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We construct a dual bosonized description of a massless Majorana fermion in $(2+1)d$. In contrast to Dirac fermions, for which a bosonized description can be constructed using a flux attachment procedure, neutral Majorana fermions call for a different approach. We argue that the dual theory is an $SO(N)_1$ Chern-Simons gauge theory with a critical $SO(N)$ vector bosonic matter field ($N \geq 3$). The monopole of the $SO(N)$ gauge field is identified with the Majorana fermion. We provide evidence for the duality by establishing the correspondence of adjacent gapped phases and by a parton construction. We also propose a generalization of the duality to N_f flavors of Majorana fermions, and discuss possible resolutions of a caveat associated with an emergent global Z_2 symmetry. Finally, we conjecture a dual description of an $\mathcal{N} = 1$ supersymmetric fixed point in $(2+1)d$, which is realized by tuning a single flavor of Majorana fermions to an interacting (Gross-Neveu) critical point.

I. INTRODUCTION

Recently, there has been a resurgence of interest in dualities in $(2 + 1)d$ quantum field theories[1–6], building on prior work on particle-vortex dualities of bosons[7,8] and dualities between Dirac fermions and bosonic theories[9–15]. In this paper we will discuss how analogous dualities for Majorana fermions can be constructed.

Let us first recall the well known $(1 + 1)d$ case of the quantum Ising model which admits a dual description in terms of fermionic variables. The fermions may be viewed as the bound state of a local spin flip and a domain wall topological defect. These new variables allow for a direct solution of the Ising model in terms of free Majorana fermions. In particular, the critical point separating the Ising ordered and disordered phases can be described either as a gapless Majorana fermion, or in terms of bosonic variables as a critical point of a real scalar field. Schematically:

$$(\partial_\mu \phi)^2 + \phi^4 = \mathcal{L}_b^{(1+1)d} \leftrightarrow \mathcal{L}_f^{(1+1)d} = \bar{\xi}\gamma_\mu\partial_\mu\xi$$

which is understood to hold in the infrared (long distance/time) limit and the absence of mass terms on both sides represent tuning to the critical point. This may be viewed as a bosonization of $(1 + 1)d$ free Majorana fermions. How does this generalize to 2+1 dimensions? In $(2 + 1)d$, a Majorana mode $\mathcal{L}_f^{(2+1)d}$ when described in terms of a bosonic field will require a Chern-Simons (CS) gauge field to implement the statistical transmutation. We will argue that an $SO(N)$ gauge theory coupled to a bosonic matter field, and supplemented by a level one CS term provides the requisite dual description. In the dual language, the monopoles of the gauge theory, which have a Z_2 character, will be identified with the original Majorana fermions.

A. Review of bosonic dual of $(2+1)d$ Dirac fermions

It is well known that in $(2 + 1)d$ a CS gauge theory allows a "bosonic" description of fermions. A CS gauge theory binds together a particle and a gauge flux, thereby changing the statistics of the particle. This flux-attachment procedure certainly applies when the particles are gapped, but it may also imply a dual description of certain critical points. For instance, it was proposed that a single two-component Dirac fermion in $(2 + 1)d$

$$\mathcal{L}_\psi = \bar{\psi}\gamma_\mu(\partial_\mu - iA_\mu)\psi + m\bar{\psi}\psi \qquad (1)$$

has an equivalent description[5,6,9–11]:

$$\begin{aligned}
\mathcal{L}_\phi &= |(\partial_\mu - ia_\mu)\phi|^2 + r|\phi|^2 + g|\phi|^4 \\
&+ \frac{i}{4\pi}(a - A)d(a - A) + 2\mathrm{CS}_g \qquad (2)
\end{aligned}$$

where ϕ is a complex boson, a_μ is a dynamical $U(1)$ gauge field, and A_μ is a background $U(1)$ gauge field. The term CS_g (with coefficient 1) is a gravitational Chern-Simons term (see appendix B) encoding chiral central charge $c_- = 1/2$. The Dirac fermion ψ in $\mathcal{L}_\psi$ is regularized in such a way that for $m > 0$ one gets a trivial insulator with $\sigma_{xy}^A = c_- = 0$, and for $m < 0$ a Chern insulator with Chern-number 1, *i.e.* $\sigma_{xy}^A = c_- = 1$. Such a regularization can be obtained by starting with two gapless Dirac cones on a $2d$ lattice and gapping one of them out.

The strong interpretation of the duality is that $\mathcal{L}_\phi$ in Eq. (2) at $r = 0$ flows in the infra-red to a conformal field theory which we call the $U(1)$-Wilson-Fisher-Chern-Simons (WFCS) fixed point, and it is identical to a single noninteracting massless Dirac fermion $\mathcal{L}_\psi$ at $m = 0$. Though not proven, this conjecture is consistent with various observations:

(*i*) The Hilbert spaces and symmetries of the two theories match. The operator dictionary between the theories $\mathcal{L}_\phi$ and $\mathcal{L}_\psi$ is as follows: the electric current of fermions $\bar{\psi}\gamma^\mu\psi$ maps to the flux current of a, $\frac{1}{2\pi}\epsilon^{\mu\nu\lambda}\partial_\nu a_\lambda$. Moreover, the fermion operator ψ is the flux 2π space-time monopole of a, which indeed creates A charge of 1. To see that this object is a fermion, imagine that the system is placed on a sphere S^2, with a uniform flux 2π of a piercing the sphere. Because of the self-CS term for a, this configuration carries a charge 1 (in addition to A charge -1). Since a is a dynamical gauge field, any state in a finite volume must carry zero a charge. The a charge can be neutralized by adding a boson ϕ to the system. Because of the presence of the flux, the angular momentum of ϕ will be half-odd-integer. Thus, this state has the same A charge and angular moment as the fermion ψ.

(*ii*) The mean field phase diagram of (2) matches the phase diagram of the Dirac fermion (1). Let's recall the phase diagram of $\mathcal{L}_\phi$. For $r > 0$, ϕ is gapped, and a is gapped due to the CS term. Since the level of CS term of a is 1, there is no intrinsic topological order: a attaches flux 2π to ϕ turning it into a fermion, which is identified with ψ - so there are no anyons present. Furthermore, integrating a out, we obtain the Hall conductivity $\sigma_{xy}^A = 0$, and also see that the background gravitational CS term in $\mathcal{L}_\phi$ is precisely cancelled out, so the final gravitational response is $c_- = 0$. Thus, the $r > 0$ phase exactly matches the trivial insulator realized by $\mathcal{L}_\psi$ for $m > 0$. Turning to $r < 0$, the ϕ field condenses, so a Higgs mass will be generated for a; at low energy, one may effectively set $a_\mu = 0$ in $\mathcal{L}_\phi$. Thus, we are left with a level $k = 1$ CS term for the background gauge field A and a gravitational response with $c_- = 1$. This is a real Chern insulator, which matches the $m < 0$ phase of $\mathcal{L}_\psi$.

