Analysis and synthesis of cascaded metasurfaces using wave matrices

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Analysis and Synthesis of Cascaded Metasurfaces Using Wave Matrices

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Various matrix representations are used to analyze the propagation of electromagnetic waves through stratified (layered) media or cascaded circuit networks. These include ABCD matrices, scattering matrices, impedance matrices and hybrid matrices. A less known network representation is the wave matrix. In this work, a brief review of wave matrices is presented, and their relation to other network representations derived. Wave matrices are found for common interfaces such as boundaries between dielectric media, dielectric slabs, as well as electric, magnetic, and magneto-electric sheet boundaries (generalized sheet transition conditions). These results are then used to develop an analytical synthesis approach for cascaded metasurfaces: metasurfaces consisting of a cascade of sheets separated by dielectric spacers. This is in contrast to earlier works which relied on numerical solvers or optimization methods to design such structures. A few design examples are presented to demonstrate the utility of the synthesis approach.

I. INTRODUCTION

In recent years, metasurfaces with electric and magnetic responses have gained significant attention [1, 2]. In particular, their ability to manipulate the phase and polarization of wavefronts across subwavelength distances has sparked strong interest in the engineering and physics communities. It has been shown that by cascading anisotropic sheets with electric responses one can realize arbitrary (passive and reciprocal) bianisotropic metasurfaces. A systematic design method was developed using an ABCD matrix formalism in [3–5].

The propagation of electromagnetic waves along cascaded structures or in multilayered media has a rich history, and numerous methods have been proposed for analytically and numerically calculating reflection and transmission coefficients. In general, a linear medium can exhibit a bianisotropic response: electric, magnetic, and magneto-electric responses. This type of medium has received considerable attention given its potential to control polarization and phase. Various techniques have been used to solve for the reflection and transmission coefficients of a bianisotropic medium [6–8].

In optics, a 2×2 matrix representation was developed by Abeles to describe light propagation in stratified isotropic media. Later, this work was extended to layered anisotropic media and a generalized 4×4 form of Abeles 2×2 matrices was developed [9–11]. In this formulation, reflections are taken into account. Similarly, Jones calculus is a powerful tool which describes the polarization state of light with a vector and the operation of a device with a 2×2 matrix. The total Jones matrix of cascaded devices is found by multiplying the matrices of the cascaded elements [12]. Alternatively, the polarization properties of light can be described by Stokes parameters and the device transformation matrix by Muller Matrices, which describes the depolarization of the device. Muller matrices are 4×4 extensions of Jones matrices. As with Jones matrices, the Muller matrix of a cascade of optical elements is a product of the matrices representing the elements. The main goal of polarimetry is to extract the 16 elements of the Muller matrix from experimental data [13].

In microwave engineering, network parameters such as Transfer matrices (ABCD matrices), Impedance matrices (Z-matrices), Scattering matrices, and Hybrid matrices are used by researchers to analyze cascaded electromagnetic structures. S-matrices are appealing since they relate scattered and incident fields to network parameters. The drawback of using S-matrices is that the scattering matrices of cascaded networks cannot be simply multiplied to find the overall response. On the other hand, ABCD matrices can be multiplied together to find the overall ABCD matrix of a cascaded network. However, ABCD matrices relate network parameters to total fields, not scattered fields. Wave matrices (WMs) are yet another approach to analyzing cascaded structures. These matrices have been employed in the past to analyze multilayered dielectric media [14]. WMs relate the forward and backward propagating fields in one region to those of the next region, and are closely related to S-matrices. WMs provide the best of both worlds: they relate network parameters to scattered and incident fields, and can be multiplied together to model cascaded structures. As a result, they are ideally suited for the synthesis of cascaded metasurfaces that realize prescribed S-parameters.

In this work, a brief review of WMs is presented, and their relation to other matrix representations is derived. Their utility in the design of cascaded metasurfaces is explained. The cascaded metasurfaces that will be considered here consist of patterned metallic sheets separated by dielectric spacers, as shown in Fig.1. Here, an analytic technique is developed to design metasurfaces operating at normal incidence [3–5, 15–18], which can be easily ex-

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tended to oblique incidence [19–21]. It is assumed that the sheets are made of subwavelength metallic patterns. Therefore, higher order Floquet modes are bound to the sheets. As a result, evanescent wave interaction between the sheets can be neglected if the thickness of the dielectric spacers is larger than periodicity of metallic patterns. Moreover, the subwavelength texturing of the sheets allows the sheets to be modeled as homogenized impedance boundary conditions [5].

First, wave matrices are found for elements comprising typical metasurfaces [3–5, 15–18]. Once the WMs of the constituent elements are determined, the cascaded metasurfaces can be analyzed. In the analysis, closed-form expressions are derived for the electric sheet admittances needed to realize a targeted scattering matrix. Previous works focused on deriving expressions for isotropic admittances that realize a given phase delay, for normal [3, 4] or oblique [20, 21] incidence. The WM formalism presented here allows one to solve for the tensor sheet admittances that realize a given bianisotropic response. It can be shown that the formulation reduces to previously derived expressions for isotropic admittances [3, 4]. Using WMs also results in compact expressions for the tensor sheet admittances. This is in contrast to the impedance matrix approach [17].

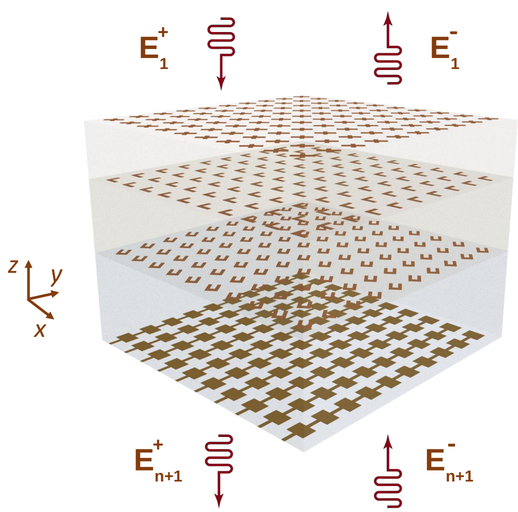


FIG. 1: A metasurface made of cascaded patterned metallic sheets. The periodic structures on the sheets are subwavelength enough to treat the sheets as homogenized boundary condition. The total structure is a four-port network that can be excited by x and y polarized waves from region 1 and $n+1$.

II. WAVE MATRICES AND THEIR RELATION TO OTHER NETWORK REPRESENTATIONS

Let us consider the cascaded structure depicted in Fig. 2, consisting of sheet admittances (patterned metallic

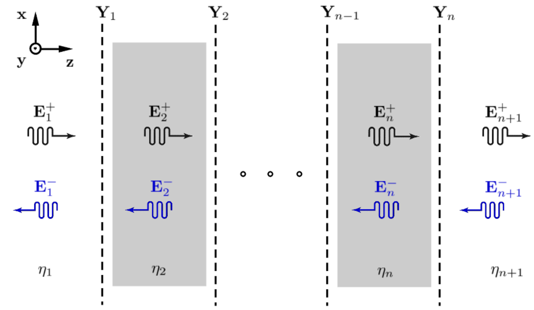


FIG. 2: Side view of a cascaded structure consisting of dielectric spacers and n sheet admittances. Total electric field in each region is the sum of forward and backward propagating fields. $\mathbf{E}_i^+$ and $\mathbf{E}_i^-$ represent the fields at the leftmost boundary of region i.

sheets) and dielectric spacers. It is representative of many cascaded metasurfaces reported to date [4, 5]. Different matrix approaches have been employed to relate surface parameters to transmission and reflection coefficients of such structures. In this work, we use WMs to analyze the structure shown in Fig.2.

First, let us establish a relation between WMs and other network representations. Once these relations are found, any network representation can be converted to its WM representation. The forward (incident) and backward (reflected) traveling waves in region i is represented by $\mathbf{E}_i^+ = [\mathrm{E}_{ix}^+, \mathrm{E}_{iy}^+]^{\mathrm{T}}$ and $\mathbf{E}_i^- = [\mathrm{E}_{ix}^-, \mathrm{E}_{iy}^-]^{\mathrm{T}}$. The reference plane for $\mathbf{E}_i^+$ and $\mathbf{E}_i^-$ is located at the leftmost boundary of region i, except for region 1 where $\mathbf{E}_1^+$ and $\mathbf{E}_1^-$ are evaluated at the rightmost boundary of region 1 (interface 1). $\mathbf{E}_i^+$ and $\mathbf{E}_i^+$ are only functions of the z coordinate, since the structure has subwavelength periodicity in x-y plane. The corresponding magnetic fields are,

$$\mathbf{H}_i^+ = \frac{1}{\eta_i}\mathbf{n}\mathbf{E}_i^+ , \quad \mathbf{H}_i^- = -\frac{1}{\eta_i}\mathbf{n}\mathbf{E}_i^- , \tag{1}$$

where $\mathbf{n}$ is a 90° rotation matrix,

$$\mathbf{n} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{2}$$

Therefore, the total electric and magnetic fields in region i are,

$$\mathbf{E}_i = \mathbf{E}_i^+ + \mathbf{E}_i^- ,$$

$$\mathbf{H}_i = \mathbf{H}_i^+ + \mathbf{H}_i^- = \frac{1}{\eta_i}\mathbf{n}(\mathbf{E}_i^+ - \mathbf{E}_i^-). \tag{3}$$

The cascaded structure can be treated as a four-port network terminated by regions 1 and $n+1$, given that there are x and y polarizations. A microwave network representation [22] of a four-port network is a 4×4 matrix that relates $\mathbf{E}_1^+$, $\mathbf{E}_1^-$, $\mathbf{E}_{n+1}^+$, and $\mathbf{E}_{n+1}^-$ (Fig.2). For example, S-matrices relate the incoming waves to outgoing waves in regions 1 and $n+1$ (note the convention used for the incoming and outgoing waves),

$$\begin{pmatrix} \mathbf{E}_1^- \\ \mathbf{E}_{n+1}^+ \end{pmatrix} = \begin{pmatrix} \mathbf{S}_{11} & \mathbf{S}_{12} \\ \mathbf{S}_{21} & \mathbf{S}_{22} \end{pmatrix} \begin{pmatrix} \mathbf{E}_1^+ \\ \mathbf{E}_{n+1}^- \end{pmatrix}. \tag{4}$$

Transfer matrices (ABCD) relate the total electric and magnetic fields in region 1 to those in region $n+1$,

$$\begin{pmatrix} \mathbf{E}_1^+ + \mathbf{E}_1^- \\ \frac{1}{\eta_1}\mathbf{n}(\mathbf{E}_1^+ - \mathbf{E}_1^-) \end{pmatrix} = \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{pmatrix} \begin{pmatrix} \mathbf{E}_{n+1}^+ + \mathbf{E}_{n+1}^- \\ \frac{1}{\eta_{n+1}}\mathbf{n}(\mathbf{E}_{n+1}^+ - \mathbf{E}_{n+1}^-) \end{pmatrix}. \tag{5}$$

Impedance matrices relate the total electric fields in regions 1 and $n+1$ to the magnetic fields in those regions,

$$\begin{pmatrix} \mathbf{E}_1^+ + \mathbf{E}_1^- \\ \mathbf{E}_{n+1}^+ + \mathbf{E}_{n+1}^- \end{pmatrix} = \begin{pmatrix} \mathbf{Z}_{11} & \mathbf{Z}_{12} \\ \mathbf{Z}_{21} & \mathbf{Z}_{22} \end{pmatrix} \begin{pmatrix} \frac{1}{\eta_1}\mathbf{n}(\mathbf{E}_1^+ - \mathbf{E}_1^-) \\ \frac{1}{\eta_{n+1}}\mathbf{n}(\mathbf{E}_{n+1}^+ - \mathbf{E}_{n+1}^-) \end{pmatrix}. \tag{6}$$

