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Many-body topological invariants in fermionic symmetry protected topological phases: Cases of point group symmetries

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We propose the definitions of many-body topological invariants to detect symmetry-protected topological phases protected by point group symmetry, using partial point group transformations on a given short-range entangled quantum ground state. Here, partial point group transformations g_D are defined by point group transformations restricted to a spatial subregion D, which is closed under the point group transformations and sufficiently larger than the bulk correlation length ξ. By analytical and numerical calculations, we find that the ground state expectation value of the partial point group transformations behaves generically as $\langle GS|g_D|GS\rangle \sim \exp\left[i\theta + \gamma - \alpha \frac{\mathrm{Area}(\partial D)}{\xi^{d-1}}\right]$. Here, Area$(\partial D)$ is the area of the boundary of the subregion D, and α is a dimensionless constant. The complex phase of the expectation value θ is quantized and serves as the topological invariant, and γ is a scale-independent topological contribution to the amplitude. The examples we consider include the $\mathbb{Z}_8$ and $\mathbb{Z}_{16}$ invariants of topological superconductors protected by inversion symmetry in $(1+1)$ and $(3+1)$ dimensions, respectively, and the lens space topological invariants in $(2+1)$-dimensional fermionic topological phases. Connections to topological quantum field theories and cobordism classification of symmetry-protected topological phases are discussed.

CONTENTS

I. INTRODUCTION

A. Topological phases in the presence of symmetries and topological quantum field theories

Topological phases of matter are gapped quantum phases which cannot be adiabatically connected to a trivial state, i.e., tensor product state. Topological phases can be discussed in the presence of various symmetries, such as time-reversal, charge conjugation, and/or space group symmetry. More specifically, it may occur that a gapped quantum phase, which can be adiabatically connected to a trivial state in the absence of symmetries, cannot be turned into a trivial phase once a certain set of symmetries are enforced. Such gapped quantum phases are called symmetry-protected topological (SPT) phases. Topological insulators and superconductors are celebrated examples of SPT phases of fermions.[1–16] Other examples of bosonic SPT phases have been also widely discussed.[15,17–25] On the other hand, topologically ordered phases[26–36] are phases of matter which are topologically distinct from a trivial state even in the absence of symmetries. Topologically ordered phases can be enriched by the presence of symmetries; They can exhibit a particular pattern of symmetry fractionalization, which can be used to distinguish and characterize different topologically ordered phases with symmetries. Topologically ordered phases of this kind are called symmetry-enriched.[37–45] Our main focus below will be SPT phases, although some of our discussion should be readily applicable to symmetry-enriched topological phases as well.

For gapped phases of matter, it is reasonable to expect that their low-energy and long-wavelength physics is captured by topological quantum field theories (TQFTs) of some sort. More specifically, let us consider topological phases protected/enriched by a set of global symmetries, which form a symmetry group $\widetilde{G}$. It is convenient to decompose the symmetry group $\widetilde{G}$ into the part which consists of unitary on-site (or "internal") symmetries ($= G$), and the part which consists of symmetry transformations which reverse the orientation of spacetime manifolds, such as time-reversal and parity. (Other spatial symmetries, such as point group symmetries, will be discussed momentarily, but for now, we focus on orientation-reversing symmetries, such as parity, or reflection in one direction.) If the underlying system includes fermions, it is convenient to include the fermion number parity to the latter part.

For the purpose of detecting (topological) properties of the gapped quantum phases, it is well-advised to couple them to a background G gauge field. One can then integrate over all matter degrees of freedom (matter fields $= \{\phi_i\}$) of the topological phases. This procedure leads to the partition function

$$Z(X, \eta, A) = \int \prod_i D\phi_i e^{-S_X(\{\phi_i\}, \eta, A)}. \tag{1.1}$$

Here, X is a closed $(d+1)$-dimensional spacetime manifold, $\{\phi_i\}$ includes all matter fields, A is the background G gauge field which couples to the matter fields. On the other hand, η is a "structure" endowed to the manifold such as an orientation, a Spin or Spin^c structure for real or complex fermions, respectively.[46] In short, the symmetry group $\widetilde{G}$ of a given gapped (topological) phase enters into the corresponding partition function as the input data (X, η, A).

For gapped phases of matter, when the correlation length of the system is much shorter than the system size, the resulting partition functions are expected not to depend very sensitively on the details of the spacetime manifold X, and the background gauge field A. In particular, for topologically non-trivial gapped phases, the partition function $Z(X, \eta, A)$, or equivalently the effective action $S_{eff}(X, \eta, A) = -\ln Z(X, \eta, A)$, may have a topological term – a $U(1)$ phase of the partition function (the imaginary part of the Euclidean effective action), which is insensitive to the metric on X (i.e., it is invariant under diffeomorphisms) as well as small variations of the background gauge field. In the limit of zero-correlation length, the partition function consists solely of a topological term.

Put differently, the partition function $Z(X, \eta, A)$ is expected to define a TQFT, or more precisely, the so-called G-equivariant TQFT (or G-equivariant Spin TQFT if we are interested in fermionic condensed matter systems).[45,47–54] For a given closed manifold X with background structures specified by (η, A), the TQFT yields a topological invariant.

When the gapped phase in question has no topological order as in an SPT phase, i.e., it has a unique ground state,[55] it was further proposed that the corresponding partition function depends only on the certain equivalence classes of the background manifolds (with structure η and the background gauge field), the cobordism class of (X, η, A).[25] Here, two $(d+1)$-dimensional spacetime manifolds $X_{1,2}$ (with structure $\eta_{1,2}$ and background fields $A_{1,2}$) are called cobordant when one can find a $(d+2)$-dimensional manifold Y with an appropriate background gauge field that can interpolate (X_1, η_1, A_1) and (X_2, η_2, A_2), i.e., $\partial Y = X_1 \sqcup X_2$. This relation can be used to define equivalence classes of $(d+1)$-dimensional manifolds (with a given structure and background gauge field configurations). It should be noted that the claim that the partition function is cobordism invariant is more stringent than topological (diffeomorphism) invariance.

One can further introduce an Abelian group structure to the equivalence classes of (X, η, A) by taking the disjoint union as an operation. The resulting group is called the cobordism group and denoted by $\Omega^{\mathrm{str}}_{d+1}(BG)$, which is Abelian (e.g., $\mathbb{Z}_n$, $\mathbb{Z}_n \oplus \mathbb{Z}_m$). Here, BG is the classifying space of G.[56,57] When there is no symmetry, we simply put a single point as BG, $BG = pt$.

In topological theories in which the partition function is given by a pure phase, $Z(X, \eta, A)$ can be thought of as a homomorphism

$$Z : \Omega^{\text{str}}_{d+1}(BG) \to U(1), \quad (X, \eta, A) \mapsto Z(X, \eta, A). \tag{1.2}$$

Thus, a useful way to classify possible topological $U(1)$ phases of $Z(X, \eta, A)$ is by using the cobordism group classification of manifolds with structures.[11,15,25] It was proposed that the torsion part of the cobordism group Tor $\Omega^{\text{str}}_{d+1}(BG)$ provides a possible classification of topological phases of matter.

B. SPT invariants in terms of ground state wave functions

The purpose of the paper is to construct topological invariants, i.e., quantities which take the same value for all points (Hamiltonians) in a given gapped quantum phase and can be used as "order parameters" of topological phases. In particular, in the condensed matter context, we wish to define and compute the topological invariants for a given Hamiltonian or a ground state, i.e., within the operator formalism. (We will mainly focus on ground states of fermionic SPT phases, which are unique. The topological invariants in this context are often called SPT invariants.) The above consideration suggests that the topological $U(1)$ phase of the path integral evaluated on a suitable manifold with structures $[X, \eta, A]$ can be thought of as a meaningful topological invariant (SPT invariant). Our task is then to find a way to "simulate" the path integral (1.1) defined for the data $[X, \eta, A]$. (It should be stressed that we are here to construct *many-body* topological invariants, as opposed to *single-particle* topological invariants, which have been commonly discussed in the literature.)

Of particular importance for our purposes is the generator $[X, \eta, A]$ of the cobordism group $\Omega^{\text{str}}_{d+1}(BG)$, which we simply call the *generating manifold* in the following. It is on this manifold that evaluating the path integral (the partition function) gives rise to a "least possible" or "most fundamental" topological $U(1)$ phase; The topological $U(1)$ phases for other possible manifolds $[X, \eta, A]$ are given as an integer multiple of the topological $U(1)$ phase for the generating manifold.

Our proposal to define/construct the topological invariants can be summarized by the following set up and operations (i)-(v):

(i) closed d-dimensional space manifolds M on which the Hamiltonian is defined,

(ii) orientation or spin structures η_M on M,

(iii) twisted boundary conditions (background flat gauge fields A_M) on M by on-site unitary symmetry G,

(iv) symmetry operations $\hat{g}$ on the many-body Hilbert space,

(v) the *partial (point group) operation* $\hat{g}_D$ on a subregion D of the space manifold M.

Let $|GS(M, A_M, \eta_M)\rangle$ be the ground state of the Hamiltonian $H(\eta_M, A_M)$ with the spin structure η_M and the twisted boundary condition A_M. From $(i - iv)$, we can extract a set of $U(1)$ phases $\{e^{i\Phi(M,\eta_M,A_M,g)}\}$ by

$$\hat{g}\,|GS(M, \eta_M, A_M)\rangle = e^{i\Phi(M,\eta_M,A_M,g)}\,\big|GS\big(M, g(\eta_M), g(A_M)\big)\big\rangle, \tag{1.3}$$

where $g(\eta_M)$ and $g(A_M)$ are the spin structure and twisted boundary condition mapped by symmetry transformation g. In particular, when $|GS(M, \eta_M, A_M)\rangle$ and $\big|GS\big(M, g(\eta_M), g(A_M)\big)\big\rangle$ are in the same Hilbert space, we can extract the $U(1)$ phases by

$$\big\langle GS\big(M, g(\eta_M), g(A_M)\big)\big|\,\hat{g}\,|GS(M, \eta_M, A_M)\rangle = e^{i\Phi(M,\eta_M,A_M,g)}. \tag{1.4}$$

The wave function overlap (1.4), and hence the $U(1)$ phase, can be readily interpreted as a spacetime path integral on the spacetime which takes the form of a mapping torus,

$$X = M \times_f S^1 := M \times [0, 1]/\big((x, 0) \sim (f(x), 1)\big), \tag{1.5}$$

where $f : M \to M$ is a diffeomorphism on X induced by the action of $\hat{g}$. For instance, it can be simply the identity $f = \text{id}$ or the space inversion $f : x \to -x$. Another example in $(2 + 1)d$ is the case when M is the 2-torus T^2 and f is chosen as modular transformations acting on T^2. In Refs. 58–64, the action of the modular transformations on the ground state(s) is discussed, as a method to extract data of topological phases, i.e., representations of the mapping class group of the space manifold.

C. Partial point group operation and spacetime path-integral

In some symmetry classes, the subset of the above ingredients (i)-(iv) is sufficient to define the topological invariant for the generating manifold (see, for example, Refs. 59, 62, 65–70). For example, the generating manifold of Spin cobordism group $\Omega_2^{\mathrm{Spin}}(pt) = \mathbb{Z}_2$, which is relevant for the topological classification of class D superconductors in $(1+1)$ dimensions, is the 2-torus T^2 with periodic boundary conditions in both time and space directions.[47,71] On the other hand, there are other symmetry classes for which the generating manifold cannot be realized as a mapping torus. For example, the generating manifold of Pin^- cobordism group $\Omega_2^{\mathrm{Pin}^-}(pt) = \mathbb{Z}_8$, which is relevant for the topological classification of class D superconductors with reflection symmetry with $R^2 = (-1)^F$ in $(1+1)$ dimensions, is the real projective plane $\mathbb{R}P^2$.[11,71] (Here $(-1)^F$ is the fermion number parity.) Since $\mathbb{R}P^2$ is not a mapping torus, the topological invariant cannot be constructed by using (i)-(iv).

In these cases, the last ingredient (v), partial symmetry operations which act on a given subregion D of the total space manifold M, is necessary to construct topological invariants. For previous studies using partial symmetry operations to detect properties of topological phases, see for example, Refs. 51, 66, 67, 72, and 73. In particular, our approach, which applies to fermionic SPT phases in arbitrary dimensions, is partly motivated by Ref. 67. In Ref. 67, Pollmann and Turner showed that the $\mathbb{Z}_2$ invariant for the inversion-symmetric Haldane chain (a bosonic SPT phase) can be detected by the partial inversion[74] on the ground state. In the path-integral picture, the Pollmann-Turner invariant is interpreted as the spacetime path-integral on the projective plane, $\mathbb{R}P^2$.[51] Tu et al.[72] also showed that the partial lattice translation contains useful information on (2+1)-dimensional topologically ordered phases such as the central charge and topological spins.

Let us now give a more detailed definition of partial symmetry operations. Specifically, we consider a Hamiltonian which is invariant under $\widetilde{G}$, $\hat{g}H\hat{g}^{-1} = H$, $g \in \widetilde{G}$. The symmetry operation $\hat{g}$ acts on the underlying fermionic operators as

$$\hat{g}\psi_i^\dagger(\boldsymbol{x})\hat{g}^{-1} = \psi_j^\dagger(g\cdot\boldsymbol{x})U_{ji}, \quad \hat{g}\left|0\right\rangle_\psi = \left|0\right\rangle_\psi, \quad g \in \widetilde{G}, \tag{1.6}$$

where $\psi_i^\dagger(\boldsymbol{x})$ is a fermion creation operator at $\boldsymbol{x} \in M$, and i, j represents internal degrees of freedom, and $\left|0\right\rangle_\psi$ is the Fock vacuum of the $\psi_i(\boldsymbol{x})$ fermion. (We focus on fermionic systems here. A similar definition applies to bosonic topological phases). We now choose a subregion D of M which is closed under the action of the group $\widetilde{G}$, i.e., $\boldsymbol{x} \in D \Leftrightarrow g\cdot\boldsymbol{x} \in D$ for any $g \in \widetilde{G}$. We define the partial transformation by restricting the transformation $\hat{g}$ to the subregion D as

$$\hat{g}_D\psi_i^\dagger(\boldsymbol{x})\hat{g}_D^{-1} = \begin{cases} \psi_j^\dagger(g\cdot\boldsymbol{x})U_{ji} & (\boldsymbol{x}\in D) \\ \psi_j^\dagger(\boldsymbol{x}) & (\boldsymbol{x}\notin D) \end{cases}, \qquad \hat{g}_D\left|0\right\rangle_\psi = \left|0\right\rangle_\psi, \tag{1.7}$$

for $g \in \widetilde{G}$. We take the subregion D such that the length scale of D is sufficiently larger than the correlation length ξ. For a given ground state $|GS\rangle$ on M, the topological invariant associated with the symmetry $\widetilde{G}$ is given by the expectation value $\langle GS|\hat{g}_D|GS\rangle$ of the partial symmetry transformation.

Of particular importance is the case when $\hat{g}$ or $\hat{g}_D$ is a point group operation. In this case, $\langle GS|\hat{g}_D|GS\rangle$ can be interpreted, in the path-integral picture, as a path-integral on the spacetime manifold which may not be obtained as a mapping torus. For example, let us consider partial reflection $\hat{R}_I$ acting on a segment I of the total $(1+1)d$ system. The action of partial reflection $\hat{R}_I$ on a ground state (represented here by using a (fermionic) matrix product state) is shown schematically in Fig. 1 [a]. In the path-integral representation (Fig. 1 [b]), $\langle GS|\hat{R}_I|GS\rangle$ can be interpreted as a path-integral on a manifold, which is obtained from the original spacetime by first introducing a slit $[-\epsilon, \epsilon] \times [0, L]$ at time $t = 0$ and then applying reflection on the slit. This procedure is topologically equivalent to introducing a cross-cap in the spacetime torus as shown in Fig. 1 [c]. For example, as we will show in Sec. III (see also Ref. 73), for $(1+1)d$ topological superconductors protected by reflection symmetry, the ground state expectation value of $\hat{R}_I$ is given by

$$\langle GS|\hat{R}_I|GS\rangle \longrightarrow \frac{e^{\pm\frac{\pi i}{4}}}{\sqrt{2}} \quad (L, N-L \gg \xi), \qquad I = [0, L], \tag{1.8}$$

where $I = [0, L]$ is an interval on the circle with N sites, and ξ is the correlation length. The $U(1)$ phase $e^{\pm\frac{\pi i}{4}}$ correctly reproduces the known the $\mathbb{Z}_8$ topological classification of $(1+1)d$ topological superconductors protected by reflection symmetry. Furthermore, we will confirm that the $U(1)$ phase agrees with the result from the TQFT; the corresponding (spin) TQFT path-integral on $\mathbb{R}P^2$ is given by (or computes) the $\mathbb{Z}_8$ Brown invariant of manifolds with

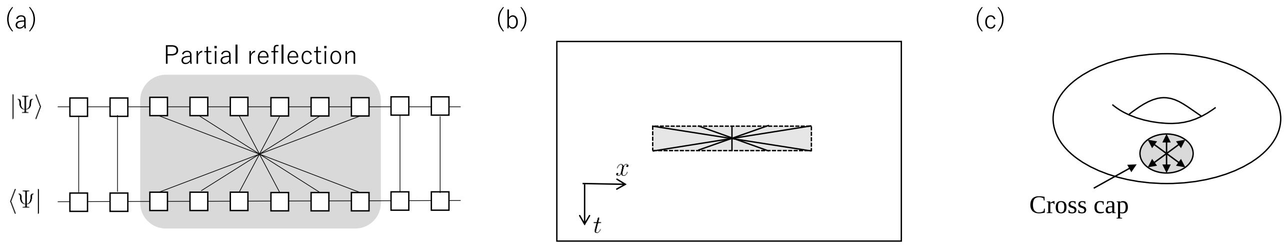


FIG. 1. [a] Fermionic matrix product representation of the partial reflection. [b] Path integral representation of the partial reflection. [c] One cross-cap on the torus.

Pin^- structure,[11,71]

$$Z(\mathbb{R}P^2,\eta_\pm)=e^{\pm\frac{\pi i}{4}}, \tag{1.9}$$

where $\eta_\pm$ represent two different Pin^- structures associated with the non-contractible loop on $\mathbb{R}P^2$. (In the operator formalism, the different Pin^- structures correspond to the presence or absence of the fermion parity operator $(-1)^F$ in the partial reflection – see Sec. III for details.)

Technically, the spacetime manifold created by the partial reflection (1.8) [Fig. 1 (c)] agrees with the real projective plane $\mathbb{R}P^2$ only up to a genus. This is however not a problem if phases of our interest are SPT phases, i.e., if the underlying TQFT is invertible. More precisely, instead of $Z(\mathbb{R}P^2,\eta_\pm)$, the expectation value of the partial reflection is related to the partition function of a Pin^- TQFT on the connected sum of $\mathbb{R}P^2$ and the 2-torus,

$$\langle GS_a|\hat{R}_I|GS_a\rangle \sim Z\big((\mathbb{R}P^2,\eta_\pm)\#(S^1_a\times S^1_{ns})\big), \qquad a=ns,r, \tag{1.10}$$

where the subscript $a=r,ns$ specifies the boundary condition of the circle S^1; r/ns represents the periodic/anti-periodic boundary condition, respectively; $X\#Y$ means the connected sum of X and Y. Note that, by the invertiblity assumption, only a single ground state $|GS_a\rangle$ appears on the LHS of (1.10). Since one can recast the RHS as $Z\big((\mathbb{R}P^2,\eta_\pm)\#(S^1_a\times S^1_{ns})\big)=Z(\mathbb{R}P^2,\eta_\pm)Z(S^1_a\times S^1_{ns})/Z(S^2)=Z(\mathbb{R}P^2,\eta_\pm)$, we conclude $\langle GS_a|\hat{R}_I|GS_a\rangle\sim Z(\mathbb{R}P^2,\eta_\pm)$, i.e., the equivalence between (1.8) and (1.9). (In this discussion, as noted earlier, the equivalence between the expectation value of the partial reflection and the TQFT partition function holds only up to the amplitude (modulus) – see (1.8); As for the partition function $Z(S^1_a\times S^1_{ns})=1$, see (2.12) for details.) The above argument also shows that the boundary condition imposed on the total space is not important. The same argument can be applied to the partial point group transformation in any space dimensions as far as we consider SPT phases (invertible TQFT).

For generic partial symmetry transformations, from numerical and analytical calculations, we find the following behavior

$$\langle GS|\hat{g}_D|GS\rangle=\exp\left[i\theta+\gamma-\alpha\frac{\mathrm{Area}(\partial D)}{\xi^{d-1}}+\cdots\right] \tag{1.11}$$

as a leading contribution to the expectation value for $\mathrm{Area}(\partial D)/\xi^{d-1}\gg 1$.[75] Here, $\mathrm{Area}(\partial D)$ is the area of the boundary of the subregion D, and α is a dimensionless constant. The complex phase θ is well quantized for a sufficiently large region D and represents a topological invariant. In addition, γ is a scale-independent part of the amplitude, and can be thought of as yet another topological contribution to the ground state expectation value of the partial symmetry operation. It is natural to expect that the topological $U(1)$ phase $e^{i\theta}$ is the same as the TQFT partition function $Z(X,\eta,A)$.

In the following sections, we will show that this equivalence between the ground state expectation value of partial point group transformations and TQFT partition functions holds quite generically, in any dimensions and for various point group symmetries. For examples, in $(2+1)$-dimensional topological phases, the ground state expectation of the partial rotation gives rise to the partition functions on the lens space. In $(3+1)$ dimensions, the partial inversion effectively induces a cross-cap and gives rise to the TQFT partition function on the 4-dimensional real projective space $\mathbb{R}P^4$.

It should be noted that the cobordism classification of SPT phases using TQFTs, at least in its original form, does not discuss point group operations other than those that reverse the orientation of the spacetime. For orientation-reversing symmetries, the corresponding partial transformations give rise to generating manifolds, and hence the desired SPT invariants (at least for the examples studied in this paper). On the other hand, for partial transformations for other point group symmetries, their connection to the cobordism classification of SPT phases using TQFTs is less obvious. Even so, since the low-energy TQFTs descriptions of the form (1.1) are often Lorentz invariant, we expect that the orientation-preserving point group operations, at least within the low-energy TQFTs, can be implemented as on-site unitary operations. For example, n-fold discrete rotations (C_n rotations) can be implemented as $\mathbb{Z}_n$ discrete unitary on-site symmetries. If so, the effects of point group symmetries can be incorporated by introducing a proper background gauge field, e.g., by including the $\mathbb{Z}_n$ symmetries by taking $G = \mathbb{Z}_n$. (For similar claims, see Ref. 76 and 77.) We will verify this claim for selected examples, non-chiral topological insulators in $(2+1)d$ protected by C_n symmetry. Following the above discussion, the relevant cobordism group is $\Omega_3^{\mathrm{Spin}^c}(B\mathbb{Z}_n)$. The generating manifold is the lens space $L(n,1)$. We will show that, by using partial C_n rotations, we can simulate the path-integral on the generating manifold $L(n,1)$. Our calculations for explicit models strongly suggest that the partial point group operations give rise to SPT invariants for SPT protected by the point group symmetries.

D. Bulk-boundary correspondence and the reduced density matrix

We confirm numerically the area law for the expectation value of the partial symmetry operation, Eq. (1.11), in various examples. Intuitively, the area law also follows naturally from the bulk-boundary correspondence, which we will make use of to develop our analytical calculations. Let us consider the reduced density matrix ρ_D for the subregion D,

$$\rho_D = \mathrm{Tr}_{M\setminus D}\big[\,|GS\rangle\,\langle GS|\,\big] = \frac{e^{-H_E}}{\mathrm{Tr}_D\big[e^{-H_E}\big]}, \tag{1.12}$$

where H_E is the entanglement Hamiltonian. By the bulk-boundary correspondence of a SPT phase, we then expect that the entanglement Hamiltonian H_E is given by, in the low energy subspace of the Hilbert space on the subregion D, the *physical* Hamiltonian $H_{\partial D}$ describing low-energy gapless boundary degrees of freedom [66,78,79]:

$$H_E \sim \frac{\xi}{v} H_{\partial D}. \tag{1.13}$$

Here, the effective temperature in the reduced density matrix is given in terms of the bulk correlation length ξ, and v is the velocity of the gapless excitation. The partial point group transformation $\hat{g}_D$ induces a symmetry action $\hat{g}_{\partial D}$ on the Hilbert space of the boundary theory; The point group transformation on the gapless theory $H_{\partial D}$ can be specified by explicitly expressing the boundary low-energy excitations of D in terms of the bulk degrees of freedom $\psi_i(\boldsymbol{x})$. Then, the ground state expectation (1.11) can be written in terms of $H_{\partial D}$ as

$$\langle GS|\hat{g}_D|GS\rangle = \mathrm{Tr}_D\big[\hat{g}_D\rho_D\big] \sim \frac{\mathrm{Tr}_{\partial D}\big[\hat{g}_{\partial D}e^{-\frac{\xi}{v}H_{\partial D}}\big]}{\mathrm{Tr}_{\partial D}\big[e^{-\frac{\xi}{v}H_{\partial D}}\big]}. \tag{1.14}$$

As an example, we will use this bulk-boundary correspondence to compute the expectation values of partial symmetry operations in (2+1)-dimensional SPT phase by using the corresponding (1+1)-dimensional CFTs. We will also present similar calculations for higher dimensions using free theories as the boundary theories. With these analytical calculations, together with numerics, we will confirm the formula (1.11).

E. Outline

In this paper, we construct and evaluate many-body topological (SPT) invariants for various fermionic SPT phases. In particular, for the cases with point group symmetries and unitary symmetries. Table I is the list of many-body topological invariants for fermionic SPT phases studied in the present paper.

Partly overlapping results were already reported in our previous paper, Ref. 73. In Ref. 73, we reported the construction of topological invariants, taking as examples, (1+1) and (3+1)-dimensional topological superconductors protected by an orientation-reversing symmetry (an inversion or time-reversal). (We shall recapitulate some essential points of Ref. 73 with regard to the partial reflection in $(1+1)d$ SPT phases.) The result for the case of time-reversal symmetry is one of the main differences, between Ref. 73 and the present work, i.e., here we focus on point group symmetries and unitary symmetries. Because of the anti-unitary nature of time-reversal, the construction of many-body topological invariants is somewhat more complicated, and involves the so-called partial transpose (or its proper extension to fermionic systems).

TABLE I. List of many-body topological invariants for fermionic SPT phases in the present paper. The first column specifies symmetry classes. "A" and "D" represent Altland-Zirnbauer symmetry classes.[80] R, C, $(-1)^F$, C_n, and I are reflection, charge conjugation, fermion parity, n-fold rotation, and inversion symmetries, respectively.

Symmetry class	Space dim.	Topological classification	Spacetime manifold	Topological invariant	Comment	Sec.
D	1	$\Omega_2^{\mathrm{Spin}}(pt)=\mathbb{Z}_2$	T^2	$\langle GS_r \,\vert\, (-1)^F \,\vert\, GS_r\rangle$	$\vert GS_r\rangle$ is the ground state with PBC.	II A
D + R, $R^2=1$	1	$\Omega_2^{\mathrm{Pin}^+}(pt)=\mathbb{Z}_2$	Klein bottle	$\langle GS_r \,\vert\, R \,\vert\, GS_r\rangle$	Full reflection on the ground state with PBC	II B
A + C, $C^2=1$	1	$\Omega_2^{\mathrm{Spin}^{\tilde{c}+}}(pt)\ni\mathbb{Z}_2$	T^2	$\dfrac{\langle GS(\pi)\,\vert\, C\,\vert\, GS(\pi)\rangle}{\langle GS(0)\,\vert\, C\,\vert\, GS(0)\rangle}$	$\vert GS(\theta)\rangle$ is the ground state with the twisted boundary condition by θ.	II C
D + R, $R^2=(-1)^F$	1	$\Omega_2^{\mathrm{Pin}^-}(pt)=\mathbb{Z}_8$	$\mathbb{R}P^2$	$\langle GS\,\vert\, R_I\,\vert\, GS\rangle$	Partial reflection	III A
A + R	1	$\Omega_2^{\mathrm{Pin}^c}(pt)=\mathbb{Z}_4$	$\mathbb{R}P^2$	$\langle GS\,\vert\, R_I\,\vert\, GS\rangle$	Bond center partial reflection. A $U(1)$ phase associated with R is chosen so that $R^2=(-1)^F$.	III B
D + R, $R^2=1$	2	$\Omega_3^{\mathrm{Pin}^+}(pt)=\mathbb{Z}_2$	Klein bottle $\times S^1$	$\prod_\eta \langle GS(\eta)\,\vert\, R\,\vert\, GS(\eta)\rangle$	η runs over $\{(r,r), (r,ns), (ns,r), (ns,ns)\}$ sectors.	II D
chiral $D+C_n$, $(C_n)^n=(-1)^F$	2	Spin TQFT with background framing	Lens space $L(n,1)$	$\langle GS\,\vert\, C_{n,D}\,\vert\, GS\rangle$	Partial n-fold rotation	IV B
nonchiral $D+C_2$, $(C_2)^2=(-1)^F$	2	$\Omega_3^{\mathrm{Spin}}(B\mathbb{Z}_2)=\mathbb{Z}_8$	$\mathbb{R}P^3$	$\langle GS\,\vert\, C_{2,D}\,\vert\, GS\rangle$	Partial 2-fold rotation	IV C
nonchiral $A+C_n$	2	$\Omega_3^{\mathrm{Spin}^c}(B\mathbb{Z}_n)$	Lens space $L(n,1)$	$\langle GS\,\vert\, C_{n,D}\,\vert\, GS\rangle$	Partial n-fold rotation	IV D 2
D + I, $I^2=(-1)^F$	3	$\Omega_4^{\mathrm{Pin}^+}(pt)=\mathbb{Z}_{16}$	$\mathbb{R}P^4$	$\langle GS\,\vert\, I_D\,\vert\, GS\rangle$	Partial inversion	V A
A + I	3	$\Omega_4^{\mathrm{Pin}^c}(pt)=\mathbb{Z}_8\oplus\mathbb{Z}_2$	$\mathbb{R}P^4$ for subgroup $\mathbb{Z}_8$	$\langle GS\,\vert\, I_D\,\vert\, GS\rangle$	Partial inversion. A $U(1)$ phase associated with I is chosen so that $I^2=(-1)^F$.	V B

Another difference between Ref. 73 and this work is that in Ref. 73 the path-integral representation of many-body topological invariant was developed, which can be applied to a given Hamiltonian without the knowledge of the ground state wave function. There, the spacetime manifold is discretized by using the Suzuki-Trotter decomposition of the thermal density matrix. We showed how to introduce a cross-cap in the spacetime path integral by using orientation-reversing symmetries. In the present paper, however, we focus on the operator formalism, in which the input data for the many-body topological invariants is a ground state wave function of SPT phases.

The rest of the paper is organized as follows:

- We start, in Sec. II, by first introducing examples of topological invariants generated by mapping tori for fermionic topological phases. For these cases, partial transformations are not necessary to discuss their topological invariants.

- In Sec. III, partial reflection in $(1+1)$ dimensions is discussed. There, we illustrate that for some symmetry classes it is not sufficient to use full symmetry transformations to topological phases. We show that the ground state expectation value of partial reflection faithfully captures the $\mathbb{Z}_8$ topological invariant for class D topological superconductors with reflection symmetry.

- In Sec. IV, we discuss partial C_n rotations in $(2+1)$-dimensional topological phases. By using the modular properties of CFTs, the ground state expectation values of partial rotations will be interpreted as partition functions of the $(2+1)$-dimensional TQFTs on the lens space. The relation between partial rotations and the Spin^c cobordism classification for on-site $\mathbb{Z}_n$ symmetry will be discussed.

- In Sec. V, we show that partial inversion can be used to construct the topological invariant of $(3+1)$-dimensional topological phases protected by inversion symmetry. We also discuss the general formula for the ground state expectation of partial inversion on topological superconductor and insulator in any even spacetime dimension, by using their boundary Dirac fermion theory on a sphere.

- Finally, we conclude in Sec. VI with some discussion and outlooks. We also explain technical details in four appendices.

II. SYMMETRY TRANSFORMATION ON TWISTED GROUND STATES

In this section, we discuss fermionic SPT phases that can be detected by "full" (as opposed to partial) symmetry transformation on the twisted ground states, such as $(1+1)d$ class D topological superconductors. It is sufficient for detecting the topological invariants not to use the partial symmetry transformations, for example, $(1+1)d$ bosonic SPT phases with on-site unitary symmetry[47,51,67] and $(1+1)d$ class D superconductor[47]. In bosonic SPT phases with on-site symmetry, Hung-Wen[59] generalized this approach to those in higher spacetime dimensions, where they also discussed modular transformations on the space manifold. However, for fermionic topological phases, there are few literatures for this approach to detect topological invariants, thus, it might be useful to explain details.

A. $(1+1)d$ topological superconductors ($\Omega_2^{\mathrm{Spin}}(pt)=\mathbb{Z}_2$)

We start with the definition of the many-body $\mathbb{Z}_2$ topological invariant for $(1+1)d$ superconductors. The topological classification is given by the spin cobordism group

$$\Omega_2^{\mathrm{Spin}}(pt)=\mathbb{Z}_2. \tag{2.1}$$

The generating manifold is the 2-torus T^2 with periodic boundary conditions for time and space directions.[71] In the operator formalism, the path integral on the generating manifold corresponds to the expectation value of the fermion parity with respect to the ground state on a closed space circle with the periodic boundary condition.[47,81,82]

1. *The Kitaev Majorana chain and the many-body $\mathbb{Z}_2$ invariant*

Our construction of the SPT invariant can be best explained by taking an example – the Kitaev Majorana chain.[83] Let us consider a closed chain with N sites. Let f_j be complex fermions defined on the j-site. Then, the Hamiltonian of the Kitaev chain is given by

$$H_{r/ns} = \frac{1}{2}\sum_{j=1}^{N-1}\big[-f_j^\dagger f_{j+1} - f_j f_{j+1} + h.c.\big] \pm \frac{1}{2}\big[-f_N^\dagger f_1 - f_N f_1 + h.c.\big]. \tag{2.2}$$

This Hamiltonian is fine-tuned such that it is at a renormalization group fixed point with zero correlation length. The subscript r and ns represent the periodic boundary condition (the "Ramond" sector) $f_{N+1} = f_1$, and the anti-periodic boundary condition (the "Neveu-Schwarz" sector) $f_{N+1} = -f_1$, respectively. The fermion parity operator $(-1)^F$ is defined by

$$(-1)^F := (-1)^{\sum_{j=1}^N f_j^\dagger f_j}. \tag{2.3}$$

The anti-periodic boundary condition can be thought of as a symmetry twist by the fermion parity $(-1)^F$. (This symmetry twist can be interpreted as an introduction of a topological defect, i.e. the location of the defect can move along the closed chain by a local unitary transformation.)