B. Parton approach to Dirac duality

Let us take a $2d$ lattice model of spinless fermion c_j which carries a $U(1)$ global symmetry that we label $U(1)_A$, and represent the fermion c_j with the standard parton (slave particle) construction: $c_j = f_j b_j$, where f_j and b_j are slave fermion and boson operators. All operators have the correct commutation relation, with the following constraint on the Hilbert space on every site: $f_j^\dagger f_j + b_j^\dagger b_j = 1$. Besides the $U(1)_A$ global symmetry, f_j and b_j must also carry a $U(1)$ gauge symmetry $U(1)_a$ due to the gauge constraint:

$$U(1)_A : \quad c_j \to e^{i\theta}c_j, \quad f_j \to e^{i\theta}f_j,$$

$$U(1)_a : \quad b_j \to e^{i\theta'}b_j, \quad f_j \to e^{-i\theta'}f_j. \tag{3}$$

Now we design the mean field band structure of f_j to be a Chern insulator with Chern number 1. Let us assume that the mean field band structure of f_j is always gapped, so we can safely integrate out f_j, and generate a CS term for $a - A$ at level 1, as well as a gravitational CS term corresponding to $c_- = 1$, as in $\mathcal{L}_\phi$ in Eq. (2). The slave boson b_j can be either in a Mott insulator, or condense. Coarse-graining b_j into a continuum field ϕ, the transition between these two phases is described by $\mathcal{L}_\phi$. While we have already discussed the bulk phase diagram of $\mathcal{L}_\phi$, the parton construction gives us additional insight into edge physics. At the mean field level, a Chern insulator of f_j has a chiral edge mode with chiral central charge $c_- = 1$. When b_j is gapped, after coupling the edge to the dynamical $U(1)$ gauge field a, there are no gauge invariant degrees of freedom left at the boundary (or in technical terms the coset conformal field theory at the boundary is trivial): this is consistent with the conclusion that the $r > 0$ phase is a trivial insulator of the physical fermion c_j. On the other hand, when b_j condenses, the gauge field a is Higgsed, so the edge mode of f_j survives; in fact, when b is condensed, $c_j \sim f_j$, so this is just a $c_- = 1$ mode of c_j as should be present in a Chern-insulator.

The dual description (2) of a free Dirac fermion can be generalized[14,16] to a $U(N)$ gauge theory, where a is a $U(N)$ CS gauge field at level 1 and ϕ is a complex boson transforming in the fundamental of $U(N)$. Surprisingly, this theory is conjectured to be dual to a single free Dirac cone (1) for *any* N.[16] We review the details of this duality in appendix A. In fact, the dual description of a *Majorana* cone that we conjecture in this paper very closely parallels this $U(N)$ dual description of a Dirac cone. Note that this particular $U(N)$ duality is one instance of dualities between CS-matter theories with nonabelian gauge field and general level k, the duality exchanges the rank and level of the gauge group, as well as fermionic and bosonic matter. In the large-N limit with fixed N/k explicit analytical calculations at the critical point give very strong support for such dualities.[14,15]

II. DUALITY OF A SINGLE MAJORANA FERMION

The main purpose of this paper is to propose a bosonized dual description of a single two-component massless Majorana fermion in $(2+1)d$,

$$\mathcal{L}_\xi = \bar{\xi}\gamma_\mu\partial_\mu\xi + m\bar{\xi}\xi. \tag{4}$$

Let us consider the two phases of the Majorana fermion on tuning the mass term m from positive to negative value. We regularize (4) so that for $m > 0$, ξ realizes a trivial phase with $c_- = 0$, while for $m < 0$ it realizes a $p_x + ip_y$ superconductor with $c_- = 1/2$. Such a regularization can be provided by starting with a lattice model with two gapless Majorana cones and initially gapping one of them out to produce (4) with $m = 0$. Thus, the massless Majorana cone corresponds to the critical point between a trivial phase and a $p_x + ip_y$ superconductor. Now, the idea is to produce a dual theory that realizes the same two phases on tuning a parameter. Then at

the critical point we may conjecture the dual theory has the same infrared behavior as (4), given the paucity of available fixed points. Observe that we can also capture the same pair of gapped phases with the following theory of a boson coupled to an $SO(N)_1$ CS gauge field:

$$\mathcal{L}_b = |(\partial_\mu - ia_\mu)\phi|^2 + r|\phi|^2 + g|\phi|^4 + \text{CS}_{SO(N)}[a]_1 + N \cdot \text{CS}_g \tag{5}$$

Here, ϕ is an N-component real vector, a is an $SO(N)$ gauge field and

$$\text{CS}_{SO(N)}[a]_1 = \frac{i}{2 \cdot 4\pi} \text{tr}_{SO(N)} \left(a \wedge da - \frac{2i}{3} a \wedge a \wedge a \right). \tag{6}$$

The trace in Eq. (6) is in the vector representation of $SO(N)$ (for a more precise definition of the CS term, see appendix B).

Let's analyze the mean field phase diagram of (5). When $r > 0$, ϕ is gapped. The theory $SO(N)_1$ coupled to gapped bosonic matter gives a state with no intrinsic topological order; the vector boson ϕ is transmuted to a fermion by the CS field.[36] The $SO(N)_1$ CS gauge theory by itself has a chiral central charge $c_- = -N/2$, which exactly cancels the background gravitational CS term in Eq. (5). So the $r > 0$ phase is a trivial state with $c_- = 0$. On the other hand, for $r < 0$, ϕ condenses, which breaks the gauge group $SO(N)$ to $SO(N-1)$. At low energies, we can then take a to be an $SO(N-1)$ gauge field, obtaining an $SO(N-1)_1$ CS theory. Again, this is a state with no intrinsic topological order, but the background gravitational term in (5) is no longer fully cancelled, rather: $c_- = N/2 - (N-1)/2 = 1/2$. So the $r < 0$ phase is a $p_x + ip_y$ superconductor.

It may appear surprising that a strongly interacting theory such as Eq. 5 is dual to a free Majorana fermion. Let us keep in mind that a short range interaction between free Majorana fermions (and also Dirac fermions) is irrelevant in $(2+1)d$, thus a dual free Majorana fermion is unfine-tuned (the interaction between Majorana fermions becomes important when tuned to a critical point, which also has a interesting duality, and it will be discussed in section IID). The same reasoning also applies to all previous works that proposed the duality between a single free complex Dirac fermion and different strongly interacting theories. By contrast, in $(1+1)d$, short range interactions between fermions are marginal (sometimes marginally relevant), thus a bosonic theory in $(1+1)d$ is usually dual to interacting fermions in the infrared.

A. Parton approach to Majorana duality

To further motivate the dual theory (5) we utilize a parton construction. The simplest construction is: $c = \sum_{\alpha=1}^N \tilde\phi_\alpha \tilde\chi_\alpha$, where the Majorana parton $\tilde\chi_\alpha$ and slave boson $\tilde\phi_\alpha$ have an $O(N)$ redundancy. However, to recover the $SO(N)$ gauge structure from this construc-

tion requires further discussion, which we provide in appendix C. Instead, here we turn to a slightly different parton representation. Consider a lattice model for Majorana fermion c_j ($c_j^\dagger = c_j$). This time we introduce on each site j, N colors of slave Majorana fermions $\chi_{j,\alpha}$ ($\alpha = 1 \ldots N$) for odd N such that

$$c_j = (i)^{\frac{N-1}{2}} \prod_{\alpha=1}^N \chi_{j,\alpha}. \tag{7}$$

By construction χ_α is coupled to a dynamical $SO(N)$ gauge field. Now we design an identical mean field $p_x + ip_y$ superconductor band structure for each color of χ_α. At the mean field level, there are N chiral Majorana fermions at the boundary, which in total leads to chiral central charge $c_- = N/2$. However, if the $SO(N)$ gauge symmetry is unbroken, after coupling to the $SO(N)$ gauge field there will be no gauge invariant degrees of freedom left at the boundary, so we are left with $c_- = 0$. Also, integrating out χ_α would generate a CS term at level $k = 1$ for the $SO(N)$ gauge field, as well as a gravitational CS term at level N, as in Eq. (5). As already discussed, there is no topological order in the bulk, thus this state is again a trivial state of the physical Majorana fermion c_j.

Using the slave particles $\chi_{j,\alpha}$ we can also define a $SO(N)$ vector boson $\hat\phi_{j,\alpha}$[37]:

$$\hat\phi_{j,\alpha} \sim (i)^{\frac{N-1}{2}} \epsilon_{\alpha\alpha_1,\cdots\alpha_{N-1}} \chi_{j,\alpha_1} \cdots \chi_{j,\alpha_{N-1}}. \tag{8}$$

When $\hat\phi_\alpha$ condenses, it breaks the $SO(N)$ gauge group down to $SO(N-1)$, and one of the slave fermions (say χ_N) is no longer coupled to any gauge field, thus its topological band structure implies that the entire system is equivalent to one copy of $p_x + ip_y$ topological superconductor. Likewise, the edge mode associated to χ_N sees no gauge field and survives as a true $c_- = 1/2$ edge mode of a $p_x + ip_y$ superconductor. All other edge modes are, as before, eliminated by $SO(N-1)$ gauge field fluctuations. Thus, by coarse-graining $\hat\phi$ into a continuum field ϕ, we can describe a transition between a trivial state and a $p_x + ip_y$ superconductor by Eq. (5).