Hybrid matrices, again relate total fields, and are defined as follows,

$$\begin{pmatrix} \mathbf{E}_1^+ + \mathbf{E}_1^- \\ \frac{1}{\eta_{n+1}}\mathbf{n}(\mathbf{E}_{n+1}^+ - \mathbf{E}_{n+1}^-) \end{pmatrix} = \begin{pmatrix} \mathbf{H}_{11} & \mathbf{H}_{12} \\ \mathbf{H}_{21} & \mathbf{H}_{22} \end{pmatrix} \begin{pmatrix} \frac{1}{\eta_1}\mathbf{n}(\mathbf{E}_1^+ - \mathbf{E}_1^-) \\ \mathbf{E}_{n+1}^+ + \mathbf{E}_{n+1}^- \end{pmatrix}. \tag{7}$$

Finally, Wave matrices relate the forward and backward propagating fields in region 1 to those in region $n+1$, in the following manner,

$$\begin{pmatrix} \mathbf{E}_1^+ \\ \mathbf{E}_1^- \end{pmatrix} = \begin{pmatrix} \mathbf{M}_{11} & \mathbf{M}_{12} \\ \mathbf{M}_{21} & \mathbf{M}_{22} \end{pmatrix} \begin{pmatrix} \mathbf{E}_{n+1}^+ \\ \mathbf{E}_{n+1}^- \end{pmatrix}. \tag{8}$$

Relations between $\mathbb{M}$ and $\mathbb{S}$ matrices can be found by considering two separate excitations from region 1: $\mathbf{E}_1^+ = \mathbf{I}_x = [1\,,\ 0]^{\mathrm{T}}$ and $\mathbf{E}_1^+ = \mathbf{I}_y = [0\,,\ 1]^{\mathrm{T}}$. Under an x polarized excitation ($\mathbf{I}_x$) from region 1, the electric field of the reflected wave in region 1 ($\mathbf{E}_1^-$) is equal to $\mathbf{S}_{11}^x$ and the transmitted field in region $n+1$ ($\mathbf{E}_{n+1}^+$) is equal to $\mathbf{S}_{21}^x$,

$$\mathbf{E}_1^- = \mathbf{S}_{11}^x = \begin{pmatrix} S_{11}^{xx} \\ S_{11}^{yx} \end{pmatrix}, \quad \mathbf{E}_{n+1}^+ = \mathbf{S}_{21}^x = \begin{pmatrix} S_{21}^{xx} \\ S_{21}^{yx} \end{pmatrix}. \tag{9}$$

Similarly, a y polarized excitation ($\mathbf{I}_y$) yields,

$$\mathbf{E}_1^- = \mathbf{S}_{11}^y = \begin{pmatrix} S_{11}^{xy} \\ S_{11}^{yy} \end{pmatrix}, \quad \mathbf{E}_{n+1}^+ = \mathbf{S}_{21}^y = \begin{pmatrix} S_{21}^{xy} \\ S_{21}^{yy} \end{pmatrix}. \tag{10}$$

Refering to the definition of $\mathbb{M}$ in Eq.(8), we can write,

$$\begin{pmatrix} \mathbf{I}_x \\ \mathbf{S}_{11}^x \end{pmatrix} = \mathbb{M} \begin{pmatrix} \mathbf{S}_{21}^x \\ \mathbf{0} \end{pmatrix}, \quad \begin{pmatrix} \mathbf{I}_y \\ \mathbf{S}_{11}^y \end{pmatrix} = \mathbb{M} \begin{pmatrix} \mathbf{S}_{21}^y \\ \mathbf{0} \end{pmatrix}. \tag{11}$$

Using the x and y polarized excitations from region $n+1$ ($\mathbf{E}_{n+1}^- = \mathbf{I}_x$ and $\mathbf{E}_{n+1}^- = \mathbf{I}_y$), we can write,

$$\begin{pmatrix} \mathbf{0} \\ \mathbf{S}_{12}^x \end{pmatrix} = \mathbb{M} \begin{pmatrix} \mathbf{S}_{22}^x \\ \mathbf{I}_x \end{pmatrix}, \quad \begin{pmatrix} \mathbf{0} \\ \mathbf{S}_{12}^y \end{pmatrix} = \mathbb{M} \begin{pmatrix} \mathbf{S}_{22}^y \\ \mathbf{I}_y \end{pmatrix}. \tag{12}$$

Combining Eq.(11) and Eq.(12) results in,

$$\begin{pmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{S}_{11} & \mathbf{S}_{12} \end{pmatrix} = \mathbb{M} \begin{pmatrix} \mathbf{S}_{21} & \mathbf{S}_{22} \\ \mathbf{0} & \mathbf{I} \end{pmatrix}, \tag{13}$$

where,

$$\mathbf{S}_{ij} = \begin{pmatrix} \mathbf{S}_{ij}^x & \mathbf{S}_{ij}^y \end{pmatrix}, \tag{14}$$

and $\mathbf{I}$ is the 2×2 identity matrix. Eq.(13) gives the WM in terms of the scattering parameters,

$$\mathbb{M} = \begin{pmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{S}_{11} & \mathbf{S}_{12} \end{pmatrix} \begin{pmatrix} \mathbf{S}_{21} & \mathbf{S}_{22} \\ \mathbf{0} & \mathbf{I} \end{pmatrix}^{-1}. \tag{15}$$

By defining,

$$\mathbb{S}_1 = \begin{pmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{S}_{11} & \mathbf{S}_{12} \end{pmatrix}, \quad \mathbb{S}_2 = \begin{pmatrix} \mathbf{S}_{21} & \mathbf{S}_{22} \\ \mathbf{0} & \mathbf{I} \end{pmatrix}, \tag{16}$$

we can write,

$$\mathbb{M} = \mathbb{S}_1 \mathbb{S}_2^{-1}. \tag{17}$$

Conversely, to find the S-matrix in terms of WM, Eq.(13) can be rearranged as follows,

$$\begin{pmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} + \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{I} & \mathbf{0} \end{pmatrix} \mathbb{S} = \begin{pmatrix} \mathbf{M}_{11} & \mathbf{M}_{12} \\ \mathbf{M}_{21} & \mathbf{M}_{22} \end{pmatrix} \times \left(\begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{pmatrix} + \begin{pmatrix} \mathbf{0} & \mathbf{I} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \mathbb{S} \right), \tag{18}$$

Solving for the S-matrix in Eq.(18) yields,

$$\begin{pmatrix} \mathbf{S}_{11} & \mathbf{S}_{12} \\ \mathbf{S}_{21} & \mathbf{S}_{22} \end{pmatrix} = \begin{pmatrix} \mathbf{0} & \mathbf{M}_{11} \\ -\mathbf{I} & \mathbf{M}_{21} \end{pmatrix}^{-1} \begin{pmatrix} \mathbf{I} & -\mathbf{M}_{12} \\ \mathbf{0} & -\mathbf{M}_{22} \end{pmatrix}. \tag{19}$$

The transfer matrix (ABCD) relates the total electric and magnetic fields in separate regions. To find $\mathbb{M}$ in terms of the ABCD matrix, we use the following relation between total fields and forward and backward propagating fields,

$$\begin{pmatrix} \mathbf{E}_i^+ \\ \mathbf{E}_i^- \end{pmatrix} = \frac{1}{2} \begin{pmatrix} \mathbf{I} & -\eta_i \mathbf{n} \\ \mathbf{I} & \eta_i \mathbf{n} \end{pmatrix} \begin{pmatrix} \mathbf{E}_i \\ \mathbf{H}_i \end{pmatrix}. \tag{20}$$

Substituting Eq.(20) into Eq.(8), for regions 1 and $n+1$ yields,

$$\begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} \mathbf{I} & \mathbf{I} \\ \frac{1}{\eta_1}\mathbf{n} & -\frac{1}{\eta_1}\mathbf{n} \end{pmatrix} \mathbb{M} \begin{pmatrix} \mathbf{I} & -\eta_{n+1}\mathbf{n} \\ \mathbf{I} & \eta_{n+1}\mathbf{n} \end{pmatrix}, \tag{21}$$

and,

$$\mathbb{M} = \frac{1}{2} \begin{pmatrix} \mathbf{I} & -\eta_1 \mathbf{n} \\ \mathbf{I} & \eta_1 \mathbf{n} \end{pmatrix} \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{pmatrix} \begin{pmatrix} \mathbf{I} & \mathbf{I} \\ \frac{1}{\eta_{n+1}}\mathbf{n} & -\frac{1}{\eta_{n+1}}\mathbf{n} \end{pmatrix}. \tag{22}$$

The relationships between WMs and Impedance and Hybrid matrices are presented in appendix A.

III. BUILDING BLOCKS

As noted, the cascaded structure in Fig.2 is composed of sheet boundaries and dielectric spacers. The sheets can possess electric, magnetic, and magneto-electric responses, while the spacers can be anisotropic. The matrices $\mathbb{M}^{(i)}_{inter}$ and $\mathbb{M}^{(i)}_{delay}$ will denote WMs for the i-th interface and dielectric spacer, respectively. According to the definition of WMs, they relate the forward and backward propagating fields in region i to those of region $i+1$. The regions are, in general, separated by dielectric spacer and interface. Therefore,

$$\begin{pmatrix} \mathbf{E}_i^+ \\ \mathbf{E}_i^- \end{pmatrix} = \mathbb{M}^{(i)}_{delay}\mathbb{M}^{(i)}_{inter} \begin{pmatrix} \mathbf{E}_{i+1}^+ \\ \mathbf{E}_{i+1}^- \end{pmatrix}. \tag{23}$$

The total WM for the cascaded structure shown in Fig.2, consisting of $n-1$ dielectric spacers and n sheet admittances, can be written as,

$$\begin{pmatrix} \mathbf{E}_1^+ \\ \mathbf{E}_1^- \end{pmatrix} = \mathbb{M}^{(1)}_{inter}\mathbb{M}^{(2)}_{delay}\mathbb{M}^{(2)}_{inter}...\mathbb{M}^{(n)}_{delay}\mathbb{M}^{(n)}_{inter} \begin{pmatrix} \mathbf{E}_{n+1}^+ \\ \mathbf{E}_{n+1}^- \end{pmatrix}. \tag{24}$$

Once the WMs of the constituent elements are found, the WM of the total structure can be computed through matrix multiplication. $\mathbb{M}^{(i)}_{inter}$ is derived by writing the boundary conditions at the interfaces. $\mathbb{M}^{(i)}_{delay}$ is found assuming plane wave propagation within the dielectric spacers.

A. Interfaces

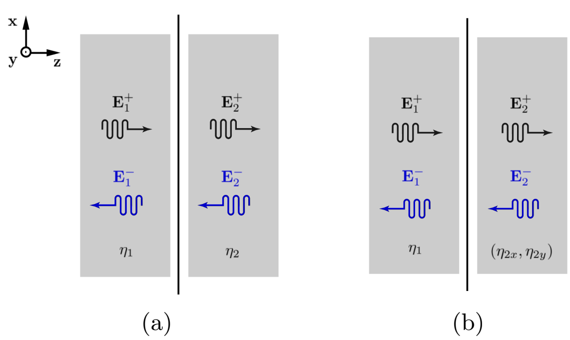


FIG. 3: Interface between an isotropic dielectric and an (a) isotropic dielectric (b) anisotropic dielectric.

