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Sachdev-Ye-Kitaev Model and Thermalization on the Boundary of Many-Body Localized Fermionic Symmetry Protected Topological States

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We consider the Sachdev-Ye-Kitaev (SYK) model[1–3] as a model for the thermalized zero-dimensional boundary of a many-body localized, Fermionic symmetry protected topological (SPT) phase in one spatial dimension. The Fermions at the boundary are always fully interacting. We find that the boundary is thermalized and investigate how its boundary anomaly, dictated by the bulk SPT order, is encoded in the quantum chaotic eigenspectrum of the SYK model. We show that depending on the SPT symmetry class, the boundary many-body level statistics cycle in a systematic manner through those of the three different Wigner-Dyson random matrix ensembles with a periodicity in the topological index that matches the interaction-reduced classification of the bulk SPT states. We consider all three symmetry classes BDI, AIII, and CII, whose SPT phases are classified in one spatial dimension by $\mathbb{Z}$ in the absence of interactions. For symmetry class BDI, we derive the eight-fold periodicity of the Wigner-Dyson statistics by using Clifford algebras.

I. INTRODUCTION

Symmetry protected topological (SPT) phases are gapped quantum systems with quantum disordered short-range entangled ground states which cannot be smoothly deformed into trivial product states without closing the gap, if the symmetry defining the SPT phase is preserved. The corresponding ground states are non-degenerate even on spatial manifolds with non-trivial topology.[4–6] Famous examples of SPT states include the ground states of the Haldane spin-1 chains[7–9] and of topological insulators and superconductors[10–19]. Short-range entanglement is manifested in a strict area-law entanglement entropy of the SPT state. As a quantum order, the SPT order is in general not expected to persist to highly-excited (finite-energy-density) states in the many-body spectrum, because highly excited states are typically thermalized according to the eigenstate thermalization hypothesis (ETH),[20–23] and have a volume-law entanglement entropy in contrast to the area-law entanglement in the SPT state.

However the phenomenon of many-body localization (MBL)[24–29] provides a class of examples where quantum many-body systems can evade thermalization in the presence of quenched disorder. MBL systems are generic non-ergodic phases of matter, which can retain the memory of local quantum information and exhibit an area-law entanglement entropy even for highly excited (finite energy-density) states. In some sense, excited states of an MBL system are like ground states[30], which enables us to extend the discussion of ground state quantum orders to highly excited (finite energy-density) states. Examples of many-body localization protected quantum order have recently been discussed in Ref. [30–37]. Here we are interested in such "MBL stabilized SPT states" (referred to hereafter as "MBL-SPT states").

In particular, we will focus on interacting Fermionic MBL-SPT states, and investigate the possibility and the consequences of thermalization at the boundary of such a state. Many-body localizability of SPT states has been discussed in Ref. [36,37]. It was shown that, at least in one spatial dimension (1D), Fermionic SPT states can be fully many-body localized in the bulk. In the strong disorder regime of the MBL system, all bulk Fermion degrees of freedom can be renormalized to local integrals of motion.[38–44] Resonances among them are suppressed by disorder, which makes the bulk stable against thermalization. However, the Fermions near the boundary of the MLB-SPT system are less protected against thermalization. Indeed, if the 1D bulk SPT order is non-trivial, its zero-dimensional (0D) boundary will host degenerate boundary ("edge") states (generalizing the Majorana Fermion zero modes in the non-interacting limit),[31] whose presence reflects a quantum anomaly[45–49] that is required by the SPT order in the bulk. For example, consider the symmetry class[16,17,50,51] BDI, whose 1D non-interacting Fermion SPT phases are classified by $\mathbb{Z}^{16–18}$; its 0D boundary can then support arbitrary many Fermion zero modes in the absence of interactions.