Notice that the integer N in the dual theory (5) needs not be odd. We could adjust our mean field construction, and take the band structure of $\chi_1, \cdots \chi_{N-1}$ to be an identical $p_x + ip_y$ superconductor, while placing χ_N into a trivial band structure. Then the mean field band structure already breaks the $SO(N)$ gauge symmetry to $O(N-1)$ gauge symmetry, and $\hat\phi_\alpha$, $\alpha = 1 \ldots N - 1$, reduces to a $SO(N-1)$ vector. The extra Z_2 subgroup of the $O(N-1)$ gauge symmetry can be broken by condensing the $SO(N-1)$ gauge singlet bosonic operator $\hat\phi_N = (i)^{\frac{N-1}{2}} \prod_{\alpha=1}^{N-1} \chi_\alpha$. Now condensing the $SO(N-1)$ vector $\hat\phi_\alpha$ also drives a transition from the trivial state of c_j to a $p_x + ip_y$ superconductor of c_j, and this time the transition is described by a $SO(N-1)$ vector field coupled to a $SO(N-1)$ gauge field with a CS term at level 1.

B. The dictionary

How do we represent the physical Majorana fermion in the dual theory (5)? In the case of Dirac duality (1) $\leftrightarrow$ (2) the physical electric charge in the Dirac theory mapped to the magnetic flux of the $U(1)$ gauge field a in the dual theory. Likewise, the fermion parity of the Majorana fermion ξ maps to the magnetic flux of the $SO(N)$ gauge field a in Eq. (5). Recall that the magnetic flux is classified by $\pi_1(SO(N)) = Z_2$. Indeed, imagine the system on a spatial sphere S^2. As usual, we place a magnetic flux through the sphere by dividing it into two hemispheres, and gluing the fields in the two hemispheres along the equator with a gauge transformation $g(\theta)$, $\theta \in [0, 2\pi]$. Such gauge transformations are classified by π_1 of the gauge group. In the case of the $SO(N)$ group, a simple representative for the single non-trivial magnetic flux sector on S^2 is obtained by considering an ordinary flux $2\pi m$ Dirac monopole in the $SO(2)$ subgroup of $SO(N)$ with $m = 1$. Note that by an $SO(N)$ rotation we can invert the magnetic flux in the $SO(2)$ subgroup, so m is, indeed, only defined modulo 2. The magnetic flux breaks the $SO(N)$ group down to $SO(2) \times SO(N-2)$ and the state on S^2 must be neutral under this reduced gauge group. As in the abelian case, the CS term in Eq. (5) leads to the monopole carrying an $SO(2)$ charge 1, so to make the monopole neutral we must act on it with the boson $\phi_1 + i\phi_2$. As before, the boson angular momentum is half-odd-integer because of the $SO(2)$ flux, so the angular momentum of the resulting state is half-odd-integer. We conclude that the $SO(N)$ monopole on S^2 carries charge under fermion parity, and identify the $SO(N)$ space-time monopole V_M with the Majorana fermion operator ξ. In particular, this discussion means that dynamical Z_2 $SO(N)$ monopoles are prohibited in the partition function of dual theory (5), as they violate fermion parity conservation.

One can also see that the $SO(N)$ monopole on S^2 will have a non-trivial fermion parity from the parton construction. Indeed, when there is a 2π flux of $SO(2)$ through the sphere, it will be seen by partons χ_1, χ_2, while χ_α, $\alpha = 3 \ldots N$ will see no flux. The ground state will then have $SO(2)$ charge 1. Since χ_α carry fermion parity, the ground state also carries $(-1)^F = -1$. The $SO(2)$ charge gets neutralized by adding a boson $\phi_1 + i\phi_2$. However, since ϕ's carry no fermion parity, $(-1)^F = -1$ is not affected.

More generally, for a spatial manifold Σ, a gauge field configuration a and boundary conditions (spin-structure) σ for the physical Majorana fermion ξ, the fermion parity in the dual theory (5) is expressed as[17,18]:

$$(-1)^F = (-1)^{\int_\Sigma W_2} (Arf[\Sigma, \sigma])^N \qquad (9)$$

Here, $W_2 \in H^2(\Sigma, Z_2)$ is the obstruction to lifting the $SO(N)$ gauge bundle to $Spin(N)$. $Arf(\Sigma, \sigma) = \pm 1$ is the Arf invariant, which computes the fermion parity of a $p_x + ip_y$ superconductor on Σ with spin-structure σ.[19] Thus, only the first term in (9) depends on the gauge

field, the second term is fixed once the boundary conditions for ξ are fixed. The first and second terms in Eq. (9) are correspondingly secretly encoded in the spin-structure dependence of CS action (6) and the gravitational CS term in (5).

Having established the equivalence of phases and operators in the free Majorana theory (4) and the dual theory (5), we conjecture that they are actually dynamically equivalent at their respective IR fixed points.

TABLE I: Duality Dictionary

Fermionic Theory	Bosonic Theory
Majorana: ξ	Monopole: V_M
"m": $\bar{\xi}\xi$	"r": $\phi \cdot \phi$

We note that the level-rank duality of Chern-Simons-matter gauge theories with $O(N)_k$ gauge group in the large-N, large-k limit has been proven in Ref. 14. Our conjecture (4) $\leftrightarrow$ (5) amounts to a statement that the duality continues to hold when $k = 1$ and N is finite (with a clarification that the precise form of the gauge group for $k = 1$ is $SO(N)$).

C. Z_2 global symmetry of the bosonic theory

One potential subtlety of the proposed duality (4) $\leftrightarrow$ (5) is that the dual theory actually has a global Z_2 symmetry. Indeed, (5) is invariant under $\phi \to O\phi$, with $O \in O(N)$; while for $O \in SO(N)$ this is a gauge transformation, $Z_2 = O(N)/SO(N)$ remains as a global symmetry. We can write down an "Ising order parameter" Φ for this global Z_2 symmetry,

$$\Phi(x) = \epsilon_{\alpha_1 \alpha_2 \ldots \alpha_N} (W(x, x_1)\phi(x_1))_{\alpha_1} (W(x, x_2)\phi(x_2))_{\alpha_2}$$
$$\ldots (W(x, x_N)\phi(x_N))_{\alpha_N} \qquad (10)$$

where $W(x, y)$ is an $SO(N)$ Wilson line and $\Phi(x)$ is obtained by taking the limit $x_i \to x$ and keeping the leading surviving terms. For instance, for $N = 3$, we have a Lorentz scalar, $\Phi = \epsilon^{\alpha\beta\gamma}\phi_\alpha (D^2\phi)_\beta (D^4\phi)_\gamma$, where $D_\mu = \partial_\mu - ia_\mu$. The presence of an extra global Z_2 symmetry in the dual theory is problematic, since the only obvious symmetry of the Majorana cone is fermion parity $(-1)^F$. Clearly, the Z_2 symmetry of the dual theory is not $(-1)^F$ since the local boson Φ transforms non-trivially under it. We conjecture the following resolution to this puzzle: the gap to Z_2 charged excitations in (5) remains finite across the critical point, i.e. Φ actually has exponentially decaying correlation functions at $r = 0$. Here's the evidence for this conjecture:

(i) Neither phase in the phase diagram has spontaneous Z_2 symmetry breaking, i.e. $\langle \Phi \rangle = 0$. Indeed, when ϕ develops an expectation value, the $O(N)$ symmetry is broken to $O(N-1)$ and the gauge symmetry $SO(N)$ to
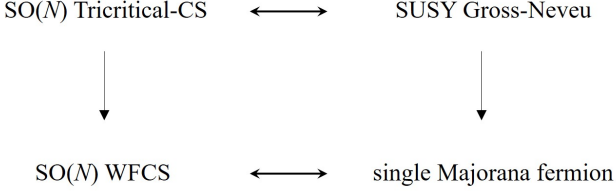
FIG. 1: The proposed duality and renormalization group flow. The double headed arrow stands for "dual to each other", while the single headed arrow represents the RG flow.

