

**The cohomology ring of product space** (see Ref. 93 page 216):

Let  $X$  and  $Y$  be arbitrary spaces. Assume  $H^k(Y; R)$  is a free and finitely generated  $R$ -module for all  $k$ . Then

$$H^*(X; R) \otimes_R H^*(Y; R) \rightarrow H^*(X \times Y; R) \quad (\text{F1})$$

is an isomorphism of graded rings. (A free  $R$ -module is a module that has a basis, or equivalently, one that is isomorphic to a direct sum of copies of the ring  $R$ .)

**The cohomology of connected sum:**

$$H^k(M^d \# N^d, \mathbb{M}) = H^k(M^d, \mathbb{M}) \oplus H^k(N^d, \mathbb{M}), \quad 0 < k < d. \quad (\text{F2})$$

**The cup product of connected sum:**

$$\begin{aligned} H^k(M^d \# N^d, \mathbb{M}) \times H^l(M^d \# N^d, \mathbb{M}) &\xrightarrow{\cup} H^{k+l}(M^d \# N^d, \mathbb{M}), \\ 0 < k, l, k+l < d : \\ (a, a') \cup (b, b') &= (a \cup b, a' \cup b'), \end{aligned} \quad (\text{F3})$$

where  $a \in H^k(M^d, \mathbb{M})$ ,  $b \in H^l(M^d, \mathbb{M})$ ,  $a' \in H^k(N^d, \mathbb{M})$ , and  $b' \in H^l(N^d, \mathbb{M})$ . The above also works for  $k+l = d$  is we identify

$$(\alpha v_{M^d}, \beta v_{N^d}) \sim (\alpha + \beta) v_{M^d \# N^d} \quad (\text{F4})$$

where  $v_{M^d}$ ,  $v_{N^d}$ , and  $v_{M^d \# N^d}$  are the generators in  $H^d(M^d, \mathbb{M})$ ,  $H^d(N^d, \mathbb{M})$ , and  $H^d(M^d \# N^d, \mathbb{M})$ .

### 1. $H^*(T^4, \mathbb{Z}_n)$

For  $M^4 = S^1 \times S^1 \times S^1 \times S^1 = T^4$ , we have

$$H^*(T^4, \mathbb{Z}_n) = \frac{\mathbb{Z}_n[a_1, a_2, a_3, a_4]}{(a_1^2, a_2^2, a_3^2, a_4^2)} \quad (\text{F5})$$

where  $a_i \in H^1(T^4, \mathbb{Z}_n)$  generate the ring. The Bockstein homomorphism all vanishes:

$$\mathcal{B}_n a_i = 0, \quad i = 1, 2, 3, 4. \quad (\text{F6})$$

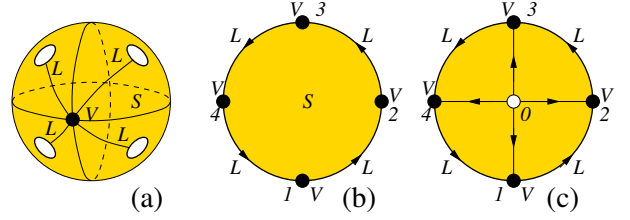


FIG. 9. (Color online) (a)  $L^2(p)$  (with  $p = 4$ ) space. (b)  $L^2(p)$  can be described by a CW-complex with a 0-cell  $V$ , a 1-cell  $L$ , and a 2-cell  $S$ . The filled dots are identified. The links (12), (23), (34), and (41) are identified. The boundary of the 2-cell  $S$  is  $p$  copies of the 1-cell  $L$ :  $\partial S = pL$  and  $L$  is a cycle  $\partial L = 0$ . (c)  $L^2(p)$  can be described by a singular-complex with two vertices (0) and  $V$ ,  $p+1$  links  $(01), \dots, (0p)$  and  $L$ ,  $p$  triangles  $(012), \dots, (0, p-1, p)$ . The sum of the  $p$  triangles gives us  $S$ , the whole  $L^2(p)$  space.

### 2. $H^*(T^2 \times S^2, \mathbb{Z}_n)$

For  $M^4 = T^2 \times S^2$  (where  $T^2 = S^1 \times S^1$ ), we have

$$H^*(T^2 \times S^2, \mathbb{Z}_n) = \frac{\mathbb{Z}_n[a_1, a_2, b]}{(a_1^2, a_2^2, b^2)} \quad (\text{F7})$$

where  $a_i \in H^1(T^2 \times S^2, \mathbb{Z}_n)$  and  $b \in H^2(T^2 \times S^2, \mathbb{Z}_n)$  generate the ring. We also have

$$\mathcal{B}_n a_i = \mathcal{B}_n b = 0, \quad i = 1, 2 \quad (\text{F8})$$

### 3. $H^*(L^2(p); \mathbb{Z}_n)$

$L^2(p)$  space is a 2-dimensional sphere with  $p$  holes removed and with the boundary of the  $p$  holes identified (see Fig. 9a). It has a CW-complex decomposition as shown in Fig. 9b. Since  $\partial S = pL$ ,  $\partial L = 0$ , we can compute explicitly that

$$\begin{aligned} H_0(L^2(p), \mathbb{Z}) &= \mathbb{Z}, & H_1(L^2(p), \mathbb{Z}) &= \mathbb{Z}_p, \\ H_2(L^2(p), \mathbb{Z}) &= 0, \end{aligned} \quad (\text{F9})$$

$$\begin{aligned} H^0(L^2(p), \mathbb{Z}) &= \mathbb{Z}, & H^1(L^2(p), \mathbb{Z}) &= 0, \\ H^2(L^2(p), \mathbb{Z}) &= \mathbb{Z}_p. \end{aligned} \quad (\text{F10})$$

