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Lattice spin models for non-Abelian Chiral Spin Liquids

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We suggest a class of two-dimensional lattice spin Hamiltonians describing non-Abelian SU(2) chiral spin liquids - spin-analogues of fractional non-Abelian quantum Hall states- with gapped bulk and gapless chiral edge excitations described by the SU(2)_n Wess-Zumino-Novikov-Witten conformal field theory. The models are constructed from an array of a generalized spin- $n/2$ ladders with multi-spin exchange interaction which are coupled by isolated spins. Such models allow a controllable analytic treatment starting from the one-dimensional limit and are characterized by a bulk gap and non-Abelian SU(2)_n gapless edge excitations.

About thirty years ago Kalmeyer and Laughlin proposed a spin-analogue of a fractional quantum Hall state (FQHS) for electrically neutral quasiparticles - dubbed Kalmeyer-Laughlin chiral spin liquid (KL CSL) state¹. This state shares basic properties of FQHS such as gapped bulk and robust gapless edge excitations^{2,3}. The latter ones are described by the SU(2)₁ Wess-Zumino-Novikov-Witten (WZNW) conformal field theory³⁻⁵. The KL CSL state is a singlet and breaks both time-reversal and parity symmetry. It is also topologically nontrivial with a vacuum degeneracy on compactified spaces⁶.

Historically, such a state was discovered by considering lattice versions of the bosonic $\nu = 1/2$ Laughlin wave function as variational candidate ground-state wave functions for the triangular lattice spin-1/2 Heisenberg model^{1,2,7}. There is now a numerical evidence for its existence in the spin-1/2 Heisenberg model on kagome lattice with the first, second and third nearest neighbor interaction⁸⁻¹⁰. Another piece of numerical evidence suggests that KL CSL exists for the spin-1/2 Heisenberg model on kagome and triangular lattices¹²⁻¹⁵ with an additional three-spin exchange interaction explicitly breaking the time-reversal symmetry. Both studies numerically demonstrate the required ground-state degeneracy and the full characterization of the underlying topological order for the KL CSL state.

Since numerics has its restrictions it is highly desirable to have microscopic models of CSLs which would allow a controlled analytical description. At the same time such controllable models should not be too unrealistic. While exact parent lattice Hamiltonians for KL CSL wave function have been constructed in Refs. 16,18,19, they are very challenging to achieve experimentally since they contain complex variables as well as long-range interactions. A complementary approach is the so-called coupled-wires construction to FQHS which starts from an array of one-dimensional (1D) wires, coupled in such a way that a two-dimensional (2D) gapped phase with 1D gapless edge excitations emerges²⁰. It opens a possibility to find microscopic models which display Abelian and

non-Abelian topologically ordered physics from arrays of quantum spin chains²¹⁻²⁴. For KL CSL such attempt was made by Gorohovsky *et.al.*²¹ who suggested a model on anisotropic triangular lattice. Unfortunately the authors have found that in any realistic situation the CSL will be destroyed by competing magnetic orders. Since this puts them in a conflict with the numerical results of Refs. 14,15 who presented a good evidence for CSL even for the isotropic lattice, things are not under control.

As was suggested in^{17,25}, the KL CSL is just one example of spin liquid among many and CSLs with non-Abelian statistics might be stabilized. We construct analytically tractable 2D microscopic models of non-Abelian SU(2) CSLs where the edge modes are described by the SU(2)_n WZNW theory. For $n = 2$ this is an analogue of the bosonic Pfaffian state. For general n these CSLs correspond to the Read-Rezayi series at filling $\nu = n/2$ with \mathbb{Z}_n parafermionic neutral states²⁶. For $n \geq 2$, these quasiparticle excitations possess non-Abelian statistics like those of the Pfaffian. The lattice models consist of an array of coupled spin- $n/2$ generalized zigzag ladders with three-spin exchange interactions which breaks the time-reversal and parity symmetries preserving their product. Although it looks unusual, such interaction may be generated either as a result of spontaneous symmetry breaking^{2,27} or, as was suggested in Ref. 11, emerges in a Mott insulator as a result of a magnetic field.