Two possibilities arise on the 0D boundary of the 1D MBL-SPT system: the boundary Fermions could be either localized or thermalized. In general, we expect random interactions among the Fermion modes on the boundary due to the strong-disorder nature of the MBL system. If the interaction is sparse or short-ranged, the boundary Fermions can be localized. An example of sparse four-fermion interaction which does not lead to thermalization was constructed in Ref. 52. On the other hand, if the interaction is dense and non-local (all fermion modes are coupled together), then due to the vanishing level-spacing between the boundary modes, the boundary Fermions can be easily thermalized. One may wonder if the MBL bulk is stable against the thermalization due to the contact with the thermalized boundaries. According renormalization group (RG) studies of the thermalization transition in one dimension[53,54], the MBL fixed-point is stable against weak thermalization: A thermal bubble (small thermalized region) in the 1D bulk cannot expand indefinitely into the MBL environment. Therefore, the thermalized boundary (which can be viewed as a thermalized bubble residing at the boundary) will not be able to thermalize the entire bulk in 1D. Then we are facing the interesting scenario of an MBL-SPT bulk with thermalized boundaries.

We use the Sachdev-Ye-Kitaev (SYK) models[1–3] (or corresponding generalizations) as concrete models to describe a class of thermalized boundary. The SYK models are 0D quantum many-body systems of Fermions with random interactions. Symmetry constraints on the boundary naturally forbid the Fermion bilinear terms in the SKY models. The extremely rich physics of SYK models is being actively explored recently.[55–62] The quantum chaotic (thermalizing) nature of SKY models have been revealed from the Lyapunov exponent of the out-of-time-ordered correlator.[3,55] Early studies[63–65] also pointed out the level statistics of the many-body spectrum as another probe for the quantum chaos in interacting Fermion models. We will study the level statistics for SYK models in this work to further explore the chaotic nature of these models in terms of their spectrum properties.

Since the MBL bulk has non-trivial SPT order, the thermalized boundary must possess a corresponding quantum anomaly that characterizes the SPT phase[6,45–48]. What is the signature of this quantum anomaly for a thermalized boundary? First of all, the presence of a protected degeneracy of every energy level in the boundary many-body spectrum[31,32] is one obvious signature. In this work, we will show that the *level statistics* of the boundary spectrum is another such signature. In particular we will show that for the thermalized boundary, the level statistics follows the Wigner-Dyson distribution of one of the three Wigner-Dyson random matrix ensembles which is in correspondence, as specified below, with the global anomaly required by the bulk SPT order: Take for example the thermalized boundary of the MBL-SPT state in symmetry class BDI. We find that its level statistics cycles through that of the Gaussian orthogonal, unitary, and symplectic ensembles (GOE, GUE, and GSE) in a systematic manner, as summarized in Tab. I with an *eight-fold periodicity* that matches the (interaction-reduced) $\mathbb{Z}_8$ classification[6,66] of the Fermionic SPT order in symmetry class BDI.

TABLE I: Eight-fold-way spectrum on the thermalized boundary of N_χ Majorana chains in symmetry class BDI ($N_\chi > 4$). qdim: quantum dimension (level degeneracy per boundary), lev. stat.: level statistics in a definite Fermion number parity sector.

$N_\chi \pmod 8$	0	1	2	3	4	5	6	7
qdim	1	$\sqrt{2}$	2	$2\sqrt{2}$	2	$2\sqrt{2}$	2	$\sqrt{2}$
lev. stat.	GOE	GOE	GUE	GSE	GSE	GSE	GUE	GOE
$\mathcal{C}\ell_{0,N_\chi-1}$	$\mathbb{R}\oplus\mathbb{R}$	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{H}\oplus\mathbb{H}$	$\mathbb{H}$	$\mathbb{C}$	$\mathbb{R}$

Subsequently, we will extend our analysis to the boundary level statistics for Fermionic MBL-SPT states in symmetry classes AIII and CII, which are the other two 1D symmetry classes that also possess a $\mathbb{Z}$ classification in the absence of interactions.[16–18] In contrast to symmetry class BDI discussed above, here we have to pay attention to the fact that Fermionic MBL-SPT states in classes AIII and CII are in general unstable to interactions in the 1D bulk[67,68] (due to the presence of charge-conjugation symmetry).[83] For this reason, we will consider in symmetry classes AIII and CII situations where interactions are solely present at the 0D boundaries, whereas the 1D bulk remains non-interacting throughout. Even though in symmetry classes AIII and CII these bulk states are not many-body localizable (when bulk interactions are turned on), we can (and will) nevertheless still discuss the spectral properties of corresponding interacting boundary Hamiltonians, which turn out to be, as mentioned, the corresponding SYK models (while always considering a corresponding non-interacting, but random bulk). We note that our discussion of level statistics also applies to bosonic MBL-SPT states, because bosonic SPT states in 1D can always be interpreted as interacting Fermionic SPT states.[69,70]