and

$$\begin{aligned} H_0(L^2(p), \mathbb{Z}_n) &= \mathbb{Z}, & H_1(L^2(p), \mathbb{Z}_n) &= \mathbb{Z}_{\langle n, p \rangle} = \{L\}, \\ H_2(L^2(p), \mathbb{Z}_n) &= \mathbb{Z}_{\langle n, p \rangle} = \left\{ \frac{n}{\langle n, p \rangle} S \right\}, \end{aligned} \quad (\text{F11})$$

$$\begin{aligned} H^0(L^2(p), \mathbb{Z}_n) &= \mathbb{Z}, & H^1(L^2(p), \mathbb{Z}_n) &= \mathbb{Z}_{\langle n, p \rangle} = \{a\}, \\ H^2(L^2(p), \mathbb{Z}_n) &= \mathbb{Z}_{\langle n, p \rangle} = \{b\}, \end{aligned} \quad (\text{F12})$$

where we have listed the generators of  $H_*(L^2(p), \mathbb{Z}_n)$  and  $H^*(L^2(p), \mathbb{Z}_n)$ .

Using the CW-complex of  $L^2(p)$ , we can compute the Bockstein homomorphism for  $\mathbb{Z}_n$  coefficient. Let  $\tilde{a} \in Z^1(L^2(p); \mathbb{Z})$  to be a generator of  $H^1(L^2(p); \mathbb{Z}_p)$ , and  $\tilde{b} \in C^2(L^2(p); \mathbb{Z})$  to be a generator of  $H^2(L^2(p); \mathbb{Z})$ :

$$\langle \tilde{a}, L \rangle = 1, \quad \langle \tilde{b}, S \rangle = 1. \quad (\text{F13})$$

We see that  $p = \langle p\tilde{a}, L \rangle = \langle \tilde{a}, pL \rangle = \langle \tilde{a}, \partial S \rangle = \langle d\tilde{a}, S \rangle$ . Thus  $d\tilde{a} = 0 \bmod p$ , confirming that  $\tilde{a}$  is a cocycle in  $H^1(L^3(p, q); \mathbb{Z}_p)$ , but  $\tilde{a}$  is not a cocycle in  $H^1(L^3(p, q); \mathbb{Z})$ . From the above calculation, we also see that  $d\tilde{a} = p\tilde{b}$ , or  $\frac{1}{p}d\tilde{a} = \tilde{b}$ . Therefore,  $\frac{1}{n}\frac{n}{\langle p, n \rangle}d\tilde{a} = \frac{p}{\langle p, n \rangle}\tilde{b}$ , or  $\frac{1}{n}d(\frac{n}{\langle p, n \rangle}\tilde{a}) = \mathcal{B}_n(\frac{n}{\langle p, n \rangle}\tilde{a}) = \frac{p}{\langle p, n \rangle}\tilde{b}$ . We note that  $\frac{n}{\langle p, n \rangle}\tilde{a}$  is an integer valued-cochain that satisfies  $d\frac{n}{\langle p, n \rangle}\tilde{a} = 0 \bmod n$ . Thus  $a = \frac{n}{\langle p, n \rangle}\tilde{a}$  is a cocycle and a generator in  $H^1(L^3(p, q); \mathbb{Z}_n)$ . Also  $b = \tilde{b}$  is a cocycle and a generator in  $H^2(L^3(p, q); \mathbb{Z}_n)$ . The Bockstein homomorphism can be written as

$$\mathcal{B}_n a = \frac{p}{\langle p, n \rangle} b. \quad (\text{F14})$$

We can calculate the cohomology ring  $H^*(L^2(p); \mathbb{Z}_n)$ , by decomposing  $L^2(p)$  into a singular-complex characterized by the vertices  $0, 1, 2, \dots, p$  (see Fig. 9c). Note that  $1, 2, \dots, p$  correspond to the same vertex. First  $a, b$  (the generators of  $H^1(L^2(p); \mathbb{Z}_n)$  and  $H^2(L^2(p); \mathbb{Z}_n)$ ) are given by

$$\begin{aligned} \langle a, (m, m+1) \rangle &= \frac{n}{\langle p, n \rangle}, \\ \langle a, (0m) \rangle &= \frac{(m-1)n}{\langle p, n \rangle}, \\ \langle b, (012) \rangle &= \langle b, (00'2) \rangle = 1, \quad \langle b, \text{others} \rangle = 0, \\ m &= 1, \dots, p. \end{aligned} \quad (\text{F15})$$

We see that

$$\langle a, L \rangle = \frac{n}{\langle p, n \rangle}, \quad \langle b, \frac{n}{\langle p, n \rangle} S \rangle = \frac{n}{\langle p, n \rangle} \quad (\text{F16})$$

where  $\frac{n}{\langle p, n \rangle} S$  is a 2-cycle  $\partial \frac{n}{\langle p, n \rangle} S = \frac{np}{\langle p, n \rangle} L = 0 \bmod n$ .