$SO(N-1)$, so $Z_2 = O(N)/SO(N) \rightarrow O(N-1)/SO(N-1)$ survives.

(ii) While Z_2 remains unbroken for both $r > 0$ and $r < 0$, these two regions could potentially realize different Z_2 symmetry protected phases. Closing of a Z_2 charge gap is then required at the transition. However, it can be shown that both $r > 0$ and $r < 0$ realize the same (trivial) Z_2 symmetry protected topological (SPT) phase. To see this, observe that our parton construction gives no Z_2 carrying edge modes on either side of the transition, which means that both phases realize a trivial Z_2 SPT (see appendix C for further details).

Given (i) and (ii), there is no requirement that the Z_2 charge gap closes at the transition, so we make the minimalist assumption that it remains finite and the effect of Z_2 global symmetry can be ignored in Eq. (5), consistent with the proposed duality.

We note that instead of considering an $SO(N)$ gauge theory in (5), we could make the gauge group $O(N)$ - i.e. gauge the extra Z_2 symmetry. Then the $r > 0$ phase will be a (trivial) Z_2 gauge theory (toric code) and the $r < 0$ phase will be a toric code + a $(p_x + ip_y)$ superconductor. The assumption that the Z_2 charge gap remains finite across the transition is equivalent to the toric code being an inert spectator. The $O(N)$ gauge theory is then conjectured to be dual to a Majorana cone (4) with a decoupled gapped toric code.

D.   Dual of the Gross-Neveu-Yukawa theory

The critical point of the $SO(N)$-WFCS theory is an infrared (IR) fixed point of an ultraviolet (UV) fixed point, which we call the $SO(N)$-Tricritical-CS theory. This theory corresponds to tuning g in Eq. (5) to a critical value g_c: at mean field $g_c = 0$ (we assume that the action is still bounded from below due to the existence of higher order terms in the polynomial of $|\phi|$), which is analogous to the tricritical Ising fixed point.

On the fermion side, a natural UV fixed point that flows to the free Majorana fermion in the IR is the Gross-Neveu-Yukawa fixed point:

$$\mathcal{L} = \bar{\xi}\gamma_\mu\partial_\mu\xi + (\partial_\mu\sigma)^2 + \lambda\sigma\bar{\xi}\xi + \frac{\tilde{\lambda}^2}{4}\sigma^4. \tag{11}$$

where σ is a real scalar. The relevant perturbation $s\sigma^2$ in Eq. (11) is dual to $(g - g_c)|\phi|^4$ in Eq. (5): when $s, g - g_c > 0$, Eq. (11) and the $SO(N)$-Tricritical-CS theory respectively flow to Eqs. (4) and (5). On the other hand, when $s, g - g_c < 0$, both theories have a first order transition between a trivial phase and a $p_x + ip_y$ superconductor.

An exact renormalization flow of Eq. (11) is difficult to compute, but if there is only one fixed point with nonzero λ and $\tilde{\lambda}$, then this fixed point must have $\lambda^* = \tilde{\lambda}^*$, and it is a supersymmetric $\mathcal{N} = 1$ conformal field theory[20–22]. Thus, our construction also conjectures that this $\mathcal{N} = 1$ supersymmetric conformal field theory is dual to the $SO(N)$-Tricritical-CS theory. The supersymmetry makes the following prediction about the scaling dimensions of the $SO(N)$-Tricritical-CS theory:

$$\Delta[V_M] - \Delta[|\phi|^2] = 1/2. \tag{12}$$

An analogous duality between the Dirac fermion Gross-Neveu fixed point and the $U(1)$-Tricritical-CS fixed point was conjectured in Ref. 5.

III.   DUALITY OF N_f FLAVORS OF MAJORANA FERMIONS

There is a generalization of our proposed duality to the case of N_f Majorana fermions with $SO(N_f)$ flavor symmetry. Take N_f Majorana fermions,

$$\mathcal{L} = \sum_I (\bar{\xi}_I\gamma_\mu\partial_\mu\xi_I + m\bar{\xi}_I\xi_I) \tag{13}$$

where $I = 1 \ldots N_f$. For $m > 0$, we have a trivial insulator with no $SO(N_f)$ response and $c_- = 0$; for $m < 0$, we have chiral central charge $c_- = N_f/2$ and CS $SO(N_f)$ response at level $k = 1$:

$$\mathcal{L} = \mathrm{CS}_{SO(N_f)}[A]_1 + N_f \cdot \mathrm{CS}_g \tag{14}$$

where A is the background $SO(N_f)$ gauge field.

We propose that the above theory is dual to N_f flavors of real bosons ϕ_I^α coupled to an $SO(N)$ gauge field a_μ at level $k = 1$ (as before, $\alpha = 1 \ldots N$ is the color gauge index):

$$\mathcal{L} = \sum_I |(\partial_\mu - ia_\mu)\phi_I|^2 + r(\mathrm{tr}M) + u(\mathrm{tr}M)^2 + v\mathrm{tr}(M^2)$$
$$+ \mathrm{CS}_{SO(N)}[a]_1 + N \cdot \mathrm{CS}_g \tag{15}$$

where $M_{IJ} = \phi_I^\alpha\phi_J^\alpha$. The physical $SO(N_f)$ flavor symmetry simply acts on the flavor indices I of ϕ_I^α. We will assume $N \geq N_f + 2$. We further assume that either N is even and N_f is arbitrary, or N and N_f are both odd.[38] When $r > 0$, ϕ is not condensed, so integrating out gauge field a cancels out the gravitational CS term and gives rise to a trivial insulator.

We assume that $v > 0$ in Eq. 15.[39] In this case, in the phase $r < 0$, the energy is minimized when

$\langle \phi_I^\alpha \phi_J^\alpha \rangle = \langle M_{IJ} \rangle \sim \delta_{IJ}$ - *i.e.* this gauge invariant observable suggests that the flavor symmetry remains unbroken (as we will see more explicitly below) by the condensate of ϕ_I^α, or equivalently $\langle \phi_I^\alpha \rangle$ are N_f orthogonal $SO(N)$ vectors.

With $v > 0$, in the condensed phase of ϕ_I^α we can choose $\langle \phi_I^\alpha \rangle \sim \delta_I^\alpha$. The gauge group $SO(N)$ is broken down to $SO(N - N_f)$ (acting on $\alpha = N_f + 1 \ldots N$). Furthermore, the combination of an identical $SO(N_f)$ flavor rotation and an $SO(N_f)$ color rotation acting on $\alpha = 1 \ldots N_f$ leaves $\langle \phi_I^\alpha \rangle$ invariant, which means that the physical $SO(N_f)$ global symmetry remains unbroken. In the presence of a background $SO(N_f)$ gauge field A_μ, the components of the dynamical gauge field $a_\mu^{\alpha\beta}$ with $\alpha, \beta = 1 \ldots N_f$ get Higgsed to $a_\mu^{\alpha\beta} = A_\mu^{\alpha\beta}$, similarly $a_\mu^{\alpha\beta} = 0$ for $\alpha = 1 \ldots N_f$, $\beta = N_f + 1 \ldots N$. Finally, $a_\mu^{\alpha\beta}$ with $\alpha, \beta = N_f + 1 \ldots N$ is not Higgsed - let's refer to this non-Higgsed field as b_μ. Therefore, our effective action takes the form:

$$\mathcal{L} = \text{CS}_{SO(N-N_f)}[b]_1 + \text{CS}_{SO(N_f)}[A]_1 + N \cdot \text{CS}_g. \quad (16)$$

Integrating b out, we recover Eq. (14).