Lattice spin models. The simplest lattice model for $n = 1$ is depicted in Fig. 1 and the Hamiltonian for individual ladders in this case includes only two- and three-spin interactions:

$$\mathcal{H}_{\text{ladder}} = \mathcal{H}_J + \mathcal{H}_\chi, \quad (1)$$

$$\mathcal{H}_J = \sum_n \left[J \mathbf{S}_n \cdot \mathbf{S}_{n+2} + J' \mathbf{S}_n \cdot \mathbf{S}_{n+1} \right], \quad (2)$$

$$\mathcal{H}_\chi = \frac{\chi}{2} \sum_{i,j,k \in \Delta} \left(\mathbf{S}_i \cdot [\mathbf{S}_j \times \mathbf{S}_k] \right), \quad (3)$$

where the diagonal exchange is weaker than the exchange along the legs: $J' < J$. For the three-spin interaction the sum goes on all triangular plaquettes and indices i, j, k

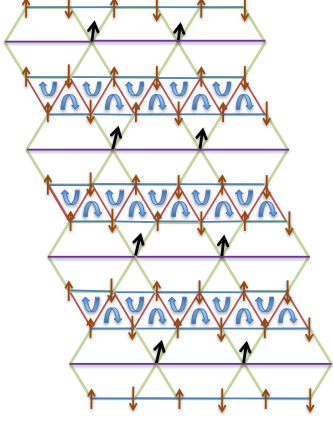


FIG. 1: The spin arrangement for the suggested model. Different link colors correspond to different exchange interactions. The blue links mark the strongest isotropic exchange J , the red ones mark weaker isotropic exchange J' , the green links correspond to the anisotropic interladder exchange and the magenta ones to the Ising-like anisotropic exchange I^{ab} between the intermediate spins. The blue arrows denote the three-spin exchange interactions. Since there is no static spin order all spin orientations depicted on the figure should be ignored.

appear in the clockwise order in each elementary triangle. The ladder model (1) allows analytic and controllable treatment in two cases. Either the interchain exchange is weak $J' \ll J^{21}$ or one is close to the integrable point: $\chi^2 = 2JJ'^{28}$. The low-energy spectrum of the $n = 1$ model (1) consists of gapless bosonic modes of opposite chirality located on different legs of the ladder^{21,28}. This fulfills the necessary condition for the coupled-wires construction which requires that neighboring wires are coupled by chiral modes of opposite chirality²⁰. The authors of Ref. 21 proceeded suggesting to couple the ladders in the fashion of triangular lattice with the conclusion that such coupling will destroy the CSL due to the competing magnetic order. This competition comes from the coupling of staggered magnetizations of the next-to-nearest neighbor chains generated by higher order virtual processes. On the triangular lattice this interaction is $J_{nnn} \sim J''(J'/J)$, where J'' is the exchange between the ladders. On the other hand, the interaction between the gapless modes of the neighboring ladders is marginally relevant and the corresponding gap is exponentially small in $1/J''$ and hence will be suppressed by the indirect exchange J_{nnn} .

In model of Fig. 1 the ladders with three-spin interactions which couple spins around triangles are separated by localized spins. The interaction with the latter spins is strongly anisotropic. Such arrangement allows one to overcome the problems related to the unfrustrated interactions between next-to-nearest spins discussed above and to make the CSL more robust, still providing a possi-

bility for analytical treatment of the model Hamiltonians. The fact that our model is formulated on a lattice allows numerical checks of our results.

In the case of generic n , the integrable model (1) for spins $S = n/2$ was found by Zvyagin²⁹. The explicit form of the Hamiltonian is given in Refs. 29,30 and it represents coupled integrable critical spin- $S = n/2$ Heisenberg chains with extra multi-spin interactions around triangular plaquettes. The most important difference is that the gapless chiral modes now are described by the $SU(2)_n$ WZNW model. A 2D lattice model of non-Abelian CSL with $SU(2)_n$ edge modes is constructed then by coupling the generalized ladders as in Fig. 1.

The continuum limit of the individual ladder model. To derive the effective field theory describing the continuum limit of the ladder model one just follows the standard approach of weakly-coupled two-leg spin ladders^{32,33}. Alternatively at the integrable point one may use the Bethe ansatz results of Refs. 28,29. The continuum limit Hamiltonian is the sum of two perturbed $SU(2)_n$ WZNW models with densities:

$$\mathcal{H}_{\text{eff}} = \mathcal{H}_R + \mathcal{H}_L + \mathcal{V} \quad (4)$$

$$\mathcal{H}_R = \frac{2\pi v}{n+2} \left[: \mathbf{J}_{R1}^2 : + : \mathbf{J}_{L2}^2 : \right] + 2\pi v g_R \mathbf{J}_{R1} \cdot \mathbf{J}_{L2} \quad (5)$$

$$\mathcal{H}_L = \frac{2\pi v}{n+2} \left[: \mathbf{J}_{L1}^2 : + : \mathbf{J}_{R2}^2 : \right] + 2\pi v g_L \mathbf{J}_{L1} \cdot \mathbf{J}_{R2} \quad (6)$$