Now, we can calculate the cup product

$$\begin{aligned} \langle a^2, (0, m, m+1) \rangle &= \langle a, (0, m) \rangle \langle a, (m, m+1) \rangle \\ &= \frac{(m-1)n}{\langle p, n \rangle} \frac{n}{\langle p, n \rangle}, \end{aligned} \quad (\text{F17})$$

or

$$\begin{aligned} \langle a^2, \frac{n}{\langle p, n \rangle} S \rangle &= \sum_{m=1}^p \frac{(m-1)n}{\langle p, n \rangle} \frac{n^2}{\langle p, n \rangle^2} = \frac{n^3 p (p-1)}{2 \langle p, n \rangle^3} \\ &= \begin{cases} \frac{n}{2}, & \text{if } p_2 > 1, \frac{n}{2^{p_2}} = \text{odd} \\ 0, & \text{otherwise} \end{cases}, \end{aligned} \quad (\text{F18})$$

where  $p_2$  is the number of prime factor 2 in  $p$ . The above

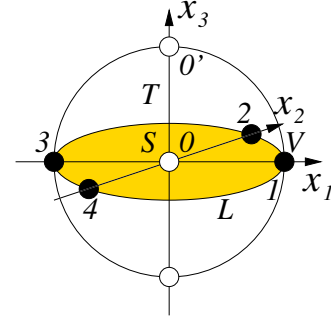


FIG. 10. (Color online) The  $S^3$  is parametrized by  $(x_1, x_2, x_3) = \frac{(\text{Re} z_1, \text{Im} z_1, \text{Re} z_2)}{1 + \text{Im} z_2}$  which is the whole  $\mathbb{R}^3$ . The open dots are the points  $(z_1, z_2) = (0, e^{i2\pi m/p})$ ,  $m = 0, \dots, p-1$ . The shaded disc is  $B_0^2$ . The north and the south hemisphere are  $B_{\pm 1}^2$ . The volume between  $B_0^2$  and  $B_1^2$  is the lens space  $L^3(p, q)|_{(p, q)=(4, 1)}$ . The lens space  $L^3(p, q)$  is described by a CW-complex with a 0-cell  $V$ , a 1-cell  $L$ , a 2-cell  $S$ , and a 3-cell  $T$ . The filled dots are identified under the quotient map and correspond to the 0-cell  $V$ . The shaded disc  $S$  is the 2-cell. The boundary of the 2-cell  $S$  is  $p$  copies of the 1-cell  $L$ :  $\partial S = pL$  and  $L$  is a cycle  $\partial L = 0$ . The 3-cell  $T$  is the half ball above the shaded disc.

implies that

$$\begin{aligned} a^2 &= \frac{n^2 p (p-1)}{2 \langle p, n \rangle^2} b \\ &= \begin{cases} \frac{\langle n, p \rangle}{2} b, & \text{if } p_2 > 1, \frac{n}{2^{p_2}} = \text{odd} \\ 0, & \text{otherwise} \end{cases}. \end{aligned} \quad (\text{F19})$$

The ring  $H^*(L^2(p); \mathbb{Z}_n)$  is determined by eqn. (F12) and eqn. (F19).

#### 4. $H^*(L^3(p, q) \times S^1, \mathbb{Z}_n)$

We know that  $S^3$  can be described by two complex numbers  $z_1, z_2$  satisfying  $|z_1|^2 + |z_2|^2 = 1$ . Let  $p$  and  $q$  be coprime integers. We can see that the action  $(z_1, z_2) \rightarrow (e^{i\frac{2\pi}{p}} z_1, e^{i\frac{2\pi q}{p}} z_2)$  is a free action on  $S^3$ . Quotient out such a free action, the resulting space is the lens space  $L^3(p, q)$ . We see that  $L^3(2, 1) = \mathbb{R}P^3$ .  $L^3(p, q_1)$  and  $L^3(p, q_2)$  are homotopically equivalent if and only if  $q_1 q_2 = \pm m^2 \bmod p$  for an integer  $m$ .

$L^3(p, q)$  is described by the CW-complex in Fig. 10 (for  $(p, q) = (4, 1)$ ), which has a 0-cell  $V$  (the 4 vertices  $1, 2, 3, 4$  are identified and correspond to  $V$ ), a 1-cell  $L$  (the 4 links  $(12), (23), (34), (41)$  are identified and correspond to  $L$ ), a 2-cell  $S$  (which is the union of  $(012), (023), (034), (041)$ ), and a 3-cell  $T$  (which is the union of  $(00'12), (00'23), (00'34), (00'41)$ ). To describe the lens space, let us first consider  $p$  points  $(z_1, z_2) = (0, e^{i2\pi m/p})$ ,  $m = 0, \dots, p-1$  (which become one point after the quotient). The 2-cell  $B_m^2$  is formed by the points  $(z_1, z_2) = \cos \theta (0, e^{i2\pi m/p}) + \sin \theta (z_1, 0)$ ,  $|z_1| = 1$

(where  $B_m^2$  and  $B_{m'}^2$  are identified by the quotient map). The volume between  $B_m^2$  and  $B_{m+1}^2$  is the lens space  $L^3(p, q)$  which is also the 3-cell  $T$ . The 0-cell is given by  $(z_1, z_2) = (e^{i2\pi m/p}, 0)$ ,  $m = 0, \dots, p-1$  (which become one point after the quotient).