Just like in the $N_f = 1$ case, let us discuss a parton construction for the case with $SO(N_f)$ flavor symmetry. On every lattice site, we use the parton decomposition:

$$c_I = (i)^{\frac{N}{2}} \chi_I \prod_{\alpha=1}^{N} \chi^\alpha, \quad (17)$$

with even integer N. The site index is hidden. Under flavor symmetry

$$O(N_f): c_I \to R_{IJ} c_J, \quad \chi_I \to R_{IJ} \chi_J, \quad \chi^\alpha \to \chi^\alpha \quad (18)$$

There is also an $O(N)$ gauge symmetry:

$$O(N): \chi^\alpha \to V^{\alpha\beta} \chi^\beta, \quad \chi^I \to (\det V) \chi^I. \quad (19)$$

In order to get Eq. (15), we still design the mean field band structure of all χ^α to be an identical $p_x + ip_y$ topological superconductor. On the other hand, χ^I are chosen to have identical trivial band structures. Then the phase transition of Eq. (15) is the order-disorder transition of the bosonic operator $\hat{\phi}_I^\alpha = i\chi_I \chi^\alpha$, which is a vector of both $O(N_f)$ and $O(N)$. The field ϕ_I^α in Eq. (15) is the coarse-grained field of operator $\hat{\phi}_I^\alpha$.

Notice that the gauge symmetry in the above construction is $O(N)$. To match with our definition of the continuum theory (15), we want to break it to $SO(N)$. Just like in the $N_f = 1$, even N case, we can simply condense $\prod_\alpha \chi^\alpha$ (and keep it condensed throughout the phase diagram of Eq. (15)). The $O(N_f)$ flavor symmetry and the $SO(N)$ part of the gauge symmetry are preserved, but the reflection part of the $O(N)$ gauge symmetry is broken, as needed. A modified mean field band structure with one of χ^αs forming a trivial band allows us to realize Eq. (15) with odd N (and odd N_f).

IV. DISCUSSION

Our dual theory Eq. (5) obviously breaks time-reversal symmetry, while the single Majorana cone (4) could preserve the time-reversal symmetry, if the system is defined on the $2d$ boundary of a $3d$ topological superconductor in class DIII (the topological phase with index $\nu = 1$ is believed to be realized by the B-phase of superfluid He^3). For the Dirac fermion, the dual U(1)-WFCS theory (2) is believed to have an emergent time-reversal symmetry in the IR, which transforms the matter field ϕ into its vortex.[6] However, this simple solution does not apply to our $SO(N)$ gauge theory, as there is no known analogue of the boson-vortex duality for an $SO(N)$ matter field with $N \geq 3$. Thus, we do not yet understand how time-reversal is hidden in the dual theory (5). Moreover, in the case of the Dirac fermion, a manifestly T-invariant description is provided by another dual theory: QED$_3$ with a single Dirac fermion matter field.[1–3,6] Again, a manifestly T-invariant dual description of a Majorana cone is currently missing. Such a theory could provide a derivation of the surface topological order of class DIII topological phases with odd ν, such as the proposed SO(3)$_3$ non-abelian topological order with just two nontrivial particles[23–25]. We recall that in the Dirac context, the pairing of composite Dirac fermions led to an explicit derivation of the T-Pfaffian state, a surface topological order for the topological insulator[1–3,26,27]. We leave these questions to future work.

Appendix A: $U(N)$ duality.

A generalization[14,16] of the dual description (2) of a free Dirac fermion (1) is the $U(N)$-WFCS theory:

$$\mathcal{L} = |(\partial_\mu - ia_\mu)\phi| + r|\phi|^2 + u|\phi|^4 + \text{CS}_{U(N)}[a-A]_1 + 2N \cdot \text{CS}[g] \quad (A1)$$

where ϕ is an N-component complex vector, a is a $U(N)$ gauge field, A as before is a background $U(1)$ gauge field, and

$$\text{CS}_{U(N)}[a]_1 = \frac{i}{4\pi}\text{tr}_N\left(a \wedge da - \frac{2i}{3}a \wedge a \wedge a\right) \quad \text{(A2)}$$

The trace above is taken in the fundamental representation of $U(N)$ and $a - A$ in Eq. (A1) should be understood as $a - A \cdot \mathbb{1}$, where $\mathbb{1}$ is the identity matrix. Surprisingly, the theory (A1) is conjectured to be dual to a single free Dirac cone (1) for *any* N.[16] Again, the symmetries and operators match: the electric current $\bar{\psi}\gamma^\mu\psi$ maps to the flux current $\frac{1}{4\pi}\epsilon^{\mu\nu\lambda}\text{tr}(f_{\mu\nu})$. Recalling that $\pi_1(U(N)) = Z$, we see that the quantization of electric charge matches, and the fermion ψ maps to the monopole of $U(N)$. The phase diagram also matches. When ϕ is gapped, we have a $U(N)$ CS gauge theory at level 1, which carries no intrinsic topological order. Integrating over a, we find that $\sigma_{xy}^A = 0$. Moreover, since the chiral central charge of $U(N)$ CS gauge theory at level 1 is $c_- = -N$, the background gravitational CS term in Eq. (A1) is exactly cancelled to give $c_- = 0$. On the other hand, when $r < 0$, ϕ condenses and $U(N)$ is broken down to a $U(N-1)$ subgroup. At low energies, we may keep only the components of a in this subgroup, so that a is now an $N - 1$ component Hermitian matrix. Then $\mathcal{L}$ effectively takes the form:

$$\mathcal{L} = \text{CS}_{U(N-1)}[a - A]_1 + \frac{i}{4\pi}AdA + 2N \cdot \text{CS}[g] \quad \text{(A3)}$$

Again, because the level of a is 1 , there is no topological order. Integrating over a, we are left with $\sigma_{xy}^A = 1$, and $c_- = N - (N-1) = 1$, *i.e.* this phase is a Chern insulator.

We can again obtain the theory (A1) through a parton construction. Represent the physical fermions $c_j = \sum_\alpha b_{\alpha,j}^\dagger f_{\alpha,j}$ where $\alpha = 1 \ldots N$. There is a $U(N)$ gauge redundancy in this description:

$$U(N)_a: \quad b_j \to U_j b_j, \quad f_j \to U_j f_j \quad \text{(A4)}$$

which will lead to the emergence of a $U(N)$ gauge field a. As before, we make f carry the $U(1)_A$ global charge (Eq. (3), first line). Now, choose an identical Chern-number 1 band structure for each f_α. Integrating f_α out, we obtain the CS term for $a - A$ in Eq. (A1) as well as a gravitational CS term with coefficient $2N$, corresponding to the chiral central charge N. The slave-boson field b can be in a gapped or condensed phase, the transition between these phases is described by (A1). While we already discussed the nature of the two phases above, further insight is provided by the edge structure. At the mean field level, the edge has N chiral $c = 1$ fermions. However, when b is gapped, there are no $U(N)$ invariant degrees of freedom on the edge, so after including fluctuations of a, the edge becomes a trivial theory with $c = 0$. On the other hand, when b condenses, for instance along the Nth direction $\langle b_\alpha\rangle \sim \delta_{\alpha,N}$, then the $U(N)$ gauge group is broken down to $U(N-1)$. As a result, the $c = 1$ chiral mode of f_N now becomes a physical electron mode, while the modes f_α, $\alpha = 1 \ldots N - 1$ are still gapped by fluctuations of a. Thus, the edge is now a $c = 1$ A-charged mode, as one expects in a Chern insulator.

Appendix B: $SO(N)$ Chern-Simons gauge theory.