$$\mathcal{V} = \gamma_{tw} \mathbf{n}_1 \cdot \overleftrightarrow{\partial}_x \mathbf{n}_2 + \gamma_{bs} (\mathbf{J}_{R1} \cdot \mathbf{J}_{L1} + \mathbf{J}_{R2} \cdot \mathbf{J}_{L2}), \quad (7)$$

where $\mathbf{J}_{R,L}$ are $SU(2)_n$ Kac-Moody chiral currents and $v \sim J$ is the spin velocity. Operators $\mathbf{n}_{1,2}$ are staggered magnetizations of the chains 1,2. They have scaling dimension $d = 3/2(n+2)$ and are related to the 2×2 matrix fields $G_{1,2}$ of the WZNW models describing individual chains^{32,33}: $\mathbf{n}_{1,2} \sim i \text{Tr}[\vec{\sigma} G_{1,2}]$ ($\vec{\sigma}$ being a vector formed by the Pauli matrices). For $n = 1$ all perturbations are marginal while for $n > 1$ the non-zero conformal spin perturbation with coupling constant γ_{tw} , dubbed twist term in Ref. 34, is strongly relevant with a scaling dimension $d_t = 1 + 3/(n+2)$. However, when the integrability condition is met, (i) $\gamma_{tw} = 0, \gamma_{bs} < 0$ and the continuum limit Hamiltonian (4) splits into two independent parts and (ii) g_R and g_L have different signs. As a consequence of (ii) at the integrable point one of the models $\mathcal{H}_{R,L}$ becomes massive and the other one is massless depending of the sign of χ with the marginally relevant dimensionless coupling $g = |g_R| \sim (J'/2J)^{1/2}$. For $g \ll 1$ the spectral gap for massive mode at the integrable point is exponentially small: $\Delta_{\text{ladder}} = Jg^{1/n} \exp(-1/g)$. However, the gapless mode remains robust independently of the value of g and hence always admits a field-theory description even when the gap is of the order of J . The field theory for the gapless mode is the one of the $SU(2)_n$ WZNW models (5, 6) which has a negative coupling constant.

According to Ref. 21 who analyzed the one-loop renormalization group flows of model (4) with $n = 1$ outside of the integrable point, the picture of the spectrum given above remains qualitatively valid in some region outside

of the integrable line. As a result one can be confident that the phase diagram of the generalized zigzag spin ladder contains a region where the right movers of (say) chain 1 are strongly coupled to the left movers of chain 2 and the left movers of chain 1 are weakly coupled to the right movers of chain 2. The part of the spectrum containing the strongly coupled modes is gapped. Such violation of parity is, of course, a consequence of the three-spin interaction. For $n > 1$ the situation is more delicate since the twist operator is relevant. Even if the gap is sufficiently large so that $\Delta_{\text{ladder}} > \gamma_{tw}^{1/(2-d_t)}$, we have to make sure that the twist operator does not ruin the quantum criticality of the edge modes. In this respect, we need to consider its projection on the low-energy sector. Since the twist operator is a spin singlet and is a sum of two $SU(2)$ singlet operators with Lorentz spin ± 1 , the projection must generate operators with the same symmetry or nothing. As described in Ref. 30, we find that the twist operator reduces to a total derivative of the trace of the $SU(2)_n$ WZNW field (see Eq. (8) below). It does not contribute to the effective low energy theory.

The generalized zigzag ladder is a critical spin nematic described by one of the $SU(2)_n$ WZNW models (5, 6), the one which has negative coupling constant. The $SU(2)$ WZNW matrix field U of this model is made from chiral components of the fields of the different chains³⁰. In this theory the local operators with power law correlation functions are binary products of staggered magnetizations from the different legs:

$$\mathbf{n}_1 \cdot \mathbf{n}_2 \sim \text{Tr } U, \quad \mathbf{n}_1 \times \mathbf{n}_2 \sim i \text{Tr}[\bar{\sigma} U], \quad (8)$$

while two-point correlators of the staggered magnetizations decay exponentially³⁰. In what follows we will assume that the parameters of the lattice Hamiltonian (1) are such that the effective couplings for the current-current interactions on a given ladder have different signs for different parities, as was described above.