Since  $\partial S = pL$ ,  $\partial L = \partial T = 0$ , we see that

$$\begin{aligned} H_0(L^3(p, q), \mathbb{Z}) &= \mathbb{Z}, & H_1(L^3(p, q), \mathbb{Z}) &= \mathbb{Z}_p, \\ H_2(L^3(p, q), \mathbb{Z}) &= 0, & H_3(L^3(p, q), \mathbb{Z}) &= \mathbb{Z}, \end{aligned} \quad (\text{F20})$$

and, by Poincaré duality,

$$\begin{aligned} H^0(L^3(p, q), \mathbb{Z}) &= \mathbb{Z}, & H^1(L^3(p, q), \mathbb{Z}) &= 0, \\ H^2(L^3(p, q), \mathbb{Z}) &= \mathbb{Z}_p, & H^3(L^3(p, q), \mathbb{Z}) &= \mathbb{Z}. \end{aligned} \quad (\text{F21})$$

Then using the universal coefficient theorem eqn. (A11) and eqn. (A8), we find that

$$\begin{aligned} H_0(L^3(p, q), \mathbb{Z}_n) &= \mathbb{Z}_n, & H_1(L^3(p, q), \mathbb{Z}_n) &= \mathbb{Z}_{\langle n, p \rangle}, \\ H_2(L^3(p, q), \mathbb{Z}_n) &= \mathbb{Z}_{\langle n, p \rangle}, & H_3(L^3(p, q), \mathbb{Z}_n) &= \mathbb{Z}_n. \end{aligned} \quad (\text{F22})$$

$$\begin{aligned} H^0(L^3(p, q), \mathbb{Z}_n) &= \mathbb{Z}_n, & H^1(L^3(p, q), \mathbb{Z}_n) &= \mathbb{Z}_{\langle n, p \rangle}, \\ H^2(L^3(p, q), \mathbb{Z}_n) &= \mathbb{Z}_{\langle n, p \rangle}, & H^3(L^3(p, q), \mathbb{Z}_n) &= \mathbb{Z}_n. \end{aligned} \quad (\text{F23})$$

$H_1(L^3(p, q), \mathbb{Z}_n)$  is generated by  $L$  and  $H_2(L^3(p, q), \mathbb{Z}_n)$  is generated by  $\frac{n}{\langle n, p \rangle} S$ .

The cohomology rings  $H^*(L^3(p, q); \mathbb{Z}_p)$  are given by (see Ref. 93 page 251)

$$\begin{aligned} H^*(L^3(p, q); \mathbb{Z}_p) \\ = \{m_0 + m_1 a + m_2 b + m_3 ab \mid a^2 = \frac{p}{2}(\langle p, 2 \rangle - 1)b\}, \\ H^*(L^3(p, q); \mathbb{Z}) = \{m_0 + m_2 b + m_3 c\}. \end{aligned} \quad (\text{F24})$$

We also have  $\mathcal{B}_p a = b$ .

In the following, we will only consider  $L^3(p, 1) \equiv L^3(p)$ . We like to calculate the cohomology ring  $H^*(L^3(p); \mathbb{Z}_n)$ , by decomposing the lens space  $L^3(p)$  into a simplicial complex characterized by the vertices  $0, 0', 1, 2, \dots, p$  (see Fig. 10). Note that  $0$  and  $0'$  correspond to the same vertex and  $1, 2, \dots, p$  correspond to the same vertex. Also note that, for example, the 2-simplices  $(012)$  and  $(0'23)$  are identified. First  $a, b$  (the generators of  $H^1(L^3(p); \mathbb{Z}_n)$  and  $H^2(L^3(p); \mathbb{Z}_n)$ ) are given by

$$\begin{aligned} \langle a, (00') \rangle &= \langle a, (m, m+1) \rangle = \frac{n}{\langle p, n \rangle}, \\ \langle a, (0m) \rangle &= \frac{(m-1)n}{\langle p, n \rangle}, \\ \langle b, (012) \rangle &= \langle b, (00'2) \rangle = 1, \quad \langle b, \text{others} \rangle = 0, \\ m &= 1, \dots, p. \end{aligned} \quad (\text{F25})$$

We see that

$$\langle a, L \rangle = \frac{n}{\langle p, n \rangle}, \quad \langle b, \frac{n}{\langle p, n \rangle} S \rangle = \frac{n}{\langle p, n \rangle} \quad (\text{F26})$$

where  $\frac{n}{\langle p, n \rangle} S$  is a 2-cycle  $\partial \frac{n}{\langle p, n \rangle} S = \frac{np}{\langle p, n \rangle} L = 0 \pmod{n}$ .

Now, we can calculate the cup product

$$\begin{aligned} \langle a^2, (0, m, m+1) \rangle &= \langle a, (0, m) \rangle \langle a, (m, m+1) \rangle \\ &= \frac{(m-1)n}{\langle p, n \rangle} \frac{n}{\langle p, n \rangle}, \end{aligned} \quad (\text{F27})$$

or

$$\begin{aligned} \langle a^2, \frac{n}{\langle p, n \rangle} S \rangle &= \sum_{m=1}^p \frac{(m-1)n}{\langle p, n \rangle} \frac{n^2}{\langle p, n \rangle^2} = \frac{n^3 p(p-1)}{2\langle p, n \rangle^3} \\ &\stackrel{n}{=} \begin{cases} \frac{n}{2}, & \text{if } p_2 > 1, \frac{n}{2^{p_2}} = \text{odd} \\ 0, & \text{otherwise} \end{cases}, \end{aligned} \quad (\text{F28})$$

where  $p_2$  is the number of prime factor 2 in  $p$ . The above implies that

$$\begin{aligned} a^2 &= \frac{n^2 p(p-1)}{2\langle p, n \rangle^2} b \\ &= \begin{cases} \frac{\langle n, p \rangle}{2} b, & \text{if } p_2 > 1, \frac{n}{2^{p_2}} = \text{odd} \\ 0, & \text{otherwise} \end{cases}. \end{aligned} \quad (\text{F29})$$