Introducing real fermion operators c_j^L, c_j^R at the j-th site by

$$c_j^L = i(f_j - f_j^\dagger), \quad c_j^R = f_j + f_j^\dagger, \tag{2.4}$$

the Hamiltonian and the fermion number parity can be rewritten as

$$H_{r/ns} = \frac{i}{2}\sum_{j=1}^{N-1} c_j^R c_{j+1}^L \pm \frac{i}{2} c_N^R c_1^L, \qquad (-1)^F = (-ic_1^L c_1^R)(-ic_2^L c_2^R)\cdots(-ic_N^L c_N^R). \tag{2.5}$$

It is also convenient to introduce complex fermions g_j living on the bond $(j, j+1)$ as

$$g_j := \frac{c_j^R + ic_{j+1}^L}{2}, \quad g_j^\dagger := \frac{c_j^R - ic_{j+1}^L}{2}. \tag{2.6}$$

In terms of $g_j, g_j^\dagger$, the Hamiltonian is written as

$$H_{r/ns} = \sum_{j=1}^{N-1}\big[g_j^\dagger g_j - \frac{1}{2}\big] \pm \big[g_N^\dagger g_N - \frac{1}{2}\big]. \tag{2.7}$$

The ground states $|GS_{r/ns}\rangle$ of $H_{r/ns}$ are given by

$$|GS_r\rangle = |0_g\rangle\,, \quad |GS_{ns}\rangle = g_N^\dagger\,|0_g\rangle\,, \tag{2.8}$$

where $|0_g\rangle$ is the Fock vacuum of g_j fermions. Explicitly, the ground states $|GS_{r/ns}\rangle$ can be written in terms of the f_j fermions and their Fock vacuum $|0_f\rangle$ as

$$\begin{aligned} |GS_{r/ns}\rangle &\sim \left[\prod_j (1+f_j^\dagger) \mp \prod_j (1-f_j^\dagger)\right]|0_f\rangle \\ &\sim \sum_{n:\mathrm{odd/even}}\ \sum_{1\le p_1<p_2<\cdots<p_n\le N} f_{p_1}^\dagger f_{p_2}^\dagger \cdots f_{p_n}^\dagger\,|0_f\rangle \end{aligned} \tag{2.9}$$

up to a normalization. Here, we used the abbreviation $\prod_j(1\pm f_j^\dagger) := (1\pm f_1^\dagger)\cdots(1\pm f_N^\dagger)$.

As advocated above, we consider the the fermion number parity of the ground state of $H_{r/ns}$. Noticing the following relation between the fermion parities defined for f_j and g_j fermions

$$\begin{aligned} &(-1)^F = -(-1)^G, \\ &(-1)^G := (-1)^{\sum_{j=1}^N g_j^\dagger g_j} = (-ic_1^R c_2^L)(-ic_2^R c_3^L)\cdots(-ic_N^R c_1^L), \end{aligned} \tag{2.10}$$

the fermion number parity of the ground states is computed as

$$(-1)^F\,|GS_r\rangle = -\,|GS_r\rangle\,, \quad (-1)^F\,|GS_{ns}\rangle = |GS_{ns}\rangle\,. \tag{2.11}$$

The odd fermion parity of the ground state for the periodic boundary condition is the hallmark of $(1+1)d$ topological superconductors. While the above result (2.11) (and hence (2.12) below) is derived for the specific model (2.2), the Kitaev chain in the limit of zero correlation length, the same result holds for any gapped $\mathbb{Z}_2$ nontrivial superconductors with a unique ground state on a closed chain, since the fermion number parity $(-1)^F$ has a definite $\mathbb{Z}_2$ value on ground states $|GS_{r/ns}\rangle$. We show, as an example, the explicit form of the ground state with a exactly solvable model with a many-body interaction and finite correlation length[84] in Sec. II A 3.

To compare these results with the $(1+1)d$ spin TQFT, we recast our result in the form of partition functions. Since our theory is gapped, the partition functions in the zero temperature limit consist only of the ground states,

$$\begin{aligned} Z(T^2,(r,r)) &= \mathrm{Tr}_r[(-1)^F] = \langle GS_r|(-1)^F|GS_r\rangle = -1, \\ Z(T^2,(ns,r)) &= \mathrm{Tr}_r[1] = \langle GS_r|GS_r\rangle = 1, \\ Z(T^2,(r,ns)) &= \mathrm{Tr}_{ns}[(-1)^F] = \langle GS_{ns}|(-1)^F|GS_{ns}\rangle = 1. \\ Z(T^2,(ns,ns)) &= \mathrm{Tr}_{ns}[1] = \langle GS_{ns}|GS_{ns}\rangle = 1. \end{aligned} \tag{2.12}$$

Here, the notation $Z(T^2,(a,b))$ means the partition function on T^2 with a-(b-)boundary condition for the time (space) direction. Recall that the periodic boundary condition for the time direction in the path integral picture corresponds to, in the operator picture, the insertion of the fermion number parity $(-1)^F$ operator in the trace. Moreover, notice that these results are modular invariant: the modular transformations on T^2 permutes (r,ns), (ns,r) and (ns,ns) sectors while the (r,r) sector is unchanged.

To further support the claim that the fermion number parity of the ground states with twisted boundary conditions is an SPT invariant, we now compare the above partition functions (2.12) of the Kitaev model with the Arf invariant of Spin structures,[11] which is a $\mathbb{Z}_2$-valued function

$$\mathrm{Arf}: \mathrm{Spin}(M) \to \mathbb{Z}_2 = \{0,1\} \tag{2.13}$$

on Spin structures of a given oriented manifold M. See Appendix A for a review of the Arf invariant, where we illustrate how to compute the Arf invariants on a given oriented 2-manifold. The Arf invariants on T^2 are summarized in (A.11), which coincide with the torus partition functions (2.12) of the Kitaev chain with twisted boundary conditions, says,

$$Z(T^2,\eta) = (-1)^{\mathrm{Arf}(\eta)}, \quad \eta \in \mathrm{Spin}(T^2). \tag{2.14}$$

2. The trivial phase

Let us consider the Hamiltonian

$$H^{\mathrm{triv}}_{r/ns} = \sum_{j=1}^{N} \left[f_j^\dagger f_j - \frac{1}{2}\right] = \frac{i}{2}\sum_{j=1}^{N} c_j^L c_j^R, \tag{2.15}$$

which realizes a trivial superconductor. There is no difference between r and ns sectors since there is no hopping term. The ground state is just the Fock vacuum of f fermions,

$$|GS^{\mathrm{triv}}_{r/ns}\rangle = |0_f\rangle\,. \tag{2.16}$$

The partition functions (SPT invariants) can be computed as

$$\begin{aligned} Z^{\mathrm{triv}}(T^2,(r,r)) &= \mathrm{Tr}_r[(-1)^F] = \langle GS^{\mathrm{triv}}_r|(-1)^F|GS^{\mathrm{triv}}_r\rangle = 1, \\ Z^{\mathrm{triv}}(T^2,(ns,r)) &= \mathrm{Tr}_r[1] = \langle GS^{\mathrm{triv}}_r|GS^{\mathrm{triv}}_r\rangle = 1, \\ Z^{\mathrm{triv}}(T^2,(r,ns)) &= \mathrm{Tr}_{ns}[(-1)^F] = \langle GS^{\mathrm{triv}}_{ns}|(-1)^F|GS^{\mathrm{triv}}_{ns}\rangle = 1. \\ Z^{\mathrm{triv}}(T^2,(ns,ns)) &= \mathrm{Tr}_{ns}[1] = \langle GS^{\mathrm{triv}}_{ns}|GS^{\mathrm{triv}}_{ns}\rangle = 1. \end{aligned} \tag{2.17}$$

There is no spin structure dependence, which is consistent with the triviality of this phase.

3. An example of interacting Majorana chains

To demonstrate that the definition of the $\mathbb{Z}_2$ SPT invariant (2.12) is applicable to interacting Majorana chains, we consider an example of the many-body ground states of an interacting Majorana chain discussed in Ref. 84. Let us consider the following Hamiltonian including an extended Hubbard interaction

$$H_{r/ns} = \sum_{j=1}^{N-1} \big[-tf_j^\dagger f_{j+1} + \Delta f_j f_{j+1} + h.c. \big] \pm \big[-tf_N^\dagger f_1 + \Delta f_N f_1 + h.c. \big] \\ - \frac{\mu}{2} \sum_{j=1}^{N} (f_j^\dagger f_j + f_{j+1}^\dagger f_{j+1} - 1) + U \sum_{j=1}^{N} \big(f_j^\dagger f_j - \frac{1}{2}\big)\big(f_{j+1}^\dagger f_{j+1} - \frac{1}{2}\big) \tag{2.18}$$

with $\mu = 4\sqrt{U^2 + tU + \frac{t^2-\Delta^2}{4}}$. The ground state is given by

$$|GS_{r/ns}(\alpha)\rangle \sim \left[\prod_j (1 + \alpha f_j^\dagger) \mp \prod_j (1 - \alpha f_j^\dagger) \right] |0_f\rangle \,, \tag{2.19}$$

where α is defined by $\alpha = \sqrt{\cot \frac{\theta}{2}}, \theta = \tan^{-1}\big(2\Delta/\mu\big)$. Clearly, the R sector consists only of odd numbers of fermions and shows the nontrivial $\mathbb{Z}_2$ invariant $\big\langle GS_r(\alpha) \,\big|\, (-1)^F \,\big|\, GS_r(\alpha) \big\rangle = -1$.

B. $(1+1)d$ superconductors protected by reflection ($\Omega_2^{\mathrm{Pin}^+}(pt) = \mathbb{Z}_2$)

Next, we consider $(1+1)d$ topological superconductors protected by reflection R with $R^2 = 1$. These topological superconductors can be thought of as a CPT dual of class DIII topological superconductors. The topological classification is given by the Pin^+ cobordism group[11,71]

$$\Omega_2^{\mathrm{Pin}^+}(pt) = \mathbb{Z}_2, \tag{2.20}$$

and the generating manifold is the Klein bottle KB. Since the Klein bottle is realized in the mapping torus

$$S^1 \times_R S^1 = S^1 \times [0,1] / \big((x,0) \sim (-x,1)\big), \tag{2.21}$$

we expect that the many-body $\mathbb{Z}_2$ invariant can be computable by using full symmetry operations on the twisted ground states.

1. *A nontrivial model and the many-body $\mathbb{Z}_2$ invariant*

We consider the reflection symmetric pair of Kitaev chains, Eq. (2.2),

$$H_{r/ns} = \frac{1}{2} \sum_{j=1}^{N-1} \big[-f_{\uparrow,j}^\dagger f_{\uparrow,j+1} - f_{\uparrow,j} f_{\uparrow,j+1} + h.c. \big] \pm \frac{1}{2} \big[-f_{\uparrow,N}^\dagger f_{\uparrow,1} - f_{\uparrow,N} f_{\uparrow,1} + h.c. \big] \\ + \frac{1}{2} \sum_{j=1}^{N-1} \big[-f_{\downarrow,j}^\dagger f_{\downarrow,j+1} + f_{\downarrow,j} f_{\downarrow,j+1} + h.c. \big] \pm \frac{1}{2} \big[-f_{\downarrow,N}^\dagger f_{\downarrow,1} + f_{\downarrow,N} f_{\downarrow,1} + h.c. \big]. \tag{2.22}$$

We consider reflection that exchanges two flavors $\uparrow$ and $\downarrow$ as

$$R f_{\uparrow,j}^\dagger R^{-1} = f_{\downarrow,N-j+1}^\dagger, \quad R f_{\downarrow,j}^\dagger R^{-1} = f_{\uparrow,N-j+1}^\dagger, \quad R\,|0_f\rangle = |0_f\rangle \,, \tag{2.23}$$

where $|0_f\rangle$ is the Fock vacuum of the $f_\uparrow$ and $f_\downarrow$ fermions. The ground states $|GS_{r/ns}\rangle$ of the R and NS sectors are given by

$$|GS_{r/ns}\rangle = \Big(\sum_{n:\mathrm{odd/even}} \sum_{1 \le p_1 < \cdots < p_n \le N} f_{\uparrow,p_1}^\dagger \cdots f_{\uparrow,p_n}^\dagger \Big) \Big(\sum_{m:\mathrm{odd/even}} i^m \sum_{1 \le q_1 < \cdots < q_m \le N} f_{\downarrow,q_1}^\dagger \cdots f_{\downarrow,q_m}^\dagger \Big) |0_f\rangle \,. \tag{2.24}$$

Then, we have

$$R\,|GS_{r/ns}\rangle = \left\{ \begin{array}{l} -|GS_r\rangle \\ |GS_{ns}\rangle \end{array} \right. . \tag{2.25}$$

Thus, the many-body $\mathbb{Z}_2$ invariant is given by the ground state expectation value of full reflection on the superconducting chain with the periodic boundary condition,

$$Z(KB,(ns,r)) = \mathrm{Tr}_r\big[R\big] = \langle GS_r|R|GS_r\rangle = -1. \tag{2.26}$$

This agrees with the cobordism classification (2.20).

C. $(1+1)d$ insulators with charge conjugation symmetry ($\Omega_2^{\mathrm{Spin}^{\tilde{c}+}}(pt)$)

Our next example concerns $(1+1)d$ topological insulators protected by charge conjugation with $C^2=1$, where the action of C on the fermionic Fock space is defined as

$$C\psi_i^\dagger(x)C^{-1} = \mathcal{C}_{ij}\psi_j(x), \quad \mathcal{C}\mathcal{C}^* = 1, \quad C\left|0\right\rangle = \left|\mathrm{full}\right\rangle, \tag{2.27}$$

where $\psi_i^\dagger(x)$ is a complex fermion operator defined on a closed ring of length L, $|0\rangle$ and $|\mathrm{full}\rangle$ are the Fock vacuum and the fully occupied state of $\psi_i^\dagger(x)$ fermions, respectively. This symmetry is relevant to, for example, the fermionic Hubbard model at half filling. At the level of non-interacting fermions, the ensemble of single-particles Hamiltonians with this symmetry is equivalent to the ensemble of BdG Hamiltonians (symmetry class D in Altland-Zirnbauer symmetry classes[80]), where C plays the role of the particle-hole symmetry (constraint) for Nambu spinors. The topological classification for this symmetry class in $(1+1)d$ is $\mathbb{Z}_2$ in the absence of interactions, and this is expected to be so even in the presence of interactions. In the following, we give the definition of the many-body $\mathbb{Z}_2$ invariant in terms of the ground state with twisted boundary conditions. The relevant spin structures here for insulators with charge-conjugation symmetry are called $\mathrm{Spin}^{\tilde{c}+}$ structures. This has to be distinguished from Spin structures for charge neutral fermions in superconductors.

1. *The $\mathbb{Z}_2$ equivariant line bundle*

For our construction of the $\mathbb{Z}_2$ many-body topological invariant we consider twsited spatial boundary conditions by symmetry of the problem. Thanks to the $U(1)$ particle number conservation, we can introduce the twisted boundary condition $\psi(x+L) = e^{i\theta}\psi(x)$. Let $|GS(\theta)\rangle$ be the ground state of the Hamiltonian with the twist $\theta \in U(1)$. Since charge conjugation C flips the $U(1)$ flux, we have a $\mathbb{Z}_2$-equivariant complex line bundle over the flux space $U(1)$:

$$C\left|GS(\theta)\right\rangle = e^{i\phi(\theta)}\left|GS(-\theta)\right\rangle, \quad e^{i\phi(\theta)}e^{i\phi(-\theta)} = 1. \tag{2.28}$$

This $\mathbb{Z}_2$-equivariant structure leads to the $\mathbb{Z}_2$ quantization of the Berry phase

$$\gamma = \exp\Big[\oint_{-\pi}^{\pi} d\theta\,\langle GS(\theta)|\partial_\theta|GS(\theta)\rangle\Big] = e^{i\phi(\pi)-i\phi(0)} = \frac{\langle GS(\pi)|C|GS(\pi)\rangle}{\langle GS(0)|C|GS(0)\rangle} \in \{\pm 1\}. \tag{2.29}$$

Here we used $\int_0^\pi d\theta\,\langle GS(\theta)|\partial_\theta|GS(\theta)\rangle = \int_0^\pi d\theta\,\langle GS(\theta)|C^{-1}\partial_\theta C|GS(\theta)\rangle\rangle = i(\phi(\pi)-\phi(0)) - \int_{-\pi}^0 d\theta\,\langle GS(\theta)|\partial_\theta GS(\theta)\rangle$. This is a candidate of the many-body $\mathbb{Z}_2$ invariant. Our remaining task is to show that the existence of a model with nontrivial $\mathbb{Z}_2$ invariant.

2. *A nontrivial model and the $\mathbb{Z}_2$ invariant*

Let us consider the following two orbital model of complex fermions a_j, b_j in a closed ring,

$$H = \sum_{j=1}^{N-1} b_j^\dagger a_{j+1} + b_N^\dagger a_1 + h.c., \tag{2.30}$$

This model is invariant under the charge conjugation symmetry defined by

$$Ca_jC^{-1} = a_j^\dagger, \quad Cb_jC^{-1} = -b_j^\dagger, \quad C\left|0\right\rangle = \left|\mathrm{full}\right\rangle. \tag{2.31}$$

We introduce the $U(1)$-twisted Hamiltonian $H(\theta)$ by

$$H(\theta) := \sum_{j=1}^{N-1} b_j^\dagger a_{j+1} + e^{-i\theta} b_N^\dagger a_1 + h.c., \quad CH(\theta)C^{-1} = H(-\theta). \tag{2.32}$$

By introducing "bond" complex fermions $c_j, d_j (j = 1, \dots N-1), c_N(\theta), d_N(\theta)$ as

$$\begin{aligned} c_j &:= \frac{b_j + a_{j+1}}{\sqrt{2}}, \quad d_j := \frac{b_j - a_{j+1}}{\sqrt{2}}, \quad (j = 1, \dots N-1), \\ c_N(\theta) &:= \frac{b_N + e^{-i\theta} a_1}{\sqrt{2}}, \quad d_N(\theta) := \frac{b_N - e^{-i\theta} a_1}{\sqrt{2}}, \end{aligned} \tag{2.33}$$

the Hamiltonian $H(\theta)$ can be written as

$$H(\theta) = \sum_{j=1}^{N-1} [c_j^\dagger c_j - d_j^\dagger d_j] + c_N^\dagger(\theta) c_N(\theta) - d_N^\dagger(\theta) d_N(\theta). \tag{2.34}$$

The ground state of $H(\theta)$ is obtained by filling d_j and $d_N(\theta)$ fermions,

$$|GS(\theta)\rangle = d_1^\dagger \cdots d_{N-1}^\dagger d_N^\dagger(\theta) |0\rangle . \tag{2.35}$$

An explicit calculation shows that the Berry phase for $|GS(\theta)\rangle$ is non-trivial, $\gamma = -1$. This can be confirmed from the $\mathbb{Z}_2$-equivariant structure on the ground state $|GS(\theta)\rangle$. By making use of

$$\begin{aligned} &C d_j^\dagger C^{-1} = -c_j \, (j = 1, \dots, N-1), \quad C d_N^\dagger(\theta) C^{-1} = -c_N(-\theta), \\ &c_N(-\theta) b_N^\dagger a_1^\dagger = -e^{i\theta} d_N^\dagger(-\theta) + b_N^\dagger a_1^\dagger c_N(-\theta), \end{aligned} \tag{2.36}$$

then we have the nontrivial $\mathbb{Z}_2$-equivariant line bundle over $U(1)$ as

$$\begin{aligned} C |GS(\theta)\rangle &\sim c_1 \cdots c_{N-1} c_N(\theta) |\text{full}\rangle \\ &\sim c_1 \cdots c_{N-1} c_N(\theta) a_1^\dagger \cdots a_N^\dagger b_1^\dagger \cdots b_N^\dagger |0\rangle \\ &\sim e^{i\theta} |GS(-\theta)\rangle . \end{aligned} \tag{2.37}$$

3. $\mathbb{Z}_2$ *anomaly on the edge state*

Let us now have a further look at the topological non-triviality of the model, from the point of view of its boundaries. By the bulk-boundary correspondence, the non-trivial bulk topological invariant ($\gamma = -1$) is expected to manifests itself, in the presence of boundaries (edges), as a quantum anomaly. Let us now consider a topological insulator with the non-trivial bulk $\mathbb{Z}_2$ invariant with open boundary conditions, and focus on the low-energy excitations of the one of the edges. There is a fermion zero mode created/annihilated by a complex fermion creation operator $a^\dagger/a$. The charge conjugation (particle-hole) operator C acts on the fermion mode as $Ca^\dagger C^{-1} = a$. The edge theory defined in terms of $a^\dagger, a$ is anomalous in the sense that both the $U(1)$ charge conservation and charge conjugation C symmetries (the total symmetry group $= U(1) \rtimes C$) cannot be imposed on a unique ground state. In fact, the ground state with the $U(1)$ symmetry (having a definite particle number) is either $|0\rangle$ or $a^\dagger |0\rangle$. However, C exchanges $|0\rangle$ and $a^\dagger |0\rangle$ (where we set $C|0\rangle \sim a^\dagger |0\rangle$). It is also easy to see that this anomaly is $\mathbb{Z}_2$ in the sense that if we stack two identical copies of the system, this anomaly disappears.

4. *Cut and glue construction and the* $\mathbb{Z}_2$ *invariant*

Once the physics at the edge of non-trivial $\mathbb{Z}_2$ topological insulators are understood, one can use the cut and glue construction[79] to give a proof that the corresponding bulk $\mathbb{Z}_2$ invariant (2.29) is nontrivial. Let us consider an interval $I = [0, N]$ of the $\mathbb{Z}_2$ nontrivial insulating chain. The low-energy excitations can be described by two sets of edge complex fermion creation/annihilation operators, $\{a_1^\dagger, a_1\}$ and $\{b_N^\dagger, b_N\}$. The action of C on these fermion opeartors is given by $Ca_1^\dagger C^{-1} = a_1$ and $Cb_N^\dagger C^{-1} = -b_N$. The unique ground state of the Hamiltonian on the closed chain can be approximated by the ground state of the following gluing Hamiltonian

$$H_{\text{glue}}(\theta) = e^{-i\theta} b_N^\dagger a_1 + h.c., \tag{2.38}$$

where we have introduced the twisted boundary condition by the $U(1)$ symmetry. A direct calculation shows that the $\mathbb{Z}_2$ invariant (2.29) is nontrivial, $\gamma = -1$.

D. $(2+1)d$ superconductor with reflection symmetry ($\Omega_3^{\text{Pin}^+}(pt) = \mathbb{Z}_2$)

In this section, we give the many-body definition of the $\mathbb{Z}_2$ topological invariant for $(2+1)$-dimensional superconductors protected by reflection symmetry with $R^2 = 1$. Topological superconductors in this symmetry class can be considered as a CPT dual of class DIII topological superconductors in $(2+1)d$. Here, the reflection acts on the fermion creation operator $\psi_i^\dagger(x,y)$ as

$$R\psi_i^\dagger(x,y)R^{-1} = \psi_j^\dagger(-x,y)\mathcal{R}_{ji}, \quad \mathcal{R}^2 = 1, \tag{2.39}$$

where (x,y) is the spatial coordinate and i,j are the flavor (orbital, spin, etc.) indices. The topological classification is given by the Pin^+ cobordism group

$$\Omega_3^{\text{Pin}^+}(pt) = \mathbb{Z}_2. \tag{2.40}$$

The generating manifold of the cobordism group $\mathbb{Z}_2$ is $KB \times S^1$ ($KB =$ the Klein bottle),[71] which is a mapping torus. In other words, in the operator formalism, the many-body $\mathbb{Z}_2$ invariant can be constructed by considering a fully symmetry action on twisted ground states.

1. *The $\mathbb{Z}_2$ equivariant line bundle*

Let us consider a superconductor on 2-torus T^2. There are four distinct spin structures: $\{(r,r),(r,ns),(ns,r),(ns,ns)\}$. Since the reflection transformation R preserves these spin structures, we have the following $\mathbb{Z}_2$ equivariant bundle over four points:

$$R\left|GS(\eta)\right\rangle = (-1)^{\nu(\eta)}\left|GS(\eta)\right\rangle, \qquad \eta \in \{(r,r),(r,ns),(ns,r),(ns,ns)\}. \tag{2.41}$$

To remove "weak indices" (2.26) arising from (1+1)d superconductors with reflection symmetry, we define the "strong" $\mathbb{Z}_2$ invariant $(-1)^\nu$ specific to $(2+1)d$ superconductors by

$$(-1)^\nu = \prod_{\eta \in \{(r,r),(r,ns),(ns,r),(ns,ns)\}} (-1)^{\nu(\eta)}. \tag{2.42}$$

In the following, we show an example of a $\mathbb{Z}_2$ nontrivial model.

2. *A nontrivial model and the many-body $\mathbb{Z}_2$ invariant*

A model Hamiltonian is a reflection symmetric pair of $(p_x + ip_y)$ and $(p_x - ip_y)$ superconductors

$$\begin{aligned} H = &\sum_{\boldsymbol{k}} \psi_\uparrow^\dagger(\boldsymbol{k})(m - \cos k_x - \cos k_y)\psi_\uparrow(\boldsymbol{k}) + \frac{1}{2}\sum_{\boldsymbol{k}}(\sin k_x + i\sin k_y)\psi_\uparrow^\dagger(\boldsymbol{k})\psi_\uparrow^\dagger(-\boldsymbol{k}) + h.c. \\ &+ \sum_{\boldsymbol{k}} \psi_\downarrow^\dagger(\boldsymbol{k})(m - \cos k_x - \cos k_y)\psi_\downarrow(\boldsymbol{k}) + \frac{1}{2}\sum_{\boldsymbol{k}}(\sin k_x - i\sin k_y)\psi_\downarrow^\dagger(\boldsymbol{k})\psi_\downarrow^\dagger(-\boldsymbol{k}) + h.c. \end{aligned} \tag{2.43}$$

The reflection R acts on the fermion fields as

$$R\psi_\uparrow^\dagger(k_x,k_y)R^{-1} = \psi_\downarrow^\dagger(k_x,-k_y), \quad R\psi_\downarrow^\dagger(k_x,k_y)R^{-1} = \psi_\uparrow^\dagger(k_x,-k_y), \quad R\left|0\right\rangle = \left|0\right\rangle, \tag{2.44}$$

where $\left|0\right\rangle$ is the Fock vacuum of the $\psi_{\uparrow,\downarrow}$ fermions. The ground state for each spin structure is given by the following BCS form:[81]

- For the R-R sector, the set of allowed momenta are $(k_x, k_y) = (\frac{2\pi n_x}{L}, \frac{2\pi n_y}{L})$, and the ground state is given by

$$|\Psi(r,r)\rangle = \Psi^{\dagger}_{\mathrm{UP}} \cdot \exp\left(\sum_{\substack{k_x>0 \\ \boldsymbol{k}\neq(0,0),(0,\pi),(\pi,0),(\pi,\pi)}} \left\{ g_+(\boldsymbol{k})\psi^{\dagger}_{\uparrow}(\boldsymbol{k})\psi^{\dagger}_{\uparrow}(-\boldsymbol{k}) + g_-(\boldsymbol{k})\psi^{\dagger}_{\downarrow}(\boldsymbol{k})\psi^{\dagger}_{\downarrow}(-\boldsymbol{k}) \right\} \right) |0\rangle \,. \tag{2.45}$$

- For the NS-NS sector, the set of allowed momenta are $(k_x, k_y) = (\frac{2\pi}{L}(n_x+\frac{1}{2}), \frac{2\pi}{L}(n_y+\frac{1}{2}))$, and the ground state is given by

$$|\Psi(ns,ns)\rangle = \exp\left(\sum_{k_x>0} \left\{ g_+(\boldsymbol{k})\psi^{\dagger}_{\uparrow}(\boldsymbol{k})\psi^{\dagger}_{\uparrow}(-\boldsymbol{k}) + g_-(\boldsymbol{k})\psi^{\dagger}_{\downarrow}(\boldsymbol{k})\psi^{\dagger}_{\downarrow}(-\boldsymbol{k}) \right\} \right) |0\rangle \,. \tag{2.46}$$

- For the R-NS sector, the set of allowed momenta are $(k_x, k_y) = (\frac{2\pi n_x}{L}, \frac{2\pi}{L}(n_y+\frac{1}{2}))$, and the ground state is given by

$$|\Psi(r,ns)\rangle = \exp\left(\sum_{k_x>0} \left\{ g_+(\boldsymbol{k})\psi^{\dagger}_{\uparrow}(\boldsymbol{k})\psi^{\dagger}_{\uparrow}(-\boldsymbol{k}) + g_-(\boldsymbol{k})\psi^{\dagger}_{\downarrow}(\boldsymbol{k})\psi^{\dagger}_{\downarrow}(-\boldsymbol{k}) \right\} \right) |0\rangle \,. \tag{2.47}$$

- For the NS-R sector, the set of allowed momenta are $(k_x, k_y) = (\frac{2\pi}{L}(n_x+\frac{1}{2}), \frac{2\pi n_y}{L})$, and the ground state is given by

$$|\Psi(ns,r)\rangle = \exp\left(\sum_{k_x>0} \left\{ g_+(\boldsymbol{k})\psi^{\dagger}_{\uparrow}(\boldsymbol{k})\psi^{\dagger}_{\uparrow}(-\boldsymbol{k}) + g_-(\boldsymbol{k})\psi^{\dagger}_{\downarrow}(\boldsymbol{k})\psi^{\dagger}_{\downarrow}(-\boldsymbol{k}) \right\} \right) |0\rangle \,. \tag{2.48}$$

Here, $g_{\pm}(\boldsymbol{k})$ is given by

$$g_{\pm}(\boldsymbol{k}) = \frac{\sqrt{\epsilon_{\boldsymbol{k}}^2 + \sin^2 k_x + \sin^2 k_y} - \epsilon_{\boldsymbol{k}}}{\sin k_x \mp i \sin k_y}, \quad \epsilon_{\boldsymbol{k}} = m - \cos k_x - \cos k_y, \tag{2.49}$$

and $\Psi^{\dagger}_{\mathrm{UP}}$ is the contribution from unpaired fermions and given by

$$\Psi^{\dagger}_{\mathrm{UP}} := \begin{cases} \psi^{\dagger}_{\uparrow}(0,0)\psi^{\dagger}_{\uparrow}(\pi,\pi)\psi^{\dagger}_{\uparrow}(0,\pi)\psi^{\dagger}_{\uparrow}(\pi,0)\psi^{\dagger}_{\downarrow}(0,0)\psi^{\dagger}_{\downarrow}(\pi,\pi)\psi^{\dagger}_{\downarrow}(0,\pi)\psi^{\dagger}_{\downarrow}(\pi,0) & (m<-2) \\ \psi^{\dagger}_{\uparrow}(0,0)\psi^{\dagger}_{\uparrow}(0,\pi)\psi^{\dagger}_{\uparrow}(\pi,0)\psi^{\dagger}_{\downarrow}(0,0)\psi^{\dagger}_{\downarrow}(0,\pi)\psi^{\dagger}_{\downarrow}(\pi,0) & (-2<m<0) \\ \psi^{\dagger}_{\uparrow}(0,0)\psi^{\dagger}_{\downarrow}(0,0) & (0<m<2) \\ 1 & (2<m) \end{cases} \tag{2.50}$$

Since the condensate of Cooper pairs in the BCS ground states is reflection symmetric, a nontrivial phase arises from unpaired fermions:

$$\begin{aligned} R\,|\Psi(r,r)\rangle &= \begin{cases} -\,|\Psi(r,r)\rangle & (-2<m<0, 0<m<2) \\ |\Psi(r,r)\rangle & (m<-2, 2<m) \end{cases}, \\ R\,|\Psi(\eta)\rangle &= |\Psi(\eta)\rangle \ \ (\eta \in \{(r,ns),(ns,r),(ns,ns)\}). \end{aligned} \tag{2.51}$$

Thus, the topological invariant is $(-1)^{\nu} = -1$ for topologically nontrivial phases $(-2<m<0, 0<m<2)$.

III. PARTIAL REFLECTIONS

We now discuss some $(1+1)d$ SPT phases, of which the detection requires the real projective plane $\mathbb{R}P^2$ as a generating manifold of the relevant cobordism group. For example, $(1+1)d$ bosonic SPT phases with reflection symmetry is classified by the unoriented cobordism group $\Omega^{O}_{2}(pt) = \mathbb{Z}_2$ with the generating manifold $\mathbb{R}P^2$.[25] Other fermionic examples are discussed in the subsequent sections. Topologically, $\mathbb{R}P^2$ is realized by introducing a cross-cap on the spacetime manifold S^2. In the operator formalism, introducing a cross-cap is intuitively equivalent to acting with a partial reflection R_I on an interval I of a closed ring S^1, as shown in Fig. 1. In $(1+1)d$ bosonic SPT phases protected by reflection symmetry, Pollmann and Turner showed that the expectation value of the partial reflection $\langle GS|R_I|GS\rangle$ on the ground state $|GS\rangle$ faithfully captures the $\mathbb{Z}_2$ SPT invariant.[67] In this section, we extend this approach to fermionic SPT phases. Some contents of this section were also discussed in Ref. 73.

A. $(1+1)d$ topological superconductors protected by reflection ($\Omega_2^{\mathrm{Pin}^-}(pt)=\mathbb{Z}_8$)

Let us consider $(1+1)d$ topological superconductors protected by reflection R with $R^2=(-1)^F$. These topological superconductors can be thought of as a CPT dual of class BDI topological superconductors. The topological classification is given by the Pin^- cobordism group[71]

$$\Omega_2^{\mathrm{Pin}^-}(pt)=\mathbb{Z}_8, \tag{3.1}$$

and the generating manifold is the real projective plane $\mathbb{R}P^2$.

1. *The Klein bottle partition function and the $\mathbb{Z}_4$ invariant*

In the following, we first show that the action of symmetry (=reflection) on the ground states in the presence of twisted boundary conditions is not sufficient to capture the $\mathbb{Z}_8$ classification. In fact, from the space-time path-integral, this corresponds to the partition function defined on the spacetime Klein bottle, which is not the generating manifold. The Klein bottle generates a modulo 4 subgroup of the $\mathbb{Z}_8$ group.

We consider the same model as (2.2) on a closed ring. This model is invariant under the following reflection symmetry

$$Rf_j^{\dagger}R^{-1}=if^{\dagger}_{N-j+1},\quad Rf_jR^{-1}=-if_{N-j+1}, \tag{3.2}$$

or equivalently,

$$Rc_j^LR^{-1}=c^R_{N-j+1},\quad Rc_j^RR^{-1}=-c^L_{N-j+1} \tag{3.3}$$

in terms of real fermions introduced in (2.4). In terms of the real fermions c_j^L,c_j^R, the reflection transformation R is explicitly written by

$$R=:\exp\Big[-\frac{\pi}{4}\sum_{j=1}^{N}c_j^Lc^R_{N-j+1}\Big]:, \tag{3.4}$$

where we introduced the normal ordering $:\cdots:$ with respect to the Fock vacuum $|0_f\rangle$ of f_j fermions to fix the overall phase of R, $R\,|0_f\rangle=|0_f\rangle$. Observe that $R^2=(-1)^F$, and hence R here generates a $\mathbb{Z}_4$ symmetry. From the concrete expressions (2.9) of the ground states $|GS_{r/ns}\rangle$, we obtain

$$\begin{aligned}R\,|GS_{r/ns}\rangle&=\sum_{n:\mathrm{odd/even}}\ \sum_{1\leq p_1<p_2<\cdots<p_n\leq N}i^n f^{\dagger}_{N-p_1+1}f^{\dagger}_{N-p_2+1}\cdots f^{\dagger}_{N-p_n+1}\,|0_f\rangle\\&=\begin{cases}i\,|GS_r\rangle\\|GS_{ns}\rangle\end{cases}\end{aligned} \tag{3.5}$$

In other words, the partition function on the Klein bottle (KB) with the periodic boundary condition for the space direction provides the $\mathbb{Z}_4$ sub group invariant:

$$Z(KB,(ns,r))=\mathrm{Tr}_r\big[R\big]=\langle GS_r|R|GS_r\rangle=i. \tag{3.6}$$

In other words, the full reflection cannot capture the $\mathbb{Z}_8$ invariant.