Finally, we would like to mention that although we focus on the SYK model as a convenient description of the thermal boundary of MBL-SPT states, the periodic behavior of the level statistics we observed should be universal as long as the boundary is thermalized, because the level statistics is a local spectral property that is only sensitive to the symmetry class.

II. SYMMETRY CLASS BDI

A. Four Fermi Interactions and Numerical Results

We will start with 1D Fermionic MBL-SPT states in symmetry class BDI, sometimes also known as the "Kitaev Majorana wire",[10] protected by a time-reversal symmetry Z_2^T which squares to the identity operator (in the single-particle setting - for a complete discussion of the action of the square of the time-reversal operator on the many-body Fock space, see[71]). In the absence of interactions, the SPT order is characterized by an integer-valued topological index $N_\chi \in \mathbb{Z}$, which counts the number N_χ (of species) of protected Majorana zero modes χ_a ($a = 1, 2, \cdots, N_\chi$) at the boundary. The operators χ_a satisfy the Clifford algebra $\{\chi_a, \chi_b\} = 2\delta_{ab}$ and $\chi_a^\dagger = \chi_a$. The (anti-unitary) time-reversal symmetry acts on these Majorana zero modes as $\mathcal{T}\chi_a\mathcal{T}^{-1} = \chi_a$. Fermion bilinear terms $\mathrm{i}\chi_a\chi_b$ are forbidden to occur in the boundary Hamiltonian H by time-reversal symmetry $\mathcal{T}$. So to lowest order in many-body terms, the boundary dynamics is governed by random four-Fermion interactions

$$H = \sum_{a<b<c<d} V_{abcd}\chi_a\chi_b\chi_c\chi_d. \tag{1}$$

Here, the interaction strengths V_{abcd} are taken to be independent random real numbers with zero mean. The randomness in the boundary Hamiltonian, Eq. (1), originates from the strong disorder in the 1D MBL bulk. The detailed probability distribution of V_{abcd} is unimportant, and we may assume it to be Gaussian. This model, Eq. (1), was introduced by Kitaev[3] as a toy model for holography. Here we would like to consider it as an effective model describing the thermalized boundary of a 1D SPT phase. From this perspective, the fact that this model contains no Fermion bilinear term (a condition imposed by hand in Kitaev's model) appears here naturally as a consequence of the symmetry requirement (in the present case, the relevant symmetry is the time-reversal symmetry in class BDI).

In general, the degeneracy at the boundary of the non-interacting system arising from the Majorana zero modes (which is vast when N_f is large) can be lifted by interactions. However, if the bulk SPT order is non-trivial, the boundary degeneracy cannot be fully lifted, because otherwise the bulk state could have been smoothly deformed into the trivial vacuum state across the boundary (which here is a boundary to vacuum). We recall that with interactions, the classification of the SPT order in symmetry class BDI is reduced from $\mathbb{Z}$ to $\mathbb{Z}_8$.[6,66] So the energy levels of the boundary many-body spectrum are non-degenerate if and only if N_χ is a multiple of eight [$N_\chi (\mathrm{mod}\, 8) = 0$]; otherwise, there is a degeneracy of every energy level of the many-body spectrum. The degeneracy of energy levels of the boundary many-body Hamiltonian can be studied numerically by exact diagonalization of the Hamiltonian in Eq. (1). In doing so we need to recall that when N_χ is odd, the low-energy Hilbert space of a single boundary is not well-defined. In that case, the "quantum dimension" (qdim) of the boundary mode is considered instead, which is defined to be the square root of the level degeneracy with both boundaries considered. Numerical results for qdim are listed on the second line of Tab. I: We see that the eight-fold periodicity of the level degeneracy matches the $\mathbb{Z}_8$ periodicity of the (global) anomaly of the boundary. The same result was also obtained in Ref. 72.