We also note that

$$\begin{aligned} \langle ab, T \rangle &= -\langle a, (0'0) \rangle \langle b, (012) \rangle \\ &= \frac{n}{\langle p, n \rangle}, \end{aligned} \quad (\text{F30})$$

which implies that

$$ab = \frac{n}{\langle p, n \rangle} c, \quad (\text{F31})$$

Thus, the cohomology ring  $H^*(L^3(p); \mathbb{Z}_n)$  is given by

$$H^*(L^3(p); \mathbb{Z}_n) = \{\zeta + \alpha a + \beta b + \gamma c\}, \quad (\text{F32})$$

$$\begin{aligned} \text{with } a^2 &= \begin{cases} \frac{\langle n, p \rangle}{2} b, & \text{if } p_2 > 1, \frac{n}{2^{p_2}} = \text{odd} \\ 0, & \text{otherwise} \end{cases} \\ ab &= \frac{n}{\langle p, n \rangle} c, \end{aligned}$$

where  $\zeta, \gamma \in \mathbb{Z}_n$  and  $\alpha, \beta \in \mathbb{Z}_{\langle p, n \rangle}$ . We also have  $\mathcal{B}_n a = \frac{p}{\langle p, n \rangle} b$ .

Notice that

$$H^*(S^1; \mathbb{Z}_n) = \frac{\mathbb{Z}_n[a_1]}{(a_1^2)} \quad (\text{F33})$$

is a free  $\mathbb{Z}_n$ -module. This allows us to compute the cohomology ring  $H^*(S^1 \times L^3(p); \mathbb{Z}_n)$ .

## 5. $H^*(F^4; \mathbb{Z}_n)$

In order for the volume-independent partition function  $Z^{\text{top}}(M^4)$  on an orientable space-time  $M^4$  to be a topological invariant, we require the Euler number and the

Pontryagin number of  $M^4$  to vanish:  $\chi(M^4) = P_1(M^4) = 0$ . We also like  $M^4$  to be complicated enough so that its second Stiefel-Whitney class  $w_2$  is non-zero. How to construct such an 4-dimensional manifold?

First, let us introduce intersection form  $Q_{M^4}: H^2(M^4; \mathbb{Z}) \times H^2(M^4; \mathbb{Z}) \rightarrow \mathbb{Z}$  defined by

$$Q_{M^4}(a, b) = \langle a \cup b, [M^4] \rangle = \int_{M^4} ab. \quad (\text{F34})$$

The intersection form has the following properties

1. Under connected sum,

$$Q_{M^4 \# N^4} = Q_{M^4} \oplus Q_{N^4}. \quad (\text{F35})$$

2. Poincaré duality implies that the intersection form  $Q_{M^4}$  is unimodular.
3. If  $M^4$  is spin, then  $Q_{M^4}(a, a) = \text{even}$  for all  $a \in H^2(M^4; \mathbb{Z})$ . If  $M^4$  is orientable and  $Q_{M^4}$  is even, then  $M^4$  is spin.
4. The signature of  $Q_{M^4}$  is one third of the Pontryagin number:  $\sigma(M^4) = \frac{1}{3}P_1(M^4)$ .
5. A smooth compact spin 4-manifold has a signature which is a multiple of 16.
6. A 4-manifold bounds a 5-manifold if and only if it has zero signature.

We know that  $Q_{\mathbb{C}P^2}$  is 1-by-1 matrix:  $Q_{\mathbb{C}P^2} = (1)$ , while  $Q_{\overline{\mathbb{C}P^2}} = (-1)$ . Thus,  $Q_{\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ . This means  $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$  is not spin and has a zero Pontryagin number.

The Euler number  $\chi(M)$  has the following properties:

1.  $\chi(S^d) = 1 + (-1)^d$ .
2.  $\chi(\mathbb{R}P^d) = \frac{1+(-1)^d}{2}$ .
3.  $\chi(\mathbb{C}P^2) = \chi(\overline{\mathbb{C}P^2}) = 3$ .
4.  $\chi(M \times N) = \chi(M)\chi(N)$ .
5.  $\chi(M^d \# N^d) = \chi(M^d) + \chi(N^d) - \chi(S^d)$ .

Using the above result, we find that

$$F^4 \equiv (S^1 \times S^3) \# (S^1 \times S^3) \# \mathbb{C}P^2 \# \overline{\mathbb{C}P^2}, \quad (\text{F36})$$

has

$$Q_{F^4} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \chi(F^4) = P_1(F^4) = 0. \quad (\text{F37})$$

We see that  $F^4$  is not spin.

The cohomology classes for  $F^4$  are

$$\begin{aligned} H^1(F^4; \mathbb{Z}_n) &= \mathbb{Z}_n^{\oplus 2}, & H^2(F^4; \mathbb{Z}_n) &= \mathbb{Z}_n^{\oplus 2}, \\ H^3(F^4; \mathbb{Z}_n) &= \mathbb{Z}_n^{\oplus 2}, & H^4(F^4; \mathbb{Z}_n) &= \mathbb{Z}_n. \end{aligned} \quad (\text{F38})$$

Let  $a_1, a_2$  be the generators of  $H^1(F^4; \mathbb{Z}_n)$ ,  $b_1, b_2$  the generators of  $H^2(F^4; \mathbb{Z}_n)$ ,  $c_1, c_2$  be the generators of  $H^3(F^4; \mathbb{Z}_n)$ , and  $v$  be the generator of  $H^4(F^4; \mathbb{Z}_n)$ :

$$H^*(F^4; \mathbb{Z}_n) = \{a_1, a_2, b_1, b_2, c_1, c_2, v\}. \quad (\text{F39})$$

We find that the non-zero cup products are given by

$$b_1^2 = -b_2^2 = a_1 c_1 = a_2 c_2 = v. \quad (\text{F40})$$

All other cup products vanish.