2. *Partial reflection and the $\mathbb{Z}_8$ invariant*

We have shown that the partial reflection faithfully defines the many-body $\mathbb{Z}_8$ invariant in Ref. 73. Here, we give a quick derivation of the $\mathbb{Z}_8$ invariant by using the "cut and glue" construction of the reduced density matrix for topological phases.[79] Let $|GS\rangle$ be a ground state belonging to the $1\in\mathbb{Z}_8$ topological phase. We wish to compute the expectation value of partial reflection $\langle GS|R_I|GS\rangle$, where $I=[1,M]$ is an interval in a whole closed chain $S^1=[0,N]$. If M and $N-M$ are sufficiently larger than the correlation length of the bulk, the reduced density matrix on the interval I, $\rho_I = \mathrm{tr}_{\bar{I}}(|GS\rangle\langle GS|)$, is approximated by the exponential of the entanglement Hamiltonian, consisting only of edge Majorana fermions:

$$\tilde{H} = \frac{i}{2} c_0^R c_1^L + \frac{i}{2} c_M^R c_{M+1}^L. \tag{3.7}$$

Defining "in" and "out" complex fermions f_{in} and f_{out} by

$$f_{in}^\dagger = \frac{c_M^R + i c_1^L}{2}, \quad f_{out}^\dagger = \frac{c_0^R + i c_{M+1}^L}{2}, \tag{3.8}$$

the ground state of $\tilde{H}$ is

$$|GS\rangle = \frac{1}{\sqrt{2}} (f_{in}^\dagger + f_{out}^\dagger) |0\rangle . \tag{3.9}$$

The partial reflection acts only on the $f_{in}^\dagger$ fermion as

$$R_I f_{in}^\dagger R_I^{-1} = i f_{in}^\dagger, \quad R_I f_{out}^\dagger R_I^{-1} = f_{out}^\dagger, \quad R_I |0\rangle = |0\rangle . \tag{3.10}$$

Hence, we obtain

$$\langle GS|R_I|GS\rangle = \mathrm{Tr}_{in}(R_I \rho_I) = \frac{1+i}{2} = \frac{1}{\sqrt{2}} e^{\frac{\pi i}{4}}. \tag{3.11}$$

The $U(1)$ phase $e^{\frac{\pi i}{4}}$ correctly captures the $\mathbb{Z}_8$ classification. In fact, the phase of $\langle GS|R_I|GS\rangle$ coincides with the Brown invariant of the Pin$^-$ structure on $\mathbb{R}P^2$.[71] (For the description of the Brown invariant, see Appendix A 2.) Furthermore, with the partial reflection, the amplitude of the wave function overlap (the partition function) is reduced from 1 to $1/\sqrt{2}$. This "loss of the amplitude" is also of topological origin, which is the quantum dimension of the edge Majorana fermion.

In the remainder of this section, we present two non-trivial applications of the partial reflection: First, we use the partial reflection to map out the phase diagram of the disordered Kitaev Majorana chain. Second, we apply it to the exact ground state of an interacting Majorana chain and show that it yields the expected $\mathbb{Z}_8$ phase.

3. *Case study 1: Robustness of partial reflection against random disorder*

In what follows, we compute the $\mathbb{Z}_8$ invariant for a microscopic realization of class D (with reflection symmetry) topological superconductors in the presence of a random chemical potential disorder. In particular, we study the robustness of the partial reflection against random disorder, and also show that we can use the many-body $\mathbb{Z}_8$ topological invariant to map out the phase diagram of the disordered Majorana chain, by reproducing the known results.[85,86] The lattice Hamiltonian consists of two parts $H = H_{\mathrm{clean}} + H_{\mathrm{dis}}$ where H_{clean} is the Hamiltonian of the Kitaev Majorana chain,

$$H = -\sum_j \left[t f_{j+1}^\dagger f_j - \Delta f_{j+1}^\dagger f_j^\dagger + \mathrm{H.c.} \right] - \mu \sum_j f_j^\dagger f_j , \tag{3.12}$$

and the disorder term is

$$H_{\mathrm{dis}} = \sum_j v_j f_j^\dagger f_j, \tag{3.13}$$

where v_j is a random number uniformly distributed over the range $[-W/2, W/2]$ and W is the disorder strength. Figure 2 shows the complex phase of the averaged partial reflection $Z = \langle GS|R_I|GS\rangle$ for various values of W over a wide range of chemical potential including the trivial and topological phases. It is interesting to note that the topological region (characterized by $\angle Z = \pi/4$) expands a little bit as the disorder strength is increased. This is similar to the disorder induced topological phase due to quadratic corrections on the lattice models, which has been discussed in the context of the 2D and 3D topological insulators.[87–89] In addition, we provide the complex phase of the partial reflection for one realization of disorder in Fig. 3. It is evident from this figure that the phase is quite robust to moderate disorder.

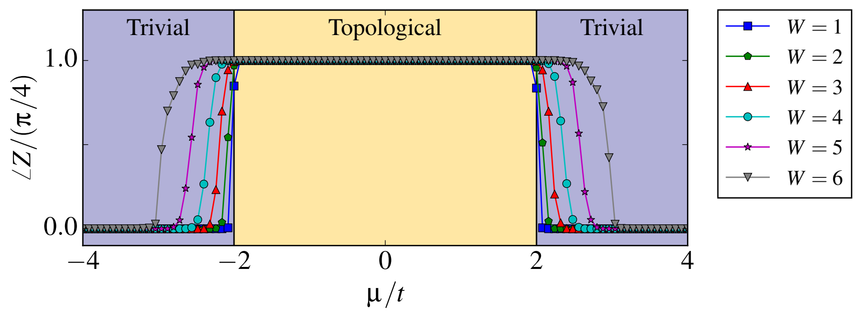


FIG. 2. (Color online) Complex phase of the partial reflection $Z = \langle GS|R_I|GS\rangle$ for the disordered Kitaev Majorana chain. Each curve represents an ensemble average over 1000 samples. Solid lines are guides for the eye. Here, we set $\Delta = t$, $N = 200$ and $N_{\rm part} = 100$.

We use the complex phase associated with the partial reflection to map out the phase diagram of the disordered Majorana chain as shown in Fig. 4(b). For reference, we compute the phase boundary between the trivial and topological phases as a function of disorder strength W, using the transfer matrix approach[85,86] (see Fig. 4(a)). In the case of our model, the Majorana chain with nearest neighbor hopping, the Lyapunov exponent can be found analytically,

$$\begin{aligned}\Lambda^{-1} &= \Big|\lim_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n}\ln|\frac{\mu+v_i}{2t}|\Big| \\ &= \left|\int_{-1/2}^{1/2}dx\,\ln|\frac{\mu+xW}{2t}|\right| \\ &= \left|\ln\left[\frac{|2\mu+W|^{\frac{\mu}{W}+\frac{1}{2}}}{|2\mu-W|^{\frac{\mu}{W}-\frac{1}{2}}}\right]-(1+\ln 2t)\right|. \end{aligned} \tag{3.14}$$

The phase boundary can then be identified as a line of critical points at which $\Lambda\to\infty$ is diverging. The diverging Λ is indicative of delocalized states at zero chemical potential which are in turn responsible for the topological phase transition. We show in Fig. 4(b) that the transfer matrix and our results are in remarkable agreement.

4. *Case study 2: Partial reflection in the interacting Majorana chain*

In this part, we consider the many-body $\mathbb{Z}_8$ invariant in the presence of interactions. We use the exact many-body ground state constructed for the interacting Majorana chain in Ref. 84:

$$H = -\sum_j[tf_{j+1}^\dagger f_j - \Delta f_{j+1}^\dagger f_j^\dagger + H.c.] - \mu\sum_j f_j^\dagger f_j + 4U\sum_j (n_j-1/2)(n_{j+1}-1/2). \tag{3.15}$$

For parameter values satisfying the condition $\mu = 4\sqrt{U^2+tU+(t^2-\Delta^2)/4}$ can be written in a closed form as

$$|\Psi_{r(ns)}\rangle = |\Psi_+^\alpha\rangle \mp |\Psi_-^\alpha\rangle\,, \tag{3.16}$$

where

$$\begin{aligned}|\Psi_\pm^\alpha\rangle &= \frac{1}{(1+\alpha^2)^{N/2}}\prod_{j=1}^{N}(1\pm\alpha f_j^\dagger)\,|0\rangle \\ &= \frac{1}{(1+\alpha^2)^{N/2}}e^{\pm\alpha f_1^\dagger}e^{\pm\alpha f_2^\dagger}\dots e^{\pm\alpha f_N^\dagger}\,|0\rangle\,, \end{aligned} \tag{3.17}$$

corresponding to odd (even) fermion parity sectors associated with (anti-)periodic boundary condition where $\alpha = \sqrt{\cot(\theta/2)}$ and $\theta = \arctan(2\Delta/\mu)$. In the following, we analytically derive that the amplitudes and complex phases of

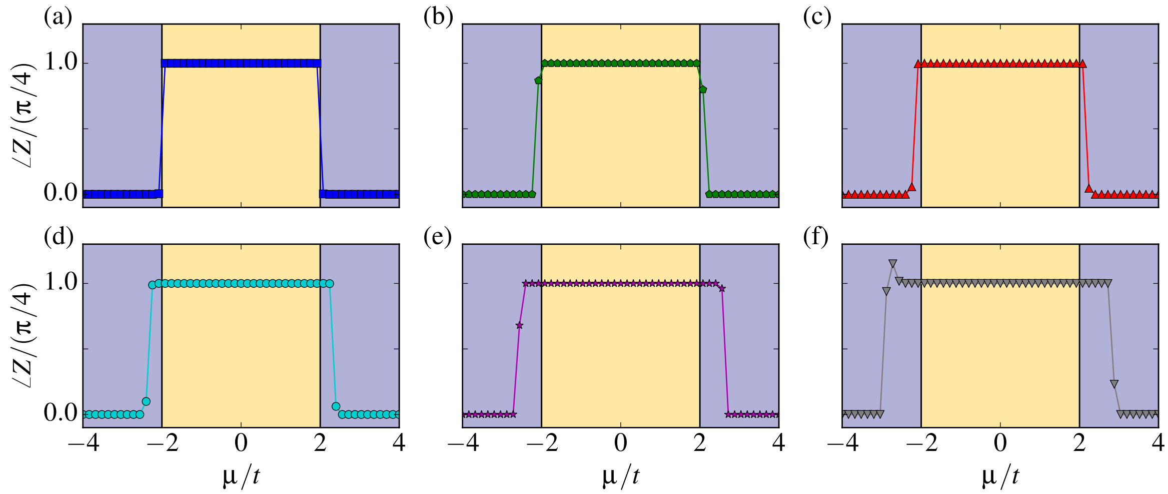


FIG. 3. (Color online) Complex phase of the partial reflection $Z = \langle GS|R_I|GS\rangle$ for one realization of the disorder potential. Panels (a)-(f) represent different disorder strength from $W = 1$ to $W = 6$ (same as the legend in Fig. 2). Solid lines are guides for the eye. Here, we set $\Delta = t$, $N = 200$ and $N_{\rm part} = 100$.

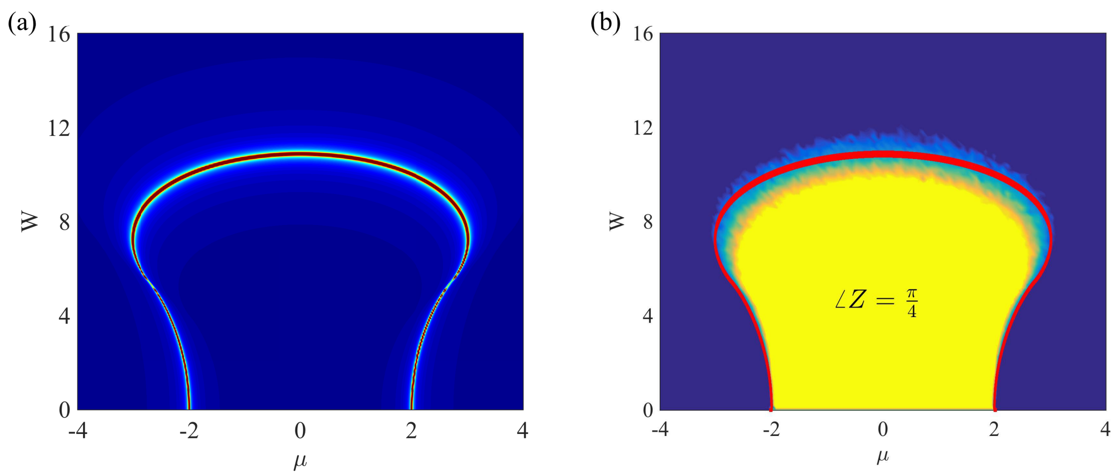


FIG. 4. (Color online) Phase diagram of the disordered Kitaev Majorana chain. (a) Color code is the Lyapunov exponent calculated using the transfer matrix approach (3.14), and (b) color code is the complex phase of the partial reflection $\langle GS|R_I|GS\rangle$. In panel (b), the red curve shows the phase boundary which is analytically determined by the transfer matrix as shown in (a). Here, we set $\Delta = t$, $N = 200$ and $N_{\rm part} = 100$.

partial reflection for a generic value of α converge to the anticipated values $1/\sqrt{2}$ and $\pi/4$ in (3.11), as we approach the long-chain limit (compared to the correlation length).

We prove our result for the case with anti-periodic boundary condition. A similar derivation can be carried out for the case of periodic boundary condition. Consider a long chain with N sites in total and M sites in the subsystem such that $N = 2M$ (we take M to be even which means reflection with respect to the central link). The wave function is given by

$$|\Psi_{ns}\rangle = \frac{1}{\sqrt{\mathcal{A}_+(N)}} \prod_{j=1}^{N} (1+\alpha f_j^\dagger)\Big|_{n\in\text{even}} |0\rangle \tag{3.18}$$

where the normalization factor is

$$\mathcal{A}_{\pm}(n) = \frac{1}{2}[(1+\alpha^2)^n \pm (1-\alpha^2)^n]. \tag{3.19}$$

For simplicity we choose the sites 1 to M to be in the subsystem. The wave function can be rewritten as

$$\begin{aligned}
|\Psi_{ns}\rangle =& \frac{1}{\sqrt{\mathcal{A}(N)}}(1+\alpha f_1^\dagger)F_{in}^\dagger(1+\alpha f_M^\dagger)F_{out}^\dagger\Big|_{n\in\text{even}}|0\rangle \\
=& \frac{1}{\sqrt{\mathcal{A}(N)}}[(1+\alpha^2 f_1^\dagger f_M^\dagger)F_{in}^{(e)\dagger}F_{out}^{(e)\dagger} + (1-\alpha^2 f_1^\dagger f_M^\dagger)F_{in}^{(o)\dagger}F_{out}^{(o)\dagger}]|0\rangle \\
&+ \frac{\alpha}{\sqrt{\mathcal{A}(N)}}[(f_1^\dagger + f_M^\dagger)F_{in}^{(e)\dagger}F_{out}^{(o)\dagger} + (f_1^\dagger - f_M^\dagger)F_{in}^{(o)\dagger}F_{out}^{(e)\dagger}]|0\rangle\,.
\end{aligned} \tag{3.20}$$

We define the new operators

$$\begin{aligned}
F_{in}^{(o/e)\dagger} &= \prod_{j=2}^{M-1}(1+\alpha f_j^\dagger)\Big|_{n\in\text{odd/even}}, \\
F_{out}^{(o/e)\dagger} &= \prod_{j=M+1}^{N}(1+\alpha f_j^\dagger)\Big|_{n\in\text{odd/even}}.
\end{aligned} \tag{3.21}$$

The partial reflection is defined by

$$R_I f_j^\dagger R_I^{-1} = i f_{M-(j-1)}^\dagger, \quad \text{for} \quad 1\le j\le M. \tag{3.22}$$

Note that

$$\begin{aligned}
\langle 0|\, F_{in}^{(o)} R_I F_{in}^{(o)\dagger}\,|0\rangle &= i\mathcal{A}_-(M-2), \\
\langle 0|\, F_{in}^{(o)} R_I F_{in}^{(o)\dagger}\,|0\rangle &= \mathcal{A}_+(M-2),
\end{aligned} \tag{3.23}$$

where $\mathcal{A}_\pm(n)$ is defined in (3.19). So, we have

$$\begin{aligned}
Z_N = \langle\Psi_{ns}|\,\mathcal{R}_I\,|\Psi_{ns}\rangle =& \frac{1}{\mathcal{A}(N)}\left[(1+\alpha^4)\mathcal{A}_+(M-2)\mathcal{A}_+(N-M) + i(1+\alpha^4)\mathcal{A}_-(M-2)\mathcal{A}_-(N-M)\right] \\
&+ \frac{2\alpha^2}{\mathcal{A}(N)}\left[i\mathcal{A}_+(M-2)\mathcal{A}_-(N-M) + \mathcal{A}_-(M-2)\mathcal{A}_+(N-M)\right]
\end{aligned} \tag{3.24}$$

In the thermodynamic limit $N\to\infty$, this becomes

$$\begin{aligned}
\lim_{N\to\infty} Z_N &= \frac{(1+\alpha^2)^{N-2}}{2(1+\alpha^2)^N}[(1+\alpha^4)(1+i) + 2\alpha^2(i+1)] \\
&= \frac{1+i}{2},
\end{aligned} \tag{3.25}$$

which is identical to (3.11). Here, we use the fact that

$$\lim_{n\to\infty}\mathcal{A}_\pm(n) = \frac{1}{2}(1+\alpha^2)^n. \tag{3.26}$$

B. $(1+1)d$ insulators protected by reflection ($\Omega^{\mathrm{Pin}^c}(pt) = \mathbb{Z}_4$)

Next, we move on to the system described by complex fermions in the presence of reflection symmetry. Specifically, we consider a reflection transformation acting on a multiplet of complex fermions operators $\psi_i(x,t)$ together with a $U(1)$ transformation as

$$U_\alpha R\psi_i^\dagger(x,t)[U_\alpha R^{-1}] = \psi_j^\dagger(-x,t)e^{-i\alpha}\mathcal{R}_{ji}. \tag{3.27}$$

This operation, when used along with twisted boundary conditions, enables us to put the theory on an unoriented manifold with the Pin^c structure. In $(1+1)$-dimensions, the corresponding cobordism group is [57]

$$\Omega_2^{\mathrm{Pin}^c}(pt) = \mathbb{Z}_4, \tag{3.28}$$

which implies there are 4 topologically distinct phases. The generating manifold is $\mathbb{R}P^2$. In this section, we will see that the expectation value of full reflection captures the $\mathbb{Z}_2$ subgroup of the $\mathbb{Z}_4$ topological classification, whereas the expectation value of partial reflection with an appropriate $U(1)$ phase as introduced in Eq. (3.27) provides the $\mathbb{Z}_4$ invariant.

1. The $\mathbb{Z}_2$ equivariant line bundle

The existence of the many-body $\mathbb{Z}_2$ topological invariant, which is a subgroup of the $\mathbb{Z}_4$ classification, can be understood by the $\mathbb{Z}_2$-equivariant structure of the line bundle over the flux space in the same way as Eqs. (2.28) and (2.29). On the closed space circle S^1, we have a $\mathbb{Z}_2$-equivariant complex line bundle

$$R\left|GS(\theta)\right\rangle = e^{i\phi(\theta)}\left|GS(-\theta)\right\rangle, \tag{3.29}$$

where $|GS(\theta)\rangle$ is the ground state under the twisted boundary condition $\psi(x+L) = e^{i\theta}\psi(x)$. Then, we have the $\mathbb{Z}_2$ quantization of the Berry phase

$$\gamma = \exp\Big[\oint_0^{2\pi} d\theta \left\langle GS(\theta)|\partial_\theta GS(\theta)\right\rangle\Big] = \frac{\langle GS(\pi)|R|GS(\pi)\rangle}{\langle GS(0)|R|GS(0)\rangle} \in \pm 1. \tag{3.30}$$

A $\mathbb{Z}_2$ nontrivial model is the same as (2.30) with the reflection symmetry defined by

$$Ra_j^\dagger R^{-1} = b_{N-j+1}^\dagger, \quad Rb_j^\dagger R^{-1} = a_{N-j+1}^\dagger. \tag{3.31}$$

One can show that $R\left|GS(\theta)\right\rangle \sim e^{i\theta}\left|\Psi(-\theta)\right\rangle$, which implies the nontrivial $\mathbb{Z}_2$ Berry phase $\gamma = -1$.

2. Partial reflection and the $\mathbb{Z}_4$ invariant

In a way similar to Sec. III A 2, we show that partial reflection provides the $\mathbb{Z}_4$ invariant. There are two issues specific to Pin^c structures: the choice of $U(1)$ phase associated with partial reflection, and the choice of the center of reflection, i.e., a lattice site or bond center.

Here, we calculate the partial reflection for the fixed point model (2.30). The ground state is given by

$$|GS\rangle = \frac{b_N^\dagger - a_1^\dagger}{\sqrt{2}}\frac{b_1^\dagger - a_2^\dagger}{\sqrt{2}}\cdots\frac{b_M^\dagger - a_{M+1}^\dagger}{\sqrt{2}}\cdots\frac{b_{N-1}^\dagger - a_N^\dagger}{\sqrt{2}}\left|0\right\rangle. \tag{3.32}$$

The partial reflection $[U_\alpha R]_I$ acts on interval $I = \{1,\dots,M\}$ as

$$[U_\alpha R]_I a_j^\dagger [U_\alpha R]_I^{-1} = e^{-i\alpha} b_{M-j+1}^\dagger, \quad [U_\alpha R]_I b_j^\dagger [U_\alpha R]_I^{-1} = e^{-i\alpha} a_{M-j+1}^\dagger, \quad [U_\alpha R]_I \left|0\right\rangle = \left|0\right\rangle, \quad (j=1,\dots,M). \tag{3.33}$$

Even M and odd M correspond to the case of reflection centered at a lattice site or at a bond, respectively. We have

$$\langle GS|[U_\alpha R]_I|GS\rangle = \frac{1}{2}(-1)^{\frac{1}{2}M(M-1)}\sin\alpha e^{-iM\alpha}. \tag{3.34}$$

To obtain the quantum dimension of the edge complex fermions, α can be chosen as $\alpha = \pm\frac{\pi}{2}$, which implies that only bond center partial reflection (odd M) provides $\mathbb{Z}_4$ phases $\pm i$.

IV. PARTIAL ROTATIONS

In this section, we introduce a kind of nonlocal operators, *partial n-fold rotations* $C_{n,D}$, acting on a subdisk region D in the 2-dimensional space manifold M. We find that the expectation value of partial rotations on the ground

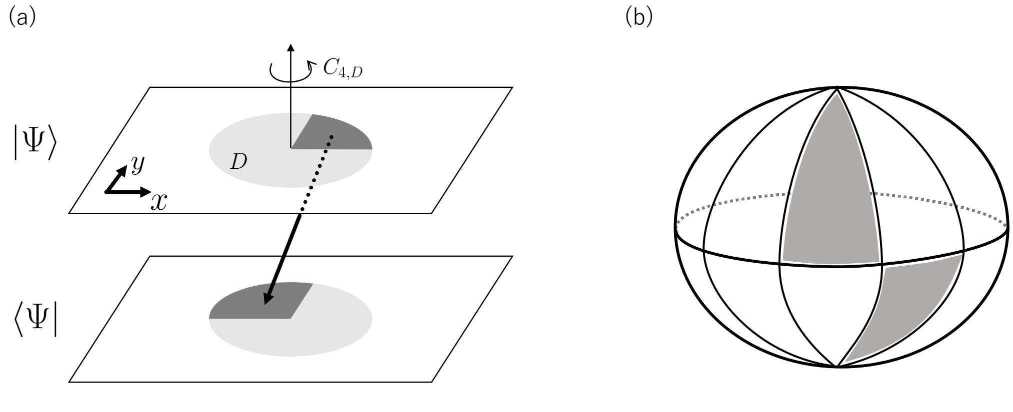


FIG. 5. (a) Partial rotation on the ground state $|\Psi\rangle$. The figure shows the partial C_4 rotation. (b) A construction of lens space $L(n,1)$. The figure shows the boundary ($\cong S^2$) of a 3-ball. The boundary of upper hemisphere is rotated by $2\pi/n$ angle, and glued into the boundary of lower hemisphere. The shadow regions are identified.

state $\langle GS\,|\,C_{n,D}\,|\,GS\rangle$ (see Fig. 5 (a)) provides scale-independent quantities which characterize topological properties of the bulk, if the length of the boundary of the subdisk D is sufficiently larger than the correlation length of bulk. Intuitively, the action of partial C_n rotation can be thought of as a generalization of the procedure introducing a cross-cap, which gives rise to lens spaces $L(n,1)$ if we act the partial rotation on S^2 (see Fig. 5 (b)). In particular, the case of $n=2$ corresponds to $\mathbb{R}P^3 = L(2,1)$.

In the following, we first temporarily depart from SPT phases and consider partial rotations in chiral topological phases, and present the calculation of their ground state expectation value using their edge CFT. In this context, the ground state expectation value of partial rotation can be used to extract a combination of modular S and T matrices: It turns out that the low energy quasi particle description is consistent with lens space partition functions of $(2+1)d$ TQFTs.[33,90] Our derivation of the formula for the partial rotation is similar to Ref. 72, where they introduced the partial lattice translation on the cylinder. The surgery construction of lens spaces from the solid torus is summarized in Appendix B.

A. $(2+1)d$ chiral topological phase and partial rotation and lens space

Let us consider a $(2+1)d$ chiral topological phase. For simplicity, we assume that the ground state is unique. We further assume the bulk is symmetric under C_n rotation symmetry. Here we consider a right moving CFT with dispersion $\varepsilon(k) \sim vk$ for the boundary theories, where $v>0$ is the velocity of excitations. The cases with left moving chiral CFT will be briefly discussed later.

We would like to estimate the expectation value of the partial rotation $\langle GS\,|\,C_{n,D}\,|\,GS\rangle$ on a subdisk D for the ground state over a space manifold M. The edge chiral CFT is described by Hamiltonian and momentum operators

$$H = \frac{2\pi v}{L}(L_0 - \frac{c}{24}), \qquad P = \frac{H}{v}, \tag{4.1}$$

where c is the central charge, and $L = |\partial D|$ is the length of the boundary. By using the cut and glue construction,[79] the reduced density matrix ρ_D for the subdisk D is approximated by the edge CFT with temperature determined by the correlation length ξ of bulk,

$$\rho_D = \frac{e^{-\frac{\xi}{v}H}}{\mathrm{Tr}\big[e^{-\frac{\xi}{v}H}\big]}. \tag{4.2}$$

Then, partial C_n rotation is nothing but translation by L/n on the edge CFT,

$$\langle GS|C_{n,D}|GS\rangle = \frac{\mathrm{Tr}\big[e^{-i\widetilde{P}\frac{L}{n}}e^{-\frac{\xi}{v}H}\big]}{\mathrm{Tr}\big[e^{-\frac{\xi}{v}H}\big]} = \frac{e^{\frac{2\pi i}{n}(\langle L_0\rangle - \frac{c}{24})}\sum_{a\in\mathrm{irreps.}}\chi_a(\frac{i\xi}{L}-\frac{1}{n})}{\sum_{a\in\mathrm{irreps.}}\chi_a(\frac{i\xi}{L})}, \tag{4.3}$$

where

$$\chi_a(\tau)=\mathrm{tr}_a[e^{2\pi i\tau(L_0-\frac{c}{24})}] \tag{4.4}$$

is the Virasoro character of the a-th irreducible representation, and the summation $\sum_{a\in\text{irreps.}}$ runs over Virasoro representations of the CFT realized in the boundary ∂D. The generator of the translation, $\widetilde{P}$, was normalized as

$$\widetilde{P}:=\frac{1}{v}(H-E_0)=\frac{2\pi}{L}\Big[L_0-\frac{c}{24}-\langle L_0-\frac{c}{24}\rangle\Big] \tag{4.5}$$

so that $\widetilde{P}\,|\text{vac}\rangle=0$ on the vacuum state $|\text{vac}\rangle$ of the CFT. Here, $E_0=\langle\text{vac}|H|\text{vac}\rangle$ is the vacuum energy.

We want to estimate (4.3) for a sufficiently large subdisc, $\xi/L\ll 1$. Using modular transformations,

$$\chi_a\left(i\ell-\frac{1}{n}\right)=S_{ab}\chi_b\left(-\frac{1}{i\ell-\frac{1}{n}}\right)=(ST^n)_{ab}\chi_b\left(-\frac{-ni\ell}{i\ell+\frac{1}{n}}\right)=(ST^nS)_{ab}\chi_b\left(\frac{i}{n^2\ell}+\frac{1}{n}\right), \tag{4.6}$$

(4.3) can be expressed as a low-temperature partition function as

$$\langle GS\,|\,C_{n,D}\,|\,GS\rangle=\frac{e^{\frac{2\pi i}{n}(\langle L_0\rangle-\frac{c}{24})}\sum_{a\in\text{irreps.}}\sum_b(ST^nS)_{ab}\chi_b(\frac{iL}{n^2\xi}+\frac{1}{n})}{\sum_{a\in\text{irreps.}}\sum_b S_{ab}\chi_b(\frac{iL}{\xi})} \tag{4.7}$$

where S and T are modular matrices. Notice that $\sum_b$ runs over all irreps. of the theory, which should be contrasted with $\sum_{a\in\text{irreps.}}$. This formula enables us to estimate the ground state expectation value of the partial rotation through the highest weight state $|h_b\rangle$,

$$\begin{aligned}\chi_b\left(\frac{iL}{n^2\xi}+\frac{1}{n}\right)&=\sum_{m=0}^{\infty}p(m)e^{2\pi i(\frac{iL}{n^2\xi}+\frac{1}{n})(h_b+m-\frac{c}{24})}\\&=e^{\frac{2\pi i}{n}(h_b-\frac{c}{24})}e^{-\frac{2\pi L}{n^2\xi}(h_b-\frac{c}{24})}\Big(1+\sum_{m\geq 1}^{\infty}p(m)e^{2\pi i(\frac{iL}{n^2\xi}+\frac{1}{n})m}\Big)\\&\sim e^{\frac{2\pi i}{n}(h_b-\frac{c}{24})}e^{-\frac{2\pi L}{n^2\xi}(h_b-\frac{c}{24})},\end{aligned} \tag{4.8}$$

where h_b is the conformal weight and $p(m)$ is the number of states with energy h_b+m. Subleading contributions are suppressed by the factor $e^{-\frac{2\pi L}{n^2\xi}}$. In the same way,

$$\chi_b\left(\frac{iL}{\xi}\right)=e^{-\frac{2\pi L}{\xi}(h_b-\frac{c}{24})}\Big(1+\sum_{m\geq 1}^{\infty}p(m)e^{-\frac{2\pi L}{n^2\xi}m}\Big)\sim e^{-\frac{2\pi L}{\xi}(h_b-\frac{c}{24})}. \tag{4.9}$$

The mapping torus $T^2\times_{ST^nS}S^1$ built from the modular transformation ST^nS is knows as the lens space $L(n,1)$.[90] Thus, the partial C_n rotation is related to the partition function on the lens space.

1. *Left mover chiral CFT*

For a left-mover chiral edge excitation with dispersion $\varepsilon(k)\sim -vk$ $(v>0)$, the momentum operator is changed to $P=-H/v$, which leads to the formula of the partial rotation for the left-mover chiral topological phases

$$\begin{aligned}\langle GS|C_{n,D}|GS\rangle&=\frac{\mathrm{Tr}\big[e^{i\widetilde{P}\frac{L}{n}}e^{-\frac{\xi}{v}H}\big]}{\mathrm{Tr}\big[e^{-\frac{\xi}{v}H}\big]}\\&=\frac{e^{\frac{2\pi i}{n}(\langle L_0\rangle-\frac{c}{24})}\sum_{a\in\text{irreps.}}\chi_a(\frac{i\xi}{L}+\frac{1}{n})}{\sum_{a\in\text{irreps.}}\chi_a(\frac{i\xi}{L})}\\&=\frac{e^{-\frac{2\pi i}{n}(\langle L_0\rangle-\frac{c}{24})}\sum_{a\in\text{irreps.}}\sum_b(ST^{-n}S)_{ab}\chi_b(\frac{iL}{n^2\xi}-\frac{1}{n})}{\sum_{a\in\text{irreps.}}\sum_b S_{ab}\chi_b(\frac{iL}{\xi})},\end{aligned} \tag{4.10}$$

This is the complex conjugate of Eq. (4.3). Thus, the non-chiral CFT without on-site symmetry cannot provide nontrivial $U(1)$ phases in partial rotations. However, if there is an on-site symmetry, we can associate the partial on-site transformation with the partial rotation, which offers a nontrivial $U(1)$ phase as shown in Sec. IV C.

B. $(2+1)d$ $(p_x - ip_y)$ chiral superconductor with rotation symmetry

For an application of partial rotations, let us consider a $(p_x - ip_y)$ superconductor:

$$H = \sum_{\boldsymbol{k}} \Big[\psi^\dagger(\boldsymbol{k})(\frac{k^2}{2m} - \mu)\psi(\boldsymbol{k}) + \frac{\Delta}{2}\psi^\dagger(\boldsymbol{k})(k_x - ik_y)\psi^\dagger(-\boldsymbol{k}) + \frac{\Delta}{2}\psi(-\boldsymbol{k})(k_x + ik_y)\psi(\boldsymbol{k})\Big] \tag{4.11}$$

where $\Delta(\boldsymbol{k}) = \Delta(k_x - ik_y)$ $(\Delta > 0)$ is the gap function and $\boldsymbol{k}$ is momentum. We introduce a polar coordinate $(x, y) = (r\cos\phi, r\sin\phi)$. This model has the following continuum rotation symmetry C_θ,

$$C_\theta \psi^\dagger(r,\phi) C_\theta^{-1} = e^{-\frac{i\theta}{2}}\psi^\dagger(r, \phi+\theta), \quad C_{2\pi} = (-1)^F. \tag{4.12}$$

Note that the 2π rotation is the fermion parity.