However the level degeneracy (or "quantum dimension") alone cannot fully resolve the eight-fold anomaly described by $\mathbb{Z}_8$. As we will now explain, we have found that the level statistics can provide an additional diagnostic. In the past, the level statistics of the many-body spectrum has been used to diagnose whether a many-body Hamiltonian is in the MBL phase or the ETH phase (see e.g.[26,73]). Here we use the level statistics of the boundary to further resolve the (global) quantum anomaly of the SPT phase, beyond the diagnostic provided by the degeneracy of all levels. In general, we collect the eigen energies $\{E_n\}$ of the Hamiltonian, and arrange them in ascending order $E_1 < E_2 < \cdots$. Let $\Delta E_n = E_n - E_{n+1}$ be the level spacing, and we evaluate the ratios of adjacent level spacings $r_n = \Delta E_n / \Delta E_{n+1}$,[26,74,75] such that the dependence on the density of states cancels out in the ratio. The distribution of the ratio r_n follows Poisson level statistics in the MBL phase,

$$\text{Poisson: } p(r) = \frac{1}{(1+r)^2}, \tag{2}$$

and Wigner-Dyson level statistics in the ETH phase (given by the "Wigner-surmise"[76]),

$$\text{Wigner-surmise: } p(r) = \frac{1}{Z} \frac{(r+r^2)^\beta}{(1+r+r^2)^{1+3\beta/2}}. \tag{3}$$

The parameters β and Z are different for GOE: $\beta = 1, Z = \frac{8}{27}$; GUE: $\beta = 2, Z = \frac{4\pi}{81\sqrt{3}}$; and GSE: $\beta = 4, Z = \frac{4\pi}{729\sqrt{3}}$. The level repulsion in the ETH spectrum manifests itself in the asymptotic behavior $p(r \to 0) \sim r^\beta$. To make clearer the contrast between different level statistics, we choose to show the probability distribution of the logarithmic ratio $\ln r$, which is given by $P(\ln r) = p(r) r$.

We now apply this analysis to the boundary Hamiltonian in Eq. (1). However, extra care should be taken regarding the Fermion parity. Levels with different Fermion parities are independent, so putting all levels together will spoil the true level statistics in each sector.[84] Therefore, the level statistics must be collected in each Fermion parity sector. Since our BDI-class Hamiltonian in Eq. (1) possesses, besides Fermion number parity, no other unitary symmetries, any remaining level degeneracies within each Fermion parity sector will be ignored, i.e. we only consider the level spacing between adjacent (non-degenerate) eigenenergies in each such sector. We have collected the probability distribution $P(\ln r) = p(r) r$ of the logarithmic ratio $\ln r$ numerically; the results are shown in Fig. 1. We see that the probability distribution varies systematically with the number N_χ of Majorana modes. First of all, in all cases Wigner-Dyson statistics is observed, which shows that the boundary is indeed in the ETH (quantum chaotic) phase. Secondly, depending on the topological index $\nu \equiv N_\chi (\mathrm{mod}\, 8)$, the data correspond to one of the three Wigner-Dyson random matrix ensembles (GOE, GUE, or GSE), as summarized on the third line of Tab. I.[85] Combining the results for the level statistics (3rd line of Tab. I) with those for the level degeneracy (2nd line of Tab. I) , the $\mathbb{Z}_8$ anomaly pattern of the thermalized boundary can be determined up to the sign of the topological index ν (i.e. ν and $-\nu$ are not distinguishable yet).[86]

B. General Hamiltonian and Analytical Results

In this section we demonstrate analytically that the 'eight-fold-way' level statistics of the boundary Hamiltonian Eq. (1) persists even after including all possible (random) higher-order interactions (see Eq. (5) below). Moreover, we show that this is related to the Bott periodicity of the real Clifford algebra $\mathcal{C}\ell_{0,N_\chi-1}$.[87] To make this