## 6. $H^*(\mathbb{R}P^d; \mathbb{Z}_2)$

Next, let us list some cohomology rings with  $\mathbb{Z}_2$  coefficient for non-orientable spaces. The cohomology ring  $H^*(\mathbb{R}P^d; \mathbb{Z}_2)$  is given by

$$H^*(\mathbb{R}P^d; \mathbb{Z}_2) = \frac{\mathbb{Z}_2[a]}{(a^{d+1})} \quad (\text{F41})$$

with  $a \in H^1(\mathbb{R}P^d; \mathbb{Z}_2)$ .  $\mathbb{R}P^d$  is non-orientable if  $d$  is even. The total Stiefel-Whitney class for  $\mathbb{R}P^d$  is given by

$$w = (1 + a)^{d+1}. \quad (\text{F42})$$

(see <https://amathew.wordpress.com/2010/12/17/the-stiefel-whitney-classes-of-projective-space/>) We see that for  $\mathbb{R}P^4$ ,  $w_1 = a$  and  $w_2 = 0$ . Thus  $\mathbb{R}P^4$  is a  $\text{pin}^+$  manifold, but not a  $\text{pin}^-$  manifold.

## 7. $H^*(F_{\text{non}}^4; \mathbb{Z}_2)$

We note that  $\mathbb{R}P^4$  has an intersection form  $Q_{\mathbb{R}P^4} = (1)$  (with  $\mathbb{Z}_2$  field),  $\sigma(\mathbb{R}P^4) = 1 \bmod 2$ , and  $\chi(\mathbb{R}P^4) = 1$ . So

$$F_{\text{non}}^4 \equiv \mathbb{R}P^4 \# \mathbb{C}P^2 \# (S^1 \times S^3) \quad (\text{F43})$$

has  $\sigma(F_{\text{non}}^4) = 0 \bmod 2$  and  $\chi(F_{\text{non}}^4) = 0$ .

The cohomology classes for  $F_{\text{non}}^4$  are

$$\begin{aligned} H^1(F_{\text{non}}^4; \mathbb{Z}_2) &= \mathbb{Z}_2^{\oplus 2}, & H^2(F_{\text{non}}^4; \mathbb{Z}_2) &= \mathbb{Z}_2^{\oplus 2}, \\ H^3(F_{\text{non}}^4; \mathbb{Z}_2) &= \mathbb{Z}_2^{\oplus 2}, & H^4(F_{\text{non}}^4; \mathbb{Z}_2) &= \mathbb{Z}_2. \end{aligned} \quad (\text{F44})$$

Let  $a^{\mathbb{R}P^4}, a^{S^1 \times S^3}$  be the generators of  $H^1(F_{\text{non}}^4; \mathbb{Z}_n)$ ,  $(a^{\mathbb{R}P^4})^2, b^{\mathbb{C}P^2}$  of  $H^2(F_{\text{non}}^4; \mathbb{Z}_n)$ ,  $(a^{\mathbb{R}P^4})^3, c^{S^1 \times S^3}$  of  $H^3(F_{\text{non}}^4; \mathbb{Z}_n)$ , and  $v$  the generator of  $H^4(F_{\text{non}}^4; \mathbb{Z}_n)$ :

$$H^*(F_{\text{non}}^4; \mathbb{Z}_n) = \{(a^{\mathbb{R}P^4})^{m=1,2,3}, a^{S^1 \times S^3}, b^{\mathbb{C}P^2}, c^{S^1 \times S^3}, v\}. \quad (\text{F45})$$

We find that the non-zero cup products are given by

$$\begin{aligned} (a^{\mathbb{R}P^4})^4 &= (b^{\mathbb{C}P^2})^2 = a^{S^1 \times S^3} c^{S^1 \times S^3} = v, \\ (a^{\mathbb{R}P^4})^2, & (a^{\mathbb{R}P^4})^3. \end{aligned} \quad (\text{F46})$$

All other cup products vanish. The first Stiefel-Whitney class for  $F_{\text{non}}^4$  is given by  $w_1 = a^{\mathbb{R}P^4}$ . Since  $\mathbb{R}P^4$ ,  $\mathbb{C}P^2$ , and  $S^1 \times S^3$  are all  $\text{pin}^+$  manifolds, their connected sum  $F_{\text{non}}^4$  is also a  $\text{pin}^+$  manifold. Thus the second Stiefel-Whitney class for  $F_{\text{non}}^4$  is  $w_2 = 0$ . Since  $w_2 + w_1^2 = (a^{\mathbb{R}P^4})^2 \neq 0$ ,  $F_{\text{non}}^4$  is not a  $\text{pin}^-$  manifold.

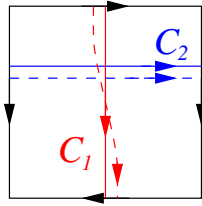


FIG. 11. (Color online) The Klein bottle: the top and bottom boundaries are identified with a twist and the left and right boundaries are identified without twist.  $H_1(K; \mathbb{Z}_2)$  is generated by  $C_1$  and  $C_2$  cycles.

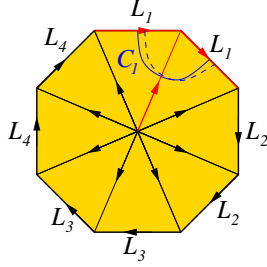


FIG. 12. (Color online) A non-orientable surface  $\Sigma_g^{\text{non}}$  with genus  $g = 4$ . All the corners are identified and the edges with the same label  $L_i$  are glued together along its direction. The Poincaré dual of the cycle  $C_1$  is  $a_1 \in H^1(\Sigma_g^{\text{non}}; \mathbb{Z}_2)$ :  $\langle a_1, \text{red link} \rangle = 1$  and  $\langle a_1, \text{black link} \rangle = 0$ . We note that  $\Sigma_1^{\text{non}} = \mathbb{RP}^2$  and  $\Sigma_2^{\text{non}} = \text{Klein bottle}$ .