Two types of interfaces are shown for the cascaded metasurface depicted in Fig.2: a sheet and a dielectric interface. The sheet interface, is represented by a sheet impedance boundary condition, and the dielectric interface results from the discontinuity in the spacer permittivities.

Let us begin with an isotropic dielectric interface (see Fig.3a). The medium on the left has an intrinsic impedance η_1, and the one on the right has an impedance η_2. Normal incidence onto the interface is assumed. Continuity of tangential electric and magnetic fields across the dielectric interface mandates,

$$\mathbf{E}_1^+ + \mathbf{E}_1^- = \mathbf{E}_2^+ + \mathbf{E}_2^-,$$

$$\frac{1}{\eta_1}\mathbf{n}\mathbf{E}_1^+ - \frac{1}{\eta_1}\mathbf{n}\mathbf{E}_1^- = \frac{1}{\eta_2}\mathbf{n}\mathbf{E}_2^+ - \frac{1}{\eta_2}\mathbf{n}\mathbf{E}_2^-. \tag{25}$$

Finding $\mathbf{E}_1^+$ and $\mathbf{E}_1^-$ in terms of $\mathbf{E}_2^+$ and $\mathbf{E}_2^-$ results in (see appendix B for details),

$$\begin{pmatrix} \mathbf{E}_1^+ \\ \mathbf{E}_1^- \end{pmatrix} = (\mathbf{t} \otimes \mathbf{I}) \begin{pmatrix} \mathbf{E}_2^+ \\ \mathbf{E}_2^- \end{pmatrix}, \tag{26}$$

where,

$$\mathbf{t} = \frac{1}{T}\begin{pmatrix} 1 & R \\ R & 1 \end{pmatrix}, \quad \mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{27}$$

and R and T are the Fresnel reflection and transmission coefficients,

$$R = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1}, \quad T = \frac{2\,\eta_2}{\eta_2 + \eta_1}. \tag{28}$$

The operator $\otimes$ denotes the Kronecker tensor product, which is defined as,

$$\mathbf{A}_{n\times m} \otimes \mathbf{B}_{p\times l} = \begin{pmatrix} a_{11}\mathbf{B} & \cdots & a_{1m}\mathbf{B} \\ \vdots & \ddots & \vdots \\ a_{n1}\mathbf{B} & \cdots & a_{nm}\mathbf{B} \end{pmatrix}_{np\times ml}. \tag{29}$$

Similarly, the WM of an anisotropic dielectric interface (Fig.3b) is given by,

$$\mathbb{M}_{inter} = \mathbf{t}_x \otimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \mathbf{t}_y \otimes \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \tag{30}$$

where,

$$\mathbf{t}_u = \frac{1}{T_u}\begin{pmatrix} 1 & R_u \\ R_u & 1 \end{pmatrix}, \tag{31}$$

with,

$$R_u = \frac{\eta_{2u} - \eta_1}{\eta_{2u} + \eta_1}, \quad T_u = \frac{2\,\eta_{2u}}{\eta_{2u} + \eta_1}. \tag{32}$$

The variable u can be either x or y, depending on the polarization of the wave.

Now, let's consider the sheet interface. In general, the sheet can possess a combination of electric, magnetic, and magneto-electric responses. The surface parameters relate electric and magnetic surface currents to the tangential components of electric and magnetic fields [5],

$$\begin{pmatrix} \mathbf{J}_{es} \\ \mathbf{J}_{ms} \end{pmatrix} = \begin{pmatrix} \mathbf{Y} & \boldsymbol{\chi} \\ \boldsymbol{\Upsilon} & \mathbf{Z} \end{pmatrix} \begin{pmatrix} \mathbf{E}_{av} \\ \mathbf{H}_{av} \end{pmatrix}, \tag{33}$$

where,

$$\mathbf{J}_{es} = \mathbf{n}(\mathbf{H}_2 - \mathbf{H}_1)\,, \quad \mathbf{J}_{ms} = -\mathbf{n}(\mathbf{E}_2 - \mathbf{E}_1)\,,$$

$$\mathbf{E}_{av} = \frac{1}{2}(\mathbf{E}_1 + \mathbf{E}_2)\,, \quad \mathbf{H}_{av} = \frac{1}{2}(\mathbf{H}_1 + \mathbf{H}_2)\,. \tag{34}$$

$\mathbf{J}_{es}$ and $\mathbf{J}_{ms}$ result from the discontinuity in tangential magnetic and electric fields, where $\mathbf{E}_i$ and $\mathbf{H}_i$ are the total tangential fields in region i. We can write the total fields ($\mathbf{E}_i$, $\mathbf{H}_i$) in Eq.(34) in terms of the forward and backward traveling fields ($\mathbf{E}_i^{\pm}$,$\mathbf{H}_i^{\pm}$) using Eq.(3). Then, we can find $\mathbf{E}_1^{\pm}$ in terms of $\mathbf{E}_2^{\pm}$ by substituting Eq.(34) into Eq.(33),

$$\begin{aligned}\begin{pmatrix}\mathbf{E}_1^+\\ \mathbf{E}_1^-\end{pmatrix} &= \mathbb{M}\begin{pmatrix}\mathbf{E}_2^+\\ \mathbf{E}_2^-\end{pmatrix}\\ &= \begin{pmatrix}\frac{\mathbf{Y}}{2} + \frac{\boldsymbol{\chi}\mathbf{n}}{2\eta_1} - \frac{\mathbf{I}}{\eta_1} & \frac{\mathbf{Y}}{2} - \frac{\boldsymbol{\chi}\mathbf{n}}{2\eta_1} + \frac{\mathbf{I}}{\eta_1}\\ \frac{\boldsymbol{\Upsilon}}{2} + \frac{\mathbf{Zn}}{2\eta_1} - \mathbf{n} & \frac{\boldsymbol{\Upsilon}}{2} - \frac{\mathbf{Zn}}{2\eta_1} - \mathbf{n}\end{pmatrix}^{-1}\\ &\times \begin{pmatrix}-\frac{\mathbf{Y}}{2} - \frac{\boldsymbol{\chi}\mathbf{n}}{2\eta_2} - \frac{\mathbf{I}}{\eta_2} & -\frac{\mathbf{Y}}{2} + \frac{\boldsymbol{\chi}\mathbf{n}}{2\eta_2} + \frac{\mathbf{I}}{\eta_2}\\ -\frac{\boldsymbol{\Upsilon}}{2} - \frac{\mathbf{Zn}}{2\eta_2} - \mathbf{n} & -\frac{\boldsymbol{\Upsilon}}{2} + \frac{\mathbf{Zn}}{2\eta_2} - \mathbf{n}\end{pmatrix}\begin{pmatrix}\mathbf{E}_2^+\\ \mathbf{E}_2^-\end{pmatrix}.\end{aligned} \tag{35}$$

Eq.(35) gives the WM for a bianisotropic sheet consisting of electric ($\mathbf{Y}$), magnetic ($\mathbf{Z}$), and magneto-electric ($\boldsymbol{\Upsilon}, \boldsymbol{\chi}$) responses.

Now let's consider the simple case where the sheet only has an electric response given by an admittance $\mathbf{Y}$ ($\mathbf{Z} = \boldsymbol{\chi} = \boldsymbol{\Upsilon} = \mathbf{0}$). The WM for such a sheet, placed between two identical isotropic media of impedance η_1 (Fig.4a), is,

$$\begin{pmatrix}\mathbf{E}_1^+\\ \mathbf{E}_1^-\end{pmatrix} = (\mathbf{I}\otimes\mathbf{I} + \frac{\eta_1}{2}\mathbf{e}\otimes\mathbf{Y})\begin{pmatrix}\mathbf{E}_2^+\\ \mathbf{E}_2^-\end{pmatrix}, \tag{36}$$

where,

$$\mathbf{e} = \begin{pmatrix}1 & 1\\ -1 & -1\end{pmatrix}, \quad \mathbf{Y} = \begin{pmatrix}Y^{xx} & Y^{xy}\\ Y^{yx} & Y^{yy}\end{pmatrix}. \tag{37}$$

Eq.(36) was derived by simplifying Eq.(35) (For details see Eq.(108) of appendix B).

Next, let's consider the case where the electric sheet is placed between two different isotropic media with wave impedances η_1 and η_2 (Fig.4b). Region 1 and 2 are separated by both an electric sheet and a dielectric interface. The total WM can be found by multiplying the WMs of the electric sheet admittance and the dielectric interface,

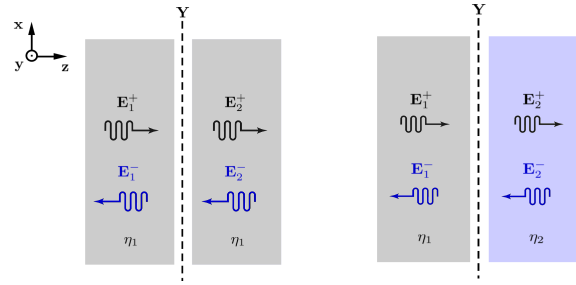
$$\mathbb{M}_{inter} = (\mathbf{I}\otimes\mathbf{I} + \frac{\eta_1}{2}\mathbf{e}\otimes\mathbf{Y})(\mathbf{t}\otimes\mathbf{I}), \tag{38}$$

which can also be expressed as,

$$\mathbb{M}_{inter} = (\mathbf{t}\otimes\mathbf{I} + \frac{\eta_1}{2}\mathbf{e}\otimes\mathbf{Y}), \tag{39}$$

by employing the identity,

$$(\mathbf{A}\otimes\mathbf{B})(\mathbf{C}\otimes\mathbf{D}) = (\mathbf{AC})\otimes(\mathbf{BD}), \tag{40}$$



(a) (b)

FIG. 4: An electric sheet admittance placed between regions that are (a) identical (b) different.

and the equality,

$$\mathbf{e}\,\mathbf{t} = \mathbf{e}. \tag{41}$$

If the sheet has a solely magnetic response $\mathbf{Z}$ ($\mathbf{Y} = \boldsymbol{\chi} = \boldsymbol{\Upsilon} = \mathbf{0}$), one can simplify Eq.(35) to (see appendix B),

$$\mathbb{M}_{inter} = (\mathbf{t}\otimes\mathbf{I} + \frac{1}{2\eta_2}\mathbf{m}\otimes(\mathbf{n}\,\mathbf{Z}\,\mathbf{n})), \tag{42}$$

with,

$$\mathbf{m} = \begin{pmatrix}1 & -1\\ 1 & -1\end{pmatrix}, \quad \mathbf{Z} = \begin{pmatrix}Z^{xx} & Z^{xy}\\ Z^{yx} & Z^{yy}\end{pmatrix}. \tag{43}$$

An isotropic sheet is simply a special case of an anisotropic admittance: $\mathbf{Y} = Y_e\mathbf{I}$ and $\mathbf{Z} = Z_m\mathbf{I}$. One can show that for an isotropic electric or magnetic sheet at a dielectric interface, we have,

$$\mathbf{M}_e = \mathbf{t} + \frac{\eta_1 Y_s}{2}\mathbf{e}\,, \quad \mathbf{M}_m = \mathbf{t} + \frac{Z_m}{2\eta_2}\mathbf{m}\,, \tag{44}$$

where the term $\mathbf{t}$ corresponds to the dielectric interface. When an isotropic sheet at a dielectric interface exhibits both electric (Y_s) and magnetic (Z_s) responses, the WM becomes,

$$\mathbf{M}_{em} = \frac{1}{1 - \frac{Y_s Z_s}{4}}\Big((1 + \frac{Y_s Z_s}{4})\mathbf{t} + \frac{\eta_1 Y_s}{2}\mathbf{e} + \frac{Z_s}{2\eta_2}\mathbf{m}\Big). \tag{45}$$

The WM for an isotropic interface is a 2×2 matrix, since the response is polarization independent.