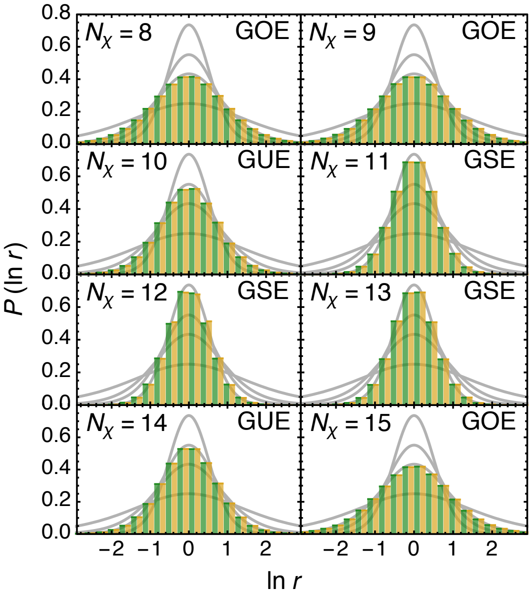

FIG. 1: Many-body level statistics (in term of the $\ln r$ distribution) of the random interaction model in Eq. (1), for $N_\chi = 8, \cdots, 15$ (a full $\mathbb{Z}_8$ period) by exact diagonalization. The background gray curves describe the "Wigner-surmise" given by Eq. (2) and Eq. (3): from wide to narrow, they correspond to Poisson, GOE, GUE, and GSE statistics respectively. The level statistics in the even (odd) Fermion parity sector is shown in green (yellow).

connection, let us first observe that the Fermion bilinear operators

$$\gamma_a = \chi_a \chi_{N_\chi} \quad (a = 1, 2, \cdots, N_\chi - 1), \tag{4}$$

where χ_{N_χ} is the "last" of the N_χ Majorana modes on the boundary, can be used to define the generators of the Clifford algebra $\mathcal{C}\ell_{0,N_\chi-1}$. We consider a real (matrix) representation in Fock space, so that we have $\chi_a^\intercal = \chi_a$ (where $^\intercal$ denotes the transposed matrix), and $\{\chi_a, \chi_b\} = 2\delta_{ab}$. Then it is easy to show, using Eq. (4), that $\gamma_a^\intercal = -\gamma_a$ and $\{\gamma_a, \gamma_b\} = -2\delta_{ab}$. So the operators γ_a indeed represent the $(N_\chi - 1)$ antisymmetric generators of the Clifford algebra $\mathcal{C}\ell_{0,N_\chi-1}$. Then it can be checked that those elements in $\mathcal{C}\ell_{0,N_\chi-1}$ which are represented by symmetric matrices (in the real representation we are currently considering – they are thus self-adjoint) are of grade $(4k-1)$ or $4k$ (for some $k \in \mathbb{Z}_+$), meaning that they can be written as products of $(4k-1)$ or $4k$ generators γ_a. It turns out that these matrices represent all possible time-reversal invariant terms that are allowed in the boundary Hamiltonian. For example, the four Fermion interaction terms in Eq. (1) are of grade 3 and 4 (corresponding to $k = 1$): $\chi_a\chi_b\chi_c\chi_{N_\chi} = -\gamma_a\gamma_b\gamma_c$ $(a, b, c < N_\chi)$ are grade-3 terms, and $\chi_a\chi_b\chi_c\chi_d = \gamma_a\gamma_b\gamma_c\gamma_d$ $(a, b, c, d < N_\chi)$ are grade-4 terms. Higher order time-reversal invariant interactions ($4k$-Fermion interactions) correspond to higher grades in the Clifford algebra $\mathcal{C}\ell_{0,N_\chi-1}$, and it is not difficult to see that these exhaust the full space of all symmetric matrices in $\mathcal{C}\ell_{0,N_\chi-1}$ (in a real representation). Therefore, if all symmetry-allowed interactions are included in the Hamiltonian,

$$H = \sum_{k=1}^{\lfloor N_\chi/4 \rfloor} \sum_{a_1 < \cdots < a_{4k}} V_{\{a_i\}}\, \chi_{a_1} \cdots \chi_{a_{4k}}, \tag{5}$$