On the disk geometry, the system supports a right-moving gapless chiral real fermion mode $\gamma(\ell)$ localized at the boundary. The fermion mode $\gamma(\ell)$ can be constructed explicitly as

$$\gamma(\frac{L\phi}{2\pi}) \sim \Big[e^{\frac{i\phi}{2}+\frac{\pi i}{4}}\psi(r,\phi) + e^{-\frac{i\phi}{2}-\frac{\pi i}{4}}\psi^\dagger(r,\phi)\Big] e^{\int^r \frac{\mu(r')}{\Delta}dr'}, \tag{4.13}$$

up to a normalization constant, where the chemical potential $\mu(r)$ is chosen such that a finite disk geometry is realized, $\mu(r) > 0$ for $r < \frac{L}{2\pi}$ and $\mu(r) < 0$ for $r > \frac{L}{2\pi}$, and L is the circumference. (See Appendix D 1 for the derivation of Eq. (4.13).) $\gamma(\ell)$ obeys the real condition $\gamma^\dagger(\ell) = \gamma(\ell)$ and the antiperiodic boundary condition $\gamma(\ell + L) = -\gamma(\ell)$, which reflects the absence of exact zero energy states since there is no π-flux inside the disk D.[91] The Hamiltonian of the edge theory is given by

$$H_{ns} = \frac{2\pi\Delta}{L}\Big(\sum_{\substack{m>0 \\ m\in\mathbb{Z}+\frac{1}{2}}} m\gamma_{-m}\gamma_m - \frac{1}{48}\Big), \quad P = \frac{H}{\Delta}. \tag{4.14}$$

The free real chiral fermion CFT is characterized by the data[92]

$$c = \frac{1}{2}, \qquad (h_1, h_\psi, h_\sigma) = (0, \frac{1}{2}, \frac{1}{16}), \tag{4.15}$$

$$S = \frac{1}{2}\begin{pmatrix} 1 & 1 & \sqrt{2} \\ 1 & 1 & -\sqrt{2} \\ \sqrt{2} & \sqrt{2} & 0 \end{pmatrix}, \qquad T = e^{-\frac{\pi i}{24}}\begin{pmatrix} 1 & & \\ & -1 & \\ & & e^{\frac{\pi i}{8}} \end{pmatrix}, \tag{4.16}$$

where c is the chiral central charge, (h_1, h_ψ, h_σ) is the set of dimensions (topological spin) in the CFT (for the vacuum, fermion, and Ising spin sectors, respectively), and the modular S and T matrices are given in the basis $(1, \psi, \sigma)$. The Virasoro representations that appear in the NS sector is $[1] \oplus [\psi]$.

1. Partial C_n rotation

Let us consider partial $C_n := C_{\theta=\frac{2\pi}{n}}$ rotation. First, we need to specify the action of the C_θ rotation on the gapless edge excitation $\gamma(\ell)$, which can be read off from the concrete expression (4.13) as

$$C_\theta \gamma(\ell) C_\theta^{-1} = \gamma(\ell + \frac{\theta L}{2\pi}). \tag{4.17}$$

This is consistent with $C_{2\pi} = (-1)^F$ because of the anti-periodic boundary condition. From the formula (4.7), the expectation value of partial C_n rotation on the $(p_x - ip_y)$ superconductor is given by

$$\langle GS|C_{n,D}|GS\rangle = \frac{e^{-\frac{2\pi i}{n}\frac{1}{48}}\sum_{a=1,\psi}\sum_b (ST^nS)_{ab}\chi_b(\frac{iL}{n^2\xi} + \frac{1}{n})}{\sum_{a=1,\psi}\sum_b S_{ab}\chi_b(\frac{iL}{\xi})}. \tag{4.18}$$

From (4.15), the matrix elements $(ST^nS)_{ab}$ are given by

$$ST^nS = e^{-\frac{n}{24}\pi i}\cdot\frac{1}{4}\begin{pmatrix} 1+(-1)^n+2e^{\frac{n}{8}\pi i} & 1+(-1)^n+2e^{\frac{n}{8}\pi i} & \sqrt{2}-(-1)^n\sqrt{2} \\ 1+(-1)^n-2e^{\frac{n}{8}\pi i} & 1+(-1)^n+2e^{\frac{n}{8}\pi i} & \sqrt{2}-(-1)^n\sqrt{2} \\ \sqrt{2}+(-1)^n\sqrt{2} & \sqrt{2}+(-1)^n\sqrt{2} & 2-2(-1)^n \end{pmatrix}, \tag{4.19}$$

and hence we have

$$e^{\frac{2\pi i}{n}(h_1-\frac{1}{24})}\sum_{a=1,\psi}(ST^nS)_{a1}=\begin{cases} e^{-\frac{(n^2+2)\pi i}{24n}} & (n:\ \text{even})\\ 0 & (n:\ \text{odd})\end{cases} \tag{4.20}$$

$$e^{\frac{2\pi i}{n}(h_\psi-\frac{1}{24})}\sum_{a=1,\psi}(ST^nS)_{a\psi}=\begin{cases} 2\cos(\frac{n\pi}{16})e^{\frac{(n^2+44)\pi i}{48n}} & (n:\ \text{even})\\ e^{\frac{(n^2+11)\pi i}{12n}} & (n:\ \text{odd})\end{cases} \tag{4.21}$$

$$e^{\frac{2\pi i}{n}(h_\sigma-\frac{1}{24})}\sum_{a=1,\psi}(ST^nS)_{a\sigma}=\begin{cases} 0 & (n:\ \text{even})\\ \frac{1}{\sqrt{2}}e^{-\frac{(n^2-1)\pi i}{24n}} & (n:\ \text{odd})\end{cases} \tag{4.22}$$

To evaluate Eq. (4.18) to the leading order, we note

$$e^{-\frac{2\pi i}{n}\frac{1}{48}}\chi_b\left(\frac{iL}{n^2\xi}+\frac{1}{n}\right)\sim e^{\frac{2\pi i}{n}(h_b-\frac{1}{24})}e^{-\frac{2\pi L}{n^2\xi}(h_b-\frac{1}{48})}. \tag{4.23}$$

There is an even odd effect: The leading contributions come from the vacuum sector $b=1$ for even n, and the $b=\sigma$ quasiparticle for odd n, whereas the $b=\psi$ quasipaticle contributes to the next leading contribution. Summarizing, we have

$$\langle GS|C_{n,D}|GS\rangle\sim\begin{cases}\exp\left[-\frac{(n^2+2)\pi i}{24n}-(1-\frac{1}{n^2})\frac{1}{48}\frac{2\pi L}{\xi}\right] & (n:\ \text{even})\\ \exp\left[-\frac{(n^2-1)\pi i}{24n}-\ln\sqrt{2}-(1+\frac{2}{n^2})\frac{1}{48}\frac{2\pi L}{\xi}\right] & (n:\ \text{odd})\end{cases} \tag{4.24}$$

For odd n, in addition to the topological $U(1)$ phase, there is a topological amplitude $e^{-\ln\sqrt{2}}$, which is an analog of the topological entanglement entropy.[93,94] Here we show some examples of partial $C_{n,D}$ rotations:

$$\langle GS|C_{2,D}|GS\rangle\sim\exp\Big[-\frac{\pi i}{8}-\frac{3}{4}\frac{2\pi L}{\xi}\frac{1}{48}\Big], \tag{4.25}$$

$$\langle GS|C_{3,D}|GS\rangle\sim\exp\Big[-\frac{\pi i}{9}-\ln\sqrt{2}-\frac{11}{9}\frac{2\pi L}{\xi}\frac{1}{48}\Big], \tag{4.26}$$

$$\langle GS|C_{4,D}|GS\rangle\sim\exp\Big[-\frac{3\pi i}{16}-\frac{15}{16}\frac{2\pi L}{\xi}\frac{1}{48}\Big], \tag{4.27}$$

$$\langle GS|C_{5,D}|GS\rangle\sim\exp\Big[-\frac{\pi i}{5}-\ln\sqrt{2}-\frac{27}{25}\frac{2\pi L}{\xi}\frac{1}{48}\Big], \tag{4.28}$$

$$\langle GS|C_{6,D}|GS\rangle\sim\exp\Big[-\frac{19}{72}\pi i-\frac{35}{36}\frac{2\pi L}{\xi}\frac{1}{48}\Big], \tag{4.29}$$

$$\langle GS|C_{7,D}|GS\rangle\sim\exp\Big[-\frac{2\pi i}{7}-\ln\sqrt{2}-\frac{51}{49}\frac{2\pi L}{\xi}\frac{1}{48}\Big], \tag{4.30}$$

$$\langle GS|C_{8,D}|GS\rangle\sim\exp\Big[-\frac{11}{32}\pi i-\frac{63}{64}\frac{2\pi L}{\xi}\frac{1}{48}\Big], \tag{4.31}$$

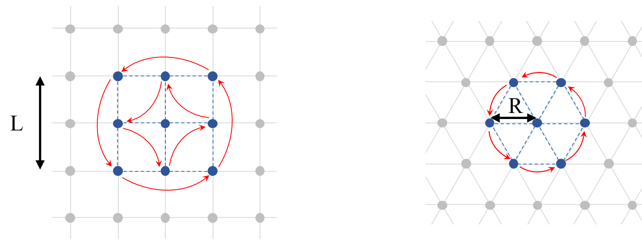
$$\langle GS|C_{9,D}|GS\rangle\sim\exp\Big[-\frac{10}{27}\pi i-\ln\sqrt{2}-\frac{83}{81}\frac{2\pi L}{\xi}\frac{1}{48}\Big]. \tag{4.32}$$

By including higher energy states in $\chi_b(\frac{iL}{n^2\xi}+\frac{1}{n})$, we can estimate the expectation value of partial rotation to higher orders in $\xi/L\ll 1$. For example, the expectation value of partial C_n rotations at next-to-leading order is given by

$$\begin{aligned}&\langle GS|C_{n,D}|GS\rangle\\ &\sim\begin{cases} e^{-\frac{(n^2+2)\pi i}{24n}}e^{-(1-\frac{1}{n^2})\frac{1}{48}\frac{2\pi L}{\xi}}+2\cos(\frac{n\pi}{16})e^{\frac{(n^2+44)\pi i}{48n}}e^{-\left[(1-\frac{1}{n^2})\frac{1}{48}+\frac{1}{2n^2}\right]\frac{2\pi L}{\xi}}, & (n:\ \text{even},\ n\neq 8(\text{mod } 16))\\ e^{-\frac{(n^2+2)\pi i}{24n}}e^{-(1-\frac{1}{n^2})\frac{1}{48}\frac{2\pi L}{\xi}}(1-e^{-\frac{1}{2}\frac{2\pi L}{\xi}}), & (n:\ \text{even},\ n=8(\text{mod } 16))\\ \frac{1}{\sqrt{2}}e^{-\frac{(n^2-1)\pi i}{24n}}e^{-(1+\frac{2}{n^2})\frac{1}{48}\frac{2\pi L}{\xi}}+e^{\frac{(n^2+11)\pi i}{12n}}e^{-\left[(1-\frac{1}{n^2})\frac{1}{48}+\frac{1}{2n^2}\right]\frac{2\pi L}{\xi}}, & (n:\ \text{odd})\end{cases}\end{aligned} \tag{4.33}$$

For example,

$$\langle GS|C_{4,D}|GS\rangle\sim e^{-\frac{3\pi i}{16}}e^{-\frac{15}{16}\frac{2\pi L}{\xi}\frac{1}{48}}(1+\sqrt{2}e^{\frac{\pi i}{2}}e^{-\frac{1}{32}\frac{2\pi L}{\xi}}). \tag{4.34}$$



(a) (b)

FIG. 6. (Color online) (a) Partial C_4 rotation on the square lattice, and (b) Partial C_6 rotation on the hexagonal lattice.

2. Partial fermion parity

In fermionic topological phases, the fermion parity symmetry is always preserved. Here, we consider the *partial fermion parity flip* $(-1)^F_D$ on the ground state of the $(p_x - ip_y)$ superconductor. The partial fermion parity flip was discussed in detection of quantum phases in some literature.[66,95] The bulk fermion parity transformation induces the edge fermion parity transformation through (4.13) as

$$(-1)^F \gamma(\ell)(-1)^F = -\gamma(\ell). \tag{4.35}$$

Then, the partial fermion parity flip on the disk D is given by

$$\begin{aligned}\langle GS|(-1)^F_D|GS\rangle &= \frac{\mathrm{Tr}\big[(-1)^F e^{-\frac{\xi}{v}H}\big]}{\mathrm{Tr}\big[e^{-\frac{\xi}{v}H}\big]} = \frac{\chi_1(\frac{i\xi}{L}) - \chi_\psi(\frac{i\xi}{L})}{\chi_1(\frac{i\xi}{L}) + \chi_\psi(\frac{i\xi}{L})} = \frac{\sum_b (S_{1b} - S_{\psi b})\chi_b(\frac{iL}{\xi})}{\sum_b (S_{1b} + S_{\psi b})\chi_b(\frac{iL}{\xi})} \\ &= \frac{\sqrt{2}\chi_\sigma(\frac{iL}{\xi})}{\chi_1(\frac{iL}{\xi}) + \chi_\psi(\frac{iL}{\xi})} \sim \exp\left[\ln\sqrt{2} - \frac{1}{16}\frac{2\pi\xi}{L}\right].\end{aligned} \tag{4.36}$$

We observe that there emerges a scale-independent, topological contribution to the amplitude, $e^{\ln\sqrt{2}}$. In topologically trivial phases there is no such topological amplitude associated with the partial fermion parity flip since the entanglement Hamiltonian is trivial. Hence, the existence of a finite topological amplitude in the partial fermion parity flip is a hallmark of topologically nontrivial phases.

3. Numerical results for lattice models

In this section, we provide numerical results for partial rotation using the lattice realizations of topological superconductors in class D. We should note that point group symmetries in two dimensional Bravais lattices are limited to four possible cases which are C_2, C_3, C_4, and C_6. Here, we study two models: $(p_x - ip_y)$ superconductors (4.11) on the square lattice and on the hexagonal lattice. The former can be furnished with C_2 or C_4 symmetry groups and the latter can have C_2, C_3, and C_6 symmetries. In the following, we verify that the partial rotation in these systems obeys the generic form of Eq. (1.11). In particular, we show that the complex phase θ and the area law coefficient α match those predicted in Eq. (4.24). We find the expectation value of the partial rotation by rearranging the position of lattice sites inside the subsystem as shown in Fig. 6 and then compute the inner product of the two wave functions $Z = \langle GS|\, C_{n,D}\, |GS\rangle$. One important fact here is that the subsystem must be invariant under C_n rotation, otherwise we cannot perform this procedure, since the full lattice after performing the partial C_n rotation will not be the same as the original one, if the subsystem is not C_n symmetric.

As the first model, we consider the tight-binding Hamiltonian

$$H = -\frac{t}{2}\sum_{\mathbf{x}}[\psi^\dagger_{\mathbf{x}+\mathbf{i}}\psi_{\mathbf{x}} + \psi^\dagger_{\mathbf{x}+\mathbf{j}}\psi_{\mathbf{x}} + h.c.] + \frac{1}{2}\sum_{\mathbf{x}}[i\Delta\psi^\dagger_{\mathbf{x}+\mathbf{i}}\psi^\dagger_{\mathbf{x}} + \Delta\psi^\dagger_{\mathbf{x}+\mathbf{j}}\psi^\dagger_{\mathbf{x}} + h.c.] - \mu\sum_{\mathbf{x}}\psi^\dagger_{\mathbf{x}}\psi_{\mathbf{x}} \tag{4.37}$$

on a square lattice with the basis vectors $\mathbf{i}$ and $\mathbf{j}$. For $-2t < \mu < 0$, this model describes a $(p_x + ip_y)$ superconductor where the long-wavelength theory around $\mathbf{k} = 0$ is given by Eq. (4.11). Figure 7(a) shows the complex phase of the partial C_2 and C_4 rotations as a function of μ. It is evident that $\angle Z = \mathrm{Im}\ln\langle GS|C_{n,D}|GS\rangle$ is zero in the trivial phase,

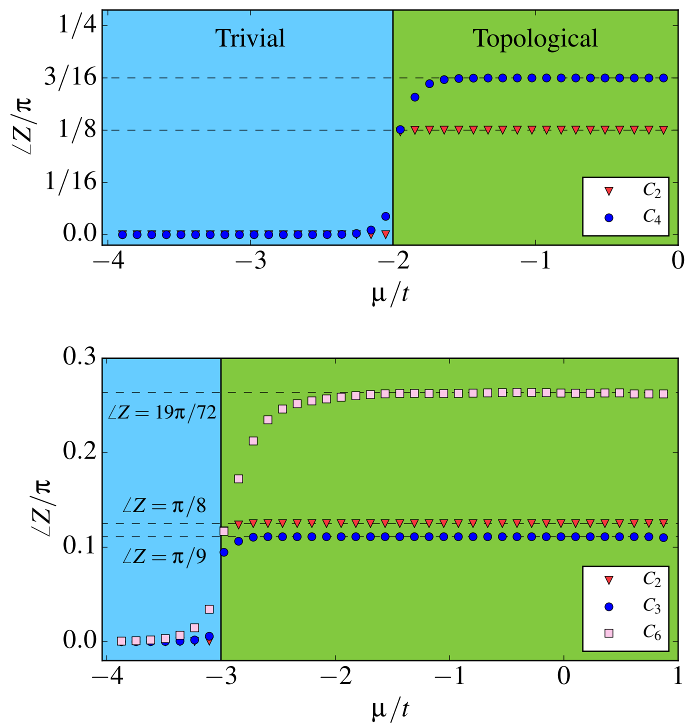


FIG. 7. (Color online) Complex phase $\angle Z = \mathrm{Im}\ln\langle GS|C_{n,D}|GS\rangle$ of the partial rotation over (a) square lattice, and (b) hexagonal lattice. We set $t = \Delta$. The total number of sites is $N = 32^2$ and the size of the transformed subsystem is $N_{\mathrm{part}} = 16^2$ for (a) and $N_{\mathrm{part}} = 217$ where $R = 8$ for (b).

Ratio	Eq. (4.24)	Numerics	Difference (%)
C_2/C_4	4/5	0.810	1.2
C_2/C_6	27/35	0.754	2.3
C_3/C_6	44/35	1.262	-0.4

TABLE II. Ratios of area-law coefficients in Eq. (4.24) for Hamiltonian (4.37). $t = \Delta$ and $\mu = -1$.

while it is quantized in the topological phase and the numerical values are in perfect agreement with the predicted values from Eq. (4.24).

In order to realize C_6 point group symmetry and its subgroups C_3 and C_2, we study the hexagonal lattice version of the $(p_x - ip_y)$ superconductor, given by the Hamiltonian

$$H = -\frac{t}{2}\sum_{\mathbf{x},m}[\psi^\dagger_{\mathbf{x}+\mathbf{e}_m}\psi_{\mathbf{x}} + h.c.] + \frac{1}{3}\sum_{\mathbf{x}}[i\Delta e^{i\theta_m}\psi^\dagger_{\mathbf{x}+\mathbf{e}_m}\psi^\dagger_{\mathbf{x}} + h.c.] - \mu\sum_{\mathbf{x}}\psi^\dagger_{\mathbf{x}}\psi_{\mathbf{x}} \tag{4.38}$$

where $m = 1,2,3$ denotes the nearest-neighbor lattice vectors $\mathbf{e}_1 = \mathbf{i}$, $\mathbf{e}_2 = \frac{1}{2}\mathbf{i} + \frac{\sqrt{3}}{2}\mathbf{j}$, and $\mathbf{e}_3 = -\frac{1}{2}\mathbf{i} + \frac{\sqrt{3}}{2}\mathbf{j}$. The phase θ_m is the angle between nearest-neighbor link and the horizontal axis, i.e. $\cos\theta_m = \mathbf{e}_m \cdot \mathbf{i}$. Notice that $\mathbf{e}_1 - \mathbf{e}_2 + \mathbf{e}_3 = 0$ which means only two vectors out of the three are independent and can be used to construct a basis for the lattice. Here, the chiral $(p_x - ip_y)$ superconductor is realized in the limit $-3t < \mu < t$ where the long wavelength theory is determined by expanding near $\mathbf{k} = 0$. The results are shown in Fig. 7(b) which conform with Eq. (4.24).

We also check the exponentiated area law behavior by looking at the amplitude of the partial rotation as a function of subsystem size. We show some typical results for the partial C_4 and C_6 rotations in Figs. 8(a) and (b), respectively. The linear behavior in terms of subsystem perimeter is evident. In addition, we look at the ratio of the area law coefficients given by Eq. (4.24) where the microscopic quantity ξ is cancelled and as a result we should get a model-independent (topological) value. Table II shows that the numerically calculated ratios are quite close to the values predicted by Eq. (4.24).

Let us now make our final remark in this section regarding the topological contribution γ in Eq. (1.11). A direct way to compute this quantity is by looking at the y-intercept of the area law plots ($\ln Z$ versus L). However, we

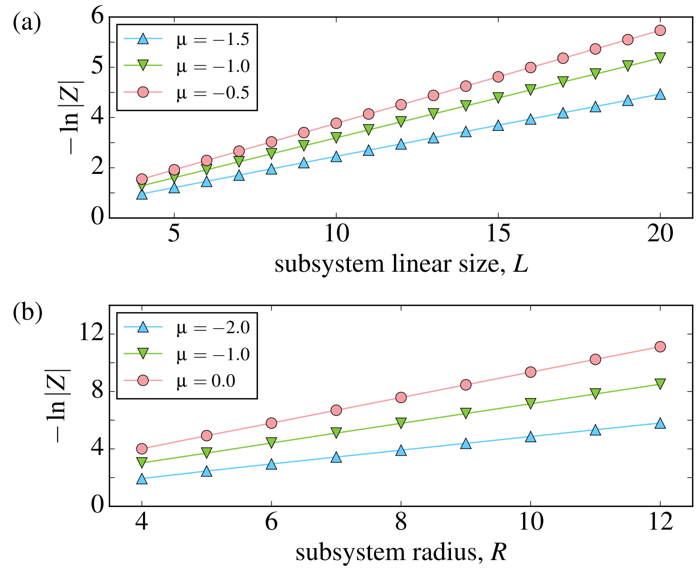


FIG. 8. (Color online) Amplitude $|Z| = |\langle GS|C_{n,D}|GS\rangle|$ of (a) partial C_4 rotation on the square lattice (4.37), and (b) partial C_6 rotation on the hexagonal lattice (4.38) for various values of the chemical potential μ. Here, we set $t = \Delta$. The total number of sites is $N = 32^2$ and the dimension of the subsystem is given in terms of L and R (see Fig. 6) in each case. Solid lines are linear fits to data points.

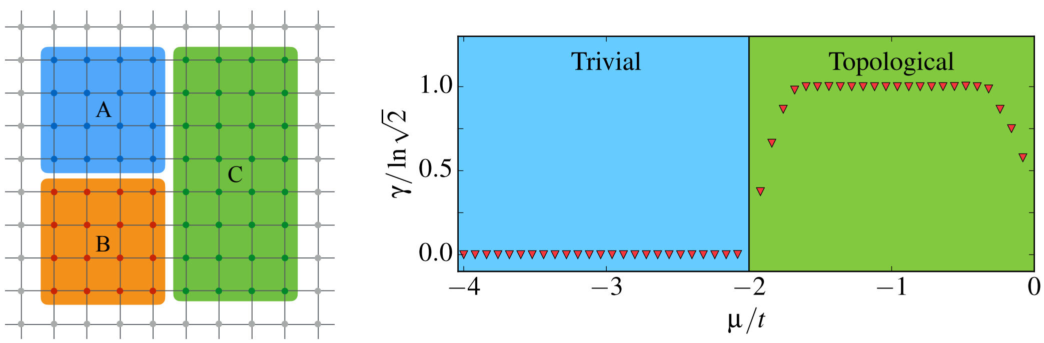


FIG. 9. (Color online) (a) The scheme to extract topological contribution γ in the partial fermion parity transformation (4.39). (b) Numerical results for the $(p_x - ip_y)$ superconductor (Eq. (4.37)) as a function of the chemical potential μ. We set $t = \Delta$. Here, the total system size is $N = 40^2$ and the size of subsystem A is $N_A = 16^2$.

should note that the expression (1.11) is derived for subsystems with smooth edges and any partitioning of a lattice inevitably results in sharp corners. Therefore, the y-intercept method is likely to fail on lattice models due to extra contributions from the corners which add to γ and make the evaluation of γ imprecise. Fortunately, there is a scheme[93,96] to overcome this issue and remove the corner terms by writing the γ in the following way

$$\gamma = \ln|Z_A| + \ln|Z_B| + \ln|Z_C| - \ln|Z_{AB}| - \ln|Z_{AC}| - \ln|Z_{BC}| + \ln|Z_{ABC}| \tag{4.39}$$

where A, B, and C are three corner sharing subsystems as shown in Fig. 9(a). We use this scheme to extract the topological contribution $\ln\sqrt{2}$ in the partial fermion parity flip as given in Eq. (4.36). We compute γ over a wide range of μ and the results are shown in Fig. 9. Away from the critical point in the topological phase, γ is quantized to $\ln\sqrt{2}$. The same procedure can be done for the hexagonal lattice which yields the same results (we do not show them here). It is worth noting that the above scheme cannot be used for partial C_n rotation since, as mentioned earlier, in order to compute the partial rotation all subsystems must be C_n-invariant and this is not the case for some segments such as C and BC.

C. $(2+1)d$ nonchiral superconductors with C_2 rotation symmetry ($\Omega_3^{\mathrm{Spin}}(B\mathbb{Z}_2)$)

Fermionic SPT phases (which are non-chiral) with on-site $\mathbb{Z}_2$ symmetry are classified by the spin cobordism group $\Omega_3^{\mathrm{Spin}}(B\mathbb{Z}_2) = \mathbb{Z}_8$ where $B\mathbb{Z}_2$ is the classifying space of $\mathbb{Z}_2$ group.[11,97,98] The generating manifold of the $\Omega_3^{\mathrm{Spin}}(B\mathbb{Z}_2)$ group is $3d$ real projective space $\mathbb{R}P^3$ with an appropriate background $\mathbb{Z}_2$ gauge field. A generating model Hamiltonian is given by the $(p_x+ip_y)_\uparrow \oplus (p_x-ip_y)_\downarrow$ superconductor

$$\begin{aligned}
&H = H_\uparrow + H_\downarrow, \\
&H_{\uparrow/\downarrow} = \sum_{\boldsymbol{k}} \psi^\dagger_{\uparrow/\downarrow}(\boldsymbol{k})(\frac{k^2}{2m}-\mu)\psi_{\uparrow/\downarrow}(\boldsymbol{k}) + \sum_{\boldsymbol{k}} \big[\frac{\Delta}{2}\psi^\dagger_{\uparrow/\downarrow}(\boldsymbol{k})(k_x \pm i k_y)\psi^\dagger_{\uparrow/\downarrow}(-\boldsymbol{k}) + h.c.\big],
\end{aligned} \tag{4.40}$$

with the on-site $\mathbb{Z}_2$ flavor symmetry

$$U\psi^\dagger_{\uparrow/\downarrow}(\boldsymbol{x})U^{-1} = \mp\psi^\dagger_{\uparrow/\downarrow}(\boldsymbol{x}). \tag{4.41}$$

That is to say, the fermion parity symmetry for each spin up and down fermions is separately preserved.

To introduce a cross-cap to have $\mathbb{R}P^3$ as the spacetime, we need to consider partial C_2 rotation in the canonical formalism (i.e., in terms of a given ground state wave function). However, there is no C_2 rotation symmetry a priori in this model. Thus, instead of imposing the non-spatial $\mathbb{Z}_2$ symmetry, we consider C_2 symmetry in advance. The breakdown of the $\mathbb{Z}$ classification of $(2+1)d$ class D non-chiral superconductors with C_2 rotation symmetry[99] is also given by $\mathbb{Z}_8$[100], which is natural from the point of view of TQFTs since C_2 rotation symmetry is an orientation preserving symmetry. The translation from the non-spatial $\mathbb{Z}_2$ symmetry to the C_2 rotation symmetry in the same topological class is as follows: We introduce a combined rotation symmetry $\tilde{C}_2 = UC_2$ from the on-site $\mathbb{Z}_2$ symmetry U and the *inherent* continuum rotation symmetry for chiral $(p_x \pm ip_y)$ superconductors $C_\theta\psi^\dagger_{\uparrow/\downarrow}(\boldsymbol{x})C_\theta^{-1} = e^{\pm i\theta/2}\psi^\dagger_{\uparrow/\downarrow}(C_\theta\boldsymbol{x})$ introduced in (4.12), where $C_2 = C_\pi$. Then, the $\tilde{C}_2$ rotation is defined as

$$\tilde{C}_2\psi^\dagger_{\uparrow/\downarrow}(\boldsymbol{x})\tilde{C}_2^{-1} = -i\psi^\dagger_{\uparrow/\downarrow}(-\boldsymbol{x}). \tag{4.42}$$

Under this $\tilde{C}_2$ rotation symmetry, the model Hamiltonian (4.40) is the generating model of the $\mathbb{Z}_8$ group. Finally, we can forget the on-site $\mathbb{Z}_2$ symmetry U and the continuum rotation symmetry C_θ: the $\mathbb{Z}_8$ classification is ensured only by the $\tilde{C}_2$ rotation symmetry.

Let us evaluate the ground state expectation value of partial $\widetilde{C}_2$ rotation. The contribution from the $(p_x - ip_y)_\downarrow$ sector is the same as (4.25). On the other hand, for the $(p_x+ip_y)_\uparrow$ sector, since the gapless edge excitation has left-moving chirality, the expectation value of partial $\widetilde{C}_2$ rotation is given by

$$\begin{aligned}
\langle GS_\uparrow|\widetilde{C}_{2,D}|GS_\uparrow\rangle &= \frac{\mathrm{Tr}\big[(-1)^F e^{i\widetilde{P}\frac{L}{2}} e^{-\frac{\xi}{v}H}\big]}{\mathrm{Tr}\big[e^{-\frac{\xi}{v}H}\big]} \\
&= \frac{e^{\pi i \frac{1}{48}}\sum_b \big\{(ST^{-2}S)_{1b} - (ST^{-2}S)_{\psi b}\big\}\chi_b(\frac{iL}{4\xi}-\frac{1}{2})}{\sum_{a=1,\psi}\sum_b S_{ab}\chi_b(\frac{iL}{\xi})} \\
&\sim e^{-\frac{\pi i}{8}} e^{-\frac{3}{4}\frac{2\pi L}{\xi}\frac{1}{48}}.
\end{aligned} \tag{4.43}$$

Therefore, the total expectation value is

$$\langle GS|\widetilde{C}_{2,D}|GS\rangle = \langle GS_\uparrow|\widetilde{C}_{2,D}|GS_\uparrow\rangle\,\langle GS_\downarrow|\widetilde{C}_{2,D}|GS_\downarrow\rangle \sim e^{-\frac{\pi i}{4}} e^{-\frac{3}{2}\frac{2\pi L}{\xi}\frac{1}{48}}. \tag{4.44}$$

Thus, the $U(1)$ phase in the expectation value of partial $\widetilde{C}_2$ rotation does capture the $\mathbb{Z}_8$ classification.

D. $(2+1)d$ topological insulators with rotation symmetry

In this section, we discuss partial rotations in $(2+1)d$ Chern insulators. In Chern insulators (particle number conserving systems), partial rotation can be combined with continuous $U(1)$ phase rotation, which should be contrasted with the case of superconductors where a partial rotation can be combined with the discrete $\mathbb{Z}_2$ fermion parity transformation. To be specific, let us consider the following simple model realizing a chiral Chern insulator

$$\begin{aligned} H &= \sum_{\boldsymbol{k}} \psi^\dagger(\boldsymbol{k})\Big[(\frac{k^2}{2m}-\mu)\sigma_z + vk_x\sigma_x + vk_y\sigma_y\Big]\psi(\boldsymbol{k}) \\ &= \sum_{\boldsymbol{k}} \Big[\psi^\dagger(\boldsymbol{k})(\frac{k^2}{2m}-\mu)\sigma_z\psi(\boldsymbol{k}) + v(k_x-ik_y)\psi_1^\dagger(\boldsymbol{k})\psi_2(\boldsymbol{k}) + h.c.\Big], \quad \psi(\boldsymbol{k}) = (\psi_1(\boldsymbol{k}),\psi_2(\boldsymbol{k}))^T. \end{aligned} \tag{4.45}$$

This model is invariant under the continuous spatial rotation and the $U(1)$ charge rotation

$$C_\theta\psi^\dagger(\boldsymbol{x})C_\theta = \psi^\dagger(C_\theta\boldsymbol{x})\begin{pmatrix} e^{-i\theta/2} & 0 \\ 0 & e^{i\theta/2}\end{pmatrix}, \qquad U_b\psi^\dagger(\boldsymbol{x})U_b^{-1} = e^{-2\pi i b}\psi^\dagger(\boldsymbol{x}), \quad b\in\mathbb{R}/\mathbb{Z}. \tag{4.46}$$

On the disk geometry, the chiral Chern insulator supports a chiral (right-moving) gapless excitations localized on the boundary, which can be created by the following complex fermion operator $\gamma(\ell)$

$$\gamma^\dagger(\frac{L\phi}{2\pi}) \sim (e^{-i\phi/2}\psi_1^\dagger(r,\phi) + ie^{i\phi/2}\psi_2^\dagger(r,\phi))e^{-\int^r dr' m(r')}, \tag{4.47}$$

where $\ell = L\phi/2\pi$ is the spatial coordinate along the boundary, and L is the circumference of the boundary. We have chosen the gauge of $\gamma(\ell)$ such that $\gamma(\ell)$ satisfies the anti-periodic boundary condition $\gamma(\ell+L) = -\gamma(\ell)$. With this boundary condition, the Hamiltonian and momentum operator for the edge mode can be written as

$$H = \frac{2\pi v}{L}\sum_{m\in\mathbb{Z}+1/2} m : \gamma_m^\dagger\gamma_m : -\frac{1}{24}, \qquad P = \frac{H}{v}, \tag{4.48}$$

where γ_m is the m-th Fourier mode of $\gamma(\ell)$. Here, the Hamiltonian and momentum operator are normal ordered with respect to the Fermi sea which is filled with $\gamma_m^\dagger(m<0)$ fermions. A derivation of (4.47) and (4.48) are summarized in Appendix D.