B. A Phase Delay

The phases of the forward and backward propagating waves vary inside the dielectric spacers. The field at the left interface of a region is related to that at the right interface by (see Fig. 5),

$$\mathbf{E}_{\mathrm{L}}^+ = \mathbf{E}_{\mathrm{R}}^+\, e^{j\varphi}, \qquad \mathbf{E}_{\mathrm{L}}^- = \mathbf{E}_{\mathrm{R}}^-\, e^{-j\varphi}. \tag{46}$$

Therefore, a phase shift matrix representing the propagation can be written as,

$$\mathbb{M}_{delay} = \begin{pmatrix} e^{j\varphi} & 0 & 0 & 0 \\ 0 & e^{j\varphi} & 0 & 0 \\ 0 & 0 & e^{-j\varphi} & 0 \\ 0 & 0 & 0 & e^{-j\varphi} \end{pmatrix} = \Phi \otimes \mathbf{I}, \tag{47}$$

where,

$$\Phi = \begin{pmatrix} e^{j\varphi} & 0 \\ 0 & e^{-j\varphi} \end{pmatrix}. \tag{48}$$

The phase shift matrices relate the forward and backward propagating fields at the leftmost and rightmost boundary of each region.

The WM of a dielectric spacer consists of matrices for the left and right interfaces and a matrix representing the phase delay within the spacer. The spacer considered is surrounded by regions with identical wave impedance ($\eta_3 = \eta_1$). Therefore, using Eq.(27) and Eq.(32), it can be shown that the WM of the second interface is the inverse of the first interface,

$$\mathbb{M}^{(2)}_{inter} = \mathbb{M}^{(1)-1}_{inter}, \tag{49}$$

Using Eq.(49), the total WM for an isotropic dielectric slab is,

$$\mathbb{M}_{iso} = \mathbb{M}^{(1)}_{inter}\mathbb{M}_{delay}\mathbb{M}^{(2)}_{inter} = \mathbb{M}^{(1)}_{inter}\mathbb{M}_{delay}\mathbb{M}^{(1)-1}_{inter}$$

$$= (\mathbf{t} \otimes \mathbf{I})(\Phi \otimes \mathbf{I})(\mathbf{t}^{-1} \otimes \mathbf{I}) = (\mathbf{t}\,\Phi\,\mathbf{t}^{-1}) \otimes \mathbf{I}. \tag{50}$$

Identity Eq.(40) has been used to simplify Eq.(50).

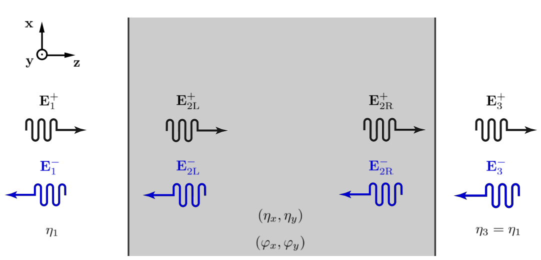


FIG. 5: Anisotropic slab with intrinsic impedances of (η_x, η_y) and phase shifts (φ_x, φ_y) for x and y polarizations.

Now, let us consider the WM of an anisotropic dielectric slab, as depicted in Fig.5. The slab's permittivity tensor is assumed to be $\bar{\epsilon}$, with eigenvectors along x, y, and z, and intrinsic impedances η_x, η_y for x and y polarized waves. As in Eq.(48), we can find the phase delay WM for an anisotropic dielectric slab,

$$\mathbb{M}_{delay} = \begin{pmatrix} e^{j\varphi_x} & 0 & 0 & 0 \\ 0 & e^{j\varphi_y} & 0 & 0 \\ 0 & 0 & e^{-j\varphi_x} & 0 \\ 0 & 0 & 0 & e^{-j\varphi_y} \end{pmatrix}, \tag{51}$$

which can be written in compact form,

$$\mathbb{M}_{delay} = \Phi_x \otimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \Phi_y \otimes \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \tag{52}$$

where,

$$\Phi_u = \begin{pmatrix} e^{j\varphi_u} & 0 \\ 0 & e^{-j\varphi_u} \end{pmatrix}. \tag{53}$$

The WM for an anisotropic dielectric interface is given by Eq.(30). The phase progression within the anisotropic slab is given by Eq.(52). Therefore, the total WM of the anisotropic slab is,

$$\begin{aligned} \mathbb{M}_{aniso} = (\mathbf{t}_x \otimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \mathbf{t}_y \otimes \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}) \\ \times (\Phi_x \otimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \Phi_y \otimes \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}) \\ \times (\mathbf{t}_x^{-1} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \mathbf{t}_y^{-1} \otimes \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}) \\ = (\mathbf{t}_x \Phi_x \mathbf{t}_x^{-1}) \otimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + (\mathbf{t}_y \Phi_y \mathbf{t}_y^{-1}) \otimes \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \end{aligned} \tag{54}$$

The variables $\mathbf{t}_u$, $\mathbf{t}_u^{-1}$, and Φ_u refer to the slab interfaces and the phase delay associated with u polarized incident wave. They are given by Eq.(31) and Eq.(53), respectively.

C. Rotation

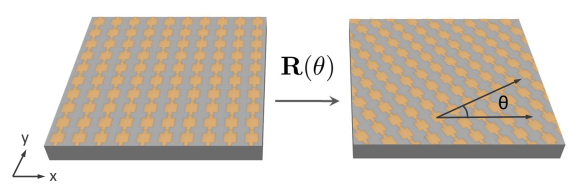


FIG. 6: Rotated sheet admittance by angle θ.

The WMs of the elements comprising the cascaded structure (shown in the Fig. 2) have been derived in the preceeding sections. The effect of rotation on the WMs of the constituent elements is explained here. Once we have the WM in a given coordinate system, we can find the WM in a rotated system. The rotation operator for a 4×4 WM is,

$$\mathbb{R}(\theta) = \mathbf{I} \otimes \mathbf{R}(\theta), \tag{55}$$

where θ is the rotation angle and $\mathbf{R}$ is 2×2 rotation matrix,

$$\mathbf{R}(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}. \tag{56}$$

Let us assume an element is rotated by an angle θ (Fig.6). The corresponding WM of the element is transformed by $\mathbb{R}$ as follows (See Ref. [16]),

$$\mathbb{M}_{rot} = \mathbb{R}\,\mathbb{M}\,\mathbb{R}^{\mathrm{T}}. \tag{57}$$

For example, consider a homogeneous electric sheet admittance that is rotated by an angle θ within an isotropic medium. The transformed WM is given by,

$$\begin{aligned}\mathbb{M}_{rot} &= (\mathbf{I}\otimes\mathbf{R})(\mathbf{I}\otimes\mathbf{I}+\frac{\eta_0}{2}\mathbf{e}\otimes\mathbf{Y})(\mathbf{I}\otimes\mathbf{R}^{\mathrm{T}})\\ &= \mathbf{I}\otimes\mathbf{I}+\frac{\eta_0}{2}\mathbf{e}\otimes(\mathbf{R}\mathbf{Y}\mathbf{R}^{\mathrm{T}}).\end{aligned} \tag{58}$$

The rotation amounts to replacing $\mathbf{Y}$ by $\mathbf{R}\mathbf{Y}\mathbf{R}^{\mathrm{T}}$. Similarly, using Eq.(54), in conjunction with Eq.(40), gives the following WM for a rotated anisotropic dielectric slab,

$$\begin{aligned}\mathbb{M}_{rot} = \;& (\mathbf{t}_x\Phi_x\mathbf{t}_x^{-1})\otimes\begin{pmatrix}\cos^2\theta & \sin\theta\cos\theta\\ \sin\theta\cos\theta & \sin^2\theta\end{pmatrix}\\ &+(\mathbf{t}_y\Phi_y\mathbf{t}_y^{-1})\otimes\begin{pmatrix}\sin^2\theta & -\sin\theta\cos\theta\\ -\sin\theta\cos\theta & \cos^2\theta\end{pmatrix}.\end{aligned} \tag{59}$$

IV. ANALYSIS AND SYNTHESIS

Metasurfaces are textured surfaces designed to exhibit tailored electromagnetic responses. Initially, metasurfaces possessed electric responses [1]. In such metasurfaces, the tangential electric field excites electric currents resulting in a discontinuity in the tangential magnetic field. Adding a magnetic response to metasurfaces has provided additional degrees of freedom in terms of phase coverage for wavefront manipulation, and reflection control [2]. Metasurfaces with magneto-electric coupling have provided control over the reflections [5, 23]. A simple approach to realizing metasurfaces with electric, magnetic, and bianisotropic responses is to cascade electric sheet admittances [3–5].

Cascaded homogeneous metasurfaces with bianisotropic responses (Fig.1) can be analyzed in a straightforward manner using WMs. The WMs of the constituent anisotropic electric sheet admittances and dielectric spacers are first computed. The overall WM is found by multiplying the constituent WMs. The reflection and transmission coefficients can then be found from the WM (Eq.(19)).

The design of a homogeneous bianisotropic metasurface begins with stipulating the scattering parameters. The synthesis problem involves finding the tensor sheets that need to be cascaded to realize the stipulated response. If one uses ABCD matrices, solving for the sheets in terms of the given S-matrix results in a set of nonlinear equations. Optimization techniques have been employed in the past to find the approximate values of the tensor admittances [5]. Here, we show that the WM representation yields a set of equations that can be analytically solved to yield the needed tensor admittances. The tensor sheet admittances are then realized with metallic patterns. The period of the patterns are assumed to be subwavelength, and the spacer's thickness larger than the cell size. Thus, near-field coupling via high-order Floquet modes can be neglected, and are not considered in the design process.

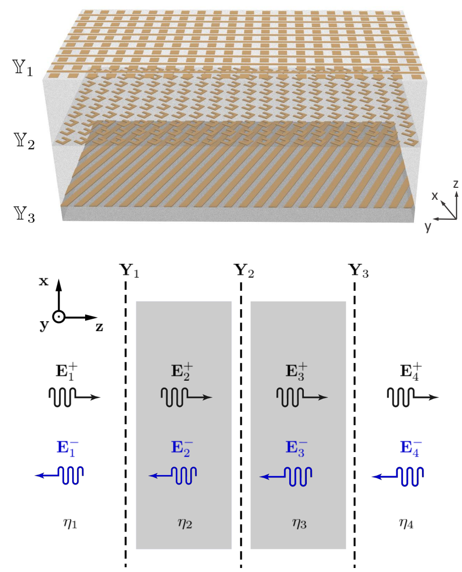


FIG. 7: A metasurface consisting of three cascaded electric sheet admittances designed to realize a stipulated S-matrix.