## 8. $H^*(K; \mathbb{Z}_2)$

The Klein bottle  $K$  has the following cohomology class

$$H^1(K; \mathbb{Z}_2) = \mathbb{Z}_2^{\oplus 2} = \{a_1, a_2\}, \quad H^2(K; \mathbb{Z}_2) = \mathbb{Z}_2 = \{b\}. \quad (\text{F47})$$

$H^1(K; \mathbb{Z}_2)$  is generated by  $a_1$  and  $a_2$  which are the Poincaré dual of  $C_1$  and  $C_2$  (see Fig. 12):

$$a_1 = C_1^*, \quad a_2 = C_2^*. \quad (\text{F48})$$

We see that  $a_1 a_2 = b$  since  $C_1$  and  $C_2$  intersect once;  $a_2^2 = 0$  since  $C_2$  does not self intersect (*i.e.*  $C_2$  and its displacement does not intersect);  $a_1^2 = b$  since  $C_1$  self intersects once (*i.e.*  $C_1$  and its displacement intersect once). Therefore  $H^*(K; \mathbb{Z}_2)$  is determined by

$$a_1^2 = a_1 a_2 = b, \quad a_2^2 = 0. \quad (\text{F49})$$

## 9. $H^*(\Sigma_g^{\text{non}}; \mathbb{Z}_2)$

The cohomology ring for non-orientable surface  $\Sigma_g^{\text{non}}$  (see Fig. 12),  $H^*(\Sigma_g^{\text{non}}; \mathbb{Z}_2)$ , is given by (see Ref. 93 page

208)

$$\begin{aligned} H^*(\Sigma_g^{\text{non}}; \mathbb{Z}_2) &= \frac{\mathbb{Z}_2[a_i]}{(a_i^3, a_i^2 - a_j^2, a_i a_j a_k, a_i a_j |_{i \neq j})} \\ &= \{\zeta + \alpha_i a_i + \beta b \mid \zeta, \alpha_i, \beta \in \mathbb{Z}_2, a_i^2 = \delta_{ij} b\}, \quad (\text{F50}) \end{aligned}$$

with  $a_i \in H^1(\Sigma_g^{\text{non}}; \mathbb{Z}_2) = \mathbb{Z}_2^{\oplus g}$ ,  $i = 1, 2, \dots, g$  and  $b \in H^2(\Sigma_g^{\text{non}}; \mathbb{Z}_2) = \mathbb{Z}_2$ .

To understand the above result, we note that the cycles  $C_i$ ,  $i = 1, \dots, g$ , generate  $H_1(\Sigma_g^{\text{non}}; \mathbb{Z}_2)$  (see Fig. 12 where only  $C_1$  is drawn). The Poincaré dual of  $C_i$ ,  $a_i = C_i^*$ , generate  $H^1(\Sigma_g^{\text{non}}; \mathbb{Z}_2)$ . We note that the self intersection number for  $C_i$  is 1. Thus  $a_i^2 = b$ .  $C_i$  and  $C_j$  does not intersect if  $i \neq j$ . Thus  $a_i a_j = 0$ .

To calculate the Stiefel-Whitney class  $w_i$ , we note that the orientation reverses as we go along the loop  $C_i$ . This implies that  $\oint_{C_i} w_1 = 1 \mod 2$ . Since  $\oint_{C_i} a_j$  is the intersection number between  $C_i$  and Poincaré dual of  $a_j$  which is  $C_j$ , we see that  $\oint_{C_i} a_j = \delta_{ij}$ . Therefore  $w_1 = \sum_{i=1}^g a_i$ . In 2-dimensions  $w_2 = w_1^2 = \sum_{i=1}^g a_i^2 = [g]_2 b$ . Thus  $\Sigma_g^{\text{non}}$  is a  $\text{pin}^+$  manifold if  $g$  is even, and it is not a  $\text{pin}^+$  manifold if  $g$  is odd.  $\Sigma_g^{\text{non}}$  is always a  $\text{pin}^-$  manifold.

We also note that the CW-complex of  $\Sigma_g^{\text{non}}$  in Fig. 12 has  $V = 2$  vertices,  $L = 3g$  links, and  $T = 2g$  triangles. Thus the Euler number  $\chi(\Sigma_g^{\text{non}}) = V - L + T = 2 - g$ . The top Stiefel-Whitney class is equal to the Euler class mod 2, regardless of the  $\mathbb{Z}$ -orientability of the manifold. In other words, every manifold is  $\mathbb{Z}_2$ -orientable. So the Euler class (with  $\mathbb{Z}_2$ -coefficients) coincides with the top Stiefel-Whitney class. This is another way to obtain  $w_2 = [g]_2 b$ .

## Appendix G: Group cohomology theory

### 1. Homogeneous group cocycle

In this section, we will briefly introduce group cohomology. The group cohomology class  $\mathcal{H}^d(G, \mathbb{M})$  is a  $\mathbb{Z}$ -model constructed from a group  $G$  and a  $\mathbb{Z}$ -module  $\mathbb{M}$  (*i.e.* a vector space over  $\mathbb{Z}$ ). Each elements of  $G$  also induce a mapping  $\mathbb{M} \rightarrow \mathbb{M}$ , which is denoted as

$$g \cdot m = m', \quad g \in G, \quad m, m' \in \mathbb{M}. \quad (\text{G1})$$

The map  $g \cdot$  is a group homomorphism:

$$g \cdot (m_1 + m_2) = g \cdot m_1 + g \cdot m_2. \quad (\text{G2})$$

The module  $\mathbb{M}$  with such a  $G$ -group homomorphism, is called a  $G$ -module.