In this appendix, we give a more careful definition of the $SO(N)$ CS gauge theory in Eq. (6).[29] Strictly speaking, Eq. (6) is only meaningful when a is a 1-form. However, there can be non-trivial $SO(N)$ bundles over our space-time manifold: we have to sum over such bundles. In general, following our parton construction below Eq. (7), we will define the CS action for the $SO(N)$ connection a as the partition function of N identical copies of a $p+ip$ superconductor coupled to an $SO(N)$ gauge field. Representing the $p + ip$ superconductor by a Majorana fermion χ with $m < 0$,

$$\mathcal{L} = \bar{\chi}(D_a + m)\chi \quad \text{(B1)}$$

where χ is an N-color Majorana fermion, $D_a = \gamma^\mu(\partial_\mu + i\omega_\mu - ia_\mu)$, and ω_μ is the spin connection, we define:

$$\exp\left(-\text{CS}_{SO(N)}[a]_1 - N \cdot \text{CS}_g\right) \equiv \lim_{m\to\infty} \frac{Z_f(-m)}{Z_f(m)}$$
$$= e^{-\pi i\eta(iD_a)} \quad \text{(B2)}$$

where

$$\eta = \frac{1}{2}(\eta(0) + N_0) \quad \text{(B3)}$$

and

$$\eta(s) = \sum_{\lambda\neq 0}\text{sgn}(\lambda)|\lambda|^{-s} \quad \text{(B4)}$$

where λ's are eigenvalues of iD_a, N_0 is the number of zero modes of iD_a and $\eta(0)$ is obtained by analytic continuation from large real s. The ratio of partition functions with $m < 0$ and $m > 0$ in (B2) is taken to cancel out any "non-topological" dependence of the action on the gauge field and the metric.

Implicitly, in defining the Dirac operator D_a, we have given the fermions χ not only an $SO(N)$ connection, but also a spin-structure. In our parton construction (7) such a spin-structure will be inherited from the physical electron c. In general, the action (B2) depends on the spin-structure, which is a sign that the underlying microscopic theory has a local electron operator.

We can use Atiyah-Patodi-Singer (APS) theorem to rewrite $\eta(iD_a)$ as,

$$\eta = \int_{X_4}\left(\frac{1}{2 \cdot (2\pi)^2}\text{tr}_{SO(N)}f \wedge f\right.$$
$$\left. + \frac{N}{8 \cdot 24\pi^2}\text{tr}R \wedge R\right) \quad (\text{mod } 2) \quad \text{(B5)}$$

Here, X_4 is a four-dimensional (Euclidean) manifold that extends our physical three-dimensional (Euclidean) manifold M, i.e. $\partial X_4 = M$. Furthermore, both a and the spin-structure on M are assumed to extend to X_4. R is the Riemann tensor on X_4. It follows from the APS theorem that the right-hand-side of Eq. (B5) is independent of the particular extension, i.e. it vanishes mod 2 when X_4 has no boundary. Another way to see this without directly appealing to the APS theorem is as follows. When X_4 is closed, we have

$$p_1 = \frac{1}{2 \cdot (2\pi)^2} \int_{X_4} \text{tr}_{SO(N)} f \wedge f \quad \text{(B6)}$$

and

$$\sigma = -\frac{1}{24\pi^2} \int_{X_4} \text{tr} R \wedge R \quad \text{(B7)}$$

p_1 is the Pontryagin number of the $SO(N)$ bundle over X_4. On a general manifold, it is an integer. However, on a spin manifold it is an even integer. σ is the signature of the manifold. On a general manifold it is an integer. However, on a spin manifold it is a multiple of 16. This implies that if our three-manifold M is endowed with a spin-structure, we can separately define

$$\text{CS}_{SO(N)}[a]_1 = \frac{i}{2(4\pi)} \int_{X_4} \text{tr}_{SO(N)} f \wedge f \quad \text{(B8)}$$

and

$$\text{CS}_g = \frac{i}{192\pi} \int_{X_4} \text{tr} R \wedge R \quad \text{(B9)}$$

Neither (B8) nor (B9) depend on the extension to X_4 as long as the spin-structure is also extended. Eq. (B8) is the standard definition of $SO(N)_1$ gauge theory; for level k - simply multiply Eq. (B8) by k. From the preceding discussion we see that the action (B8) generally depends on the spin-structure for odd k, but not even k. Physically, when the theory has bosonic a-matter fields, the odd k theories always have transparent fermions in the spectrum, while even k theories are consistent theories of microscopic bosons. In this paper, we are principally interested in the case $k = 1$: it is consistent that our dual theory has a local fermion in its spectrum.

We note that for even N and odd k one can still make the theory live in a purely bosonic Hilbert space, if one combines both the CS term for a and the gravitational CS term via the definition (B2) or equivalently through Eq. (B5). Physically, in this case one must restrict to fermionic a-matter only. Indeed, for even N, the center of $SO(N)$ is $Z_2 = \{1, -1\}$. The transition functions for the fermionic matter field χ live in the group $(SO(N) \times Spin(3))/Z_2$, i.e. χ feels only the combination of spin-structure and $SO(N)$ connection. Thus, if one includes only fermionic a-matter, one does not need to give a spin-structure as an input and we get a theory of microscopic bosons. However, if we allow for bosonic matter

ϕ (as our dual theory does), than the ϕ action depends on the $SO(N)$ connection, but not on spin-structure, so the entire action depends on both the $SO(N)$ connection and the spin-structure. So in this case, we still have microscopic fermions in the Hilbert space.

We conclude by reviewing our notation for the case $N = 3$. In this case, one may attempt to lift the $SO(3)$ bundle over M to an $SU(2)$ bundle. The lift does not always exist, so the two theories are generally different in their global properties (although local properties will be identical). When the lift does exist, the $SO(3)$ action becomes,

$$\text{CS}_{SO(3)}[a]_k = 2 \times \frac{ik}{4\pi} \int \text{tr}_{SU(2)} \left(\hat{a} \wedge d\hat{a} - \frac{2i}{3} \hat{a} \wedge \hat{a} \wedge \hat{a} \right) \quad \text{(B10)}$$

Here the trace is over the spin 1/2 representation of $SU(2)$ and $\hat{a}$ is the corresponding lift of a. (We can write the action directly in 3d, because $SU(2)$ in $d = 3$ does not admit non-trivial bundles). That is $SO(3)_k$ goes to $SU(2)_{2k}$.

Appendix C: Z_2 symmetry.

In this section, we discuss the issue of the "extra" global Z_2 symmetry of the dual theory (5) in more detail. We will argue that this symmetry is realized in the same way in both phases of (5), thus, it is natural that the Z_2 gap remains finite across the transition.

It is useful to have an explicitly Z_2 symmetric parton construction as a starting point for the analysis. Imagine we use the parton decompostion

$$c = \sum_{\alpha=1}^{N} \phi_\alpha \chi_\alpha \quad \text{(C1)}$$

with c - the electron, ϕ_α - slave bosons and χ_α - slave fermions. This decomposition has an $O(N)$ gauge symmetry. As before, we place χ_α into identical $p_x + ip_y$ superconductor band structures, and ϕ_α into a trivial paramagnet. This yields the effective theory (5) with an $O(N)$ gauge group. Now, imagine further that the system has some Ising spin degrees of freedom σ (not related to the electrons c) that transform under a global Z_2 symmetry:

$$Z_2: \quad \sigma \to -\sigma, \quad c \to c \quad \text{(C2)}$$

Imagine we condense the bosonic operator $\Phi\sigma$, with Φ defined by Eq. (10). This breaks the $O(N)$ gauge symmetry down to $SO(N)$. However, the physical global Z_2 symmetry is not broken, since its combination with an $O(N)$ reflection leaves $\Phi\sigma$ invariant. Thus, we conclude that after $\Phi\sigma$ condensation, the global Z_2 symmetry acts as an $O(N)$ reflection of ϕ_α and χ_α. This is precisely the "extra" global symmetry of theory (5).