1. *Partial rotation with $U(1)$ charge transformation*

We now calculate the expectation value of the partial C_θ rotation together with the partial $U(1)$ charge transformation. The partial rotation $C_{\theta,D}$ on the disk D combined with the partial $U(1)$ transformation $U_{b,D}$ acts on the boundary fermion operators as

$$U_bC_\theta\gamma^\dagger(\ell)(U_bC_\theta)^{-1} = e^{-2\pi i b}\gamma^\dagger(\ell+\frac{\theta L}{2\pi}). \tag{4.49}$$

These transformations are generated by the momentum and the $U(1)$ charge defined by

$$\widetilde{P} = \frac{2\pi}{L}\sum_{m\in\mathbb{Z}+1/2} m : \gamma_n^\dagger\gamma_m :, \qquad \widetilde{Q} = \sum_{m\in\mathbb{Z}+1/2} : \gamma_m^\dagger\gamma_m : . \tag{4.50}$$

Here we regularized $\widetilde{P}$ and $\widetilde{Q}$ so that $\widetilde{P}|FS\rangle = \widetilde{Q}|FS\rangle = 0$ where $|FS\rangle$ is the Fermi sea of the edge theory. The expectation value of $U_{b,D}C_{\theta,D}$ is given by

$$\langle GS|U_{b,D}C_{\theta,D}|GS\rangle \sim \frac{\mathrm{Tr}_{a=\frac{1}{2}}\left[e^{-2\pi i\widetilde{Q}b}e^{-i\widetilde{P}\frac{\theta L}{2\pi}}e^{-\frac{\xi}{v}H}\right]}{\mathrm{Tr}_{a=\frac{1}{2}}\left[e^{-\frac{\xi}{L}H}\right]} = \frac{e^{-\frac{i\theta}{24}}Z_{\frac{1}{2},b+\frac{1}{2}}(\frac{i\xi}{L}-\frac{\theta}{2\pi})}{Z_{\frac{1}{2},\frac{1}{2}}(\frac{i\xi}{L})}. \tag{4.51}$$

Here, $Z_{a,b}(\tau)$ is the partition function of the right-mover complex fermion theory defined on the spacetime torus with twisted boundary conditions

$$Z_{a,b}(\tau) = \mathrm{Tr}_a\left[e^{-2\pi i(\widetilde{Q}+a-\frac{1}{2})(b-\frac{1}{2})}e^{2\pi i\tau(L_0-\frac{1}{24})}\right] = \frac{\theta_{a-\frac{1}{2},\frac{1}{2}-b}(0|\tau)}{\eta(\tau)}, \tag{4.52}$$

where $\theta_{a,b}(z|\tau)$ and $\eta(\tau)$ is the generalized theta function and the Dedekind eta function, respectively.[101]

	$p=0$	$p=1$	$p=2$	$p=3$	$p=4$	$p=5$	$p=6$	$p=7$	$p=8$
$n=2$	$-\frac{\pi}{4}$	$\frac{\pi}{4}$							
$n=3$	$-\frac{2\pi}{9}$	$\frac{\pi}{9}$	$\frac{\pi}{9}$						
$n=4$	$-\frac{3\pi}{8}$	$-\frac{\pi}{8}$	$\frac{5\pi}{8}$	$-\frac{\pi}{8}$					
$n=5$	$-\frac{2\pi}{5}$	$-\frac{\pi}{5}$	$\frac{2\pi}{5}$	$\frac{2\pi}{5}$	$-\frac{\pi}{5}$				
$n=6$	$-\frac{19\pi}{36}$	$-\frac{13\pi}{36}$	$\frac{5\pi}{36}$	$\frac{35\pi}{36}$	$\frac{5\pi}{36}$	$-\frac{13\pi}{36}$			
$n=7$	$-\frac{4\pi}{7}$	$-\frac{3\pi}{7}$	0	$\frac{5\pi}{7}$	$\frac{5\pi}{7}$	0	$-\frac{3\pi}{7}$		
$n=8$	$-\frac{11\pi}{16}$	$-\frac{9\pi}{16}$	$-\frac{3\pi}{16}$	$\frac{7\pi}{16}$	$\frac{21\pi}{16}$	$\frac{7\pi}{16}$	$-\frac{3\pi}{16}$	$-\frac{9\pi}{16}$	
$n=9$	$-\frac{20\pi}{27}$	$-\frac{17\pi}{27}$	$-\frac{8\pi}{27}$	$\frac{7\pi}{27}$	$\frac{28\pi}{27}$	$\frac{28\pi}{27}$	$\frac{7\pi}{27}$	$-\frac{8\pi}{27}$	$-\frac{17\pi}{27}$

TABLE III. The $U(1)$ phases of the partial $U_{\frac{p}{n}}C_n$ rotation (4.58) of the topological insulator with rotational symmetry defined in (4.45) for $n=2,\dots,9$ and $p=0,\dots,8$.

We now specialize to partial n-fold rotation $C_n := C_{\frac{2\pi}{n}}$ combined with with the n-fold $U(1)$ transformation $U_{b=\frac{p}{n}}$ ($p=0,\dots,n-1$). By noting the a,b dependence and the modular transformation of $Z_{a,b}(\tau)$,

$$
\begin{aligned}
&Z_{a+1,b}(\tau) = Z_{a,b}(\tau), && Z_{a,b+1}(\tau) = e^{-2\pi i(a-\frac{1}{2})}Z_{a,b}(\tau),\\
&Z_{a,b}(\tau+1) = e^{-\pi i a^2+\frac{\pi i}{6}}Z_{a,b-a}(\tau), && Z_{a,b}(-\frac{1}{\tau}) = e^{-2\pi i(a-\frac{1}{2})(b-\frac{1}{2})}Z_{1-b,a}(\tau),
\end{aligned}
\tag{4.53}
$$

one can show

$$
\langle GS|U_{\frac{p}{n},D}C_{n,D}|GS\rangle = \frac{e^{-\frac{i\pi}{12n}}Z_{\frac{1}{2},\frac{p}{n}+\frac{1}{2}}(\frac{i\xi}{L}-\frac{1}{n})}{Z_{\frac{1}{2},\frac{1}{2}}(\frac{i\xi}{L})} = \frac{e^{-\frac{i\pi}{12n}-\frac{ni\pi}{12}+\frac{p^2\pi i}{n}}Z_{\frac{1}{2}+\frac{n}{2},\frac{1}{2}-\frac{p}{n}}(\frac{iL}{n^2\xi}+\frac{1}{n})}{Z_{\frac{1}{2},\frac{1}{2}}(\frac{iL}{\xi})}
\tag{4.54}
$$

by the same modular transformation as (4.6). When the circumference of the disk is sufficiently larger than the bulk correlation length, $L\gg\xi$, we can approximate (4.54) by taking lowest energy states. The denominator is approximated as

$$
Z_{\frac{1}{2},\frac{1}{2}}\left(\frac{iL}{\xi}\right) \sim e^{\frac{2\pi L}{\xi}\frac{1}{24}}.
\tag{4.55}
$$

As for the numerator, when $n=$ even, the unique vacuum state $|0\rangle_{\frac{1}{2}}$ gives the leading contribution

$$
Z_{\frac{1}{2}+\frac{n}{2},\frac{1}{2}-\frac{p}{n}}\left(\frac{iL}{n^2\xi}+\frac{1}{n}\right) = Z_{\frac{1}{2},\frac{1}{2}-\frac{p}{n}}\left(\frac{iL}{n^2\xi}+\frac{1}{n}\right) \sim e^{-\frac{\pi i}{12n}}e^{\frac{2\pi L}{n^2\xi}\frac{1}{24}} \qquad (n:\text{even}).
\tag{4.56}
$$

On the other hand, when n is odd, the ground state associated with the torus partition function has double degeneracy originated from the zero mode. We have

$$
\begin{aligned}
Z_{\frac{1}{2}+\frac{n}{2},\frac{1}{2}-\frac{p}{n}}\left(\frac{iL}{n^2\xi}+\frac{1}{n}\right) &= Z_{0,\frac{1}{2}-\frac{p}{n}}\left(\frac{iL}{n^2\xi}+\frac{1}{n}\right)\\
&\sim (e^{\pi ip/n}+e^{-\pi ip/n})e^{2\pi i(\frac{iL}{n^2\xi}+\frac{1}{n})(\frac{1}{8}-\frac{1}{24})} = 2\cos\frac{\pi p}{n}e^{\frac{\pi i}{6n}}e^{-\frac{\pi L}{6n^2\xi}} \qquad (n:\text{odd}).
\end{aligned}
\tag{4.57}
$$

Combining these contributions,

$$
\langle GS|U_{\frac{p}{n},D}C_{n,D}|GS\rangle \sim \begin{cases} e^{\frac{12p^2-n^2-2}{12n}\pi i}e^{-(1-\frac{1}{n^2})\frac{2\pi L}{\xi}\frac{1}{24}} & (n:\text{even})\\ 2\cos\frac{\pi p}{n}e^{\frac{12p^2-n^2+1}{12n}\pi i}e^{-(1+\frac{2}{n^2})\frac{2\pi L}{\xi}\frac{1}{24}} & (n:\text{odd})\end{cases}
\tag{4.58}
$$

The $U(1)$ phases in (4.58) for $n=2,\dots,9$ and $p=0,\dots,8$ are summarized in Table III. We have confirmed that all these results in the above table match with numerical calculations in two dimensional lattice models for $n=2,3,4$, and 6.

The expectation value of the partial $U_{\frac{p}{n}}C_n$ rotation is related to the appropriate TQFT partition function on $L(n,1)$. In the same way, the partial $(U_{\frac{p}{n}}C_n)^m$ rotation with n and m being coprime is related to the the lens space $L(n,m)$.

	$p=0$	$p=1$	$p=2$	$p=3$	$p=4$	$p=5$	$p=6$	$p=7$	$p=8$
$n=3$	$\frac{14\pi}{9}$	$\frac{2\pi}{9}$	$\frac{2\pi}{9}$						
$n=5$	0	$\frac{2\pi}{5}$	$\frac{8\pi}{5}$	$\frac{8\pi}{5}$	$\frac{2\pi}{5}$				
$n=7$	$\frac{10\pi}{7}$	$\frac{12\pi}{7}$	$\frac{4\pi}{7}$	0	0	$\frac{4\pi}{7}$	$\frac{12\pi}{7}$		
$n=9$	$\frac{50\pi}{27}$	$\frac{2\pi}{27}$	$\frac{20\pi}{27}$	$\frac{50\pi}{27}$	$\frac{38\pi}{27}$	$\frac{38\pi}{27}$	$\frac{50\pi}{27}$	$\frac{20\pi}{27}$	$\frac{2\pi}{27}$

TABLE IV. The $U(1)$ phases of the partial $(U_{\frac{p}{n}}C_n)^2$ rotation (4.59) of topological insulator with rotational symmetry defined in (4.45) for $n=3,5,7,9$ and $p=0,\dots,8$.

The expectation value of the partial $(U_{\frac{p}{n}}C_n)^m$ can be evaluated by using an appropriate modular transformation, which is determined by the fraction expansion of $\frac{n}{m}$ (see Appendix B.) For instance, for odd $n=2k+1$, the partial $(U_{\frac{p}{2k+1}}C_{2k+1})^2$ rotation is computed by using the modular transformation $ST^kST^{-2}S$ as

$$\begin{aligned}\langle GS|(U_{\frac{p}{2k+1},D}C_{2k+1,D})^2|GS\rangle &= \frac{e^{-\frac{i\pi}{6(2k+1)}}Z_{\frac{1}{2},\frac{2p}{2k+1}+\frac{1}{2}}(\frac{i\xi}{L}-\frac{2}{2k+1})}{Z_{\frac{1}{2},\frac{1}{2}}(\frac{i\xi}{L})}\\ &= \frac{e^{-\frac{i\pi k\left(12k^2+k(8-48p)+48p^2-24p-3\right)}{24k+12}}Z_{\frac{1}{2},\frac{1}{2}+\frac{k}{2}+\frac{p}{2k+1}}(\frac{iL}{(2k+1)^2\xi}-\frac{k}{2k+1})}{Z_{\frac{1}{2},\frac{1}{2}}(\frac{iL}{\xi})}\\ &\sim e^{-\frac{i\pi k\left(3k^2+k(2-12p)+12p^2-6p-1\right)}{6k+3}}e^{-(1-\frac{1}{(2k+1)^2})\frac{2\pi L}{\xi}\frac{1}{24}}.\end{aligned} \tag{4.59}$$

The $U(1)$ phases in (4.59) for $n=3,5,7,9$ and $p=0,\dots,8$ are summarized in Table IV. Notice that the boundary condition of the space direction is still anti-periodic after the modular transformation in (4.59), which implies that the partition function is approximated by the single vacuum state as in (4.56). For even $n=2k$, the partial $(U_{\frac{p}{2k}}C_{2k})^2$ transformation is the same as the partial $U_{\frac{p}{k}}C_k$ transformation computed in (4.57).

2. $(2+1)d$ non-chiral topological insulators with rotation symmetry ($\Omega_3^{\mathrm{Spin}^c}(B\mathbb{Z}_n)$)

The calculations similar to the above results can be deployed to discuss a classification of SPT phases with C_n rotation symmetry in non-chiral topological insulators: We first construct a model Hamiltonian and the C_n symmetry in a similar way to Sec. IV C; we then introduce the combined $\widetilde{C}_n$ symmetry from the model Hamiltonian with $\mathbb{Z}_n$ on-site symmetry and continuum rotation symmetry. The topological classification of non-chiral Chern insulators with $\mathbb{Z}_n$ symmetry is given by the Spin^c bordsim group $\Omega_3^{\mathrm{Spin}^c}(B\mathbb{Z}_n)$. The cobordism groups $\Omega_3^{\mathrm{Spin}^c}(B\mathbb{Z}_n)$ and their generating manifolds are derived by Bahri and Gilkey[57,102] by use of the η-invariant of Dirac operators with a Spin^c structure. In Appendix E, we briefly summarize their results and a relation to the equivariant K-theory classification at the free fermion level.

The third column in Table V shows the cobordism group $\Omega_3^{\mathrm{Spin}^c}(B\mathbb{Z}_n)$ for $n=2,\dots,9$. $(1-t)$ and $(1-t^2)$ represent generating model Hamiltonians as follows: We consider a 4 flavor non-chiral Chern insulator

$$\begin{aligned}H_{1-t^p} &= \sum_{\boldsymbol{k}}\psi_p^\dagger(\boldsymbol{k})\Big[(\frac{k^2}{2m}-\mu)\sigma_z+vk_x\sigma_x+vk_y\sigma_y\Big]\psi_p(\boldsymbol{k})\\ &\quad+\sum_{\boldsymbol{k}}\psi_0^\dagger(\boldsymbol{k})\Big[(\frac{k^2}{2m}-\mu)\sigma_z+vk_x\sigma_x-vk_y\sigma_y\Big]\psi_0(\boldsymbol{k}),\end{aligned} \tag{4.60}$$

$$\psi_s(\boldsymbol{k})=(\psi_{s,1}(\boldsymbol{k}),\psi_{s,2}(\boldsymbol{k}))^T,\quad s=p,0,\quad p\geq 1. \tag{4.61}$$

Here, subscripts of $\psi_s(\boldsymbol{x}), s\in\{0,\dots n-1\}$ mean 1-dimensional representations of the $\mathbb{Z}_n$ symmetry as $U\psi_s(\boldsymbol{x})U^{-1}=e^{-\frac{2\pi is}{n}}\psi_s(\boldsymbol{x})$. As shown in Table V, the generating models of $\Omega_3^{\mathrm{Spin}^c}(B\mathbb{Z}_n)$ for $n\geq 3$ are given by H_{1-t} and H_{1-t^2} which generate independent cyclic groups. In addition to the $\mathbb{Z}_n$ symmetry, the Hamiltonian (4.60) has the continuum rotation symmetry

$$C_\theta\psi_p^\dagger(\boldsymbol{x})C_\theta^{-1}=\psi_p^\dagger(C_\theta\boldsymbol{x})e^{-i\theta\sigma_z/2},\quad C_\theta\psi_0^\dagger(\boldsymbol{x})C_\theta^{-1}=\psi_0^\dagger(C_\theta\boldsymbol{x})e^{i\theta\sigma_z/2}. \tag{4.62}$$

	Partial $\widetilde{C}_n$ transformation		The partial $(\widetilde{C}_n)^2$ transformation		Cobordism
	H_{1-t}	H_{1-t^2}	H_{1-t}	H_{1-t^2}	$\Omega_3^{\mathrm{Spin}^c}(B\mathbb{Z}_n)$
$n=2$	$\frac{\pi}{2}$	0	0	0	$\mathbb{Z}_4[1-t]$
$n=3$	$\frac{\pi}{3}$	$\frac{\pi}{3}$	$\frac{2\pi}{3}$	$\frac{2\pi}{3}$	$\mathbb{Z}_3[1-t]\oplus\mathbb{Z}_3[1-t^2]$
$n=4$	$\frac{\pi}{4}$	π	$\frac{\pi}{2}$	0	$\mathbb{Z}_8[1-t]\oplus\mathbb{Z}_2[1-t^2]$
$n=5$	$\frac{\pi}{5}$	$\frac{4\pi}{5}$	$\frac{2\pi}{5}$	$\frac{8\pi}{5}$	$\mathbb{Z}_5[1-t]\oplus\mathbb{Z}_5[1-t^2]$
$n=6$	$\frac{\pi}{6}$	$\frac{2\pi}{3}$	$\frac{\pi}{3}$	$\frac{\pi}{3}$	$\mathbb{Z}_{12}[1-t]\oplus\mathbb{Z}_3[1-t^2]$
$n=7$	$\frac{\pi}{7}$	$\frac{4\pi}{7}$	$\frac{2\pi}{7}$	$\frac{8\pi}{7}$	$\mathbb{Z}_7[1-t]\oplus\mathbb{Z}_7[1-t^2]$
$n=8$	$\frac{\pi}{8}$	$\frac{\pi}{2}$	$\frac{\pi}{4}$	π	$\mathbb{Z}_{16}[1-t]\oplus\mathbb{Z}_4[1-t^2]$
$n=9$	$\frac{\pi}{9}$	$\frac{4\pi}{9}$	$\frac{2\pi}{9}$	$\frac{8\pi}{9}$	$\mathbb{Z}_9[1-t]\oplus\mathbb{Z}_9[1-t^2]$

TABLE V. The 2nd and 3rd (4th and 5th) columns show the $U(1)$ complex phases of the expectation value of the partial $\widetilde{C}_n$ ($(\widetilde{C}_n)^2$) rotation $\langle GS|\widetilde{C}_n|GS\rangle$ ($\langle GS|(\widetilde{C}_n)^2|GS\rangle$) on the Hamiltonian (4.60) with $p=1$ and 2, respectively. The rightmost column show the Spin^c cobordsim group classification of SPT phases with on-site $\mathbb{Z}_n$ symmetry. The notation $\mathbb{Z}_q[1-t^p]$ means that the Spin^c cobordism group $\Omega_3^{\mathrm{Spin}^c}(B\mathbb{Z}_n)$ consists of $\mathbb{Z}_q$ groups generated by the Hamiltonian H_{1-t^p} defined in (4.60).

We introduce the combined n-fold rotation symmetry by $\widetilde{C}_n := UC_n$ where $C_n = C_{\theta=\frac{2\pi}{n}}$.

Let us evaluate the expectation value of the partial $\widetilde{C}_n$ rotation with respect to the ground state of (4.60). The expectation value is expected to simulate the path integral (the partition function) on lens spaces with background Spin^c structures with a $\mathbb{Z}_n$ gauge field. The contribution from the ψ_p fermion, $\langle GS_p|\widetilde{C}_{n,D}|GS_p\rangle$, is the same as the previous section. On the other hand, for the ψ_0 fermion sector, since the gapless edge excitation has left-moving chirality, the expectation value of the partial C_n rotation is given by the complex conjugate of (4.54),

$$\langle GS_0|\widetilde{C}_{n,D}|GS_0\rangle = \frac{\mathrm{Tr}_{a=\frac{1}{2}}\left[e^{i\widetilde{P}\frac{L}{n}}e^{-\frac{\xi}{v}H}\right]}{\mathrm{Tr}_{a=\frac{1}{2}}\left[e^{-\frac{\xi}{v}H}\right]} = \langle GS|C_{n,D}|GS\rangle^* . \tag{4.63}$$

In the same way, the expectation value of the partial $(\widetilde{C}_n)^2$ rotation with respect to the ground state (4.60) is computed. The total $U(1)$ phases of the expectation value $\langle GS|\widetilde{C}_{n,D}|GS\rangle$ and $\langle GS|(\widetilde{C}_{n,D})^2|GS\rangle$ are summarized in Table V. To compare with SPT phases with on-site $\mathbb{Z}_n$ symmetry, we list the Spin^c cobordisms in the rightmost column of Table V.

It is interesting to note that, from Table V, the partial $(\widetilde{C}_n)^2$ rotation provides less information than the partial $\widetilde{C}_n$ rotation. Except for $n=2,6$, the $U(1)$ phase of the partial $(\widetilde{C}_n)^2$ rotation is twice that of the partial $\widetilde{C}_n$ rotation. As an example, let us focus on the C_3 rotation symmetry. The Spin^c cobordism with $\mathbb{Z}_3$ on-site symmetry, $\Omega^{\mathrm{Spin}^c}(B\mathbb{Z}_3)=\mathbb{Z}_3\oplus\mathbb{Z}_3$, suggests the existence of two inequivalent/independent SPT phases which are generated by Hamiltonians (4.60) with $p=1$ and 2, respectively. In general, in order to confirm that the classification of SPT phases is given by two or more Abelian groups (e.g., $\mathbb{Z}_p\oplus\mathbb{Z}_q$), multiple many-body invariants are needed. (I.e., multiple many-body invariants are required to distinguish all possible SPT phases.) However, the partial $\widetilde{C}_3$ and $(\widetilde{C}_3)^2$ rotations give rise to the same $U(1)$ phases for ground states of both Hamiltonians H_{1-t} and H_{1-t^2}, which implies that the partial $\widetilde{C}_3$ and $(\widetilde{C}_3)^2$ rotations cannot distinguish these two series of SPT phases. The ground state expectation value of the *full* $\widetilde{C}_3$ rotation[103] is a candidate for the manybody topological invariant to differentiate these two groups of SPT phases.

More generically, for even n, the expectation value of the partial $\widetilde{C}_n$ rotations gives rise to the $U(1)$ phases which are consistent with the cobordism classification of fermionic SPT phases protected by the on-site $\mathbb{Z}_n$ symmetry. On the other hand, for odd n, there is a mismatch between partial $\widetilde{C}_n$ rotations and cobordism groups: The partial $\widetilde{C}_n$ rotation gives at least one $\mathbb{Z}_{2n}$ SPT phase whereas $\Omega_3^{\mathrm{Spin}^c}(B\mathbb{Z}_n)=\mathbb{Z}_n\oplus\mathbb{Z}_n$. This mismatch suggests that either one of the following statements can be true: (i) The SPT phase protected by C_n rotation symmetry for odd n includes a $\mathbb{Z}_{2n}$ phase, which differs from the SPT phase protected by *on-site* $\mathbb{Z}_n$ symmetry. I.e., the statement/conjecture made in the end of Sec. I C is not correct in this case. (ii) The $\mathbb{Z}_{2n}$ $U(1)$ phases associated with the partial $\widetilde{C}_n$ rotation for odd n is not stable; that is, it reduces to the $\mathbb{Z}_n$ $U(1)$ phase under perturbations and/or disorder. We wish to clarify this point in a future work.

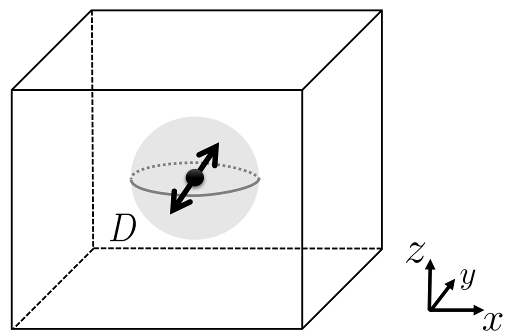


FIG. 10. Partial inversion on a ground state on 3d space torus T^3. The partial inversion transformation is performed only on inside of the 3-ball D (the shadow region).

E. $(2+1)d$ nonchiral superconductors with C_n rotation symmetry ($n \geq 3$)

Here, we briefly comment on how to construct n-fold rotation symmetry in non-chiral superconductors for $n \geq 3$. To this end, let us first consider how to realize an on-site $\mathbb{Z}_n$ symmetry in real fermion systems. Since a single-component real fermion field does not have the $U(1)$ phase degree of freedom, the $\mathbb{Z}_n$ symmetry on the one-component real fermion cannot be introduced except for $n = 2$. To define $\mathbb{Z}_n$ symmetry on real fermions, it is necessary to introduce a complex fermion operator $\phi(\boldsymbol{x})$ consisting of two real fermion operators $\chi(\boldsymbol{x}), \eta(\boldsymbol{x})$ as

$$\phi^\dagger(\boldsymbol{x}) = \chi(\boldsymbol{x}) + i\eta(\boldsymbol{x}), \quad \phi(\boldsymbol{x}) = \chi(\boldsymbol{x}) - i\eta(\boldsymbol{x}). \tag{4.64}$$

The on-site $\mathbb{Z}_n$ symmetry is now defined such that the complex fermion operator transforms as a 1-dimensional representation of $\mathbb{Z}_n$, $U\phi^\dagger(\boldsymbol{x})U^{-1} = e^{-2\pi i p/n}\phi^\dagger(\boldsymbol{x})$, $p = 0, \dots, n-1$, which is equivalent to in terms of the real fermions,

$$U\chi(\boldsymbol{x})U^{-1} = \cos\frac{2\pi p}{n}\chi(\boldsymbol{x}) + \sin\frac{2\pi p}{n}\eta(\boldsymbol{x}), \quad U\eta(\boldsymbol{x})U^{-1} = -\sin\frac{2\pi p}{n}\chi(\boldsymbol{x}) + \cos\frac{2\pi p}{n}\eta(\boldsymbol{x}). \tag{4.65}$$

As the combined transformation $\tilde{C}_2$ in Sec. IV D 2, the on-site $\mathbb{Z}_n$ phase rotation can be combined with C_n rotation to define $\widetilde{C}_n$ rotation. The computation of the ground state expectation value of the partial $\widetilde{C}_n$ rotation is recast into the calculation presented in Sec. IV D 2.

V. PARTIAL INVERSIONS

There are SPT phases in $(3+1)d$, which are protected by orientation-reversing symmetry, and the generating manifold of the relevant cobordism group is the $4d$ real projective space, $\mathbb{R}P^4$. For example, $(3+1)d$ topological superconductors with inversion/reflection symmetry, which are the CPT dual of class DIII time-reversal symmetric topological superconductors, are classified by the Pin$^+$ cobordism group, $\Omega_4^{\mathrm{Pin}^+}(pt) = \mathbb{Z}_{16}$. The abelian group $\mathbb{Z}_{16}$ is generated by $\mathbb{R}P^4$.[11,71] In this section, given a ground state wave function and symmetry actions, we aim at directly computing the many-body topological invariant associated to $\mathbb{R}P^4$. $\mathbb{R}P^4$ is not a mapping torus, and hence we need to employ a partial symmetry operation similar to partial reflection introduced in Sec. III. Topologically, $\mathbb{R}P^4$ is realized by inserting a cross-cap in S^4. The path-integral on $\mathbb{R}P^4$ is expected to be simulated by considering an expectation value of the partial inversion operator I_D defined for a subregion D, which is a three-ball of the total system (Fig. 10).

Taking $(3+1)d$ topological superconductors and insulators as an example, we will demonstrate below that the expectation value of the partial inversion correctly reproduces the known topological classification. We will evaluate the expectation value of partial inversion both numerically and analytically. For our analytical calculations, we will again make use of the cut and glue construction:[79] We assume that the reduced density matrix for the 3-ball D, obtained by taking the partial trace $\mathrm{Tr}_{\bar{D}}$ for the complimentary region $\bar{D} = T^3 \backslash D$ of the pure state $|GS\rangle\langle GS|$, is given approximately by the canonical thermal density matrix of a gapless theory (CFT) realized on the boundary

$S^2=\partial D$. The fictitious temperature is determined by the correlation length ξ of the bulk. Namely,

$$\rho_D = \mathrm{Tr}_{T^3\setminus D}\big[\,|GS\rangle\langle GS|\,\big] \sim \frac{e^{-\frac{\xi}{v}H_{S^2}}}{\mathrm{Tr}\big[e^{-\frac{\xi}{v}H_{S^2}}\big]}, \tag{5.1}$$

where v is a velocity of gapless theory on S^2. The gapless theory is defined on the spacetime manifold $S^2(R)\times S^1(\xi/v)$ where R is the radius of the 3-ball D. We assume, for simplicity, that the Hamiltonian H_{S^2} is rotation symmetric, and exclude the possibility of surface topological order.[24] The expectation value of the partial inversion is given in terms of the gapless surface theory as the expectation value of an antipodal map I_{S^2} on S^2:

$$\langle GS|I_D|GS\rangle \sim \frac{\mathrm{Tr}\big[I_{S^2}e^{-\frac{\xi}{v}H_{S^2}}\big]}{\mathrm{Tr}\big[e^{-\frac{\xi}{v}H_{S^2}}\big]}, \qquad I_{S^2}:(\theta,\phi)\mapsto(\pi-\theta,\phi+\pi), \tag{5.2}$$

where (θ,ϕ) is the polar coordinates of S^2. We will be interested in the behavior of (5.2) for sufficiently large R, $R\gg\xi$.

In this section, we deal with two examples of free theories: class D superconductors with inversion symmetry and class A insulators with inversion symmetry. It will turn out that the surface CFT calculations of partial inversions indeed provide $\mathbb{Z}_{16}$ and $\mathbb{Z}_8$ topological invariants.[10,11,14,15,100,104] We also show numerical calculations for lattice models, in which the results are consistent with the calculation of the surface CFTs. A generalization to higher spacetime dimensions will be discussed at last.

A. $(3+1)d$ superconductors with inversion symmetry ($\Omega_4^{\mathrm{Pin}^+}(pt)=\mathbb{Z}_{16}$)

Let us consider $(3+1)d$ topological superconductors protected by inversion symmetry I with $I^2=(-1)^F$. The topological classification is given by the Pin^+ cobordism group, $\Omega_4^{\mathrm{Pin}^+}(pt)=\mathbb{Z}_{16}$. Notice that the π rotation C_π of the real fermions are associated with $\pm i$ phase as shown in (4.12), which implies that the inversion transformation $I=C_\pi R$ with $I^2=(-1)^F$ in 3-space dimensions is equivalent to the reflection transformation with $R^2=1$, that is, the Pin^+ structure. The generating manifold is $\mathbb{R}P^4$.[11,71,100] A convenient model Hamiltonian, which describes the ^{3}He-B phase, is given by

$$H=\sum_{\boldsymbol{k}}\Psi^\dagger(\boldsymbol{k})\Big[\big(\frac{k^2}{2m}-\mu\big)\tau_z+\Delta\tau_x\boldsymbol{\sigma}\cdot\boldsymbol{k}\Big]\Psi(\boldsymbol{k}), \quad \Psi(\boldsymbol{k})=(\psi_\uparrow(\boldsymbol{k}),\psi_\downarrow(\boldsymbol{k}),\psi_\downarrow^\dagger(-\boldsymbol{k}),-\psi_\uparrow^\dagger(-\boldsymbol{k}))^T. \tag{5.3}$$

The model is invariant under inversion defined by

$$I\psi_\sigma^\dagger(\boldsymbol{x})I^{-1}=i\psi_\sigma^\dagger(-\boldsymbol{x}), \quad (\sigma=\uparrow,\downarrow). \tag{5.4}$$

To compute the expectation value of partial inversion using the cut and glue construction, we first look for the effective surface theory on the boundary of the 3-ball.[105] We consider the Hamiltonian (5.3) on the open 3-ball with radius R. We introduce a polar coordinate $(x,y,z)=(r\sin\theta\cos\phi, r\sin\theta\sin\phi, r\cos\theta)$. Instead of specifying a boundary condition, we consider the following Jackiw-Rebbi type domain wall one-particle Hamiltonian

$$\mathcal{H}=-i\Delta\tau_x(\sigma_x\partial_x+\sigma_y\partial_y+\sigma_z\partial_z)+\mu(r)\tau_z. \tag{5.5}$$

with $\mu(r)<0$ for $r<R$ and $\mu(r)>0$ for $r>R$. From a straightforward calculation (Appendix D 2 b), we obtain the explicit form of the complex fermion operators $\gamma^\dagger(\theta,\phi)$ creating gapless surface excitations

$$\begin{aligned}\gamma^\dagger(\theta,\phi)&\sim\Big[-e^{-i\frac{\phi}{2}}\sin\frac{\theta}{2}\big\{i\psi_\uparrow^\dagger(r,\theta,\phi)+\psi_\downarrow(r,\theta,\phi)\big\}+e^{i\frac{\phi}{2}}\cos\frac{\theta}{2}\big\{i\psi_\downarrow^\dagger(r,\theta,\phi)-\psi_\uparrow(r,\theta,\phi)\big\}\Big]e^{-\int^r\frac{\mu(r')}{\Delta}dr'},\\ \gamma(\theta,\phi)&\sim\Big[-e^{-i\frac{\phi}{2}}\cos\frac{\theta}{2}\big\{\psi_\uparrow^\dagger(r,\theta,\phi)+i\psi_\downarrow(r,\theta,\phi)\big\}-e^{i\frac{\phi}{2}}\sin\frac{\theta}{2}\big\{\psi_\downarrow^\dagger(r,\theta,\phi)-i\psi_\uparrow(r,\theta,\phi)\big\}\Big]e^{-\int^r\frac{\mu(r')}{\Delta}dr'}.\end{aligned} \tag{5.6}$$

Notice that the anti-periodic boundary condition in the ϕ direction, $\gamma^\dagger(\theta,\phi+2\pi)=-\gamma^\dagger(\theta,\phi)$, is satisfied. In terms of these fermion operators, the effective surface BdG Hamiltonian is given by

$$\begin{aligned}H_{S^2}&=\int\sin\theta d\theta d\phi\big(\gamma^\dagger(\theta,\phi),-\gamma(\theta,\phi)\big)\mathcal{H}\begin{pmatrix}\gamma(\theta,\phi)\\-\gamma^\dagger(\theta,\phi)\end{pmatrix},\\ \mathcal{H}&=\frac{\Delta}{R}\begin{pmatrix}0 & -i\partial_\theta-\frac{1}{\sin\theta}\partial_\phi-\frac{i\cot\theta}{2}\\ -i\partial_\theta+\frac{1}{\sin\theta}\partial_\phi-\frac{i\cot\theta}{2} & 0\end{pmatrix}.\end{aligned} \tag{5.7}$$

The one-particle Hamiltonian $\mathcal{H}$ is solved by using the monopole harmonics $Y^g_{l,m}(\theta,\phi)$ with $g=\pm\frac{1}{2}$ (see, for example, Ref. 106):

$$\mathcal{H}\Psi_{\pm n,m}(\theta,\phi)=\pm\frac{n\Delta}{R}\Psi_{\pm n,m}(\theta,\phi),\qquad \Psi_{\pm n,m}(\theta,\phi)=\frac{1}{\sqrt{2}}\begin{pmatrix}Y^{-\frac{1}{2}}_{n-\frac{1}{2},m}(\theta,\phi)\\ \mp iY^{\frac{1}{2}}_{n-\frac{1}{2},m}(\theta,\phi)\end{pmatrix},$$
$$n=1,2,\dots,\qquad m=-(n-\frac{1}{2}),\dots,n-\frac{3}{2},n-\frac{1}{2}. \tag{5.8}$$

There is no zero mode, which is consistent with the absence of a monopole inside of the 3-ball D. The degeneracy of the states with the eigenvalue $\pm\frac{n\Delta}{R}$ is $2|n|$. The explicit form of the monopole harmonics is[106]

$$\begin{aligned}Y^g_{l,m}(\theta,\phi)&=2^m\sqrt{\frac{(2l+1)(l-m)!(l+m)!}{4\pi(l-g)!(l+g)!}}(\sin\frac{\theta}{2})^{-(m+g)}(\cos\frac{\theta}{2})^{-(m-g)}P^{(-m-g,-m+g)}_{l+m}(\cos\theta)e^{im\phi}\\&=(-1)^{l+m}\sqrt{\frac{(2l+1)(l-m)!(l+m)!}{4\pi(l-g)!(l+g)!}}e^{im\phi}\sum_n(-1)^n\begin{pmatrix}l-g\\n\end{pmatrix}\begin{pmatrix}l+g\\g-m+n\end{pmatrix}(\sin\frac{\theta}{2})^{2l-2n-g+m}(\cos\frac{\theta}{2})^{2n+g-m}\end{aligned} \tag{5.9}$$

where $P^{\alpha,\beta}_n(x)$ is the Jacobi polynominals and sum $\sum_n$ runs over all possible integers. The periodicities of eigenstates agree with anti-periodic boundary condition of $\gamma(\theta,\phi)$, $\Psi_{\pm n,m}(\theta,\phi+2\pi)=-\Psi_{\pm n,m}(\theta,\phi)$. In terms of these eigen functions, we introduce the Bogoliubov operators $\chi^\dagger_{n,m}$ for the positive energy states by

$$\chi^\dagger_{n,m}:=\int\sin\theta d\theta d\phi\Big[\frac{1}{\sqrt{2}}Y^{-\frac{1}{2}}_{n-\frac{1}{2},m}(\theta,\phi)\gamma^\dagger(\theta,\phi)+i\frac{1}{\sqrt{2}}Y^{\frac{1}{2}}_{n-\frac{1}{2},m}(\theta,\phi)\gamma(\theta,\phi)\Big],\quad (n>0). \tag{5.10}$$

In terms of the Bogoliubov operators, the Hamiltonian H_{S^2} can be written as

$$H_{S^2}=\frac{\Delta}{R}\sum_{n\in\mathbb{Z},n>0}\ \sum_{m=-(n-\frac{1}{2}),\cdots,n-\frac{3}{2},n-\frac{1}{2}}n\chi^\dagger_{n,m}\chi_{n,m}. \tag{5.11}$$

1. *Partial inversion*

To compute the partial inversion on the surface theory (5.11), we first derive the antipodal transformation I_{S^2} on the Bogoliubov operators $\chi_{n,m}$. It is induced by the inversion transformation in the bulk (5.4) through (5.6), (5.7), and (5.10) as

$$I_{S^2}\chi^\dagger_{n,m}I^{-1}_{S^2}=i(-1)^n\chi^\dagger_{n,m},\quad I_{S^2}\chi_{n,m}I^{-1}_{S^2}=-i(-1)^n\chi_{n,m},\quad (n>0). \tag{5.12}$$

Here, we have used

$$\begin{aligned}&I\gamma^\dagger(\theta,\phi)I^{-1}=-i\gamma(\pi-\theta,\phi+\pi), && I\gamma(\theta,\phi)I^{-1}=i\gamma^\dagger(\pi-\theta,\phi+\pi),\\&Y^{\frac{1}{2}}_{n-\frac{1}{2},m}(\pi-\theta,\phi+\pi)=(-1)^niY^{-\frac{1}{2}}_{n-\frac{1}{2},m}(\theta,\phi), && Y^{-\frac{1}{2}}_{n-\frac{1}{2},m}(\pi-\theta,\phi+\pi)=(-1)^niY^{\frac{1}{2}}_{n-\frac{1}{2},m}(\theta,\phi).\end{aligned} \tag{5.13}$$

As explained around (5.2), the expectation value of the partial inversion is given by that of the antipodal map within the surface theory:

$$\langle GS|I_D|GS\rangle\sim\frac{\mathrm{Tr}\big[I_{S^2}e^{-\frac{\xi}{\Delta}H_{S^2}}\big]}{\mathrm{Tr}\big[e^{-\frac{\xi}{\Delta}H_{S^2}}\big]}=\frac{\prod_{n=1}^\infty(1+i(-q)^n)^{2n}}{\prod_{n=1}^\infty(1+q^n)^{2n}},\quad q=e^{-\frac{\xi}{R}}. \tag{5.14}$$

Here we normalized the antipodal transformation I_{S^2} so that $I_{S^2}|0_\chi\rangle=|0_\chi\rangle$ where $|0_\chi\rangle$ is the Fock vacuum of $\chi_{n,m}$ fermions.