A. The Three Sheet Problem

A number of polarization controlling devices have been implemented using cascaded metasurfaces consisting of three electric sheet admittances ($\mathbf{Y}_1$, $\mathbf{Y}_2$, $\mathbf{Y}_3$) separated by two dielectric spacers [5]. In the design of such metasurfaces, the S-parameters are stipulated and the necessary sheet admittances solved for. To find the necessary sheet admittances, a numerical solver was employed in [5]. Here, we analytically solve for the sheets given an S-matrix. The WM of a cascaded metasurface, consisting of three electric sheet admittances (Fig.7), can be written as,

$$\begin{aligned}\mathbb{M}_{casc} = &(\mathbf{t}_1\otimes\mathbf{I}+\frac{\eta_1}{2}\mathbf{e}\otimes\mathbf{Y}_1)(\Phi_2\otimes\mathbf{I})(\mathbf{t}_2\otimes\mathbf{I}+\frac{\eta_2}{2}\mathbf{e}\otimes\mathbf{Y}_2)\\ &\times(\Phi_3\otimes\mathbf{I})(\mathbf{t}_3\otimes\mathbf{I}+\frac{\eta_3}{2}\mathbf{e}\otimes\mathbf{Y}_3).\end{aligned} \tag{60}$$

The first term in parentheses corresponds to the first dielectric interface and sheet admittance. The second term represents the phase delay of the first dielectric spacer (Fig.7). Similarly, the other terms are related to successive dielectric interfaces, sheets, and delays. The total WM on the left side of Eq.(60) is then written in terms of the desired S-matrix using Eq.(17). If the parentheses on the right side of Eq.(60) are expanded, and identity Eq.(40) is used, we can write,

$$\begin{aligned}\mathbb{S}_1\mathbb{S}_2^{-1} &= (\mathbf{t}_1\Phi_2\mathbf{t}_2\Phi_3\mathbf{t}_3)\otimes\mathbf{I} + \frac{\eta_1}{2}(\mathbf{e}\Phi_2\mathbf{t}_2\Phi_3\mathbf{t}_3)\otimes\mathbf{Y}_1 + \\ &\frac{\eta_2}{2}(\mathbf{t}_1\Phi_2\mathbf{e}\Phi_3\mathbf{t}_3)\otimes\mathbf{Y}_2 + \frac{\eta_3}{2}(\mathbf{t}_1\Phi_2\mathbf{t}_2\Phi_3\mathbf{e})\otimes\mathbf{Y}_3 + \\ &\frac{\eta_1\eta_2}{4}(\mathbf{e}\Phi_2\mathbf{e}\Phi_3\mathbf{t}_3)\otimes\mathbf{Y}_1\mathbf{Y}_2 + \frac{\eta_1\eta_3}{4}(\mathbf{e}\Phi_2\mathbf{t}_2\Phi_3\mathbf{e})\otimes\mathbf{Y}_1\mathbf{Y}_3 + \\ &\frac{\eta_2\eta_3}{4}(\mathbf{t}_1\Phi_2\mathbf{e}\Phi_3\mathbf{e})\otimes\mathbf{Y}_2\mathbf{Y}_3 + \frac{\eta_1\eta_2\eta_3}{8}(\mathbf{e}\Phi_2\mathbf{e}\Phi_3\mathbf{e})\otimes\mathbf{Y}_1\mathbf{Y}_2\mathbf{Y}_3\,. \end{aligned} \tag{61}$$

Multiplying both sides of Eq.(61) by $\mathbf{e}\otimes\mathbf{I}$, from the right, eliminates the terms containing $\mathbf{Y}_3$. This yields an expression for $\mathbf{Y}_1$ in terms of $\mathbf{Y}_2$:

$$\begin{aligned}\mathbf{e}\otimes\mathbf{Y}_1 &= \frac{1}{a_1}\ \Big(\mathbb{S}_1\mathbb{S}_2^{-1}(\mathbf{e}\otimes\mathbf{I}) - (\mathbf{t}_1\Phi_2\mathbf{t}_2\Phi_3\mathbf{t}_3\mathbf{e})\otimes\mathbf{I} \\ &- \frac{\eta_2}{2}(\mathbf{t}_1\Phi_2\mathbf{e}\Phi_3\mathbf{t}_3\mathbf{e})\otimes\mathbf{Y}_2\Big)\Big(\mathbf{I}\otimes(\mathbf{I}+\frac{a_{12}}{a_1}\mathbf{Y}_2)^{-1}\Big), \end{aligned} \tag{62}$$

where a_1 and a_{12} are given by,

$$\frac{\eta_1}{2}(\mathbf{e}\Phi_2\mathbf{t}_2\Phi_3\mathbf{t}_3\mathbf{e}) = a_1\mathbf{e}, \quad \frac{\eta_1\eta_2}{4}(\mathbf{e}\Phi_2\mathbf{e}\Phi_3\mathbf{t}_3\mathbf{e}) = a_{12}\mathbf{e}, \tag{63}$$

Multiplying Eq.(61) by $\mathbf{e}\otimes\mathbf{I}$, from the left, cancels the $\mathbf{Y}_1$ terms. This yields $\mathbf{Y}_3$ in terms of $\mathbf{Y}_2$,

$$\begin{aligned}\mathbf{e}\otimes\mathbf{Y}_3 &= \frac{1}{a_3}\big(\mathbf{I}\otimes(\mathbf{I}+\frac{a_{23}}{a_3}\mathbf{Y}_2)^{-1}\big)\big((\mathbf{e}\otimes\mathbf{I})\,\mathbb{S}_1\mathbb{S}_2^{-1} \\ &- (\mathbf{e}\mathbf{t}_1\Phi_2\mathbf{t}_2\Phi_3\mathbf{t}_3)\otimes\mathbf{I} - \frac{\eta_2}{2}(\mathbf{e}\mathbf{t}_1\Phi_2\mathbf{e}\Phi_3\mathbf{t}_3)\otimes\mathbf{Y}_2\big), \end{aligned} \tag{64}$$

with,

$$\frac{\eta_3}{2}(\mathbf{e}\mathbf{t}_1\Phi_2\mathbf{t}_2\Phi_3\mathbf{e}) = a_3\,\mathbf{e}, \quad \frac{\eta_2\eta_3}{4}(\mathbf{e}\mathbf{t}_1\Phi_2\mathbf{e}\Phi_3\mathbf{e}) = a_{23}\,\mathbf{e}, \tag{65}$$

In Eq.(62) and Eq.(64) above, the fact that $\mathbf{e}^2 = 0$ (Eq.(37)) was used to eliminate the terms that contain products of admittances. Now, $\mathbf{Y}_2$ can be isolated from the outer sheet admittances by multiplying both sides of Eq.(61), from the left and from the right, by $\mathbf{e}\otimes\mathbf{I}$. This cancels the terms with $\mathbf{Y}_1$ and $\mathbf{Y}_3$, resulting in,

$$\mathbf{e}\otimes\mathbf{Y}_2 = \frac{1}{a_2}\Big((\mathbf{e}\otimes\mathbf{I})\mathbb{S}_1\mathbb{S}_2^{-1}(\mathbf{e}\otimes\mathbf{I}) - (\mathbf{e}\mathbf{t}_1\Phi_2\mathbf{t}_2\Phi_3\mathbf{t}_3\mathbf{e})\otimes\mathbf{I}\Big), \tag{66}$$

where a_2 satisfies,

$$\frac{\eta_2}{2}(\mathbf{e}\mathbf{t}_1\Phi_2\mathbf{e}\Phi_3\mathbf{t}_3\mathbf{e}) = a_2\,\mathbf{e}. \tag{67}$$

In summary, the middle sheet admittance $\mathbf{Y}_2$ can be calculated using (66). The outer sheets, $\mathbf{Y}_1$ and $\mathbf{Y}_3$, can then be found in terms of $\mathbf{Y}_2$ using Eq.(62) and Eq.(64). Using this method, we have analytically solved for the sheets in terms of the given scattering parameters. The constants a_i and a_{ij} can be computed from Eqs.(63), (65), and (67) simply by noting that for any 2×2 matrix $\mathbf{M}$ with components m_{ij},

$$\mathbf{e}\,\mathbf{M}\,\mathbf{e} = (m_{11}+m_{21}-m_{12}-m_{22})\,\mathbf{e} = a\,\mathbf{e}. \tag{68}$$

B. The Four Sheet Problem

Increasing the number of sheet admittances in a cascaded metasurface, adds degrees of design freedom. For example, one can realize a bianisotropic response with three sheets. However, adding a fourth sheet can provide a wider bandwidth, as has been shown in [5]. The synthesis procedure described earlier for a metasurface consisting of three sheets can be extended to a larger number of sheets. Here, we present the design of a metasurface consisting of four sheets. Specifically, we 'peel off' the outer sheets and find a relation between the middle sheets and the scattering parameters. The outer sheets are then solved for once the middle sheets are known. As in the case of the three sheet problem, first, the total WM is written in terms of the WMs of the constituent elements,

$$\begin{aligned}\mathbb{M}_{casc} &= (\mathbf{t}_1\otimes\mathbf{I}+\frac{\eta_1}{2}\mathbf{e}\otimes\mathbf{Y}_1)(\Phi_2\otimes\mathbf{I})(\mathbf{t}_2\otimes\mathbf{I}+\frac{\eta_2}{2}\mathbf{e}\otimes\mathbf{Y}_2) \\ &\times(\Phi_3\otimes\mathbf{I})(\mathbf{t}_3\otimes\mathbf{I}+\frac{\eta_3}{2}\mathbf{e}\otimes\mathbf{Y}_3)(\Phi_4\otimes\mathbf{I})(\mathbf{t}_4\otimes\mathbf{I}+\frac{\eta_4}{2}\mathbf{e}\otimes\mathbf{Y}_4). \end{aligned} \tag{69}$$

Expanding the parantheses, using Eq.(40), and multiplying Eq.(69) by $\mathbf{e}\otimes\mathbf{I}$, from the left and from the right, eliminates the terms containing the outer sheets: $\mathbf{Y}_1$ and $\mathbf{Y}_4$. This yields a relation between the middle sheets $\mathbf{Y}_2$ and $\mathbf{Y}_3$,

$$b_2\ \mathbf{e}\otimes\mathbf{Y}_2 + b_3\ \mathbf{e}\otimes\mathbf{Y}_3 + b_{23}\ \mathbf{e}\otimes\mathbf{Y}_2\mathbf{Y}_3 = \mathbb{F}, \tag{70}$$

The constants b_2, b_3, and b_{23} and matrix $\mathbb{F}$ are,

$$\frac{\eta_2}{2}(\mathbf{e}\mathbf{t}_1\Phi_2\mathbf{e}\Phi_3\mathbf{t}_3\Phi_4\mathbf{t}_4\mathbf{e}) = b_2\,\mathbf{e},$$

$$\frac{\eta_3}{2}(\mathbf{e}\mathbf{t}_1\Phi_2\mathbf{t}_2\Phi_3\mathbf{e}\Phi_4\mathbf{t}_4\mathbf{e}) = b_3\,\mathbf{e},$$

$$\frac{\eta_2\eta_3}{4}(\mathbf{e}\mathbf{t}_1\Phi_2\mathbf{e}\Phi_3\mathbf{e}\Phi_4\mathbf{t}_4\mathbf{e}) = b_{23}\,\mathbf{e},$$

$$\mathbb{F} = (\mathbf{e}\otimes\mathbf{I})(\mathbb{S}_1\mathbb{S}_2^{-1})(\mathbf{e}\otimes\mathbf{I}) - (\mathbf{e}\mathbf{t}_1\Phi_2\mathbf{t}_2\Phi_3\mathbf{t}_3\Phi_4\mathbf{t}_4\mathbf{e})\otimes\mathbf{I}\,. \tag{71}$$

If $\mathbf{Y}_2$ is stipulated, then $\mathbf{Y}_3$ can be found using Eq.(70),

$$\mathbf{e}\otimes\mathbf{Y}_3 = \frac{1}{b_3}\big(\mathbf{I}\otimes(\mathbf{I}+\frac{b_{23}}{b_3}\ \mathbf{Y}_2)^{-1}\big)\left(\mathbb{F} - b_2\ \mathbf{e}\otimes\mathbf{Y}_2\right). \tag{72}$$