We would like to show that from the point of view of this global Z_2 symmetry, both $r > 0$ and $r < 0$ phases of

(5) realize the same SPT. To argue this, let us look at the edge of the system: a Z_2 SPT would possess non-trivial Z_2 carrying gapless edge modes.[30] At the mean field level, χ_α has N chiral $c = 1/2$ modes f_α transforming under the $O(N)$ symmetry. However, after coupling to the $SO(N)$ gauge field, all these modes are eliminated. Therefore, the $r > 0$ phase must realize a trivial Z_2 SPT. Let's now turn to $r < 0$. Suppose ϕ condenses along the N'th direction, $\langle \phi_\alpha \rangle = \delta_{\alpha N}$. The $SO(N)$ group is broken to $SO(N-1)$ acting on $\alpha = 1 \ldots N-1$. Moreover, the global Z_2 symmetry (after potentially combining with an $SO(N)$ gauge rotation) now acts as $O(N-1)$ on $\alpha = 1 \ldots N-1$. Therefore, the edge mode f_N is not coupled to the $SO(N-1)$ gauge field and, moreover, is neutral under the global Z_2. On the other hand, the edge modes f_α with $\alpha = 1 \ldots N-1$ are eliminated by fluctuations of the $SO(N-1)$ gauge field. So for $r < 0$, there are no Z_2 carrying edge modes, which means that this phase is also trivial from the Z_2 SPT standpoint.

Another way to diagnose the presence of a Z_2 SPT is to look at the bulk response to Z_2 fluxes, e.g. at the partition function of the theory on some closed manifold in the presence of background Z_2 gauge field b. Let us consider the case of odd N, so $O(N) = Z_2 \times SO(N)$. For $r > 0$, the Z_2 symmetry simply acts on partons χ_α as $Z_2 : \chi_\alpha \to -\chi_\alpha$. This coincides with the action of fermion parity on χ_α. But we know that our system is a trivial $c_- = 0$ superconductor. Therefore, its partition function does not depend on spin-structure, and consequently, on the background Z_2 gauge field b. So the $r > 0$ phase realizes a trivial Z_2 SPT.

Next, we turn to the phase $r < 0$. As we already discussed, the fermion χ_N now forms a $p_x + ip_y$ superconductor and is neutral under Z_2. It, therefore, does not contribute any b dependence to the partition function. We are thus left with the contribution of fermions χ_α, $\alpha = 1 \ldots N-1$ coupled to an $SO(N-1)$ gauge field and charged under the $O(N-1)/SO(N-1)$ global symmetry. We note that because $N-1$ is even (and all bosonic charge matter is trivial and gapped), this system can effectively be thought as living in a bosonic microscopic Hilbert space (see appendix B). Therefore, as a Z_2 SPT, it must realize either a trivial phase or the Chen-Liu-Wen-Levin-Gu (CLWLG) bosonic SPT phase.[31,32] These can be distinguished by their partition functions on RP^3 in the presence of a background Z_2 flux. In section C 1, we will explicitly compute this partition function for $N-1 = 2$ and show that, indeed, a trivial Z_2 SPT phase is realized. For other N, we don't have an explicit bulk partition function computation and have to rely on the edge argument above.

Despite the above arguments, from a mean field viewpoint it appears surprising that the operator (10) can have exponentially decaying correlation functions at the critical point $r = 0$. We now give an example of a theory where a similar phenomenon takes place. Consider the WFCS theory (2), which is an $SO(2)$ cousin of our non-Abelian theory (5). This theory has a $Z_2 = O(2)/SO(2)$

symmetry, which acts as $\phi \to \phi^\dagger$. On the dual Dirac fermion side (1), this symmetry is precisely the charge-conjugation symmetry $Z_2 : \psi \to C\bar\psi^T$. The gap to this symmetry vanishes at the critical point: for instance, the physical electric current operator $\bar\psi \gamma^\mu \psi$ is odd under C. Now, imagine perturbing the Dirac fermion by a superconducting mass term (which breaks the global $U(1)$ symmetry):

$$
\begin{aligned}
H &= \psi^\dagger(-i\partial_x \sigma^x - i\partial_y \sigma^z)\psi + m\psi^\dagger \sigma^y \psi + \\
&\quad + \frac{1}{2}\Delta \psi^\dagger \sigma^y \psi^* + \frac{1}{2}\Delta^* \psi^T \sigma^y \psi
\end{aligned}
$$

Here, we've written the real-time Hamiltonian and made a choice of γ matrices. For $\Delta = \Delta^*$, the charge-conjugation symmetry $Z_2 : \psi \to \psi^\dagger$ is preserved. We see that the Dirac transition splits into two Majorana transitions. Indeed, writing $\psi = \xi_1 + i\xi_2$,

$$
\begin{aligned}
H &= \xi_1^T(-i\partial_x \sigma^x - i\partial_y \sigma^z)\xi_1 + (m+\Delta)\xi_1^T \sigma^y \xi_1 \\
&\quad + \xi_2^T(-i\partial_x \sigma^x - i\partial_y \sigma^z)\xi_2 + (m-\Delta)\xi_2^T \sigma^y \xi_2
\end{aligned}
$$

The Majorana fermion ξ_1 becomes massless at $m = -\Delta$, and ξ_2 becomes massless at $m = \Delta$. The Z_2 symmetry acts as, $\xi_1 \to \xi_1$, $\xi_2 \to -\xi_2$. Since at finite Δ, ξ_1 and ξ_2 are never simultaneously gapless, all bosons charged under Z_2 remain gapped throughout the phase diagram. Now, on the WFCS side, the superconducting mass perturbation maps to a double monopole of a. Therefore, in this perturbed WFCS theory, the Z_2 charge gap (for bosons) also never closes, just as we expect in the non-Abelian theory (5). Of course, in the perturbed Abelian theory, an even more dramatic effect is that the transition at $r = 0$ splits into two; in the non-Abelian case (5) there is no reason to expect such a splitting. We note that the perturbed Abelian WFCS theory can be thought as arising by starting with the $N = 3$ non-Abelian WFCS theory (5), going to the $r < 0$ phase where $SO(3)$ gauge symmetry is broken to $SO(2)$; the Abelian WFCS transition then corresponds to further breaking the $SO(2)$ gauge group to trivial. Note that in such a construction, the $SO(2)$ theory will be naturally perturbed by double monopole operators, as they are allowed in the original $SO(3)$ theory. Our assumption is that the gap to $Z_2 = O(3)/SO(3)$ bosons remains finite throughout the above sequence of phase transitions.

1. $O(2)_1$ theory.

In this section, we focus on the theory (5) with $N = 3$ in the regime $r < 0$ and compute its partition function on RP^3 in the presence of a background Z_2 flux b. This will serve as a check that this phase is not a Z_2 SPT. Indeed, as we argued above, the $r < 0$ phase (apart from a Z_2 neutral $p_x + ip_y$ superconductor) is at most a CLWLG bosonic Z_2 SPT. The partition function of the CLWLG phase on a manifold M in the presence of a background

Z_2 flux $b \in H^1(M, Z_2)$ is given by[29,33]

$$Z[b] = \exp\left(\pi i \int_M b^3\right) \qquad \text{(C3)}$$

On RP^3, $H^1(RP^3, Z_2) = Z_2$, and for the non-trivial b, we have $Z[b] = -1$. On the other hand, for a trivial Z_2 SPT, we will have $Z[b] = 1$ on any manifold.