Coupling the ladders. When zigzag ladders are arranged in a 2D array, the critical nematic modes couple. Depending on the nature of this coupling the system may either order as a nematic or remain a CSL. The order prevails if the matrix order parameters U_y from different ladders couple directly. As discussed above, one needs to isolate the ladders from each other more thoroughly to stabilize a CSL state. The arrangement shown on Fig. 1 makes it more difficult for the nematic order parameters to couple. In that case the chiral gapless modes from neighboring ladders $y, y+1$ interact indirectly with each other through the intermediate spins interacting with a ferromagnetic Ising-like interaction. We will demonstrate that this arrangement produces a spin gap in the bulk. As will be shown the gap magnitude increases when the exchange with the intermediate spins is anisotropic. Meanwhile the exchange within the ladders must remain $SU(2)$ invariant to preserve the Kac-Moody algebra of the currents. Taking into account that each leg of an individual ladder contains just one chiral gapless mode and that the in-chain staggered magnetizations

are short-ranged fields at energies $\ll \Delta_{\text{ladder}}$, we arrive to the following low-energy model which describes the neighboring ladders $y, y+1$ as a Kondo lattice model with a forward scattering:

$$\begin{aligned} \mathcal{H} = & \frac{2\pi v}{n+2} \int dx \left[: \mathbf{J}_{R,y+1}^2 : + : \mathbf{J}_{L,y}^2 : \right] \\ & + \sum_l \eta_a \left[J_{L,y}^a(x_l) + J_{R,y+1}^a(x_l) \right] S_{y+1/2}^a(x_l) \\ & - \sum_l I^{ab} S_{y+1/2}^a(x_l) S_{y+1/2}^b(x_{l+1}). \end{aligned} \quad (9)$$

We will consider the case $\eta^z \gg \eta^x = \eta^y$. According to Ref.³⁷ where somewhat similar system was studied, Hamiltonian (9) must be augmented by the RKKY interaction between the localized spins generated by the coupling to irrelevant operators. The strongest of those is $\eta_z \partial_x (n_y^z + n_{y+1}^z) S_{y+1/2}^z$ in our anisotropic model. Since correlation functions of the staggered magnetization decay exponentially, the second-order perturbation theory in η_a yields the short range interaction $\mathcal{J}_r^{ab} S_n^a S_{n+r}^b$, where the exchange \mathcal{I}_r decays exponentially with distance³⁰. This interaction is antiferromagnetic and competes with the bare one. We will consider the case when the net interaction $I_r^{ab} = -I^{ab} \delta_{r,1} + \mathcal{J}_r^{ab}$ is small. For $n=1$ model (9) with $I_r=0$ was considered in Refs. 35,36 and it was concluded that the spectrum is gapped and has a nonlocal order parameter. Using the Abelian bosonization approach, one can bosonize the chiral $SU(2)_1$ currents in terms of chiral bosons $\Phi_{R,L}$ and recast the Lagrangian density for model (9) as (from now on we set $v=1$)

$$\begin{aligned} \mathcal{L} = & \frac{1}{2} [(\partial_\tau \Phi)^2 + (\partial_x \Phi)^2] + \rho_s \left\{ \frac{i}{2} S^z \partial_\tau \psi + \frac{\eta_z}{\sqrt{2\pi}} \partial_x \Phi S^z \right. \\ & + \left. \frac{\eta_\perp}{2\pi a_0} \cos(\sqrt{2\pi} \Phi) (S^+ e^{-i\sqrt{2\pi} \Theta} + S^- e^{i\sqrt{2\pi} \Theta}) \right\} \\ & + \sum_{n,r} I_r^{ab} S_n^a S_{n+r}^b, \end{aligned} \quad (10)$$

where we assume that the local spins can be located at arbitrary points along the chains with spin density ρ_s , and not necessarily in a regular fashion as depicted on Fig. 1. Here $\Theta = \Phi_L - \Phi_R$ is the field dual to $\Phi = \Phi_L + \Phi_R$ and $S^\pm = e^{\pm i\psi} \sin \theta$, $S^z = \cos \theta$. After the transformation $\psi \rightarrow \psi + \sqrt{2\pi} \Theta$ the coupling in the second term changes to $\eta_z \rightarrow \eta_z - 2\pi$ and the Hamiltonian becomes

$$\begin{aligned} \mathcal{H} = & \sum_{n,r} I_r^{ab} \tilde{S}_n^a \tilde{S}_{n+r}^b + \eta_\perp \sum_n \tilde{S}_n^x \cos(\sqrt{2\pi} \Phi_n) \\ & + \frac{\eta_z - 2\pi}{\sqrt{2\pi}} \sum_n \partial_x \Phi_n \tilde{S}_n^z + \frac{1}{2} \int dx [(\partial_x \Theta)^2 + (\partial_x \Phi)^2], \end{aligned} \quad (11)$$