the Hamiltonian H will be a general (real) symmetric matrix in (the real representation of) the Clifford algebra $\mathcal{C}\ell_{0,N_\chi-1}$. Hence, when the real coefficients $V_{\{a\}}$ are random, H will be a general random symmetric matrix in $\mathcal{C}\ell_{0,N_\chi-1}$, and its level statistics will fall into the matrix ensemble determined by the real representation of $\mathcal{C}\ell_{0,N_\chi-1}$. The representations of the Clifford algebra $\mathcal{C}\ell_{0,N_\chi-1}$ are known and are listed in the last line of Tab. I, where $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$ stand respectively for the set of all $m \times m$ matrices with real, complex and quaternion entries with certain matrix dimensions m, which are not written out explicitly; the Hermitian such matrices (symmetric in a real representation of the corresponding Clifford algebra) correspond, respectively, to the Hamiltonians in the three Wigner-Dyson random matrix ensembles GOE, GUE and GSE. The numerical results reported in the previous sections of this paper for a Hamiltonian of the form of Eq. (5), which contains solely four-Fermion interactions, show that restricting the generic Hamiltonian in Eq. (5) to one that contains only four-Fermion interactions (as in Eq. (1)), does not affect the level statistics. Because the level spacing is a local property of the spectrum, it should depend only on the symmetry class of the Hamiltonian matrix, not on its sparsity. However, as the SYK model is relatively sparse compared to the Gaussian random matrix H in Eq. (5), their global spectral properties, such as the spectral density and the long-time correlator, will not agree with each other.

In the remainder of this paper we extend the above discussion for the MBL-SPT state in symmetry class BDI to the other two symmetry classes that also have a $\mathbb{Z}$ classification in 1D in the absence of interactions: These are symmetry classes AIII and CII which we will now discuss in turn. As already mentioned in the introduction, here we need to pay attention to the fact that, in contrast to symmetry class BDI, there are no many-body localized Fermionic SPT phases in symmetry classes AIII and CII in the 1D bulk.[88] For this reason, in symmetry classes AIII and CII we consider situations in which interactions are only present at the 0D boundaries, the 1D bulk system remaining throughout a non-interacting Anderson-localized insulator. The non-interacting Anderson-localized 1D *bulk* systems in these two symmetry classes are known to possess a $\mathbb{Z}$ classification which applies[30] also to excited states (at finite energy density), in complete analogy with the usual MBL systems (in which interactions are present). Here the interactions, localized solely at the boundary, reduce the $\mathbb{Z}$ classification of the 0D boundary Hamiltonian to $\mathbb{Z}_4$ in class AIII, and to $\mathbb{Z}_2$ in class CII. The following analyses of the level statistics (which are confirmed numerically) of the resulting SYK systems at the 0D boundaries in symmetry classes AIII and CII must then follow as long as the respective protecting symmetries are not broken spontaneously in (0+1) dimensions.

III. SYMMETRY CLASS AIII

Ground states of 1D SPT phases in symmetry class AIII can be viewed as being protected by U(1) $\times$ $Z_2^{\mathcal{S}}$ symmetry, where the superscript $^{\mathcal{S}}$ stands for *chiral* symmetry $\mathcal{S}$.[16,17,19,50,51] The 1D SPT phases protected by this symmetry are classified by $\mathbb{Z}_4$ in the presence of interactions. In the sense explained above, the resulting boundary anomaly will determine the properties of all states of the random boundary Hamiltonian in this class. The corresponding 0D boundary degrees of freedom are complex Fermion modes c_a $(a = 1, 2, \cdots, N_c)$, where N_c labels the number of complex Fermion mode species. The U(1) symmetry is naturally implemented as $c_a \to e^{\mathrm{i}\theta} c_a$. The *chiral* symmetry, an anti-unitary symmetry operation when acting on the many-body Fermion Fock space, can be taken to act as $\mathcal{S} c_a \mathcal{S}^{-1} = c_a^\dagger, \quad \text{and } \mathcal{S} c_a^\dagger \mathcal{S}^{-1} = c_a$ on canonical Fermion annihilation and creation operators at the boundary[89]. Fermion bilinear terms are again forbidden at the boundary by the U(1) $\times$ $Z_2^{\mathcal{S}}$ symmetry which protects that SPT order, so that the boundary Hamiltonian only contains charge-conserving interactions of fourth and higher order in Fermion operators that are invariant under the action of the chiral symmetry. The boundary Hamilonian thus roughly reads $H = \sum V_{abcd} c_a^\dagger c_b^\dagger c_c c_d + \cdots$, or more precisely