Now that the middle sheets are determined, we can find the outer sheets in terms of $\mathbf{Y}_2$ and $\mathbf{Y}_3$. Multiplying both sides of Eq.(69) by $\mathbf{e}\otimes\mathbf{I}$, from the right, cancels the $\mathbf{Y}_4$ terms. This gives $\mathbf{Y}_1$ in terms of middle sheets,

$$\begin{aligned}\mathbf{e}\otimes\mathbf{Y}_1 &= \mathbb{G}(\mathbf{Y}_2,\mathbf{Y}_3)\big(b_1\,\mathbf{I}\otimes\mathbf{I} + b_{12}\,\mathbf{I}\otimes\mathbf{Y}_2 \\ &+ b_{13}\,\mathbf{I}\otimes\mathbf{Y}_3 + b_{123}\,\mathbf{I}\otimes\mathbf{Y}_2\mathbf{Y}_3\big)^{-1}, \end{aligned} \tag{73}$$

Multiplying both sides of Eq.(69) by $\mathbf{e}\otimes\mathbf{I}$, from the left, cancels the $\mathbf{Y}_1$ terms. This gives $\mathbf{Y}_4$ in terms of the middle sheets,

$$\mathbf{e}\otimes\mathbf{Y}_4=\big(b_4\,\mathbf{I}\otimes\mathbf{I}+b_{24}\,\mathbf{I}\otimes\mathbf{Y}_2+b_{34}\,\mathbf{I}\otimes\mathbf{Y}_3$$
$$+\,b_{234}\,\mathbf{I}\otimes\mathbf{Y}_2\mathbf{Y}_3\big)^{-1}\,\mathbb{H}(\mathbf{Y}_2,\mathbf{Y}_3). \tag{74}$$

The constants b_i, b_{ij}, and b_{ijk} and matrices $\mathbb{G}$ and $\mathbb{H}$ in Eqs.(73) and (74) are given by,

$$\begin{aligned}
\frac{\eta_1}{2}(\mathbf{e}\Phi_2\mathbf{t}_2\Phi_3\mathbf{t}_3\Phi_4\mathbf{t}_4\mathbf{e})&=b_1\,\mathbf{e},\\
\frac{\eta_4}{2}(\mathbf{e}\mathbf{t}_1\Phi_2\mathbf{t}_2\Phi_3\mathbf{t}_3\Phi_4\mathbf{e})&=b_4\,\mathbf{e},\\
\frac{\eta_1\eta_2}{4}(\mathbf{e}\Phi_2\mathbf{e}\Phi_3\mathbf{t}_3\Phi_4\mathbf{t}_4\mathbf{e})&=b_{12}\,\mathbf{e},\\
\frac{\eta_1\eta_3}{4}(\mathbf{e}\Phi_2\mathbf{t}_2\Phi_3\mathbf{e}\Phi_4\mathbf{t}_4\mathbf{e})&=b_{13}\,\mathbf{e},\\
\frac{\eta_2\eta_4}{4}(\mathbf{e}\mathbf{t}_1\Phi_2\mathbf{e}\Phi_3\mathbf{t}_3\Phi_4\mathbf{e})&=b_{24}\,\mathbf{e},\\
\frac{\eta_3\eta_4}{4}(\mathbf{e}\mathbf{t}_1\Phi_2\mathbf{t}_2\Phi_3\mathbf{e}\Phi_4\mathbf{e})&=b_{34}\,\mathbf{e},\\
\frac{\eta_1\eta_2\eta_3}{8}(\mathbf{e}\Phi_2\mathbf{e}\Phi_3\mathbf{e}\Phi_4\mathbf{t}_4\mathbf{e})&=b_{123}\,\mathbf{e},\\
\frac{\eta_2\eta_3\eta_4}{8}(\mathbf{e}\mathbf{t}_1\Phi_2\mathbf{e}\Phi_3\mathbf{e}\Phi_4\mathbf{e})&=b_{234}\,\mathbf{e},
\end{aligned} \tag{75}$$

$$\mathbb{G}(\mathbf{Y}_2,\mathbf{Y}_3)=(\mathbb{S}_1\mathbb{S}_2^{-1})(\mathbf{e}\otimes\mathbf{I})-(\mathbf{t}_1\Phi_2\mathbf{t}_2\Phi_3\mathbf{t}_3\Phi_4\mathbf{t}_4\mathbf{e})\otimes\mathbf{I}$$
$$-\frac{\eta_2}{2}(\mathbf{t}_1\Phi_2\mathbf{e}\Phi_3\mathbf{t}_3\Phi_4\mathbf{t}_4\mathbf{e})\otimes\mathbf{Y}_2-\frac{\eta_3}{2}(\mathbf{t}_1\Phi_2\mathbf{t}_2\Phi_3\mathbf{e}\Phi_4\mathbf{t}_4\mathbf{e})\otimes\mathbf{Y}_3$$
$$-\frac{\eta_2\eta_3}{4}(\mathbf{t}_1\Phi_2\mathbf{e}\Phi_3\mathbf{e}\Phi_4\mathbf{t}_4\mathbf{e})\otimes\mathbf{Y}_2\mathbf{Y}_3, \tag{76}$$

$$\mathbb{H}(\mathbf{Y}_2,\mathbf{Y}_3)=(\mathbf{e}\otimes\mathbf{I})(\mathbb{S}_1\mathbb{S}_2^{-1})-(\mathbf{e}\mathbf{t}_1\Phi_2\mathbf{t}_2\Phi_3\mathbf{t}_3\Phi_4\mathbf{t}_4)\otimes\mathbf{I}$$
$$-\frac{\eta_2}{2}(\mathbf{e}\mathbf{t}_1\Phi_2\mathbf{e}\Phi_3\mathbf{t}_3\Phi_4\mathbf{t}_4)\otimes\mathbf{Y}_2-\frac{\eta_3}{2}(\mathbf{e}\mathbf{t}_1\Phi_2\mathbf{t}_2\Phi_3\mathbf{e}\Phi_4\mathbf{t}_4)\otimes\mathbf{Y}_3$$
$$-\frac{\eta_2\eta_3}{4}(\mathbf{e}\Phi_2\mathbf{e}\Phi_3\mathbf{e}\Phi_4\mathbf{t}_4)\otimes\mathbf{Y}_2\mathbf{Y}_3. \tag{77}$$

Here, we have shown realization of bianisotropic metasurfaces with a cascaded of four electric sheet admittances. In contrast to three sheets problem, additional degrees of freedom requires us to stipulate one of the middle sheets, for example $\mathbf{Y}_2$, then, using Eqs.(72), (73), and (74) we can find all of necessary sheet admittances ($\mathbf{Y}_3$, $\mathbf{Y}_1$, $\mathbf{Y}_4$) that realize a given S-matrix.

V. EXAMPLES

The synthesis procedure presented earlier allows us to find, in closed form, the tensor sheet admittances needed to realize a given scattering performance. Next, a few designs examples are shown to verify the synthesis approach, and show its utility:

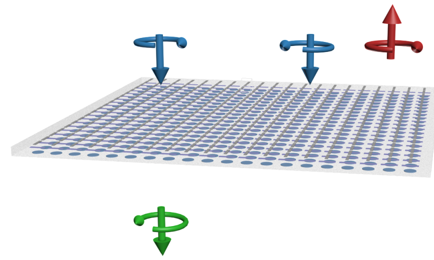


FIG. 8: Asymmetric transmission of light through the metasurface: the incident RHCP is fully transmitted and converted to LHCP, and the incident LHCP is reflected.

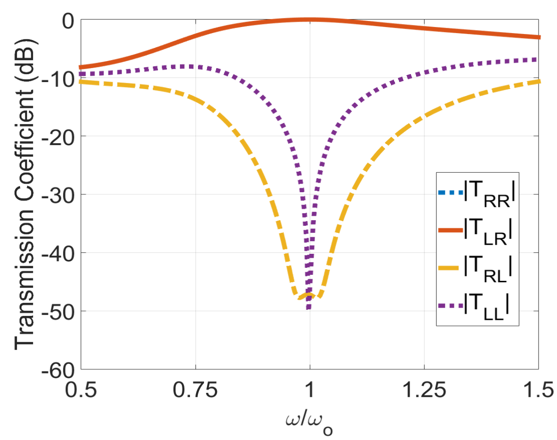


FIG. 9: The frequency response of cascaded sheet admittances that realize an asymmetric circular polarizer. Subscripts R and L denote the right-handed and left-handed circularly polarized waves. For example, T_{LR} indicates the transmission coefficient from an RHCP to an LHCP wave.

A. Asymmetric Circular Polarizer

Let's begin with the design of an asymmetric circular polarizer. The device converts an incident right-handed circularly polarized wave to a left-handed circularly polarized wave, upon transmission. On the other hand, it reflects a left-handed circularly polarized incident wave (Fig.8). The asymmetric polarizer of interest has the following Jones matrix,

$$\mathbf{S}_{21}=\frac{e^{j\phi}}{2}\begin{pmatrix}1 & j\\ j & -1\end{pmatrix}, \tag{78}$$

where the variable ϕ represents the average phase delay of the response. The device is assumed to be reciprocal, lossless, and symmetric, which requires $\mathbf{S}_{12}$, $\mathbf{S}_{11}$ and $\mathbf{S}_{22}$ to be,

$$\begin{aligned}
\mathbf{S}_{12}=\mathbf{S}_{21}^{\mathrm{T}}&=\frac{e^{j\phi}}{2}\begin{pmatrix}1 & j\\ j & -1\end{pmatrix},\\
\mathbf{S}_{11}=\mathbf{S}_{22}&=\frac{e^{j\phi}}{2}\begin{pmatrix}1 & -j\\ -j & -1\end{pmatrix}.
\end{aligned} \tag{79}$$

Now that the scattering parameters of the device have been defined, the next step is to find the overall WM of the device. According to Eq.(17), we can find the WM representation of the given S-matrix if,

$$\det(\mathbb{S}_2) \neq 0 \ , \tag{80}$$

which is equivalent to,

$$\det(\mathbf{S}_{21}) \neq 0 \ . \tag{81}$$

Since Eq.(81) is not satisfied for this device, we can simply perturb its S-matrix such that its determinant is nonzero. The perturbed S-matrix should still closely resemble the ideal Jones matrix. Here, we choose the following matrix,

$$\mathbf{S}_{21} = \mathbf{S}_{12}^{\mathrm{T}} = \frac{1}{2}\begin{pmatrix} 1 & j \\ j & -e^{j1^\circ} \end{pmatrix} . \tag{82}$$

The average phase delay, ϕ, is assumed to be zero. The next step in the design is to stipulate the dielectric spacer properties. The isotropic dielectric constant of the spacers is assumed $\epsilon_2 = \epsilon_3 = 5$, and the corresponding electrical thicknesses: $\varphi_2 = \varphi_3 = 2\pi/5$. Using Eq.(27), the WMs of the dielectric interfaces are computed to be,

$$\mathbf{t}_1 = \begin{pmatrix} 1.62 & -0.62 \\ -0.62 & 1.62 \end{pmatrix}, \quad \mathbf{t}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$\mathbf{t}_3 = \begin{pmatrix} 0.72 & 0.28 \\ 0.28 & 0.72 \end{pmatrix} . \tag{83}$$

From Eq.(48), the phase delay WMs are,

$$\Phi_2 = \Phi_3 = \begin{pmatrix} e^{j\frac{2\pi}{5}} & 0 \\ 0 & e^{-j\frac{2\pi}{5}} \end{pmatrix} . \tag{84}$$

Substituting the S-matrix, dielectric interface WMs and phase delay WMs into Eq.(66) provides the middle sheet admittance,

$$\mathbf{Y}_2 = \frac{j}{\eta_0}\begin{pmatrix} 1268.31 & 5.52 \\ 5.52 & 1.43 \end{pmatrix} . \tag{85}$$

Eqs.(62) and (64) yield the outer sheet admittances,

$$\mathbf{Y}_1 = \mathbf{Y}_3 = \frac{j}{\eta_0}\begin{pmatrix} 0.73 & 1.00 \\ 1.00 & 0.72 \end{pmatrix} . \tag{86}$$

It should be emphasized that the sheet admittances were analytically found.