Equation (5.14) can be evaluated in the same way as in Ref. 107. Here we briefly sketch the method. What we want to compute are

$$I_1(q)=\sum_{n=1}^\infty n\ln(1+q^n),\quad I_2(q)=\sum_{n=1}^\infty n\ln(1+i(-q)^n) \tag{5.15}$$

for $\delta \ll 1$ with $q = e^{-\delta}$. In terms of $I_{1,2}(q)$, the partial inversion is given by

$$\langle GS|I_D|GS\rangle \sim \exp\Big[2I_2(e^{-\frac{\xi}{R}}) - 2I_1(e^{-\frac{\xi}{R}})\Big]. \tag{5.16}$$

By using the Cahen-Mellin integral

$$e^{-y} = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} y^{-s}\Gamma(s)ds \tag{5.17}$$

for $c > 0$, $\mathrm{Re}(y) > 0$ and y^{-s} on the principal branch ($\Gamma(s)$ is the Gamma function), $I_1(q)$ is written as

$$\begin{aligned}
I_1(q = e^{-\delta}) &= -\sum_n n \sum_{r=1}^{\infty} r^{-1}(-1)^r e^{-\delta n r} \\
&= -\sum_n n \sum_{r=1}^{\infty} r^{-1}(-1)^r \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} ds(\delta n r)^{-s}\Gamma(s) \quad (c \gg 0) \\
&= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} ds\delta^{-s}\Big[\Gamma(s)\zeta(s-1)(1-2^{-s})\zeta(s+1)\Big],
\end{aligned} \tag{5.18}$$

where c is sufficiently far to the right. $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ is the Riemann zeta function. From the contour integral, one finds the contributions from the poles of the integrand $F(s) = \Gamma(s)\zeta(s-1)(1-2^{-s})\zeta(s+1)$ as

$$I_1(q = e^{-\delta}) = \sum_{s\in\text{poles}} \frac{\mathrm{Res}(F, s)}{\delta^s}. \tag{5.19}$$

(There is no multiple pole in this case.) To estimate this for small $\delta > 0$, it is sufficient to include only poles with $\mathrm{Re}(s) \geq 0$. Furthermore, the pole at $s = 0$ is scale independent, i.e., it is a topological contribution. There are two simple poles at $s = 0$ and $s = 2$. (Recall that $\zeta(s)$ has a single pole at $s = 1$ with residue 1, $\Gamma(s)$ has single poles at integers $n \leq 0$ with residue $\frac{(-1)^n}{n!}$, and $\zeta(s)$ has zeros at negative even integers. One of the poles at $s = 0$ is canceled by zero of $(1 - 2^{-s})$.) One can show

$$I_1(q = e^{-\delta}) = \frac{3}{4}\zeta(3)\delta^{-2} - \frac{1}{12}\ln(2) + \frac{\delta^2}{960} + \cdots. \tag{5.20}$$

Similarly, we obtain (see Appendix F)

$$\begin{aligned}
I_2(q = e^{-\delta}) &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} ds\delta^{-s}\Big[-2^{1-s}\Gamma(s)\zeta(s-1)\mathrm{Li}_{s+1}(-i) - (1-2^{1-s})\Gamma(s)\zeta(s-1)\mathrm{Li}_{s+1}(i)\Big] \\
&= \frac{3}{32}\zeta(3)\delta^{-2} - \frac{\pi i}{16} - \frac{1}{24}\ln(2) + \frac{\delta^2}{480} + \cdots.
\end{aligned} \tag{5.21}$$

Here, $\mathrm{Li}_s(z) := \sum_{n=1}^{\infty} \frac{z^n}{n^s}$ is the polylogarithm function. Let us consider the imaginary part of $I_2(q)$,

$$\mathrm{Im}\big[I_2(q = e^{-\delta})\big] = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} ds\delta^{-s}\Gamma(s)\zeta(s-1)(2^{2-s}-1)\beta(s+1), \tag{5.22}$$

where $\beta(s) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^s}$ is the Dirichlet beta function. There is a dramatic cancellation: Since $\beta(s+1)$ has zeros at even negative integers $s = -2, -4, -6, \ldots$, all the poles from $\Gamma(s)$ are canceled with zeros! Moreover, the zero from $(2^{2-s} - 1)$ at $s = 2$ is canceled with the pole from $\zeta(s-1)$. Eventually, there remains only one pole at $s = 0$, which implies an exactly scale-independent value

$$\mathrm{Im}\big[I_2(q = e^{-\delta})\big] = -\frac{\pi i}{16}. \tag{5.23}$$

Finally, we get the formula of partial inversion

$$\langle GS|I_D|GS\rangle = |\langle GS|I_D|GS\rangle| e^{i\theta_{\text{top}}}, \quad \theta_{\text{top}} = -\frac{\pi}{8}, \tag{5.24}$$

$$|\langle GS|I_D|GS\rangle| \sim \exp\Big[\frac{1}{12}\ln(2) - \frac{21}{16}\zeta(3)\left(\frac{R}{\xi}\right)^2\Big]. \tag{5.25}$$

The topological $U(1)$ phase $e^{-\frac{\pi i}{8}}$ is indeed consistent with the cobordism classification $\Omega_4^{\mathrm{Pin}^+}(pt) = \mathbb{Z}_{16}$. Also, observe that in addition to the topological $U(1)$ phase, a topological amplitude $e^{\frac{1}{12}\ln(2)}$ appears.

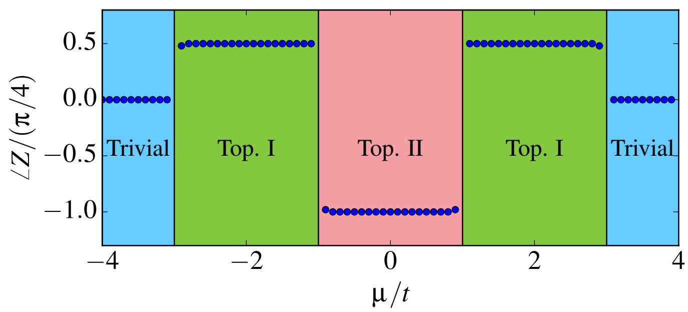


FIG. 11. (Color online) Complex phase of the partial inversion $\angle Z = \mathrm{Im} \ln \langle GS|I_D|GS\rangle$ computed for 3D inversion symmetric topological superconductor (class D). Top. I (II) corresponds to the phase with odd (even) number of gapless Majorana surface states. Here, we set $t = \Delta$. The size of total system and subsystem are $N = 12^3$ and $N_{\mathrm{part}} = 6^3$, respectively.

2. Numerical results for lattice systems

In this section, we provide a direct numerical evidence for the partial inversion of the three-dimensional lattice models. A generating model in class D is given by the BdG Hamiltonian

$$\hat{H} = \frac{1}{2}\sum_{\mathbf{k}} \Psi^\dagger(\mathbf{k}) h(\mathbf{k}) \Psi(\mathbf{k}), \tag{5.26}$$

on a cubic lattice, where

$$h(\mathbf{k}) = \left[-t(\cos k_x + \cos k_y + \cos k_z) - \mu\right]\tau_z + \Delta\left[\sin k_x \tau_x \sigma_x + \sin k_y \tau_x \sigma_y + \sin k_z \tau_x \sigma_z\right]. \tag{5.27}$$

in which the τ and σ matrices act on particle-hole and spin subspaces, respectively. As mentioned earlier, the above Hamiltonian also describes the ^{3}He-B phase. The inversion symmetry in this model is defined as in Eq. (5.4). This model exhibits three different topological phases depending on the chemical μ potential as follows:

1. $|\mu| < t$: Top. II. This phase supports an even number of 2d gapless Majorana surface states. It is topologically equivalent to a stack of 2d topological superconductors in the same symmetry class.

2. $t < |\mu| < 3t$: Top. I. This phase hosts a 2d gapless Majorana surface states.

3. $|\mu| > 3t$: Trivial. No topological surface states.

Figure 11 shows the calculated complex phases $\angle Z = \mathrm{Im} \ln \langle GS|I_D|GS\rangle$ of the partial inversion for various values of μ. This quantity is computed in a similar fashion to the two-dimensional case that is to calculate the inner product $Z = \langle GS|\, I_D \,|GS\rangle$ after rearranging the lattice sites in the subsystem (to get $I_D\,|GS\rangle$). Remarkably, the partial inversion gives the correct $\mathbb{Z}_{16}$ and $\mathbb{Z}_8$ phases in the topological phases characterized by odd and even number of gapless Majorana surface modes, respectively. We should note that the latter case is topologically equivalent to stacking two dimensional reflection symmetric class D SPT layers which obey a $\mathbb{Z}_8$ classification.

3. Other partial symmetry operations

In addition to partial inversion, we now discuss three additional partial symmetry operations: partial fermion parity, partial reflection, and partial π-rotation. In the partial fermion parity flip and the partial reflection, we have a logarithmic term $\log(R/\xi)$ as will be shown in Eq. (5.31). The partial π-rotation gives rise to a scale-independent contribution (5.39).

a. Partial fermion parity Following our discussion on $(2+1)d$ topological superconductors in Sec. IV B 2, let us consider the partial fermion parity flip on $(3+1)d$ topological superconductors. Since the partial fermion parity flip induces the fermion parity flip on the surface fermions as $(-1)^F \chi_{n,m} (-1)^F = -\chi_{n,m}$, we have

$$\langle GS|(-1)^F_D|GS\rangle \sim \frac{\mathrm{Tr}\left[(-1)^F e^{-\frac{\xi}{\Delta}H_{S^2}}\right]}{\mathrm{Tr}\left[e^{-\frac{\xi}{\Delta}H_{S^2}}\right]} = \frac{\prod_{n=1}^\infty (1-q^n)^{2n}}{\prod_{n=1}^\infty (1+q^n)^{2n}} = \exp\left[2I_3(q=e^{-\frac{\xi}{R}}) - 2I_1(q=e^{-\frac{\xi}{R}})\right] \tag{5.28}$$

with

$$I_3(q=e^{-\delta}) = \sum_{n=1}^{\infty} n\ln(1-q^n) = -\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} ds\delta^{-s}\Big[\Gamma(s)\zeta(s-1)\zeta(s+1)\Big]. \tag{5.29}$$

The integrand has a double pole at $s=0$, which leads to an algebraic power law for $\delta = \frac{\xi}{R} \sim 0$,

$$I_3(q=e^{-\delta}) \sim -\zeta(3)\delta^{-2} - \frac{1}{12}\ln(\delta) - \frac{1}{12} + \ln(A), \tag{5.30}$$

where $A \cong 1.2824\dots$ is the Glaisher-Kinkelin constant. Finally, we obtain the following formula for the expectation value of the partial fermion parity in the $(3+1)d$ topological superconductor

$$\langle GS|(-1)^F_D|GS\rangle \sim \exp\Big[-\frac{1}{6} + \frac{1}{6}\ln(2) + 2\ln(A) + \frac{1}{6}\ln\left(\frac{R}{\xi}\right) - \frac{7}{4}\zeta(3)\left(\frac{R}{\xi}\right)^2\Big]. \tag{5.31}$$

It should be noted that the logarithmic term appears in addition to the area law term. A similar logarithmic contribution to the the entanglement entropy in gapped phases is discussed in Ref. 108, where the authors pointed out that the non-flatness of the curvature of the boundary of a region D is a necessary condition to give a constant part of the entanglement entropy.

b. Partial reflection The topological classification of $(3+1)d$ superconductors with reflection symmetry with $R^2 = 1$ is the same as the topological classification of inversion symmetric superconductors discussed in Sec. V A, i.e., it is given by $\Omega_4^{\mathrm{Pin}^+}(pt) = \mathbb{Z}_{16}$, and the generating manifold is $\mathbb{R}P^4$. However, it seems difficult to make a 4d analog of the cross-cap leading to $\mathbb{R}P^4$ by using the reflection. Let us consider a general partial point group transformation on a subregion D in a 3-space manifold. In order to make the resulting 4-manifold free from a singularity, the partial transformation should act freely on the boundary ∂D. In the case of the reflection $x \mapsto -x$, only the partial reflection on the subregion in the form of $[-L, L] \times M_2$ (M_2 is a 2-space manifold) meets this condition, which leads to the 4-manifold $\mathbb{R}P^2 \times M_2$, not $\mathbb{R}P^4$. Here, we discuss the partial reflection on a 3-ball D. Let us consider the reflection symmetry R_z for the model (5.3),

$$R_z \psi_i^\dagger(x,y,z) R_z^{-1} = \psi_j^\dagger(x,y,-z)[\sigma_z]_{ji}. \tag{5.32}$$

By using the following relations

$$\begin{aligned} &R_z\gamma^\dagger(\theta,\phi)R_z^{-1} = i\gamma(\pi-\theta,\phi), && R_z\gamma(\theta,\phi)R_z^{-1} = -i\gamma^\dagger(\pi-\theta,\phi), \\ &Y^{\frac{1}{2}}_{n-\frac{1}{2},m}(\pi-\theta,\phi) = (-1)^{n+\frac{1}{2}-m}Y^{-\frac{1}{2}}_{n-\frac{1}{2},n-\frac{1}{2}-m}(\theta,\phi), && Y^{-\frac{1}{2}}_{n-\frac{1}{2},m}(\pi-\theta,\phi) = (-1)^{n+\frac{1}{2}-m}Y^{\frac{1}{2}}_{n-\frac{1}{2},n-\frac{1}{2}-m}(\theta,\phi), \end{aligned} \tag{5.33}$$

We see that the Bogoliubov operators are transformed as

$$R_z\chi^\dagger_{n,m}R_z^{-1} = (-1)^{n+\frac{1}{2}-m}\chi^\dagger_{n,m}, \qquad R_z\chi_{n,m}R_z^{-1} = (-1)^{n+\frac{1}{2}-m}\chi_{n,m}, \quad (n>0). \tag{5.34}$$

The half of degenerate energy states labeled by n have the negative reflection parity $R_z = -1$. Then, the partial reflection on the 3-ball D is given by

$$\langle GS|R_{z,D}|GS\rangle \sim \frac{\mathrm{Tr}\big[R_z e^{-\frac{\xi}{\Delta}H_{S^2}}\big]}{\mathrm{Tr}\big[e^{-\frac{\xi}{\Delta}H_{S^2}}\big]} = \frac{\prod_{n=1}^{\infty}(1-q^n)^n}{\prod_{n=1}^{\infty}(1+q^n)^n}, \quad q = e^{-\frac{\xi}{R}}, \tag{5.35}$$

which is exactly the square root of the partial fermion parity flip (5.28). This is real positive number and has no information for the $\mathbb{Z}_{16}$ classification.

c. Partial π-rotation Finally, we consider the π-rotation symmetry of the model (5.3),

$$C_z\psi_i^\dagger(x,y,z)C_z^{-1} = -i\psi_j^\dagger(-x,-y,z)[\sigma_z]_{ji}. \tag{5.36}$$

From the point of view of TQFTs, the existence of π-rotation symmetry is equivalent to the on-site $\mathbb{Z}_2$ symmetry, thus we expect the topological classification is given by the Spin cobordism $\Omega_4^{\mathrm{Spin}}(B\mathbb{Z}_2) = \mathbb{Z}$.[11] Here, the integer cobordism group means that there is a topological action parameterized by $a \in U(1)$ through $\mathrm{Hom}\big[\Omega_4^{\mathrm{Spin}}(B\mathbb{Z}_2), U(1)\big] = U(1)$. $a \in U(1)$ is a material parameter determined by a model Hamiltonian similar to axion θ term in $(3+1)d$ insulators.[109] In the context of SPT phases, the topological action parameterized by $U(1)$ does not mean the existence of a nontrivial

SPT phase since all phases labeled by $a \in U(1)$ are adiabatically connected unless there is a symmetry fixing $a \in U(1)$ to a discrete value.[25]

Here we compute the partial π-rotation on a 3-ball D. One can show

$$\begin{aligned} &C_z\gamma^\dagger(\theta,\phi)C_z^{-1} = \gamma^\dagger(\theta,\phi+\pi), && C_z\gamma(\theta,\phi)C_z^{-1} = \gamma(\theta,\phi+\pi), \\ &Y^{\frac{1}{2}}_{n-\frac{1}{2},m}(\pi,\phi+\pi) = -i(-1)^{m+\frac{1}{2}}Y^{\frac{1}{2}}_{n-\frac{1}{2},n-\frac{1}{2}-m}(\theta,\phi), && Y^{-\frac{1}{2}}_{n-\frac{1}{2},m}(\theta,\phi+\pi) = -i(-1)^{m+\frac{1}{2}}Y^{-\frac{1}{2}}_{n-\frac{1}{2},n-\frac{1}{2}-m}(\theta,\phi), \end{aligned} \tag{5.37}$$

then, we have

$$C_z\chi^\dagger_{n,m}C_z^{-1} = i(-1)^{m+\frac{1}{2}}\chi^\dagger_{n,m}, \quad C_z\chi_{n,m}C_z^{-1} = i(-1)^{m+\frac{1}{2}}\chi_{n,m}, \quad n>0). \tag{5.38}$$

The expectation value of the partial π-rotation on the 3-ball D is given by

$$\begin{aligned} \langle GS|C_{z,D}|GS\rangle &\sim \frac{\mathrm{Tr}\left[C_z e^{-\frac{\xi}{\Delta}H_{S^2}}\right]}{\mathrm{Tr}\left[e^{-\frac{\xi}{\Delta}H_{S^2}}\right]} = \frac{\prod_{n=1}^\infty(1+iq^n)^n(1-iq^n)^n}{\prod_{n=1}^\infty(1+q^n)^{2n}} = \exp\left[I_1(q=e^{-\frac{2\xi}{R}}) - 2I_1(q=e^{-\frac{\xi}{R}})\right] \\ &\sim \exp\left[\frac{1}{12}\ln(2) - \frac{21}{16}\zeta(3)\left(\frac{R}{\xi}\right)^2\right]. \end{aligned} \tag{5.39}$$

Note that this coincides with the amplitude part of the partial inversion (5.25).

B. $(3+1)d$ insulators with inversion symmetry ($\Omega_4^{\mathrm{Pin}^c}(pt) = \mathbb{Z}_8 \oplus \mathbb{Z}_2$)

In this section, we consider $(3+1)d$ topological insulators protected by inversion symmetry. The topological classification is given by the Pin^c cobordism[57]

$$\Omega_4^{\mathrm{Pin}^c}(pt) = \mathbb{Z}_8 \oplus \mathbb{Z}_2. \tag{5.40}$$

The latter direct summand $\mathbb{Z}_2$ arises from bosonic SPT phases corresponding to one of $\mathbb{Z}_2$ of the unoriented cobordism group $\Omega_4^O(pt) = \mathbb{Z}_2 \oplus \mathbb{Z}_2$.[110] Our focus here is on the former direct summand $\mathbb{Z}_8$. This part is generated by $\mathbb{R}P^4$, and the following four-orbital free fermion model

$$\begin{aligned} &H = \sum_{\boldsymbol{k}} \psi^\dagger(\boldsymbol{k})\mathcal{H}(\boldsymbol{k})\psi(\boldsymbol{k}), \quad \psi(\boldsymbol{k}) = \{\psi_{\tau,\sigma}\}_{\tau,\sigma=1,2}, \\ &\mathcal{H}(\boldsymbol{k}) = (\frac{k^2}{2m} - \mu)\tau_z + \Delta\tau_x\boldsymbol{k}\cdot\boldsymbol{\sigma}, \quad (m,\mu,\Delta>0), \end{aligned} \tag{5.41}$$

which is equivalent to the two copies of the $(3+1)d$ superconductor (5.3). The inversion symmetry is defined by

$$I\psi^\dagger(\boldsymbol{x})I^{-1} = \psi^\dagger(-\boldsymbol{x})\tau_z. \tag{5.42}$$

In addition to the inversion symmetry, there is the $U(1)$ charge conservation symmetry,

$$U_b\psi^\dagger(\boldsymbol{x})U_b^{-1} = e^{-2\pi ib}\psi^\dagger(\boldsymbol{x}). \tag{5.43}$$

As in the case of $(3+1)d$ topological superconductors protected by inversion, the expectation value of the partial inversion is a candidate of the $\mathbb{Z}_8$ SPT invariant. The $U(1)$ phase in the expectation value of the partial inversion is simply twice that of the $(3+1)d$ superconductors. On the other hand, in the topological insulator system, there is the additional charge $U(1)$ symmetry, which can be combined with the partial inversion to introduce $I_DU_{b,D}$, where I_D and $U_{b,D}$ is the partial inversion and partial $U(1)$ transformation for the 3-ball D. We will focus on the role of this $U(1)$ twist.

1. *Partial inversion with $U(1)$ transformation*

We first give an analytical evaluation of $\langle GS|I_DU_{b,D}|GS\rangle$. In the same way as in Sec. V A, we have a surface entanglement Hamiltonian on S^2 (see Appendix D 2 a)

$$H_{S^2} = \sum_{n=1}^\infty \sum_{m=-(n-\frac{1}{2})}^{n-\frac{1}{2}} \frac{\Delta n}{R}\left[\chi^\dagger_{n,m}\chi_{n,m} + \chi_{-n,m}\chi^\dagger_{-n,m}\right]. \tag{5.44}$$

Here, we have normal-ordered the fermion operators with respect to the fermi sea $|FS\rangle$, which is defined by fully occupying states created by $\chi^\dagger_{-n,m}(n>0)$. In general, an entanglement chemical potential μ_e can be added, which is determined by the geometries of the ball D and other region. Here, for simplicity, we assume $\mu_e=0$. The quasiparticles operators $\chi^\dagger_{n,m}$ are obtained from the surface Dirac fermion operators in the spherical coordinate $\gamma_{1,2}(\theta,\phi)$ as

$$\begin{aligned}\chi^\dagger_{n,m} &:= \int \sin\theta d\theta d\phi\Big[\frac{1}{\sqrt{2}}Y^{-\frac{1}{2}}_{n-\frac{1}{2},m}(\theta,\phi)\gamma^\dagger_1(\theta,\phi) - i\frac{1}{\sqrt{2}}Y^{\frac{1}{2}}_{n-\frac{1}{2},m}(\theta,\phi)\gamma^\dagger_2(\theta,\phi)\Big],\\ \chi^\dagger_{-n,m} &:= \int \sin\theta d\theta d\phi\Big[\frac{1}{\sqrt{2}}Y^{-\frac{1}{2}}_{n-\frac{1}{2},m}(\theta,\phi)\gamma^\dagger_1(\theta,\phi) + i\frac{1}{\sqrt{2}}Y^{\frac{1}{2}}_{n-\frac{1}{2},m}(\theta,\phi)\gamma^\dagger_2(\theta,\phi)\Big],\end{aligned} \tag{5.45}$$

where $n>0$. The surface Dirac fermion operators are related to the bulk fermion operators as

$$\begin{aligned}\gamma^\dagger_1(\theta,\phi) \sim \Big[&-e^{-i\frac{\phi}{2}}\sin\frac{\theta}{2}\big\{i\psi^\dagger_{\tau=1,\sigma=1}(r,\theta,\phi)+\psi^\dagger_{\tau=2,\sigma=1}(r,\theta,\phi)\big\}\\ &+e^{i\frac{\phi}{2}}\cos\frac{\theta}{2}\big\{i\psi^\dagger_{\tau=1,\sigma=2}(r,\theta,\phi)+\psi^\dagger_{\tau=2,\sigma=2}(r,\theta,\phi)\big\}\Big]e^{-\int^r\frac{m(r')}{v}dr'},\end{aligned} \tag{5.46}$$

$$\begin{aligned}\gamma^\dagger_2(\theta,\phi) \sim \Big[&e^{-i\frac{\phi}{2}}\cos\frac{\theta}{2}\big\{\psi^\dagger_{\tau=1,\sigma=1}(r,\theta,\phi)+i\psi^\dagger_{\tau=2,\sigma=1}(r,\theta,\phi)\big\}\\ &+e^{i\frac{\phi}{2}}\sin\frac{\theta}{2}\big\{\psi^\dagger_{\tau=1,\sigma=2}(r,\theta,\phi)+i\psi^\dagger_{\tau=2,\sigma=2}(r,\theta,\phi)\big\}\Big]e^{-\int^r\frac{m(r')}{v}dr'}.\end{aligned} \tag{5.47}$$

From the transformation law

$$I\gamma^\dagger_1(\theta,\phi)I^{-1}=\gamma^\dagger_2(\pi-\theta,\phi+\pi),\qquad I\gamma^\dagger_2(\theta,\phi)I^{-1}=-\gamma^\dagger_1(\pi-\theta,\phi+\pi), \tag{5.48}$$

and Eq. (5.13), we note the partial inversion and partial $U(1)$ charge transformation act on the quasiparticle operators $\chi^\dagger_{n,m}$ as

$$\begin{aligned}&I\chi^\dagger_{n,m}I^{-1}=(-1)^n\chi^\dagger_{n,m},\quad I\chi^\dagger_{-n,m}I^{-1}=-(-1)^n\chi^\dagger_{-n,m},\quad (n>0),\\ &U_b\chi^\dagger_{n,m}U_b^{-1}=e^{-2\pi ib}\chi^\dagger_{n,m}.\end{aligned} \tag{5.49}$$

We fix constant phases associated with the antipodal map I and $U(1)$ transformation so that $I|FS\rangle = mbU_b|FS\rangle = |FS\rangle$. Then, the expectation value of the combined partial inversion with partial $U(1)$ charge transformation can be evaluated as

$$\langle GS|U_bI|GS\rangle \sim \frac{\mathrm{Tr}\Big[U_bIe^{-\frac{\xi}{\Delta}\widetilde{H}_{S^2}}\Big]}{\mathrm{Tr}\Big[e^{-\frac{\xi}{\Delta}\widetilde{H}_{S^2}}\Big]} = \frac{\prod_{n=1}^\infty(1+e^{-2\pi ib}(-q)^n)^{2n}(1-e^{2\pi ib}(-q)^n)^{2n}}{\prod_{n=1}^\infty(1+q^n)^{2n}(1+q^n)^{2n}}. \tag{5.50}$$

From Appendix F, the phase and amplitude of this expectation value, $\langle GS|U_bI|GS\rangle = |\langle GS|U_bI|GS\rangle|e^{i\theta_{\mathrm{top}}}$, are evaluated as

$$\theta_{\mathrm{top}} = \begin{cases}\frac{\pi}{4} & (0<b<\frac{1}{2})\\ -\frac{\pi}{4} & (-\frac{1}{2}<b<0)\end{cases}, \tag{5.51}$$

$$\begin{aligned}|\langle GS|U_bI|GS\rangle| = \exp\Big[&-\Big(3\zeta(3)+\frac{1}{4}\big\{\mathrm{Li}_3(e^{4\pi ib})+\mathrm{Li}_3(e^{-4\pi ib})\big\}\Big)\Big(\frac{R}{\xi}\Big)^2-\frac{1}{6}\ln\Big|\frac{\sin(2\pi b)}{2}\Big|\\ &+\frac{3+\cos(4\pi b)}{480\sin^2(2\pi b)}\Big(\frac{\xi}{R}\Big)^2+\frac{125+68\cos(4\pi b)-\cos(8\pi b)}{96768\sin^4(2\pi b)}\Big(\frac{\xi}{R}\Big)^4+\cdots\Big],\end{aligned} \tag{5.52}$$

for $e^{-2\pi ib}\neq\pm1$. Notice that the result of the $U(1)$ phase part $e^{i\theta_{\mathrm{top}}}$ is exact, which is independent of the scale ξ/R. The quantized scale-independent $U(1)$ phase (5.51) is somewhat unexpected from the viewpoint of Pin^c structure. In the Pin^c structure on $\mathbb{R}P^4$, the holonomy associated to the $\mathbb{Z}_2$ nontrivial loop threading the cross-cap is quantized to $\pm i$. However, (5.51) means that even if the holonomy is not properly chosen to be $e^{-2\pi ib}=\pm i$, the $U(1)$ phase of the partial inversion is well quantized. This agrees with the numerical calculation (see Fig. 13), where the plateau structure of the $U(1)$ phase becomes sharper as one increases the sizes of the subsystem. As b approaches the "phase transition" points $e^{-2\pi ib}=\pm1$, the higher-order terms proportional to $(\xi/\sin(2\pi b)R)^{2\ell}$ in (5.52) contribute to the

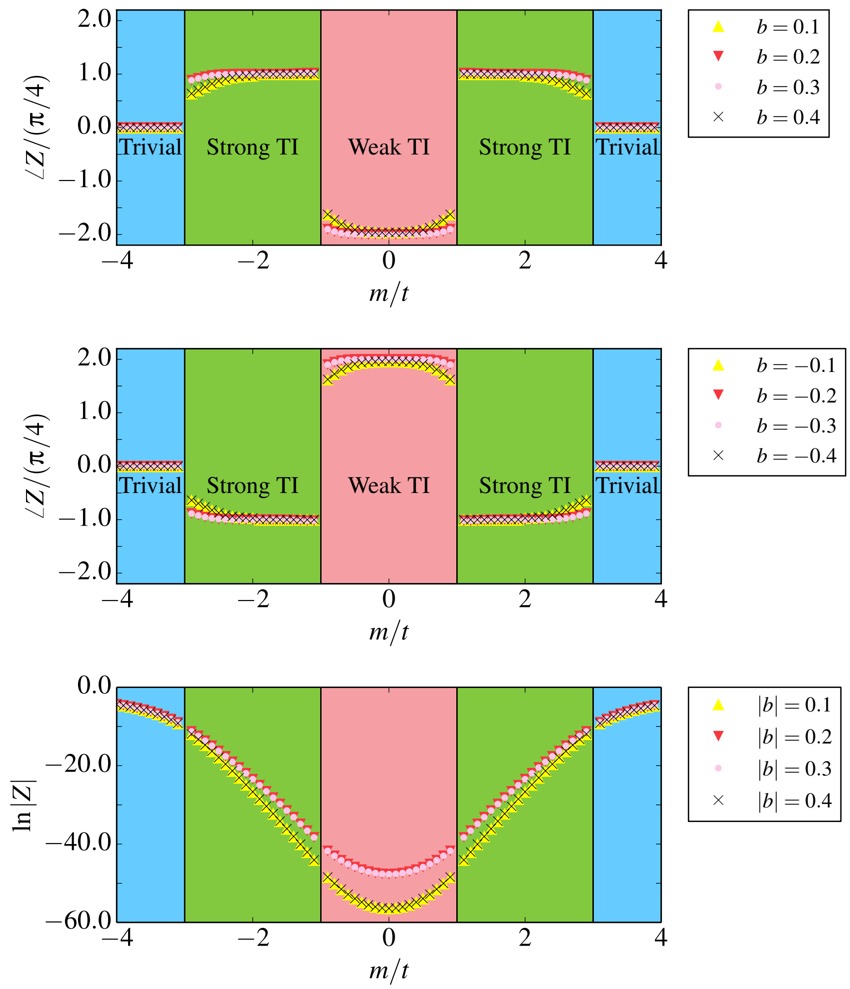


FIG. 12. (Color online) The phase ($\angle Z$) and modulus ($|Z|$) of the expectation value of the partial inversion, $Z = \langle GS|I_D|GS\rangle$, computed for the 3D inversion symmetric topological insulator (class A) as a function of the mass parameter m for various values of the $U(1)$ phase transformation b defined in Eq. (5.43). Strong (weak) TI refers to the phase with odd (even) number of Dirac surface states. Here, we set $t = r$. The size of total system and subsystem are $N = 12^3$ and $N_{\text{part}} = 6^3$, respectively.

amplitude. At the points $e^{-2\pi i b} = \pm 1$, there appears a double pole at $s = 0$ and it gives an algebraic correction to the amplitude, in addition to the area law decay, as

$$
\begin{aligned}
&|\langle GS|U_{b=0,1/2} I|GS\rangle| \\
&= \exp\Big[-\frac{7\zeta(3)}{2}\Big(\frac{R}{\xi}\Big)^2 + \frac{1}{6}\ln\Big(\frac{R}{\xi}\Big) + \frac{12\ln(A) - 1 + \ln(2)}{6} - \frac{1}{720}\Big(\frac{\xi}{R}\Big)^2 - \frac{1}{25920}\Big(\frac{\xi}{R}\Big)^4 - \cdots\Big],
\end{aligned} \tag{5.53}
$$

c.f., (F.6) and (5.30).