$$
\begin{aligned}
H = \sum_{a<b,c<d} V_{abcd} [& (c_a^\dagger c_d - \tfrac{1}{2}\delta_{ad})(c_b^\dagger c_c - \tfrac{1}{2}\delta_{bc}) \\
& - (c_a^\dagger c_c - \tfrac{1}{2}\delta_{ac})(c_b^\dagger c_d - \tfrac{1}{2}\delta_{bd})] \\
& + h.c. + \cdots ,
\end{aligned} \tag{6}
$$

where the coefficients V_{abcd} are complex numbers. In the above equation the interaction terms are written in a way that makes their invariance under chiral symmetry obvious. As in the BDI case, randomness in the complex coefficients V_{abcd} is induced by the randomness in the 1D bulk in symmetry class AIII (which, as discussed above, is here non-interacting). Possible higher order random interactions are not written out explicitly in Eq. (6). This model was first introduced by Sachdev and Ye[1], and was revisited[2] recently in view of its close analogy with the model in Eq. (1), considered by Kitaev. We find that the level statistics of the Hamiltonian in Eq. (6) exhibits a four-fold periodicity in N_c, matching the $\mathbb{Z}_4$ global anomaly on the boundary characteristic of class AIII with interactions. Let us explain our findings. Due to the U(1) symmetry, the Hamiltonian can be block-diagonalized in each U(1) charge sector, where the charge operator (with eigenvalue q) reads

$$
Q = \sum_{a=1}^{N_c} (c_a^\dagger c_a - 1/2). \tag{7}
$$

Therefore, the level statistics must be collected in each charge sector separately. It turns out that there is an interplay between the charge quantum number q (= eigenvalue of Q) and the level statistics, as can be seen from our results shown in Tab. II(a): First, observe that there is an even-odd effect for the charge q, depending on N_c: q takes integer values if N_c is even, and half-integer values if N_c is odd (due to charge fractionalization occuring at the boundary of the 1D SPT). So the charge neutral sector $(q = 0)$ exists only for even N_c. Second, in the $q = 0$ sectors, the level statistics is that of GOE for $N_c (\mathrm{mod}\, 4) = 0$ and of GSE for $N_c (\mathrm{mod}\, 4) = 2$. (We can think[90] of the statistics as being inherited from the $N_\chi (\mathrm{mod}\, 8) = 0, 4$ cases of class BDI via the correspondence $N_\chi = 2N_c$ - compare Table I.). Furthermore, in any $q \neq 0$ sector, the chiral symmetry operation $\mathcal{S}$ connects the q and $-q$ sectors which turns out to result in GUE level statistics (see the discussion below).

TABLE II: Level statistics on the thermalized boundary of AIII and CII class MBL-SPT states. q is the U(1) charge quantum number. The Fermion flavor number must be sufficiently large for the result to be universal.

(a) AIII class

$N_c(\mathrm{mod}\,4)$	0	1	2	3
$q=0$	GOE		GSE	
$q=\pm1/2$		GUE		GUE
$q\neq 0$	GUE	GUE	GUE	GUE