The device performance versus frequency is shown in Fig.9. It shows that an RHCP incident wave is fully transmitted and converted to an LHCP ($\mathrm{T_{LR}} = 0\,\mathrm{dB}$) and an LHCP incident wave is reflected. The frequency has been normalized with respect to the frequency of operation. In this plot, the sheet admittances are assumed to obey Foster's reactance theorem. The positive susceptance has capacitive frequency response,

$$B_c(\omega) = j\frac{\omega}{\omega_0}B_c \, , \tag{87}$$

while the negative susceptance shows an inductive frequency response,

$$B_l(\omega) = j\frac{B_l}{\frac{\omega}{\omega_0}} \, . \tag{88}$$

B. Polarization Rotator

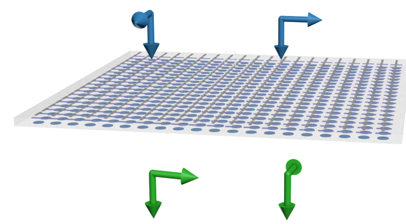


FIG. 10: A metasurface that rotates the polarization of the incident wave by 90°.

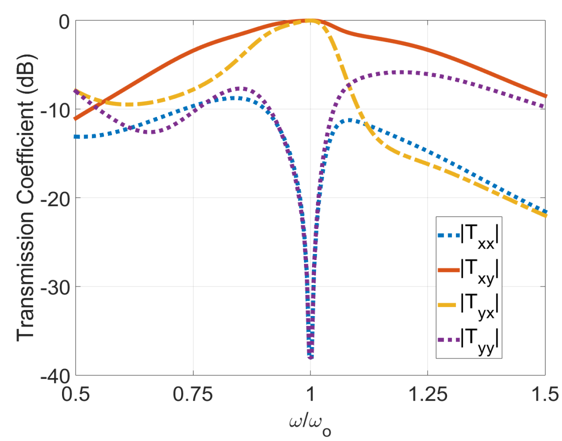


FIG. 11: Frequency response of the polarization rotator, realized by cascading of four electric sheet admittances. $\mathrm{T_{ij}}$ indicates the transmission coefficient from an incident j polarized wave to an i polarized wave.

The next device that is considered is a polarization rotator. This device rotates the polarization of an incident, linearly polarized wave by 90° (Fig.10). The Jones matrix for this device is,

$$\mathbf{S}_{21} = e^{j\phi}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} . \tag{89}$$

The variable ϕ represents the average transmitted phase delay. Having a lossless and reciprocal device requires that,

$$\mathbf{S}_{11} = \mathbf{S}_{22} = \mathbf{0}\,, \qquad \mathbf{S}_{12} = \mathbf{S}_{21}^{\mathrm{T}}\,. \tag{90}$$

Here, we realize this response with a cascade of four electric sheet admittances. First, the overall phase delay of the response is stipulated: $\phi = \pi/4.5$. Dielectric spacers with dielectric constants $\epsilon_2 = \epsilon_3 = \epsilon_4 = 3.5$ are assumed. The electrical thickness of the spacers are $\varphi_2 = \varphi_3 = \varphi_4 = 2\pi/10$. The interface and phase delay WMs are given by Eq.(27) and Eq.(48),

$$\mathbf{t}_1 = \mathbf{t}_4^{-1} = \begin{pmatrix} 1.43 & -0.43 \\ -0.43 & 1.43 \end{pmatrix}, \qquad \mathbf{t}_2 = \mathbf{t}_3 = \mathbf{I}\,,$$

$$\Phi_2 = \Phi_3 = \Phi_4 = \begin{pmatrix} e^{j\frac{2\pi}{10}} & 0 \\ 0 & e^{-j\frac{2\pi}{10}} \end{pmatrix}, \tag{91}$$

The sheet admittance $\mathbf{Y}_2$ is assumed to be,

$$\mathbf{Y}_2 = \frac{j}{\eta_0}\begin{pmatrix} 9.30 & 0 \\ 0 & 1.00 \end{pmatrix}, \tag{92}$$

Using Eq.(72) $\mathbf{Y}_3$ is found,

$$\mathbf{Y}_3 = \frac{j}{\eta_0}\begin{pmatrix} 7.59 & -7.77 \\ -7.77 & 2.71 \end{pmatrix}, \tag{93}$$

Eqs.(73) and (74) provide the outer electric sheet admittances ($\mathbf{Y}_1$ and $\mathbf{Y}_4$),

$$\mathbf{Y}_1 = \frac{j}{\eta_0}\begin{pmatrix} 5.01 & 0.77 \\ 0.77 & 0.13 \end{pmatrix}, \quad \mathbf{Y}_4 = \frac{j}{\eta_0}\begin{pmatrix} 2.57 & -1.30 \\ -1.30 & 2.57 \end{pmatrix}. \tag{94}$$

The frequency response for this device is shown in Fig.11. The frequency response of sheet admittances is assumed to obey the Foster's reactance theorem. The plot indicates that an incident x polarized wave is fully transmitted and converted to a y polarized wave ($\mathrm{T}_{yx} = 0\,\mathrm{dB}$). Similarly, an incident y polarized wave is fully transmitted and converted to an x polarized wave ($\mathrm{T}_{xy} = 0\,\mathrm{dB}$).

VI. CONCLUSION

In this work, wave matrices were reviewed and expanded upon. Their advantages in the design of cascaded metasurfaces were outlined, and their relation to other network parameters derived. Wave matrices of interfaces, sheet admittances, and dielectric spacers were reported. The wave matrix formalism was applied to the synthesis of cascaded metasurfaces in order to develop an analytic design approach. The approach allows the sheet admittances of cascaded metasurfaces to be found in terms of a stipulated scattering matrix. The synthesis approach will find broad application in the design of flat optical devices.

VII. ACKNOWLEDGEMENT

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VIII. APPENDICES

A. Impedance and Hybrid Matrices

Here, we derive the relationships between WMs, impedance matrices as well as Hybrid matrices. First, Impedance and Hybrid matrices are found in terms of the WM. This is achieved by considering two separate cases. In the first case, we set the forward propagating electric field to unity and the backward propagating field to zero in region $n+1$ (see Fig.2). Inserting $\mathbf{E}_{n+1}^{+} = \mathbf{I}$ and $\mathbf{E}_{n+1}^{-} = \mathbf{0}$, into Eq.(8) results in,

$$\mathbf{E}_1^{+} = \mathbf{M}_{11}\,, \quad \mathbf{E}_1^{-} = \mathbf{M}_{21}\,. \tag{95}$$

In the second case, we assume $\mathbf{E}_{n+1}^{+} = \mathbf{0}$ and $\mathbf{E}_{n+1}^{-} = \mathbf{I}$ in Eq.(8), which yields,

$$\mathbf{E}_1^{+} = \mathbf{M}_{12}\,, \quad \mathbf{E}_1^{-} = \mathbf{M}_{22}\,. \tag{96}$$

Now, we have two sets of electric field vectors in terms of WM components. We can insert them into Eq.(6) and Eq.(7) to find the Impedance and Hybrid matrices as a function of WM components. This results in the following expression for the Impedance matrix,

$$\begin{pmatrix} \mathbf{Z}_{11} & \mathbf{Z}_{12} \\ \mathbf{Z}_{21} & \mathbf{Z}_{22} \end{pmatrix} = \begin{pmatrix} \mathbf{M}_{11} + \mathbf{M}_{21} & \mathbf{M}_{12} + \mathbf{M}_{22} \\ \mathbf{I} & \mathbf{I} \end{pmatrix}$$

$$\times \begin{pmatrix} \frac{1}{\eta_1}\mathbf{n}(\mathbf{M}_{11} - \mathbf{M}_{21}) & \frac{1}{\eta_1}\mathbf{n}(\mathbf{M}_{12} - \mathbf{M}_{22}) \\ \frac{1}{\eta_{n+1}}\mathbf{n} & -\frac{1}{\eta_{n+1}}\mathbf{n} \end{pmatrix}^{-1}. \tag{97}$$

Similarly, for the Hybrid matrix we find,

$$\begin{pmatrix} \mathbf{H}_{11} & \mathbf{H}_{12} \\ \mathbf{H}_{21} & \mathbf{H}_{22} \end{pmatrix} = \begin{pmatrix} \mathbf{M}_{11} + \mathbf{M}_{21} & \mathbf{M}_{12} + \mathbf{M}_{22} \\ \frac{1}{\eta_{n+1}}\mathbf{n} & -\frac{1}{\eta_{n+1}}\mathbf{n} \end{pmatrix}$$

$$\times \begin{pmatrix} \frac{1}{\eta_1}\mathbf{n}(\mathbf{M}_{11} - \mathbf{M}_{21}) & \frac{1}{\eta_1}\mathbf{n}(\mathbf{M}_{12} - \mathbf{M}_{22}) \\ \mathbf{I} & \mathbf{I} \end{pmatrix}^{-1}. \tag{98}$$

Conversely, to find the WM in terms of the impedance matrix, we can rearrange Eq.(97) as,

$$\begin{pmatrix} \mathbf{Z}_{11} & \mathbf{Z}_{12} \\ \mathbf{Z}_{21} & \mathbf{Z}_{22} \end{pmatrix} \left[\begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \frac{1}{\eta_{n+1}}\mathbf{n} & -\frac{1}{\eta_{n+1}}\mathbf{n} \end{pmatrix} + \begin{pmatrix} \frac{1}{\eta_1}\mathbf{n} & -\frac{1}{\eta_1}\mathbf{n} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \mathbb{M} \right]$$

$$= \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{I} & \mathbf{I} \end{pmatrix} + \begin{pmatrix} \mathbf{I} & \mathbf{I} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \mathbb{M}\,. \tag{99}$$

Solving for $\mathbb{M}$ yields,

$$\mathbb{M} = \begin{pmatrix} -\mathbf{I} + \frac{1}{\eta_1}\mathbf{Z}_{11}\mathbf{n} & -\mathbf{I} - \frac{1}{\eta_1}\mathbf{Z}_{11}\mathbf{n} \\ \frac{1}{\eta_1}\mathbf{Z}_{21}\mathbf{n} & -\frac{1}{\eta_1}\mathbf{Z}_{21}\mathbf{n} \end{pmatrix}^{-1}$$

$$\times \begin{pmatrix} -\frac{1}{\eta_{n+1}}\mathbf{Z}_{12}\mathbf{n} & \frac{1}{\eta_{n+1}}\mathbf{Z}_{12}\mathbf{n} \\ \mathbf{I} - \frac{1}{\eta_{n+1}}\mathbf{Z}_{22}\mathbf{n} & \mathbf{I} + \frac{1}{\eta_{n+1}}\mathbf{Z}_{22}\mathbf{n} \end{pmatrix}. \quad (100)$$