The situation simplifies when $\eta^z = 2\pi$. As we have said, we will always consider the case when I_r is sufficiently small which can be always achieved by tuning the interactions. In the simplest case $I_r = 0$ we are left with the

interaction term

$$V = \frac{\eta_{\perp} \rho_s}{2\pi a_0} \cos(\sqrt{2\pi}\Phi) \tilde{S}^x, \quad (12)$$

where \tilde{S}^a are transformed spin components which includes the staggered factor^{35,36}. Then since the operators $\tilde{S}^x(x)$ commute with the Hamiltonian, they can be replaced by constants. The ground state configuration corresponds to all $\tilde{S}^x(x)$ being equal. In this sector model (12) is reduced to the integrable sine-Gordon model with a spectral gap $\Delta \sim (\rho_s \eta_{\perp})^{2/3}$. The dimensional analysis indicates that for general $\eta_z \gg \eta_{\perp}$ the gap is $\Delta \sim \eta_{\perp}^{4\pi/3\eta_z}$. So the bulk excitations are gapped and the boundary chiral modes are gapless as is expected for a FQHS. The latter ones are described by the $SU(2)_1$ WZNW theory. A similar Toulouse approach can be performed in the general $n > 1$ case, as described in Ref. 30, with the emergence of a bulk gapped phase with gapless edge excitations described by the $SU(2)_n$ WZNW conformal field theory.

The gap Δ provides us with the window of tolerance for the magnitude of I_r : this exchange interaction should not exceed Δ . We find it interesting to take a closer look at the bulk excitations in the $n = 1$ case. At the integrable point $\eta_z = 2\pi, I_r = 0$ model (11) has two kinds of excitations: the massive mobile sine-Gordon triplet modes and immobile defects corresponding to flips of \tilde{S}^x . Away from this point the latter defects become mobile solitons; each soliton carries spin 1/2 from the ladders and a zero Majorana mode from the interladder spins. To see this we need to consider a perturbation around $\eta^z = 2\pi$. Integrating out the massive Φ modes in the leading order in $\delta\eta = \eta_z - 2\pi$ changes I_r by the amount $\delta I_r \sim (\delta\eta/v)^2 (\Delta/v) \exp(-\Delta|r|/v)$. Treating the exponentially decaying Ising exchange as the nearest neighbor interaction we arrive to the Ising model in a transverse dynamical field. The Jordan-Wigner transformation brings us to the model of Majorana fermions which mass term changes sign when $\cos(\sqrt{2\pi}\Phi)$ does. Such model has been considered in Ref. 38 and in the limit when the Majoranas are slower than kinks possesses rich physics. It certainly has Majorana zero modes riding on kinks. There are also traces of the two-channel Kondo physics suggested for this case in Ref. 12 in the sense that the Majorana zero modes on kinks are similar to static Majorana zero modes on isolated spins in the two-channel Kondo effect³¹. So we see that the bulk of our model is gapped and the excitations are nontrivial. A half of the spinon modes of the ladders are gapped inside of each ladder. These are spin-1/2 solitons. The remaining half are gapped due to the interladder coupling through

the intermediate spins. These are non-Abelian kinks (vibrons?). Such fractionalized excitations cannot propagate in the transverse direction. However, there are probably their bound states with integer quantum numbers and these ones can. If the applied magnetic field exceeds the bulk gap we expect the bulk to become gapless with central charge $c = 3/2$ and the velocity $v \sim M$ (the magnetization). This may lead to interesting T/M scaling in the thermodynamics.

In order to demonstrate the robustness of the described state we have to consider a possibility of interladder couplings other than the ones already described. In the low-energy theory such couplings are generated by high energy virtual processes and their magnitudes are proportional to powers of the couplings of the bare lattice Hamiltonian³⁹. In the given case one has to expect, for instance, the appearance of the effective interactions between the local (or Kondo) spins from different rows which may lead to their ordering. This process competes with the Kondo screening preventing the gap formation. Such unwanted interactions constitute a general problem for all quantum Hall constructions based on coupling of wires. Starting from the original idea by Teo and Kane²⁰ all these scenarios envisage that the original lattice model can at low energy be reduced to 1D critical models coupled by marginally relevant current-current interactions. The difficulty is that the gaps generated by the marginal interactions are exponentially small in the *direct* interwire exchange and the ones generated by high energy virtual processes have power law dependence on the *same* couplings of the lattice model. In the model with the intermediate spin we suggest this difficulty is avoided since the spin currents from neighboring ladders are not coupled directly which would generate exponentially small gap, but via the anisotropic exchange with intermediate spins.

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