From the parton construction below Eq. (C1) the background flux b and the dynamical $SO(2)$ gauge field a combine to an $O(2) = SO(2) \rtimes Z_2$ gauge field: the action is the partition function of the partons χ_1, χ_2 coupled to this $O(2)$ gauge field - it is given by Eq. (B2). (Here, we ignore the third parton χ_3, which sees no dynamical gauge field or background Z_2 gauge field). Thus, the partition function takes the form:

$$Z[b] = \frac{1}{Vol(\mathcal{G})} \sum_{H^2(M,\tilde{Z}_b)} \int Da \, e^{-\pi i \eta[a]} \qquad \text{(C4)}$$

Once the Z_2 gauge field b is fixed, a becomes a $u(1)$ gauge field, but its transition functions satisfy a cocycle condition twisted by b. Topologically distinct $u(1)$ gauge fields correspond to elements of $H^2(X, \tilde{Z}_b)$, where the coefficients are in the local system twisted by b. The path integral is over gauge fields a in each such class and the factor in front, $Vol(\mathcal{G})^{-1}$, corresponds to the volume of (twisted) $u(1)$ gauge group. $\eta[a]$ is the η-invariant (B3) of the Dirac operator in the background of the $O(2)$ gauge field a. A standard computation gives,

$$Z[b] = \frac{e^{2\pi i \eta(*d)/8}}{\sqrt{|T|}} \sum_{a_{cl} \in T} e^{-\pi i \eta[a_{cl}]} \qquad \text{(C5)}$$

Here a_{cl} are topologically non-trivial flat $u(1)$ bundles twisted by b - these are in one-to-one correspondence with the torsion subgroup $T = Tor(H^2(M, \tilde{Z}_b))$ and $|T|$

denotes the order of T. $\eta(*d)$ is the η-invariant of the operator $*d$ acting on (twisted) 1-forms and 3-forms - it vanishes for manifolds with an orientation-reversing isometry, such as RP^3.

Let's compute the partition function on RP^3. As already noted, $H^1(RP^3, Z_2) = Z_2$, so there is only one non-trivial b-flux sector. In the untwisted ($b = 0$) sector, we have $H^2(RP^3, Z) = Z_2$ which correspond to two spin-structures for slave-fermions χ_α on RP^3. Explicitly, working on the double-cover S^3 and using the stereographic coordinates, $(\frac{4\vec{u}}{u^2+4}, \frac{u^2-4}{u^2+4})$, we have under the anti-podal map:

$$\chi_\alpha(\vec{u}) = \pm \frac{iu^i\gamma^i}{|\vec{u}|} \chi_\alpha\left(\frac{-4\vec{u}}{u^2}\right) \qquad \text{(C6)}$$

Note that χ_1 and χ_2 have the same spin-structure. Now, one of the spin-structures has $\eta = 1/8$ and the other $\eta = -1/8$ (for a single fermion, say χ_1). Adding contributions of χ_1 and χ_2, $\eta = \pm 1/4$. So, we get

$$Z_0(RP^3) = \frac{1}{\sqrt{2}}(e^{\pi i/4} + e^{-\pi i/4}) = 1 \qquad \text{(C7)}$$

This is the correct partition function of a trivial superconductor on RP^3, as needed.

Now, in the twisted sector, $H^2(RP^3, \tilde{Z}) = Z_1$, there is just a single twisted $u(1)$ bundle that can be chosen to be:

$$\chi_\alpha(\vec{u}) = \tau^z_{\alpha\beta} \frac{iu^i\gamma^i}{|\vec{u}|} \chi_\beta\left(\frac{-4\vec{u}}{u^2}\right) \qquad \text{(C8)}$$

So χ_1 and χ_2 now have opposite spin-structures, so together they give $\eta = 0$. So, we get

$$Z_{\text{twist}}(RP^3) = 1 \qquad \text{(C9)}$$

Since the partition function in the twisted sector is trivial, this phase must realize a trivial Z_2 SPT as claimed.

[1] D. T. Son, Phys. Rev. X **5**, 031027 (2015).
[2] C. Wang and T. Senthil, Phys. Rev. X **5**, 041031 (2015).
[3] M. A. Metlitski and A. Vishwanath, Phys. Rev. B **93**, 245151 (2016).
[4] D. F. Mross, J. Alicea, and O. I. Motrunich, Phys. Rev. Lett. **117**, 016802 (2016).
[5] A. Karch and D. Tong, Phys. Rev. X **6**, 031043 (2016), URL http://link.aps.org/doi/10.1103/PhysRevX.6.031043.
[6] N. Seiberg, T. Senthil, C. Wang, and E. Witten, Annals of Physics **374**, 395 (2016).
[7] M. E. Peskin, Annals Phys. **122**, 113 (1978).
[8] C. Dasgupta and B. I. Halperin, Phys. Rev. Lett. **47**, 1556 (1981).
[9] A. M. Polyakov, Mod. Phys. Lett. A **3**, 325 (1988).
[10] W. Chen, M. P. A. Fisher, and Y.-S. Wu, Phys. Rev. B **48**, 13749 (1993).
[11] M. Barkeshli and J. McGreevy, Phys. Rev. B **89**, 235116 (2014).
[12] S. Giombi, S. Minwalla, S. Prakash, S. P. Trivedi, and S. R. Wadia, Eur.Phys.J. **C72**, 2112 (2012).
[13] O. Aharony, G. Gur-Ari, and R. Yacoby, JHEP **1203**, 037 (2012).
[14] O. Aharony, G. Gur-Ari, and R. Yacoby, JHEP **1212**, 028 (2012).
[15] O. Aharony, JHEP **1602**, 093 (2016).
[16] P.-S. Hsin and N. Seiberg, arXiv:1607.07457 (2016).
[17] J. A. Jenquin (2005), arXiv:math/0504524.
[18] J. A. Jenquin (2006), arXiv:math/0605239.
[19] A. Kapustin, R. Thorngren, A. Turzillo, and Z. Wang, JHEP **1512**, 052 (2015).
[20] T. Grover and A. Vishwanath, arXiv:1206.1332 (2012).

[21] T. Grover, D. N. Sheng, and A. Vishwanath, Science **344**, 280 (2014).

[22] L. Fei, S. Giombi, I. R. Klebanov, and G. Tarnopolsky, arXiv:1607.05316 (2016).

[23] L. Fidkowski, X. Chen, and A. Vishwanath, Phys. Rev. X **3**, 041016 (2013).

[24] S. Sahoo, Z. Zhang, and J. C. Y. Teo, Phys. Rev. B **94**, 165142 (2016).

[25] C. Wang and M. Levin, arXiv:1610.08478 (2016).

[26] X. Chen, L. Fidkowski, and A. Vishwanath, Phys. Rev. B **89**, 165132 (2014).

[27] P. Bonderson, C. Nayak, and X.-L. Qi, J. Stat. Mech. p. P09016 (2013).

[28] O. Aharony, F. Benini, P.-S. Hsin, and N. Seiberg, arXiv:1611.07874 (2016).

[29] R. Dijkgraaf and E. Witten, Commun. Math. Phys. **129**, 393 (1990).

[30] Z.-C. Gu and M. Levin, arXiv:1304.4569 (2013).

[31] X. Chen, Z.-X. Liu, and X.-G. Wen, Phys. Rev. B **84**, 235141 (2011).

[32] M. Levin and Z.-C. Gu, Phys. Rev. B **86**, 115109 (2012).

[33] A. Kapustin and R. Thorngren, arXiv:1404.3230 (2014).

[34] X.-G. Wen, Phys. Rev. B **60**, 8827 (1999), URL `http://link.aps.org/doi/10.1103/PhysRevB.60.8827`.

[35] E. Vicari, PoS(Lattie 2007):023 (2007).

[36] The $SO(3)_1$ example might be the most familiar: this theory is the same as $SU(2)_2 = \{1, \sigma, f\}$ restricted to integer spin, *i.e.* to $\{1, f\}$. The chiral central charge is $c_- = -3/2$. Notice that for $SO(N)_1$ CS theory, we forbid excitations with projective representation (such as spinor) of $SO(N)$, otherwise the bulk would have topological order, like the case studied in Ref. 34.

[37] Strictly speaking, $\hat{\phi}_\alpha$ satisfy $\prod_{i=1}^N \hat{\phi}_\alpha = 1$, but this constraint is assumed to be softened in the infrared.

[38] The case of odd N and even N_f requires working with an $O(N)$ gauge group.

[39] If the gauge field is ignored, with large N and fixed N_f, there does exists a stable fixed point with $u > 0$ and $v > 0$.[35].