Similarly, Eq.(98) can be rearranged to find the WM in terms of the hybrid matrix,

$$\begin{pmatrix} \mathbf{H}_{11} & \mathbf{H}_{12} \\ \mathbf{H}_{21} & \mathbf{H}_{22} \end{pmatrix} \left[\begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{I} & \mathbf{I} \end{pmatrix} + \begin{pmatrix} \frac{1}{\eta_1}\mathbf{n} & -\frac{1}{\eta_1}\mathbf{n} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \mathbb{M} \right]$$

$$= \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \frac{1}{\eta_{n+1}}\mathbf{n} & -\frac{1}{\eta_{n+1}}\mathbf{n} \end{pmatrix} + \begin{pmatrix} \mathbf{I} & \mathbf{I} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \mathbb{M}. \quad (101)$$

Solving for $\mathbb{M}$ using the equation above leads to,

$$\mathbb{M} = \begin{pmatrix} \frac{1}{\eta_1}\mathbf{H}_{11}\mathbf{n} - \mathbf{I} & -\frac{1}{\eta_1}\mathbf{H}_{11}\mathbf{n} - \mathbf{I} \\ \frac{1}{\eta_1}\mathbf{H}_{21}\mathbf{n} & -\frac{1}{\eta_1}\mathbf{H}_{21}\mathbf{n} \end{pmatrix}^{-1}$$

$$\times \begin{pmatrix} -\mathbf{H}_{12} & -\mathbf{H}_{12} \\ \frac{\mathbf{n}}{\eta_{n+1}} - \mathbf{H}_{22} & -\frac{\mathbf{n}}{\eta_{n+1}} - \mathbf{H}_{22} \end{pmatrix}. \quad (102)$$

B. WM Derivations

Here, we derive the WMs reported in section III, using block matrix inversion rules. Assume a 4×4 nonsingular matrix $\mathbb{R}$ that is partitioned into 2×2 submatrices of $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, and $\mathbf{D}$,

$$\mathbb{R} = \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{pmatrix}. \quad (103)$$

If $\mathbf{A}$ and $\mathbf{D}$ are both invertible matrices, we have the following expression for $\mathbb{R}^{-1}$ (see Ref.[24] for more details),

$$\mathbb{R}^{-1} =$$

$$\begin{pmatrix} (\mathbf{A} - \mathbf{B}\mathbf{D}^{-1}\mathbf{C})^{-1} & -\mathbf{A}^{-1}\mathbf{B}(\mathbf{D} - \mathbf{C}\mathbf{A}^{-1}\mathbf{B})^{-1} \\ -\mathbf{D}^{-1}\mathbf{C}(\mathbf{A} - \mathbf{B}\mathbf{D}^{-1}\mathbf{C})^{-1} & (\mathbf{D} - \mathbf{C}\mathbf{A}^{-1}\mathbf{B})^{-1} \end{pmatrix}. \quad (104)$$

The WM of an interface between two isotropic media is derived by writing boundary conditions at the interface (see Eq.(25)),

$$\mathbb{M} = \begin{pmatrix} \mathbf{I} & \mathbf{I} \\ \frac{1}{\eta_1}\mathbf{n} & -\frac{1}{\eta_1}\mathbf{n} \end{pmatrix}^{-1} \begin{pmatrix} \mathbf{I} & \mathbf{I} \\ \frac{1}{\eta_2}\mathbf{n} & -\frac{1}{\eta_2}\mathbf{n} \end{pmatrix}. \quad (105)$$

Using Eq.(104), $\mathbb{M}$ can be simplified to,

$$\mathbb{M} = \left(\frac{\eta_2 + \eta_1}{2\eta_2}\right) \begin{pmatrix} \mathbf{I} & (\frac{\eta_2 - \eta_1}{\eta_2 + \eta_1})\mathbf{I} \\ (\frac{\eta_2 - \eta_1}{\eta_2 + \eta_1})\mathbf{I} & \mathbf{I} \end{pmatrix} = \mathbf{t} \otimes \mathbf{I}, \quad (106)$$

which is the same as Eq.(26).

Now, let's assume an interface only has an electric response: $\boldsymbol{\chi} = \mathbf{0}$, $\boldsymbol{\Upsilon} = \mathbf{0}$, and $\mathbf{Z} = \mathbf{0}$. Therefore, the WM given by Eq.(35) simplifies to,

$$\mathbb{M} = \begin{pmatrix} \frac{\mathbf{Y}}{2} - \frac{\mathbf{I}}{\eta_1} & \frac{\mathbf{Y}}{2} + \frac{\mathbf{I}}{\eta_1} \\ -\mathbf{n} & -\mathbf{n} \end{pmatrix}^{-1} \begin{pmatrix} -\frac{\mathbf{Y}}{2} - \frac{\mathbf{I}}{\eta_1} & -\frac{\mathbf{Y}}{2} + \frac{\mathbf{I}}{\eta_1} \\ -\mathbf{n} & -\mathbf{n} \end{pmatrix}. \quad (107)$$

Taking the inverse of the first matrix, and performing the multiplications results in,

$$\mathbb{M} = \begin{pmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{pmatrix} + \frac{\eta_1}{2} \begin{pmatrix} \mathbf{Y} & \mathbf{Y} \\ -\mathbf{Y} & -\mathbf{Y} \end{pmatrix}, \quad (108)$$

which is the same expression as Eq.(36). Similarly, the WM for a magnetic sheet ($\boldsymbol{\chi} = \mathbf{0}$, $\boldsymbol{\Upsilon} = \mathbf{0}$, and $\mathbf{Y} = \mathbf{0}$) is,

$$\mathbb{M} = \begin{pmatrix} -\frac{\mathbf{I}}{\eta_1} & \frac{\mathbf{I}}{\eta_1} \\ \frac{\mathbf{Z}\mathbf{n}}{2\eta_1} - \mathbf{n} & -\frac{\mathbf{Z}\mathbf{n}}{2\eta_1} - \mathbf{n} \end{pmatrix}^{-1} \begin{pmatrix} -\frac{\mathbf{I}}{\eta_1} & \frac{\mathbf{I}}{\eta_1} \\ -\frac{\mathbf{Z}\mathbf{n}}{2\eta_1} - \mathbf{n} & \frac{\mathbf{Z}\mathbf{n}}{2\eta_1} - \mathbf{n} \end{pmatrix}. \quad (109)$$

Finally, Matrix $\mathbb{M}$ can be simplified to,

$$\mathbb{M} = \begin{pmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{pmatrix} + \frac{1}{2\eta_2} \begin{pmatrix} \mathbf{nZn} & -\mathbf{nZn} \\ \mathbf{nZn} & -\mathbf{nZn} \end{pmatrix}. \quad (110)$$

IX. REFERENCES

[1] N. Yu, P. Genevet, M. Kats, F. Aieta, J. Tetienne, F. Capasso and Z. Gaburro, "Light Propagation with Phase Discontinuities: Generalized Laws of Reflection and Refraction", Science, vol. 334, no. 6054, pp. 333-337, 2011.

[2] C. Pfeiffer and A. Grbic, "Metamaterial Huygens' Surfaces: Tailoring Wave Fronts with Reflectionless Sheets", Phys. Rev. Lett., vol. 110, no. 19, 2013.

[3] F. Monticone, N. Estakhri and A. Alù, "Full Control of Nanoscale Optical Transmission with a Composite Metascreen", Phys. Rev. Lett., vol. 110, no. 20, 2013.

[4] C. Pfeiffer and A. Grbic, "Millimeter-Wave Transmitarrays for Wavefront and Polarization Control", IEEE Transactions on Microwave Theory and Techniques, vol. 61, no. 12, pp. 4407-4417, 2013.

[5] C. Pfeiffer and A. Grbic, "Bianisotropic Metasurfaces for Optimal Polarization Control: Analysis and Synthesis",

Phys. Rev. Applied, vol. 2, no. 4, 2014.

[6] S. He, Y. Hu and S. Strom, "Electromagnetic reflection and transmission for a stratified bianisotropic slab", IEEE Trans. Antennas Propagat., vol. 42, no. 6, pp. 856-858, 1994.

[7] D. Holmes and D. Feucht, "Electromagnetic Wave Propagation in Birefringent Multilayers", Journal of the Optical Society of America, vol. 56, no. 12, p. 1763, 1966.

[8] Ning and E. Tan, "Hybrid Matrix Method for Stable Analysis of Electromagnetic Waves in Stratified Bianisotropic Media",IEEE Microwave and Wireless Components Letters, vol. 18, no. 10, pp. 653-655, 2008.

[9] S. Teitler and B. Henvis, "Refraction in Stratified, Anisotropic Media", Journal of the Optical Society of America, vol. 60, no. 6, p. 830, 1970.

[10] D. Berreman, "Optics in Stratified and Anisotropic Media: 4×4-Matrix Formulation," J. Opt. Soc. Am. 62, 502-510 (1972).

[11] P. Yeh, "Optics of anisotropic layered media: A new 4 × 4 matrix algebra",Surface Science Letters, vol. 96, no. 1-3, p. A211, 1980.

[12] R. Jones, "A New Calculus for the Treatment of Optical Systems I. Description and Discussion of the Calculus," J. Opt. Soc. Am. 31, 488-493 (1941).

[13] R. Azzam and N. Bashara, Ellipsometry and polarized light, 1st ed. Amsterdam: North-Holland Pub. Co., 1977.

[14] R. Collin, Field theory of guided waves. New York: McGraw-Hill, 1960.

[15] Y. Zhao, M. Belkin and A. Alù, "Twisted optical metamaterials for planarized ultrathin broadband circular polarizers",Nature Communications, vol. 3, p. 870, 2012.

[16] A. Askarpour, Y. Zhao and A. Alù, "Wave propagation in twisted metamaterials",Physical Review B, vol. 90, no. 5, 2014.

[17] M. Selvanayagam and G. Eleftheriades, "Design And Measurement of Tensor Impedance Transmitarrays For Chiral Polarization Control", IEEE Transactions on Microwave Theory and Techniques, pp. 1-15, 2015.

[18] M. Li and N. Behdad, "Wideband True-Time-Delay Microwave Lenses Based on Metallo-Dielectric and All-Dielectric Lowpass Frequency Selective Surfaces",IEEE Transactions on Antennas and Propagation, vol. 61, no. 8, pp. 4109-4119, 2013.

[19] M. Al-Joumayly and N. Behdad, "Wideband Planar Microwave Lenses Using Sub-Wavelength Spatial Phase Shifters", IEEE Transactions on Antennas and Propagation, vol. 59, no. 12, pp. 4542-4552, 2011.

[20] J. Wong, A. Epstein and G. Eleftheriades, "Reflectionless Wide-Angle Refracting Metasurfaces",IEEE Antennas and Wireless Propagation Letters, vol. 15, pp. 1293-1296, 2016.

[21] N. Mohammadi Estakhri and A. Alù, "Wave-front Transformation with Gradient Metasurfaces",Physical Review X, vol. 6, no. 4, 2016.

[22] D. Pozar, Microwave engineering, 1st ed. Hoboken, NJ: J. Wiley, 2005.

[23] V. Asadchy, Y. Ra'di, J. Vehmas and S. Tretyakov, "Functional Metamirrors Using Bianisotropic Elements", Physical Review Letters, vol. 114, no. 9, 2015.

[24] T. Lu and S. Shiou, "Inverses of 2 × 2 block matrices", Computers and Mathematics with Applications, vol. 43, no. 1-2, pp. 119-129, 2002.