2. *Numerical results for lattice systems*

In this section, we study the standard Wilson-Dirac Hamiltonian on a cubic lattice as a simple model of the three-dimensional inversion-symmetric TI[109,111]

$$
H = \frac{1}{2}\sum_{\substack{\mathbf{x}\\ s=1,2,3}} \Big[\psi^\dagger_{\mathbf{x}+\mathbf{e}_s}(it\alpha_s - r\beta)\psi_{\mathbf{x}} + h.c.\Big] + m\sum_{\mathbf{x}} \psi^\dagger_{\mathbf{x}}\beta\psi_{\mathbf{x}} \tag{5.54}
$$

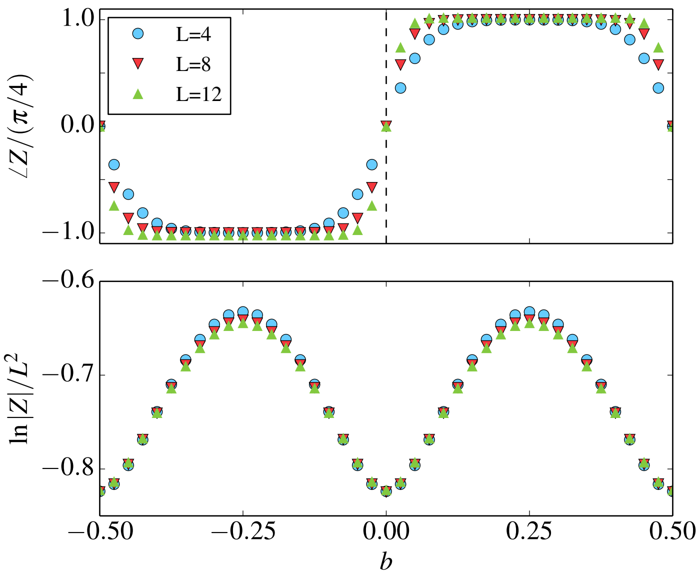


FIG. 13. (Color online) The phase ($\angle Z$) and modulus ($|Z|$) of the expectation value of the partial inversion, $Z = \langle GS|I_D|GS\rangle$, computed for the 3D inversion symmetric topological insulator as a function of the $U(1)$ phase transformation b defined in Eq. (5.43). Here, we set $t = r$ and $m = 2$. The sizes of the whole system and the subsystem are $N = 16^3$ and $N_{\text{part}} = L^3$, respectively.

where the Dirac matrices are given by

$$\alpha_s = \tau_1 \otimes \sigma_s = \begin{pmatrix} 0 & \sigma_s \\ \sigma_s & 0 \end{pmatrix}, \quad \beta = \tau_3 \otimes 1 = \begin{pmatrix} \mathbb{I} & 0 \\ 0 & -\mathbb{I} \end{pmatrix}.$$

In this convention the σ and τ matrices act on the spin and orbital degrees of freedom respectively. Transforming to reciprocal space, the Bloch Hamiltonian reads

$$h(\mathbf{k}) = \sum_{s=1,2,3} \Big[t\alpha_s \sin k_s - r\beta \cos k_s \Big] + m\beta.$$

This model can exhibit a non-trivial 3D TI phase protected by the inversion symmetry which is defined by Eq. (5.42). In fact, as the mass parameter m is varied, the Hamiltonian shows the following phases:

1. $|m| < r$: weak TI with an even number of Dirac cones on each boundary surface.
2. $r < |m| < 3r$: strong TI with a single Dirac cone on each boundary surface.
3. $|m| > 3r$: trivial phase equivalent to the atomic limit.

As shown in Fig. 12, we compute the complex phase $\angle Z = \operatorname{Im} \ln \langle GS|U_b I|GS\rangle$ of the partial inversion for various values of the $U(1)$ phase $e^{i2\pi b}$ for $t = r$. The calculation procedure here is very similar to the two-dimensional case where in order to get $U_b I\,|GS\rangle$ we relocate the lattice points inside the subsystem according to the inversion symmetry operator I and multiply the states by the $U(1)$ phase given by U_b and finally the inner product $Z = \langle GS|\, U_b I\,|GS\rangle$ is computed. In particular, we observe that as b changes from negative values to positive values the complex phase transitions from $-\pi/4$ to $\pi/4$ (see Fig. 13). It is worth noting that as the subsystem is made larger, the transition becomes sharper and sharper indicating that this change will turn into a discontinuity in the thermodynamic limit. All these observations conform with our analytical results in the previous parts.

C. General even spacetime dimensions

The topological $U(1)$ phase emerging in the expectation value of the partial inversion for inversion symmetric topological superconductors can be generalized to any even spacetime dimensions. Let us consider the following BdG

Hamiltonians which describe odd parity topological superconductors in $(2p+1)$-space dimensions:

$$\begin{aligned} &H=\frac{1}{2}\sum_{\boldsymbol{k}}\Psi^\dagger(\boldsymbol{k})\mathcal{H}(\boldsymbol{k})\Psi(\boldsymbol{k}), \quad \Psi(\boldsymbol{k})=(\psi(\boldsymbol{k}),\psi^\dagger(-\boldsymbol{k}))^T, \\ &\mathcal{H}(\boldsymbol{k})=\Delta\sum_{\mu=1}^{2p+1}k_\mu\Gamma_\mu+m\Gamma_{2p+2}, \quad \{\Gamma_\mu,\Gamma_\nu\}=2\delta_{\mu\nu} \quad (\mu,\nu=1,\dots,2p+2). \end{aligned} \tag{5.55}$$

Here, we consider the gamma matrices having the minimum possible dimension to satisfy the reality condition[112]

$$\begin{aligned} &p\equiv 0,1 \ (\mathrm{mod}\ 4)\longrightarrow 2^{p+1}\times 2^{p+1}, \\ &p\equiv 2,3 \ (\mathrm{mod}\ 4)\longrightarrow 2^{p+2}\times 2^{p+2}. \end{aligned} \tag{5.56}$$

Inversion acts on the BdG Hamiltonian and the fermion operators as

$$IHI^{-1}=H, \quad I\psi^\dagger(\boldsymbol{x})I^{-1}=i\psi^\dagger(-\boldsymbol{x}), \tag{5.57}$$

Note that inversion (5.57) is CPT dual of time-reversal of class DIII for odd p and class BDI for even p: The π rotation of the real fermions is associated with $\pm i$ phase as shown in (4.12), which implies that the inversion transformation with $I^2=(-1)^F$ in $(2p+1)$-dimensions is equivalent to the reflection transformation with $R^2=1$ for odd p and $R^2=(-1)^F$ for even p. Also, $R^2=1$ and $R^2=(-1)^F$ correspond to the TRS with $T^2=(-1)^F$ and $T^2=1$, respectively, due to the Wick rotation.[11]

We consider the reduced density matrix for a $(2p+1)$-dimensional ball of radius R, which can be described by the surface theory on the sphere S^{2p} (the Dirac Hamiltonian on S^{2p})[113]

$$\mathcal{H}_{\mathrm{surf}}=\frac{\Delta}{R}\Big[\big(-i\partial_\theta-\frac{(2p-1)i}{2}\cot\theta\big)\gamma_{2p}+\frac{-i}{\sin\theta}\gamma_i(\tilde{e}_i-\frac{1}{2}\tilde{\omega}_{ijk}\Sigma^{jk})\Big], \tag{5.58}$$

where we have used the polar coordinates. The summations over i,j,k are implicit. θ is the angle from the north pole, $\{\tilde{e}_j\}_{j=1}^{2p-1}$ is a vielbein on S^{2p-1}, $\tilde{\omega}_{ijk}=\frac{1}{2}(\tilde{C}_{ijk}-\tilde{C}_{ikj}-\tilde{C}_{jki})$ with $[\tilde{e}_i,\tilde{e}_j]=\sum_{k=1}^{2p}\tilde{C}_{ijk}\tilde{e}_k$, and $\Sigma_{jk}=\frac{1}{4}[\Gamma_j,\Gamma_k]$. See Ref. 113 for more details. The dimension of gamma matrices $\{\gamma_i\}_{i=1}^{2p}$ is a half of (5.57). Eigenvalues of $\mathcal{H}_{\mathrm{surf}}$ are given by[113]

$$E_{n,p,\pm}=\pm\frac{\Delta}{R}(n+p) \quad (n=0,1,\dots), \tag{5.59}$$

with degeneracy

$$D_{2p}(n)=\frac{2^p(n+2p-1)!c_p}{n!(2p-1)!}, \quad c_p:=\begin{cases}1, & p\equiv 0,1 \ (\mathrm{mod}\ 4)\\ 2, & p\equiv 2,3 \ (\mathrm{mod}\ 4)\end{cases} \tag{5.60}$$

Similar to our discussion on Sec. V A 1, the $U(1)$ phase of the expectation value of partial inversion is given by

$$\begin{aligned} &\mathrm{Im}\big[\ln\langle GS\,|\,I_D\,|\,GS\rangle\big]=\mathrm{Im}\ I_{2p}(q=e^{-\frac{\xi}{R}}), \\ &I_{2p}(q)=\ln\prod_{n=0}^\infty(1+i(-q)^{n+p})^{D_{2p}(n)}=\sum_{n=0}^\infty D_{2p}(n)\ln(1+i(-q)^{n+p}). \end{aligned} \tag{5.61}$$

One can show

$$\mathrm{Im}\ I_{2p}(q=e^{-\delta})=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}ds\frac{2^pc_p}{(2p-1)!}\delta^{-s}\Gamma(s)f_p(s-1)\beta(s+1), \tag{5.62}$$

where

$$f_p(s-1):=\sum_{n=1}^\infty(n^2-1)(n^2-2^2)\cdots(n^2-(p-1)^2)(-1)^nn^{-s+1}, \tag{5.63}$$

and $\beta(s)=\sum_{n=0}^\infty\frac{(-1)^n}{(2n+1)^s}$ is the Dirichlet beta function. All poles except for $s=0$ are canceled, and there is a single pole at $s=0$. We get a scale-independent exact result

$$\mathrm{Im}\big[\ln\langle GS\,|\,I_D\,|\,GS\rangle\big]=(-1)^p(2\pi)\times\begin{cases}2^{-(p+3)}, & p\equiv 0,1 \ (\mathrm{mod}\ 4)\\ 2^{-(p+2)}, & p\equiv 2,3 \ (\mathrm{mod}\ 4)\end{cases} \tag{5.64}$$

This implies $\mathbb{Z}_{2^{p+3}}$ classification for $p \equiv 0, 1 \pmod 4$ and $\mathbb{Z}_{2^{p+2}}$ classification for $p \equiv 2, 3 \pmod 4$, which are consistent with Refs. 11, 100, and 104. Once again, note that inversion symmetry (5.57) is CPT dual of time-reversal of class DIII for odd p and class BDI for even p.

The partial inversion on class A insulators with inversion symmetry in even spacetime dimensions is given by adding the contribution from the negative energy eigenstates in (5.59), which leads to the twice of $p \equiv 0, 1 \pmod 4$ cases in (5.64),

$$\mathrm{Im}\big[\ln\langle GS\,|\,I_D\,|\,GS\rangle\big]|_{\mathrm{class A}+I} = (-1)^p(2\pi) \times 2^{-(p+2)}. \tag{5.65}$$

This is also consistent with Refs. 100 and 104 and the free fermion part of Pin^c cobordism group[102] $\mathbb{Z}_{2^{p+2}} \subset \Omega^{\mathrm{Pin}^c}_{2p+2}(pt)$.

VI. CONCLUSION

In this paper, we developed an approach to detect interacting fermionic SPT phases by introducing non-local order parameters, the expectation value of partial point group transformations with respect to a given ground state wave function on a closed space manifold. From the point of view of TQFTs, the interacting SPT invariants are defined as partition functions (path-integrals) on generating spacetime manifolds of cobordism group. In order to simulate the path integrals, in the operator formalism, on various generating manifolds such as real projective spaces and lens spaces, we showed that the partial point group transformations provide a unified way for this purpose: The SPT topological invariants emerge as the complex $U(1)$ phases of the expectation value of the partial point group transformations, if the subregion D is sufficiently larger than the bulk correlation length. In addition to the topological $U(1)$ phases, we find that the amplitude part also includes scale-independent contributions, which is another signature of nontrivial SPT phases.

We confirm these results both by analytic calculations using gapless surface theories and by numerics in lattice models. It is worth emphasizing that the definition of the partial point group transformation depends only on the symmetry of the problem, which contrasts with the modular transformation on the ground states on the 2-torus T^2.[59–61,63,64]

Let us close by mentioning a number of interesting future directions.

– First, we focused in this paper on unitary symmetries. The definition of SPT invariants for SPT phases protected by time-reversal symmetry (and more general anti-unitary symmetries) is not fully understood. In $(1+1)d$ SPTs with time-reversal symmetry, it is known that the partial transpose plays the role of "gauging time-reversal symmetry" and yields real projective plane and Klein bottle for bosons[67] and fermions.[73] The construction of SPT invariant for more general higher space dimensions is an open problem.

– We note that our formula (1.11) can be applied to symmetry-enriched topological (SET) phases by point group symmetry. In topologically ordered phases where there are ground state degeneracies depending on the space manifold M, the ground state $|GS\rangle$ in the formula (1.11) is replaced by a linear combinations of degenerate ground states as $|GS\rangle = \sum_i c_i\,|GS_i\rangle$, $c_i \in \mathbb{C}$, $\sum_i |c_i|^2 = 1$, where $|GS_i\rangle$ is the ground state of the topological sector labeled by i. We leave the detailed studies of SET phase for the future.

– Throughout this paper, we assumed that the entanglement chemical potential μ_e associated with the reduced density matrix of the sub region D is zero. However, in general, μ_e can be nonzero, that depends on the geometry of the region D and other details. The agreement between TQFT partition functions and the numerical calculation of the partial point group transformations suggests that effect of a finite entanglement chemical potential can be neglected in the boundary theory.

– Due to the lattice translational symmetry, numerical calculations of partial rotations in this paper are limited into C_2, C_3, C_4, and C_6 rotations. It is an interesting problem to compute the partial rotations for rotation symmetries which can not be defined on translational symmetric lattice systems such as C_5 symmetry.

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Appendix A: Arf and Brown invariants

The purpose of this appendix is to introduce the Arf invariant[115] $(-1)^{\mathrm{Arf}(\eta)}$ of Spin structures $\eta \in \mathrm{Spin}(\Sigma)$ on given 2-dimensional oriented manifolds Σ,[116,117] and the Brown invariant[118] for Pin$^-$ structures on unoriented 2-manifolds.[11,71,119]

1. The Arf invariant

The obstruction to give a Spin structure on Σ is measured by the 2nd Stiefel-Whitney class $w_2(T\Sigma) \in H^2(\Sigma, \mathbb{Z}_2)$. For oriented 2-manifolds, i.e., Riemann surfaces Σ_g with genus g, $w_2(\Sigma_g)$ always disappears, hence one can define Spin structures on Σ_g. The set of spin structures $\mathrm{Spin}(\Sigma_g)$ on Σ_g is equivalent to $H^1(\Sigma_g, \mathbb{Z}_2)$ as a set. Here, $H^1(\Sigma_g, Z_2) \cong \mathrm{Hom}(\pi_1(\Sigma_g), \mathbb{Z}_2)$ has 2^{2g} elements and can be thought of as the space of $\mathbb{Z}_2$-Wilson lines (background nontrivial $\mathbb{Z}_2$ gauge fields) on Σ_g. It is known that $H^1(\Sigma_g, \mathbb{Z}_2)$ acts on $\mathrm{Spin}(\Sigma_g)$ freely and transitively (i.e. $\mathrm{Spin}(\Sigma_g)$ is a $H^1(\Sigma_g, \mathbb{Z}_2)$-Torsor), which means any spin structures $\eta \in \mathrm{Spin}(\Sigma_g)$ can be given by an action of a Wilson line $a \in H^1(\Sigma_g, \mathbb{Z}_2)$ on a some "reference" Spin structure η_0. Note that there is no canonical choice of the reference spin structure η_0, which contrasts with $\mathbb{Z}_2$-Wilson lines where there is the zero flux in $H^1(\Sigma_g, \mathbb{Z}_2)$. The absence of any reference elements is a feature of "Torsor". See, for example, Ref. 120 for details, where the equivalence between the Spin structures and the Kasteleyn orientations in the dimer model is also explained.

The Arf invariant is defined in a pure algebraic manner. For a given vector space V over the field $\mathbb{Z}_2$ and a fixed bilinear form $\phi : V \times V \to \mathbb{Z}_2$, one can define the $\mathbb{Z}_2$-valued Arf invariant of the *quadratic forms* $Q_2(V, \phi)$, which will be described in Sec. A 1 b.

It is known that the spin structures $\mathrm{Spin}(\Sigma)$ on an oriented 2 manifold Σ is equivalent to the quadratic forms $Q_2(H^1(\Sigma, \mathbb{Z}_2), \int x \cup y)$ on $H^1(\Sigma, \mathbb{Z}_2)$ with the intersection form $\int x \cup y$ as a $H^1(\Sigma, \mathbb{Z}_2)$-Torsor. (We do not explain this equivalence in the present paper. See Ref. 120 for details.) This implies that one can define the Arf invariant of the spin structures $\mathrm{Spin}(\Sigma)$, which is nothing but the $\mathbb{Z}_2$ topological invariant of the Spin TQFT in 2 spacetime dimensions.[11,47,71]

a. *Simplicial calculus*

To describe $\mathbb{Z}_2$-Wilson lines belonging to $H^1(\Sigma, \mathbb{Z}_2)$ and the intersection form $\int x \cup y$ in a rigorous way, we introduce the simplicial calculus, according to Appendix A in Ref. 121.

Let K be a triangulation of Σ and G be an Abelian group. A p-cochain f is a function over p-simplices $\{(v_0, \dots, v_p)\}$ to G. We assume the vertices of K are ordered in some way. The space of p-chains is denoted by $C^p(K, G)$. The differential operator $\delta : C^p(K, G) \to C^{p+1}(K, G)$ is defined by

$$(\delta f)(v_0, \dots, v_{p+1}) = \sum_{i=0}^{p+1} (-1)^i f(v_0, \dots, \hat{v}_i, \dots, v_{p+1}) \tag{A.1}$$

for a $(p+1)$ simplex $(v_0, v_1, \dots, v_{p+1})$, where $\hat{v}_i$ means that v_i is excluded. One can show $\delta^2 = 0$ as

$$\begin{aligned}(\delta^2 f)(v_0, \dots, v_{p+2}) &= \sum_{i=0}^{p+2} (-1)^i (\delta f)(v_0, \dots, \hat{v}_i, \dots, v_{p+2}) \\ &= \sum_{j<i}^{p+2} (-1)^{i+j} f(v_0, \dots, \hat{v}_j, \dots, \hat{v}_i, \dots, v_{p+2}) + \sum_{i<j}^{p+2} (-1)^{i+j-1} f(v_0, \dots, \hat{v}_i, \dots, \hat{v}_j, \dots, v_{p+2}) \\ &= 0.\end{aligned} \tag{A.2}$$

We define the coboundary $B^p(K, G) = \mathrm{Im}[\delta : C^{p-1}(K, G) \to C^p(K, G)]$ and the cocycle $Z^p(K, G) := \mathrm{Ker}[\delta : C^p(K, G) \to C^{p+1}(K, G)]$ and the cohomology $H^p(K, G) := Z^p(K, G)/B^p(K, G)$.

If G is a commutative ring (for example, a cyclic group $\mathbb{Z}_p$), we can define the cup product $\cup : C^p(K, G) \times C^q(K, G) \to C^{p+q}(K, G)$ by

$$(f \cup g)(v_0, \dots, v_{p+q}) = f(v_0, \dots, v_p) g(v_p, \dots, v_{p+q}). \tag{A.3}$$

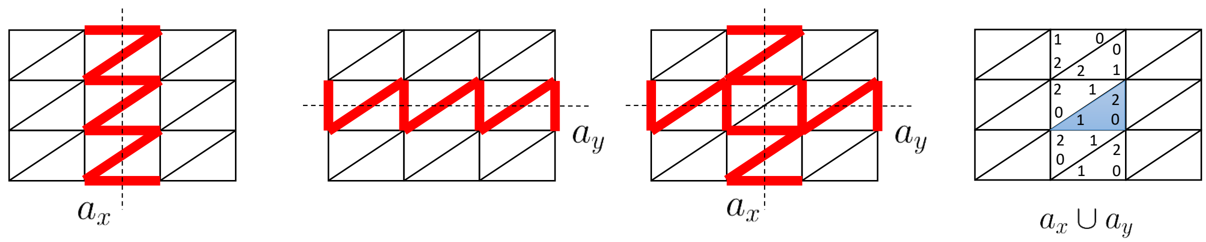

FIG. 14. (Color online) Wilson lines and the intersection forms.

(Here, the r.h.s. is the product of the ring G.) The cup product satisfies the Leibniz rule

$$\delta(f \cup g) = \delta f \cup g + (-1)^p f \cup \delta g. \tag{A.4}$$

Obviously, the cup product is well-defined in the cohomology $H^p(K, G)$ since $\delta(f \cup g) = 0$ for $\delta f = \delta g = 0$.

b. A quadratic form and the $\mathbb{Z}_2$ *Arf invariant*

Let V be a finite dimensional vector space over the field $\mathbb{Z}_2$, and let $\phi : V \times V \to \mathbb{Z}_2$ be a fixed bilinear form. A quadratic form q on (V, ϕ) is a *function* $q : V \to \mathbb{Z}_2$ (not a linear form) which satisfies

$$q(x + y) = q(x) + q(y) + \phi(x, y). \tag{A.5}$$

Note that the difference (= sum) of two quadratic forms $q_1 + q_2$ on (V, ϕ) is a linear form on V because $2\phi(x, y) = 0$. Therefore, the set $Q_2(V, \phi)$ of quadratic forms on (V, ϕ) is a V^*-torsor, i.e. all quadratic forms q is given by the action of a linear form $f : V \to \mathbb{Z}_2$ as $(f \cdot q)(x) := q(x) + f(x)$. The Arf invariant $\mathrm{Arf}(q) \in \{0, 1\}$ on the quadratic forms $Q_2(V, \phi)$ is defined by

$$(-1)^{\mathrm{Arf}(q)} := \frac{1}{\sqrt{|V|}} \sum_{x \in V} (-1)^{q(x)}, \qquad q \in Q_2(V, \phi), \tag{A.6}$$

where $|V|$ is the number of elements in V.

c. The Arf invariant for Spin *structures on* T^2

Let T^2 be the 2-torus. Let us consider the quadratic forms $Q_2\big(H^1(T^2, \mathbb{Z}_2), \int_{T^2} x \cup y\big)$. Here $H^1(T^2, \mathbb{Z}_2) = \mathbb{Z}_2 \oplus \mathbb{Z}_2 = \{0, a_x, a_y, a_x + a_y\}$ is generated by two different Wilson lines a_x and a_y along the x and y-directions. Intersection forms are given as

$$\int_{T^2} a_x \cup a_x = \int_{T^2} a_y \cup a_y = 0, \qquad \int_{T^2} a_x \cup a_y = 1. \tag{A.7}$$

This is the even/odd parity of the number of intersections between Wilson lines. See Fig. 14. There are $2^4 = 16$ kinds of functions $q : H^1(T^2, \mathbb{Z}_2) = \mathbb{Z}_2 \oplus \mathbb{Z}_2 \to \mathbb{Z}_2$. The quadratic forms have to obey

$$\begin{cases} q(0) = q(0) + q(0), \\ q(a_x) = q(a_x) + q(0), \\ q(a_y) = q(a_y) + q(0), \\ q(a_x + a_y) = q(a_x) + q(a_y) + 1. \end{cases} \tag{A.8}$$

Then, $q(0) = 0$ and $q(a_x + a_y) = q(a_x) + q(a_y) + 1$. Solving these equations, we have $|H^1(T^2, \mathbb{Z}_2)| = 4$ distinct quadratic forms:

$$\begin{cases} q_{ns,ns}(a_x) = 0, \ q_{ns,ns}(a_y) = 0 \\ q_{ns,r}(a_x) = 0, \ q_{ns,r}(a_y) = 1 \\ q_{r,ns}(a_x) = 1, \ q_{r,ns}(a_y) = 0 \\ q_{r,r}(a_x) = 1, \ q_{r,r}(a_y) = 1 \end{cases} \tag{A.9}$$

These four different quadratic forms correspond to spin structures $\{(ns, ns), (ns, r), (r, ns), (r, r)\}$, where r (ns) represents the periodic (anti-periodic) boundary condition of the real fermion along the x or y-directions. Through the bijection

$$\mathrm{Spin}(T^2) \cong Q_2\big(H^1(T^2, \mathbb{Z}_2), \int_{T^2} x \cup y\big), \tag{A.10}$$

we identify a quadratic form $q \in Q_2\big(H^1(T^2, \mathbb{Z}_2), \int_{T^2} x \cup y\big)$ with a spin structure $\eta \in \mathrm{Spin}(T^2)$. Finally, we obtain the Arf invariant for each spin structures $\mathrm{Spin}(T^2)$ as

$$\begin{aligned} &(-1)^{\mathrm{Arf}(ns,ns)} = (-1)^{\mathrm{Arf}(ns,r)} = (-1)^{\mathrm{Arf}(r,ns)} = 1, \\ &(-1)^{\mathrm{Arf}(r,r)} = -1. \end{aligned} \tag{A.11}$$

This is the same as the partition function on T^2 of the Kitaev chain model (2.12).

2. A Quadratic form and the $\mathbb{Z}_8$ Brown invariant

In the same way, the Brown $\mathbb{Z}_8$ invariant[118] is constructed as follows. Let V be a finite dimensional vector space over the field $\mathbb{Z}_2$ and let $\phi : V \times V \to \mathbb{Z}_2$ be a fixed bilinear form. We consider a quadratic form q on (V, ϕ) is a function $q : V \to \mathbb{Z}_4$ which satisfies

$$q(x + y) = q(x) + q(y) + 2\phi(x, y). \tag{A.12}$$

Note that the difference (= sum) of two quadratic forms q_1, q_2 on (V, ϕ) is a linear form on V since $4\phi(x, y) = 0$. The set $Q_4(V, \phi)$ of quadratic forms $q : V \to \mathbb{Z}_4$ is an V^*-torsor by the action $(f \cdot q)(x) = q(x) + 2f(x)$ of $f : V \to \mathbb{Z}_2$ on $q \in Q_4(V, \phi)$. The Brown invariant $\beta(q) \in \mathbb{Z}_8$ is defined by

$$e^{2\pi i\beta(q)/8} := \frac{1}{\sqrt{|V|}} \sum_{x \in V} i^{q(x)}, \qquad q \in Q_4(V, \phi), \tag{A.13}$$

where $|V|$ is the number of elements in V.

a. *The Brown invariant for* Pin$^-$ *structures on* $\mathbb{R}P^2$

There is a canonical 1 to 1 correspondence between Pin$^-$ structures on an unoriented surface Σ and quadratic forms $q : H^1(\Sigma, \mathbb{Z}_2) \to \mathbb{Z}_4$.[71] Let $\mathbb{R}P^2$ be the real projective plane. Let us consider $\mathbb{Z}_4$-valued quadratic forms $q : H^1(\mathbb{R}P^2, \mathbb{Z}_2) \to \mathbb{Z}_4$ with the intersection form $\int_{\mathbb{R}P^2} x \cup y$. Here $H^1(\mathbb{R}P^2, \mathbb{Z}_2) = \mathbb{Z}_2 = \{0, a\}$ is generated by the Wilson line a for noncontractible loop of $\mathbb{R}P^2$. The intersection form is given by

$$\int_{\mathbb{R}P^2} a \cup a = 1 \tag{A.14}$$

See Fig. 15. There are $4^2 = 16$ kinds of functions $q : H^1(\mathbb{R}P^2, \mathbb{Z}_2) \to \mathbb{Z}_4$. The quadratic form $q : H^1(\mathbb{R}P^2, \mathbb{Z}_2) \to \mathbb{Z}_4$ should satisfy

$$q(0) = q(0) + q(0), \qquad q(a + a) = q(a) + q(a) + 2, \tag{A.15}$$

which implies that there are two distinct quadratic forms:

$$q_+(a) = 1, \qquad q_-(a) = 3. \tag{A.16}$$

We identify these quadratic forms with two different Pin$^-$ structures η_+, η_- on $\mathbb{R}P^2$, respectively. Finally, we have the $\mathbb{Z}_8$ Brown invariant for each Pin$^-$ structure as

$$e^{2\pi i\beta(\eta_+)/8} = e^{\pi i/4}, \qquad e^{2\pi i\beta(\eta_-)/8} = e^{-\pi i/4}. \tag{A.17}$$

The former one is the $U(1)$ phase part which appeared in (3.11).

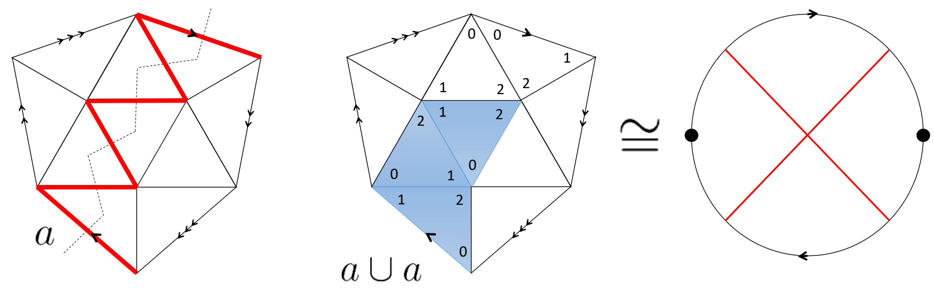

FIG. 15. (Color online) The Wilson line and the intersection form on $\mathbb{R}P^2$.

Appendix B: Lens space

The lens space $L(p,q) = S^3/\mathbb{Z}_p$ is defined by the quotient

$$(z_1, z_2) \sim (e^{\frac{2\pi i}{p}} z_1, e^{\frac{2q\pi i}{p}} z_2), \tag{B.1}$$

where (z_1, z_2) with $|z_1|^2 + |z_2|^2 = 1$ represent the 3-sphere S^3, and p and q are coprime.

The surgery construction of the lens space $L(p,q)$ is given by the following modular transformation[90]

$$ST^{m_{t-1}} ST^{m_{t-2}} \cdots ST^{m_1}, \tag{B.2}$$

on the boundary of a solid torus in S^3. Here, $(m_1, \ldots, m_{t-1})$ is determined by the fraction expansion of (p,q)

$$-\frac{p}{q} = m_{t-1} - \frac{1}{m_{t-2} - \frac{1}{\cdots - \frac{1}{m_1}}}. \tag{B.3}$$

For example, the surgery by $ST^n S$ leads to $L(-n, 1)$.

Appendix C: Eta and theta functions

In this section, we summarize the properties of the Dedekind eta function $\eta(\tau)$ and the generalized theta function $\theta_{a,b}(z|\tau)$ used in the main text. For $q = e^{2\pi i \tau}$ (Im $\tau > 0$), the Dedekind eta function is defined as

$$\eta(\tau) := q^{1/24} \prod_{n=1}^{\infty} (1 - q^n). \tag{C.1}$$

The theta function is defined as

$$\begin{aligned} \theta_{a,b}(z|\tau) &:= \sum_{n \in Z} e^{i\pi\tau(n+a)^2 + 2\pi i(n+a)(z+b)} \\ &= e^{2\pi i a(z+b)} q^{\frac{1}{2}a^2} \prod_{n=1}^{\infty} (1 - q^n)\Big(1 + e^{2\pi i(z+b)} q^{n+a-\frac{1}{2}}\Big)\Big(1 + e^{-2\pi i(z+b)} q^{n-a-\frac{1}{2}}\Big), \end{aligned} \tag{C.2}$$

where we noted the Jacobi's triple identity

$$\sum_{n \in \mathbb{Z}} e^{\pi i \tau n^2 + 2\pi i n v} = \prod_{m=1}^{\infty} (1 - e^{2m\pi i\tau})(1 + e^{(2m-1)\pi i\tau + 2\pi i v})(1 + e^{(2m-1)\pi i \tau - 2\pi i v}). \tag{C.3}$$

$\theta_{a,b}(z|\tau)$ obeys the following periodicities and modular transformations

$$\theta_{a+1,b}(z|\tau) = \theta_{a,b}(z|\tau), \qquad \theta_{a,b+1}(z|\tau) = e^{2\pi i a}\theta_{a,b}(z|\tau), \tag{C.4}$$

$$\theta_{a,b}(z|\tau+1) = e^{-\pi i(a^2+a)}\theta_{a,b+a+1/2}(z|\tau), \qquad \theta_{a,b}(\frac{z}{\tau}|-\frac{1}{\tau}) = \sqrt{-i\tau} e^{2\pi i ab} e^{\frac{i\pi}{\tau}z^2} \theta_{b,-a}(z|\tau), \tag{C.5}$$

$$\eta(\tau+1) = e^{\pi i/12}\eta(\tau), \qquad \eta(-\frac{1}{\tau}) = \sqrt{-i\tau}\,\eta(\tau). \tag{C.6}$$

Appendix D: Derivation of boundary theories

Here, we summarize the derivation of gapless boundary theories in $(2+1)d$ and $(3+1)d$ free fermion topological phases. We employ the same approach as Ref. 105; We first solve the radial part of the eigenvalue equation of single-particle Hamiltonians, and then construct an effective single-particle boundary Hamiltonians for directions tangential to the boundary.