A homogeneous  $d$ -cochain is a function  $\nu_d : G^{d+1} \rightarrow \mathbb{M}$ , that satisfies

$$\nu_d(g_0, \dots, g_d) = g \cdot \nu_d(gg_0, \dots, gg_d), \quad g, g_i \in G. \quad (\text{G3})$$

We denote the set of  $d$ -cochains as  $\mathcal{C}^d(G, \mathbb{M})$ . Clearly  $\mathcal{C}^d(G, \mathbb{M})$  is an abelian group.



Let us define a mapping  $d$  (group homomorphism) from  $\mathcal{C}^d(G, \mathbb{M})$  to  $\mathcal{C}^{d+1}(G, \mathbb{M})$ :

$$(d\nu_d)(g_0, \dots, g_{d+1}) = \sum_{i=0}^{d+1} (-)^i \nu_d(g_0, \dots, \hat{g}_i, \dots, g_{d+1}) \quad (\text{G4})$$

where  $g_0, \dots, \hat{g}_i, \dots, g_{d+1}$  is the sequence  $g_0, \dots, g_i, \dots, g_{d+1}$  with  $g_i$  removed. One can check that  $d^2 = 0$ . The homogeneous  $d$ -cocycles are then the homogeneous  $d$ -cochains that also satisfy the cocycle condition

$$d\nu_d = 0. \quad (\text{G5})$$

We denote the set of  $d$ -cocycles as  $\mathcal{Z}^d(G, \mathbb{M})$ . Clearly  $\mathcal{Z}^d(G, \mathbb{M})$  is an abelian subgroup of  $\mathcal{C}^d(G, \mathbb{M})$ .

Let us denote  $\mathcal{B}^d(G, \mathbb{M})$  as the image of the map  $d : \mathcal{C}^{d-1}(G, \mathbb{M}) \rightarrow \mathcal{C}^d(G, \mathbb{M})$  and  $\mathcal{B}^0(G, \mathbb{M}) = \{0\}$ . The elements in  $\mathcal{B}^d(G, \mathbb{M})$  are called  $d$ -coboundary. Since  $d^2 = 0$ ,  $\mathcal{B}^d(G, \mathbb{M})$  is a subgroup of  $\mathcal{Z}^d(G, \mathbb{M})$ :

$$\mathcal{B}^d(G, \mathbb{M}) = \{d\nu_{d-1} | \nu_{d-1} \in \mathcal{C}^{d-1}(G, \mathbb{M})\} \subset \mathcal{Z}^d(G, \mathbb{M}). \quad (\text{G6})$$

The group cohomology class  $\mathcal{H}^d(G, \mathbb{M})$  is then defined as

$$\mathcal{H}^d(G, \mathbb{M}) = \mathcal{Z}^d(G, \mathbb{M}) / \mathcal{B}^d(G, \mathbb{M}). \quad (\text{G7})$$

We note that the  $d$  operator and the cochains  $\mathcal{C}^d(G, \mathbb{M})$  (for all values of  $d$ ) form a so called cochain complex,

$$\dots \xrightarrow{d} \mathcal{C}^d(G, \mathbb{M}) \xrightarrow{d} \mathcal{C}^{d+1}(G, \mathbb{M}) \xrightarrow{d} \dots \quad (\text{G8})$$

which is denoted as  $C(G, \mathbb{M})$ . So we may also write the group cohomology  $\mathcal{H}^d(G, \mathbb{M})$  as the standard cohomology of the cochain complex  $H^d[C(G, \mathbb{M})]$ .

## 2. Inhomogeneous group cocycle

The above definition of group cohomology class can be rewritten in terms of inhomogeneous group cochains/cocycles. An inhomogeneous group  $d$ -cochain is a function  $\omega_d : G^d \rightarrow M$ . All  $\omega_d(g_1, \dots, g_d)$  form  $\mathcal{C}^d(G, \mathbb{M})$ . The inhomogeneous group cochains and the homogeneous group cochains are related as

$$\nu_d(g_0, g_1, \dots, g_d) = \omega_d(a_{01}, \dots, a_{d-1,d}), \quad (\text{G9})$$

with

$$g_0 = 1, \quad g_1 = g_0 a_{01}, \quad g_2 = g_1 a_{12}, \quad \dots \quad g_d = g_{d-1} a_{d-1,d}. \quad (\text{G10})$$

Now the  $d$  map has a form on  $\omega_d$ :

$$\begin{aligned} (d\omega_d)(a_{01}, \dots, a_{d,d+1}) &= a_{01} \cdot \omega_d(a_{12}, \dots, a_{d,d+1}) \\ &+ \sum_{i=1}^d (-)^i \omega_d(a_{01}, \dots, a_{i-1,i}, a_{i,i+1}, \dots, a_{d,d+1}) \\ &+ (-)^{d+1} \omega_d(a_{01}, \dots, \tilde{a}_{d-1,d}) \end{aligned} \quad (\text{G11})$$

This allows us to define the inhomogeneous group  $d$ -cocycles which satisfy  $d\omega_d = 0$  and the inhomogeneous group  $d$ -coboundaries which have a form  $\omega_d = d\mu_{d-1}$ . In the following, we are going to use inhomogeneous group cocycles to study group cohomology. Geometrically, we may view  $g_i$  as living on the vertex  $i$ , while  $a_{ij}$  as living on the link connecting the two vertices  $i$  to  $j$ .

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