1. Edge theory on the boundary of disc

Let us consider the following single-particle Hamiltonian describing a $(2+1)d$ bulk topological phase defined on the disc:

$$\mathcal{H} = -i\sigma_x\partial_x - i\sigma_y\partial_y + m(r)\sigma_z. \tag{D.1}$$

Here, the profile of the position-dependent $m(r)$ ($r = \sqrt{x^2+y^2}$) is chosen to create the disk geometry: $m(r) < 0$ for $r < R$ and $m(r) > 0$ for $r > R$, where R is the radius of the disc. Introducing the polar coordinate $(x, y) = (r\cos\phi, r\sin\phi)$, $\mathcal{H}$ can be written as

$$\begin{aligned}
&\mathcal{H} = \mathcal{H}_r + \mathcal{H}_\phi,\\
&\mathcal{H}_r = e^{-i\frac{\phi}{2}\sigma_z}\big[-i\sigma_x\partial_r + m(r)\sigma_z\big]e^{i\frac{\phi}{2}\sigma_z}, \qquad \mathcal{H}_\phi = e^{-i\frac{\phi}{2}\sigma_z}\frac{1}{r}\sigma_y(-i\partial_\phi - \frac{\sigma_z}{2})e^{i\frac{\phi}{2}\sigma_z}.
\end{aligned} \tag{D.2}$$

In the following, we approximately derive the wave function of the edge state and effective edge Hamiltonian. Since gapless edge excitations are exponentially localized at $r \sim R \gg \xi$ (ξ is the correlation length of bulk which is determined by the gap $m(r)$), the edge state wave function is approximated by solving $\mathcal{H}_r$ as

$$\mathcal{H}_r\phi_{\text{edge}}(r,\phi) = 0, \quad \phi_{\text{edge}}(r,\phi) \sim u(\phi)\cdot e^{-i\frac{\phi}{2}\sigma_z}\begin{pmatrix}1\\ i\end{pmatrix}e^{-\int^r m(r')dr'}, \quad u(\phi+2\pi) = -u(\phi). \tag{D.3}$$

Here $u(\phi)$ is a complex-valued function representing the ϕ-dependence and obeys anti-periodic boundary condition to account for the factor $e^{i\frac{\phi}{2}\sigma_z}$. Since $\phi_{\text{edge}}(r,\phi)$ is well localized at $r = R$, we can replace $1/r$ by $1/R$. Then, the effective single-particle Hamiltonian for the edge excitations is given by

$$\mathcal{H}_{edge} = \int rdr[\phi_{\text{edge}}(r,\phi)]^\dagger\mathcal{H}_\phi\phi_{\text{edge}}(r,\phi) \sim \frac{1}{R}u^*(\phi)(-i\partial_\phi)u(\phi). \tag{D.4}$$

In the following, we use the above result for the single-particle bulk and the corresponding edge Hamiltonians to construct (second quantized) fermionic operators creating/annihilating edge excitations and the Hamiltonians. We will discuss topological insulators and superconductors separately.

a. *Chern insulator*

Let $\psi(\boldsymbol{x}) = (\psi_1(\boldsymbol{x}), \psi_2(\boldsymbol{x}))^T$ be a two-orbital complex fermion and consider a Chern insulator on the disk

$$H = \int d^2\boldsymbol{x}\,\psi^\dagger(\boldsymbol{x})\Big[-i\sigma_x\partial_x - i\sigma_y\partial_y + m(r)\sigma_z\Big]\psi(\boldsymbol{x}). \tag{D.5}$$

From (D.3), the complex fermion annihilation operator $\gamma(\phi)$ for edge excitations is given by

$$\gamma^\dagger(\phi) \sim \big[e^{-i\frac{\phi}{2}}\psi_1^\dagger(r,\phi) + ie^{i\frac{\phi}{2}}\psi_2^\dagger(r,\phi)\big]e^{-\int^r m(r')dr'}, \quad \gamma(\phi+2\pi) = -\gamma(\phi) \tag{D.6}$$

with the Hamiltonian

$$H = \frac{1}{R}\int d\phi\,\gamma^\dagger(\phi)(-i\partial_\phi)\gamma(\phi). \tag{D.7}$$

b. $(p_x - ip_y)$ superconductor

Let us consider the following model of a $(p_x - ip_y)$ superconductor,

$$H = \int d^2\boldsymbol{x}\, \Psi^\dagger(\boldsymbol{x})\Big[-i\sigma_x\partial_x - i\sigma_y\partial_y + m(r)\sigma_z\Big]\Psi(\boldsymbol{x}), \quad \Psi(\boldsymbol{x}) = (\psi(\boldsymbol{x}), \psi^\dagger(\boldsymbol{x}))^T. \tag{D.8}$$

From (D.3), the Majorana fermion operator $\gamma(\phi)$, $\gamma^\dagger(\phi) = \gamma(\phi)$, for edge excitations is given by

$$\gamma(\phi) \sim \big[e^{-i\frac{\phi}{2}-\frac{\pi}{4}i}\psi^\dagger(r,\phi) + e^{i\frac{\phi}{2}+\frac{\pi}{4}i}\psi(r,\phi)\big]e^{-\int^r m(r')dr'}, \quad \gamma(\phi + 2\pi) = -\gamma(\phi) \tag{D.9}$$

with the Hamiltonian

$$H = \frac{1}{R}\int d\phi\, \gamma(\phi)(-i\partial_\phi)\gamma(\phi). \tag{D.10}$$

2. Surface theory on the boundary of ball

Let us consider the following single-particle Hamiltonian

$$\mathcal{H} = -i\tau_x(\sigma_x\partial_x + \sigma_y\partial_y + \sigma_z\partial_z) + m(r)\tau_z \tag{D.11}$$

representing a bulk topological phase on a 3-ball. We assume $m(r) < 0$ for $r < R$ and $m(r) > 0$ for $r > R$, where R is the radius of the ball. By introducing polar coordinate $(x, y, z) = (r\sin\theta\cos\phi, r\sin\theta\sin\phi, r\cos\theta)$, $\mathcal{H}$ can be written as

$$\begin{aligned}
&\mathcal{H} = \mathcal{H}_r + \mathcal{H}_{\theta,\phi},\\
&\mathcal{H}_r = e^{-i\frac{\phi}{2}\sigma_z}e^{-i\frac{\theta}{2}\sigma_y}\big[-i\tau_x\sigma_z\partial_r + m(r)\tau_z\big]e^{i\frac{\theta}{2}\sigma_y}e^{i\frac{\phi}{2}\sigma_z},\\
&\mathcal{H}_{\theta,\phi} = e^{-i\frac{\phi}{2}\sigma_z}e^{-i\frac{\theta}{2}\sigma_y}\Big[\frac{1}{r}\tau_x\sigma_x(-i\partial_\theta - \frac{\sigma_y}{2}) + \frac{1}{r\sin\theta}\tau_x\sigma_y(-i\partial_\phi - \frac{\cos\theta}{2}\sigma_z - \frac{\sin\theta}{2}\sigma_x)\Big]e^{i\frac{\theta}{2}\sigma_y}e^{i\frac{\phi}{2}\sigma_z}.
\end{aligned} \tag{D.12}$$

Two-component wave functions of boundary gapless excitations $\phi_{\mathrm{surf}}(r,\theta,\phi)$ are well approximated by solving the radial part as

$$\begin{aligned}
&\mathcal{H}_r\phi_{\mathrm{surf}}(r,\theta,\phi) = 0,\\
&\phi_{\mathrm{surf}}(r,\theta,\phi) \sim e^{-i\frac{\phi}{2}\sigma_z}e^{-i\frac{\theta}{2}\sigma_y}\left(u_1(\theta,\phi)\begin{pmatrix} i\\ 1\end{pmatrix}_\tau \otimes \begin{pmatrix} 0\\ 1\end{pmatrix}_\sigma, u_2(\theta,\phi)\begin{pmatrix} 1\\ i\end{pmatrix}_\tau \otimes \begin{pmatrix} 1\\ 0\end{pmatrix}_\sigma\right)e^{-\int^r m(r')dr'},\\
&u_i(\theta,\phi + 2\pi) = -u_i(\theta,\phi), \quad (i = 1,2).
\end{aligned} \tag{D.13}$$

Here $u_i(\theta,\phi)$ $(i = 1,2)$ are scalar functions representing the (θ,ϕ)-dependence and obey anti-periodic boundary condition in ϕ to account for the factor $e^{i\frac{\phi}{2}\sigma_z}$. Since $\phi_{\mathrm{surf}}(r,\phi)$ is well localized at $r = R$, we can approximate $1/r$ by $1/R$. The effective single-particle Hamiltonian for the surface is given by

$$\begin{aligned}
\mathcal{H}_{surf} &= \int r^2dr[\phi_{\mathrm{surf}}(r,\phi)]^\dagger\mathcal{H}_{\theta,\phi}\phi_{\mathrm{surf}}(r,\phi)\\
&\sim (u_1^*(\theta,\phi), u_2^*(\theta,\phi))\frac{1}{R}\begin{pmatrix} 0 & -i\partial_\theta - \frac{1}{\sin\theta}\partial_\phi - \frac{i\cot\theta}{2}\\ -i\partial_\theta + \frac{1}{\sin\theta}\partial_\phi - \frac{i\cot\theta}{2} & 0\end{pmatrix}\begin{pmatrix} u_1(\theta,\phi)\\ u_2(\theta,\phi).\end{pmatrix}
\end{aligned} \tag{D.14}$$

a. $(3+1)d$ topological insulator

Let $\psi(\boldsymbol{x}) = \{\psi_{\tau\sigma}(\boldsymbol{x})\}_{\tau,\sigma=1,2}$ be a four-orbital complex fermion. We consider a massive Dirac Hamiltonian defined on a ball of radius R,

$$H = \int d^3\boldsymbol{x}\, \psi^\dagger(\boldsymbol{x})\Big[-i\tau_x\boldsymbol{\sigma}\cdot\boldsymbol{\partial} + m(r)\tau_z\Big]\psi(\boldsymbol{x}). \tag{D.15}$$

From (D.13), gapless surface excitations $\gamma_i(\theta,\phi)(i=1,2)$ are given by

$$\begin{aligned}\gamma_1^\dagger(\theta,\phi) &\sim \Big[-e^{-i\frac{\phi}{2}}\sin\frac{\theta}{2}\big\{i\psi^\dagger_{\tau=1,\sigma=1}(r,\theta,\phi)+\psi^\dagger_{\tau=2,\sigma=1}(r,\theta,\phi)\big\}\\ &\qquad +e^{i\frac{\phi}{2}}\cos\frac{\theta}{2}\big\{i\psi^\dagger_{\tau=1,\sigma=2}(r,\theta,\phi)+\psi^\dagger_{\tau=2,\sigma=2}(r,\theta,\phi)\big\}\Big]e^{-\int^r m(r')dr'},\\ \gamma_2^\dagger(\theta,\phi) &\sim \Big[e^{-i\frac{\phi}{2}}\cos\frac{\theta}{2}\big\{\psi^\dagger_{\tau=1,\sigma=1}(r,\theta,\phi)+i\psi^\dagger_{\tau=2,\sigma=1}(r,\theta,\phi)\big\}\\ &\qquad +e^{i\frac{\phi}{2}}\sin\frac{\theta}{2}\big\{\psi^\dagger_{\tau=1,\sigma=2}(r,\theta,\phi)+i\psi^\dagger_{\tau=2,\sigma=2}(r,\theta,\phi)\big\}\Big]e^{-\int^r m(r')dr'}.\end{aligned} \tag{D.16}$$

and obey the boundary condition

$$\gamma_i(\theta,\phi+2\pi)=-\gamma_i(\theta,\phi)\quad (i=1,2). \tag{D.17}$$

The effective Hamiltonian on S^2 is given by

$$H=\frac{1}{R}\int\sin\theta d\theta d\phi\,(\gamma_1^\dagger(\theta,\phi),\gamma_2^\dagger(\theta,\phi))\begin{pmatrix}0 & -i\partial_\theta-\frac{1}{\sin\theta}\partial_\phi-\frac{i\cot\theta}{2}\\ -i\partial_\theta+\frac{1}{\sin\theta}\partial_\phi-\frac{i\cot\theta}{2} & 0\end{pmatrix}\begin{pmatrix}\gamma_1(\theta,\phi)\\ \gamma_2(\theta,\phi)\end{pmatrix}, \tag{D.18}$$

b. $(3+1)d$ *topological superconductor*

Let $\psi(\boldsymbol{x})=\{\psi_\sigma(\boldsymbol{x})\}_{\sigma=\uparrow,\downarrow}$ be a two-orbital complex fermion. We consider a topological superconductor on a ball of radius R,

$$H=\int d^3\boldsymbol{x}\,\Psi^\dagger(\boldsymbol{x})\Big[-i\tau_x\boldsymbol{\sigma}\cdot\boldsymbol{\partial}+m(r)\tau_z\Big]\Psi(\boldsymbol{x}),\quad \Psi(\boldsymbol{x})=(\psi_\uparrow(\boldsymbol{x}),\psi_\downarrow(\boldsymbol{x}),\psi_\downarrow^\dagger(\boldsymbol{x}),-\psi_\uparrow^\dagger(\boldsymbol{x}))^T. \tag{D.19}$$

From (D.13), complex fermion operators $\gamma_i^\dagger(\theta,\phi)(i=1,2)$ for gapless surface excitations are given by

$$\begin{aligned}\gamma_1^\dagger(\theta,\phi) &\sim \Big[-e^{-i\frac{\phi}{2}}\sin\frac{\theta}{2}\big\{i\psi_\uparrow^\dagger(r,\theta,\phi)+\psi_\downarrow(r,\theta,\phi)\big\}+e^{i\frac{\phi}{2}}\cos\frac{\theta}{2}\big\{i\psi_\downarrow^\dagger(r,\theta,\phi)-\psi_\uparrow(r,\theta,\phi)\big\}\Big]e^{-\int^r m(r')dr'},\\ \gamma_2^\dagger(\theta,\phi) &\sim \Big[e^{-i\frac{\phi}{2}}\cos\frac{\theta}{2}\big\{\psi_\uparrow^\dagger(r,\theta,\phi)+i\psi_\downarrow(r,\theta,\phi)\big\}+e^{i\frac{\phi}{2}}\sin\frac{\theta}{2}\big\{\psi_\downarrow^\dagger(r,\theta,\phi)-i\psi_\uparrow(r,\theta,\phi)\big\}\Big]e^{-\int^r m(r')dr'}.\end{aligned} \tag{D.20}$$

They satisfy $\gamma_2^\dagger(\theta,\phi)=-\gamma_1(\theta,\phi)$ and the boundary condition,

$$\gamma_1(\theta,\phi+2\pi)=-\gamma_1(\theta,\phi). \tag{D.21}$$

The effective Hamiltonian on S^2 is given by

$$H=\frac{1}{R}\int\sin\theta d\theta d\phi\big(\gamma_1^\dagger(\theta,\phi),-\gamma_1(\theta,\phi)\big)\begin{pmatrix}0 & -i\partial_\theta-\frac{1}{\sin\theta}\partial_\phi-\frac{i\cot\theta}{2}\\ -i\partial_\theta+\frac{1}{\sin\theta}\partial_\phi-\frac{i\cot\theta}{2} & 0\end{pmatrix}\begin{pmatrix}\gamma_1(\theta,\phi)\\ -\gamma_1^\dagger(\theta,\phi)\end{pmatrix}, \tag{D.22}$$

Appendix E: Spin^c cobordism group $\Omega_3^{\mathrm{Spin}^c}(B\mathbb{Z}_n)$

Spin^c cobordism groups $\Omega_d^{\mathrm{Spin}^c}(BG)$ for a cyclic group $G=\mathbb{Z}_n$ for any dimensions are computed in Ref. 102. The result is

$$\widetilde{\Omega}_*^{\mathrm{Spin}^c}(B\mathbb{Z}_n)\cong\{A_*(n)\otimes\mathbb{Z}[\mathbb{C}P^2,\mathbb{C}P^4,\dots]\}\oplus\ker_*(\eta,n), \tag{E.1}$$

$$\ker_m(\eta,n)\cong\bigoplus_{j<m}\mathrm{Tor}(\Omega_j^{\mathrm{Spin}^c}),\quad A_{2k+1}(n)\cong(1-t)/((1-t)^{k+2}),\quad A_{2k}(n)=0, \tag{E.2}$$

where $\widetilde{\Omega}_*^{\mathrm{Spin}^c}(B\mathbb{Z}_n)$ is the kernel of the symmetry forgetting functor of $\mathbb{Z}_n$ symmetry, says,

$$\Omega_*^{\mathrm{Spin}^c}(B\mathbb{Z}_n)\cong\widetilde{\Omega}_*^{\mathrm{Spin}^c}(B\mathbb{Z}_n)\oplus\Omega_*^{\mathrm{Spin}^c}(pt). \tag{E.3}$$

$(1-t)$ and $((1-t)^{k+2})$ are $R(\mathbb{Z}_n)$-modules defined by ,

$$(1-t) = \{(1-t)f(t)|f(t) \in R(\mathbb{Z}_n)\}, \quad \big((1-t)^{k+2}\big) = \{(1-t)^{k+2}f(t)|f(t) \in R(\mathbb{Z}_n)\}. \tag{E.4}$$

$R(\mathbb{Z}_n) = \mathbb{Z}[t]/(1-t^n)$ is the representation ring of $\mathbb{Z}_n$. $\mathbb{Z}[\mathbb{C}P^2, \mathbb{C}P^4, \dots]$ represents contributions from bosonic SPT phases. As will be explained in Sec. E 2, the ideal $(1-t)$ means non-chiral SPT phases of free fermions and the quotient $(1-t)/((1-t)^k)$ is interpreted as the breakdown of free fermionic classification by interactions.

For our purposes, only $\Omega_3^{\mathrm{Spin}^c}(B\mathbb{Z}_n)$ is needed, in which there is no contribution from bosonic SPT phases. We have

$$\Omega_3^{\mathrm{Spin}^c}(B\mathbb{Z}_n) \cong (1-t)/((1-t)^3). \tag{E.5}$$

Calculations for some n are illustrated in Sec. E 3.

1. Smith homomorphism

Pin^c cobordism groups $\Omega_{2k}^{\mathrm{Pin}^c}(pt)$ in even spacetime dimensions are isomorphic to the Spin^c cobordism group in one-higher spacetime dimensions with $\mathbb{Z}_2$ on-site symmetry. The Smith homomorphism provides an isomorphism[102]

$$\widetilde{\Omega}_{2k+1}^{\mathrm{Spin}^c}(B\mathbb{Z}_2) \xrightarrow{\cong} \widetilde{\Omega}_{2k}^{\mathrm{Pin}^c}(pt) \tag{E.6}$$

in the present case. The l.h.s. means $(2k+1)d$ SPT phases of complex fermions with onsite $\mathbb{Z}_2$ symmetry, which can be computed by the formula (E.1, E.2). The r.h.s. represents $(2k)d$ SPT phases of complex fermions with orientation reversing symmetry (class A with reflection symmetry or class AIII, say). We show some examples in low dimensions

$$\widetilde{\Omega}_2^{\mathrm{Pin}^c}(pt) \cong \widetilde{\Omega}_3^{\mathrm{Spin}^c}(B\mathbb{Z}_2) = \mathbb{Z}_4[A_3] = \mathbb{Z}_4, \tag{E.7}$$

$$\widetilde{\Omega}_4^{\mathrm{Pin}^c}(pt) \cong \widetilde{\Omega}_5^{\mathrm{Spin}^c}(B\mathbb{Z}_2) = \mathbb{Z}_8[A_5] \oplus (\mathbb{Z}_2[A_1] \otimes \mathbb{Z}[\mathbb{C}P^2]) = \mathbb{Z}_8 \oplus \mathbb{Z}_2, \tag{E.8}$$

$$\widetilde{\Omega}_6^{\mathrm{Pin}^c}(pt) \cong \widetilde{\Omega}_7^{\mathrm{Spin}^c}(B\mathbb{Z}_2) = \mathbb{Z}_{16}[A_7] \oplus (\mathbb{Z}_4[A_3] \otimes \mathbb{Z}[\mathbb{C}P^2]) = \mathbb{Z}_{16} \oplus \mathbb{Z}_4, \tag{E.9}$$

$$\begin{aligned}\widetilde{\Omega}_8^{\mathrm{Pin}^c}(pt) \cong \widetilde{\Omega}_9^{\mathrm{Spin}^c}(B\mathbb{Z}_2) &= \mathbb{Z}_{32}[A_9] \oplus (\mathbb{Z}_8[A_5] \otimes \mathbb{Z}[\mathbb{C}P^2]) \oplus (\mathbb{Z}_2[A_1] \otimes \mathbb{Z}[\mathbb{C}P^4]) \oplus (\mathbb{Z}_2[A_1] \otimes \mathbb{Z}[(\mathbb{C}P^2)^2]) \\ &= \mathbb{Z}_{32} \oplus \mathbb{Z}_8 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2.\end{aligned} \tag{E.10}$$

Here we used the relation

$$(1-t)/\big((1-t)^{k+2}\big) = (1-t)/\big(2^{k+1}(1-t)\big) = \mathbb{Z}_{2^{k+1}} \tag{E.11}$$

in the $R(\mathbb{Z}_2)$-module. (E.11) is the breakdown formula of the free fermion topological phases in class A with reflection symmetry or class AIII, which is consistent with Refs. 100 and 104.

2. K-theory classification

The K-theory classification of $(2d+1)$-dimensional class A topological insulators with on-site G symmetry is given by the G-equivariant complex K-theory[9,122,123]

$$K_G(S^{2d}) \cong R(G) \otimes K(S^{2d}) = R(G) \otimes (\mathbb{Z} \oplus \mathbb{Z}) \cong R(G) \oplus R(G), \tag{E.12}$$

where G trivially acts on S^{2d}. We introduce the reduced K-theory to remove the trivial contribution from a point in S^{2d} as

$$\widetilde{K}_G(S^{2d}) = R(G), \quad K_G(S^{2d}) = \widetilde{K}_G(S^{2d}) \oplus K_G(pt) \tag{E.13}$$

Here, $R(G)$ is the representation ring of G, which is generated by irreps. of G and multiplicative structure is induced by the tensor product of representations.

For our purposes, we consider $\widetilde{K}_{\mathbb{Z}_n}(S^2) \cong R(\mathbb{Z}_n)$ that represents the classification for $\mathbb{Z}_n$ symmetry in 2 space dimensions. The representation ring is given by

$$R(\mathbb{Z}_n) = \mathbb{Z}[t]/(1-t^n) = \{m_0 + m_1 t + \dots + m_{n-1}t^{n-1}|m_p \in \mathbb{Z}\}. \tag{E.14}$$

A base element $t^p \in R(\mathbb{Z}_n)$ corresponds to the following chiral Chern insulator, in which the complex fermion operator is in the the $e^{-2\pi ip/n}$ representation of $\mathbb{Z}_n$ symmetry:

$$H_{t^p} = \sum_{\boldsymbol{k}} \psi^\dagger(\boldsymbol{k})\big[k_x\sigma_x + k_y\sigma_y + (m-k^2)\sigma_z\big]\psi(\boldsymbol{k}), \quad (m>0), \tag{E.15}$$

$$\psi(\boldsymbol{k}) = (\psi_1(\boldsymbol{k}), \psi_2(\boldsymbol{k}))^T, \quad U\psi^\dagger(\boldsymbol{k})U^{-1} = e^{-2\pi ip/n}\psi^\dagger(\boldsymbol{k}). \tag{E.16}$$

For general insulators specified by $f(t) = m_0 + m_1 t + \cdots + m_{n-1}t^{n-1} \in R(\mathbb{Z}_n)$, the 1st Chern number is given by the sum $\mathrm{ch}_1\big(f(t)\big) = f(1) = m_0 + m_1 + \cdots + m_{n-1}$. Thus, the subgroup representing non-chiral phases is characterized by $f(1) = 0$, which is the ideal $(1-t)$,

$$(1-t) = \{(1-t)f(t) | f(t) \in R(\mathbb{Z}_n)\}, \tag{E.17}$$

where the basis is spanned by

$$\{1-t, 1-t^2, \dots, 1-t^{n-1}\}. \tag{E.18}$$

Here $1-t^p \in R(\mathbb{Z}_n)$ represent the following non-chiral topological insulators,

$$\begin{aligned} H_{1-t^p} = &\sum_{\boldsymbol{k}} \psi_\uparrow^\dagger(\boldsymbol{k})\big[k_x\sigma_x + k_y\sigma_y + (m-k^2)\sigma_z\big]\psi_\uparrow(\boldsymbol{k}) \\ &+ \sum_{\boldsymbol{k}} \psi_\downarrow^\dagger(\boldsymbol{k})\big[k_x\sigma_x - k_y\sigma_y + (m-k^2)\sigma_z\big]\psi_\downarrow(\boldsymbol{k}), \quad (m>0), \end{aligned} \tag{E.19}$$

where

$$\begin{aligned} &\psi_s(\boldsymbol{k}) = (\psi_{s,1}(\boldsymbol{k}), \psi_{s,2}(\boldsymbol{k}))^T, \quad (s = \uparrow, \downarrow), \\ &U\psi_\uparrow^\dagger(\boldsymbol{k})U^{-1} = \psi_\uparrow^\dagger(\boldsymbol{k}), \quad U\psi_\downarrow^\dagger(\boldsymbol{k})U^{-1} = e^{-2\pi ip/n}\psi_\downarrow^\dagger(\boldsymbol{k}). \end{aligned} \tag{E.20}$$

3. Calculations of $(1-t)/((1-t)^3)$

In this section, we illustrate the computation of the breakdown formula

$$\Omega_3^{\mathrm{Spin}^c}(B\mathbb{Z}_n) \cong (1-t)/((1-t)^3). \tag{E.21}$$

We show it for $n = 2, 3, 4$.

a. $n = 2$

Since $(1-t)^3 = 4 - 4t$ in $R(\mathbb{Z}_2)$, we have

$$\Omega_3^{\mathrm{Spin}^c}(B\mathbb{Z}_2) \cong (1-t)/((1-t)^3) = (1-t)/(4-4t) \cong \mathbb{Z}_4[1-t]. \tag{E.22}$$

Thus, the topological classification is given by $\mathbb{Z}_4$ and it is generated by the non-chiral topological insulator H_{1-t}.

b. $n = 3$

In the representation ring $R(\mathbb{Z}_3) = \mathbb{Z}[t]/(1-t^3)$, $\big((1-t)^3\big)$ reads $\big((1-t)^3\big) = (-3t + 3t^2)$, which is spanned by

$$\{-3t + 3t^2, t(-3t+3t^2)\} = \{3(1-t) - 3(1-t^2), 3(1-t^2)\} \sim \{3(1-t), 3(1-t^2)\}. \tag{E.23}$$

This leads to

$$\Omega_3^{\mathrm{Spin}^c}(B\mathbb{Z}_2) \cong (1-t)/((1-t)^3) = \mathbb{Z}_3[1-t] \oplus \mathbb{Z}_3[1-t^2]. \tag{E.24}$$

H_{1-t} and H_{1-t^2} provide two independent generators of the $\mathbb{Z}_3$ classification.

c. $n = 4$

$((1-t)^3) = (1 - 3t + 3t^2 - t^3)$ is spanned by

$$
\begin{aligned}
&\{1 - 3t + 3t^2 - t^3 . t(1 - 3t + 3t^2 - t^3), t^2(1 - 3t + 3t^2 - t^3)\} \\
&= \{3(1-t) - 3(1-t^2) + (1-t^3), -(1-t) + 3(1-t^2) - 3(1-t^3), (1-t) - (1-t^2) + 3(1-t^3)\} \\
&\sim \{3(1-t) - 3(1-t^2) + (1-t^3), 8(1-t), 2(1-t^2)\},
\end{aligned}
\tag{E.25}
$$

then,

$$
(1-t)/((1-t)^3) = \mathbb{Z}_8[1-t] \oplus \mathbb{Z}_2[1-t^2]. \tag{E.26}
$$

The $\mathbb{Z}_8$ and $\mathbb{Z}_2$ direct summands are generated by the non-chiral topological phases, H_{1-t} and H_{1-t^2}, respectively.

Appendix F: Calculations of $\prod_{n=1}^{\infty}(1 + e^{i\phi}(-q)^n)^n$

In this appendix, we compute

$$
I(q = e^{-\delta}, z) = \sum_{n=1}^{\infty} n \ln(1 + z(-q)^n) = \sum_{n=1}^{\infty} (2n) \ln(1 + zq^{2n}) + \sum_{n=1}^{\infty} (2n-1) \ln(1 - zq^{2n-1}) \tag{F.1}
$$

with a pure phase $z = e^{i\phi}$. Using the Cahen-Mellin integral,

$$
\begin{aligned}
\sum_{n=1}^{\infty} (2n) \ln(1 + zq^{2n}) &= -\sum_{n=1}^{\infty} (2n) \sum_{r=1}^{\infty} r^{-1} (-z)^r q^{2nr} \\
&= -\sum_{n=1}^{\infty} (2n) \sum_{r=1}^{\infty} r^{-1} (-z)^r \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} ds (2\delta n r)^{-s} \Gamma(s) \\
&= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} ds \delta^{-s} \Big[-2^{1-s} \Gamma(s) \zeta(s-1) \mathrm{Li}_{s+1}(-z) \Big], \\
\sum_{n=1}^{\infty} (2n-1) \ln(1 - zq^{2n-1}) &= -\sum_{n=1}^{\infty} (2n-1) \sum_{r=1}^{\infty} r^{-1} z^r q^{(2n-1)r} \\
&= -\sum_{n=1}^{\infty} (2n-1) \sum_{r=1}^{\infty} r^{-1} z^r \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} ds (\delta(2n-1)r)^{-s} \Gamma(s) \\
&= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} ds \delta^{-s} \Big[-\Gamma(s)(1 - 2^{1-s}) \zeta(s-1) \mathrm{Li}_{s+1}(z) \Big].
\end{aligned}
\tag{F.2}
$$

We used

$$
\zeta(s) = \frac{1}{1 - 2^{-s}} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^s} \quad (\mathrm{Re}(s) > 1), \tag{F.3}
$$

and introduced the polylogarithm function

$$
\mathrm{Li}_s(z) := \sum_{n=1}^{\infty} \frac{z^n}{n^s}. \tag{F.4}
$$

Then,

$$
I(q = e^{-\delta}, z) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} ds \delta^{-s} \Big[-2^{1-s} \Gamma(s) \zeta(s-1) \mathrm{Li}_{s+1}(-z) - (1 - 2^{1-s}) \Gamma(s) \zeta(s-1) \mathrm{Li}_{s+1}(z) \Big]. \tag{F.5}
$$

The integrand has a double pole at $s=0$ at $z=\pm 1$ since $\mathrm{Li}_{s+1}(1)=\zeta(s+1)$. From the contour integral, for small δ, we have following expansions:

$$\begin{aligned} I(q=e^{-\delta},z=1) &= -\frac{1}{8}\zeta(3)\delta^{-2}+\frac{1}{12}-\ln(A)+\frac{1}{12}\ln(\delta)+\frac{17}{2880}\delta^2+\cdots, \\ I(q=e^{-\delta},z=-1) &= -\frac{1}{8}\zeta(3)\delta^{-2}-\frac{1}{6}-\frac{1}{12}\ln(2)+2\ln(A)-\frac{1}{6}\ln(\delta)-\frac{13}{2880}\delta^2+\cdots, \end{aligned} \tag{F.6}$$

for $z=\pm 1$, and for $z\neq \pm 1$

$$\begin{aligned} I(q=e^{-\delta},z=e^{i\phi}) &= -\frac{1}{8}\mathrm{Li}_3(e^{2i\phi})\delta^{-2}+\frac{1}{12}\big\{\ln(1-e^{i\phi})-2\ln(1+e^{i\phi})\big\}+\frac{1-15\cos\phi}{480\sin^2\phi}\delta^2+\cdots \\ &= -\frac{1}{8}\mathrm{Li}_3(e^{2i\phi})\delta^{-2}+\frac{1}{12}\big\{\ln|2\sin\frac{\phi}{2}|-2\ln|2\cos\frac{\phi}{2}|\big\}+\frac{1}{12}\cdot\left\{\begin{array}{ll} -\frac{\pi i}{2}-\frac{i\phi}{2} & (0<\phi<\pi) \\ \frac{\pi i}{2}-\frac{i\phi}{2} & (-\pi<\phi<0) \end{array}\right\}+\frac{1-15\cos\phi}{480\sin^2\phi}\delta^2+\cdots. \end{aligned} \tag{F.7}$$

Here, $A\cong 1.2824\dots$ is the Glaisher-Kinkelin constant. Especially,

$$\begin{aligned} &I(q=e^{-\delta},z=\pm i)=\frac{3}{32}\zeta(3)\delta^{-2}-\frac{1}{24}\ln(2)\mp\frac{\pi i}{16}+\frac{\delta^2}{480}+\cdots \\ &I(q=e^{-\delta},z=e^{i\phi})+I(q=e^{-\delta},z=-e^{-i\phi}) \\ &\quad=-\frac{1}{8}\big\{\mathrm{Li}_3(e^{2i\phi})+\mathrm{Li}_3(e^{-2i\phi})\big\}\delta^{-2}-\frac{1}{12}\ln|2\sin\phi|+\left\{\begin{array}{ll} -\frac{\pi i}{8} & (0<\phi<\pi) \\ \frac{\pi i}{8} & (-\pi<\phi<0) \end{array}\right\}+\frac{1}{240\sin^2\phi}\delta^2+\cdots. \end{aligned} \tag{F.8}$$

1. The imaginary part of $I(q,e^{2\pi ix})$

If we look at the imaginary part of $I(q,e^{2\pi ix})$, we obtain the following formula:

$$\begin{aligned} \mathrm{Im}\big[I(q=e^{-\delta},z=e^{2\pi ix})\big] &= \frac{1}{2i}\big[I(q=e^{-\delta},z=e^{2\pi ix})-I(q=e^{-\delta},z=e^{-2\pi ix})\big] \\ &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}ds\delta^{-s}\Big[-2^{1-s}\Gamma(s)\zeta(s-1)\frac{\mathrm{Li}_{s+1}(-e^{2\pi ix})-\mathrm{Li}_{s+1}(-e^{-2\pi ix})}{2i} \\ &\qquad -(1-2^{1-s})\Gamma(s)\zeta(s-1)\frac{\mathrm{Li}_{s+1}(e^{2\pi ix})-\mathrm{Li}_{s+1}(e^{-2\pi ix})}{2i}\Big]. \end{aligned} \tag{F.9}$$

In this expression, poles of $\Gamma(s)$ at odd negative integers are canceled with zeros of $\zeta(s-1)$, and also poles of $\Gamma(s)$ at even negative integers are also canceled since

$$\mathrm{Li}_{-n}(z)+(-1)^n\mathrm{Li}_{-n}(z^{-1})=0\quad(n=1,2,3,\dots). \tag{F.10}$$

The integrand has poles only at $s=0,2$ and we get an exact result

$$\begin{aligned} \mathrm{Im}\big[I(q=e^{-\delta},z=e^{2\pi ix})\big] &= \left\{\begin{array}{ll} -\frac{\pi}{6}B_1(x+\frac{1}{2})+\frac{\pi}{12}B_1(x+1)-\frac{\pi^3}{12}B_3(2x+1)\delta^{-2} & (-1/2<x<0) \\ -\frac{\pi}{6}B_1(x+\frac{1}{2})+\frac{\pi}{12}B_1(x)-\frac{\pi^3}{12}B_3(2x)\delta^{-2} & (0<x<1/2) \end{array}\right. \\ &= \left\{\begin{array}{ll} -\frac{\pi}{12}(x-\frac{1}{2})-\frac{2\pi^3}{3}x(x+\frac{1}{4})(x+\frac{1}{2})\delta^{-2} & (-1/2<x<0) \\ -\frac{\pi}{12}(x+\frac{1}{2})-\frac{2\pi^3}{3}x(x-\frac{1}{4})(x-\frac{1}{2})\delta^{-2} & (0<x<1/2) \end{array}\right. \end{aligned} \tag{F.11}$$

Here, $B_1(x)=x-\frac{1}{2}, B_3(x)=x(x-\frac{1}{2})(x-1)$ are the Bernoulli polynomials. At $x=\pm\frac{1}{4}$, $\mathrm{Im}\big[I(q=e^{-\delta},z=e^{2\pi ix})\big]$ is δ-independent, which is the origin of the very sharp plateau structure in the expectation value of partial inversion in $(3+1)d$ superconductors in Fig. 11.

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