(b) CII class

$N_f(\mathrm{mod}\,2)$	0	1
$q \in$ even	GOE	GSE
$q \in$ odd	GSE	GOE

The interplay of level statistics and symmetries can be understood from the analysis of the projective representations of the chiral symmetry on the boundary. In general, the anti-unitary operator implementing the chiral symmetry on the many-body Fock space can always we written as the complex conjugation operator $\mathcal{K}$ followed by a unitary operator $\mathcal{U}$ on the Fock space, i.e. $\mathcal{S} = \mathcal{U}\mathcal{K}$. The unitary operator $\mathcal{U}$ can be found by considering its action on the boundary Fermions as follows. Let us first represent the Fermion operators c_a as "qubit operators" $c_a = (\prod_{b<a} \sigma_b^z)(\sigma_a^x + \mathrm{i}\sigma_a^y)/2$ using a Jordan-Wigner type transformation. In this representation (i.e. in this basis of Fock space), c_a and $c_a^\dagger$ are hence both represented by real matrices (using the standard convention for Pauli matrices). Both are therefore invariant under complex conjugation, $\mathcal{K} c_a \mathcal{K}^{-1} = c_a$, and $\mathcal{K} c_a^\dagger \mathcal{K}^{-1} = c_a^\dagger$, in this representation (i.e. in this basis of the Fermion Fock space). Therefore, to implement the chiral transformation on the Fermion operators, $\mathcal{S} c_a \mathcal{S}^{-1} = \mathcal{U} c_a \mathcal{U}^\dagger = c_a^\dagger$

and similarly $\mathcal{U}c_a^\dagger\mathcal{U}^\dagger = c_a$, one only needs to set

$$\mathcal{U} \equiv e^{i\pi N_c Q}\prod_{a=1}^{N_c}\xi_a, \quad \text{where} \quad \xi_a = \mathrm{i}(c_a^\dagger - c_a) \tag{8}$$

are Majorana Fermion operators satisfying $\{\xi_a, \xi_b\} = 2\delta_{ab}$, $\xi_a^\dagger = \xi_a$, as well as $\mathcal{K}\xi_a\mathcal{K}^{-1} = -\xi_a$. Here Q is the U(1) charge operator defined in Eq. (7), which satisfies $\mathcal{S}Q\mathcal{S}^{-1} = -Q$. Using these algebraic relations, it is straightforward to verify that

$$\mathcal{S}^2 = \mathcal{U}\mathcal{U}^* = \begin{cases} +\mathbf{1} & \text{if } N_c \bmod 4 = 0, 1, \\ -\mathbf{1} & \text{if } N_c \bmod 4 = 2, 3, \end{cases} \tag{9}$$

where $\mathcal{U}^* = \mathcal{K}\mathcal{U}\mathcal{K}^{-1}$. Note that since this result for $\mathcal{S}^2$ is invariant under a change of basis of the many-body Fock space, it holds true in any such basis (even though it was initially derived in a representation in which both c_a and $c_a^\dagger$ are real).[91] Chiral symmetry leaves the charge neutral sector ($q = 0$) of the Hamiltonian invariant and is thus a symmetry of the Hamiltonian in this sector. Specifically, in this sector, chiral symmetry of the Hamiltonian $H_{q=0}$ amounts to

$$\mathcal{S}H_{q=0}\mathcal{S}^{-1} = \mathcal{U}H_{q=0}^*\mathcal{U}^{-1} = H_{q=0}. \tag{10}$$

When $N_c (\bmod 4) = 0$ we have $\mathcal{U}\mathcal{U}^* = +\mathbf{1}$, and so one can choose a basis of the many-body Fock space in which $\mathcal{U} = \mathbf{1}$. Then Eq. (10) implies that $H_{q=0} \in \mathbb{R}$ is a real symmetric matrix, which should exhibit GOE level statistics in the ETH phase. When $N_c (\bmod 4) = 2$ we have $\mathcal{U}\mathcal{U}^* = -\mathbf{1}$, and so one can choose a basis of the many-body Fock space in which $\mathbf{U} = \left(\begin{smallmatrix} 0 & +\mathbf{1} \\ -\mathbf{1} & 0 \end{smallmatrix}\right)$. Then Eq. (10) implies that $H_{q=0} \in \mathbb{H}$ is a quaternion Hermitian matrix, which should consequently exhibit GSE level statistics in the ETH phase. Since, as mentioned above, N_c must be even when $q = 0$ this exhausts all possibilities for the $q = 0$ sector. However for $q \neq 0$, the chiral symmetry transformation $\mathcal{S}$ connects the two charge sectors $\pm q$. In block-matrix form, we have

$$Q = \left(\begin{smallmatrix} -q & 0 \\ 0 & +q \end{smallmatrix}\right), \; \mathcal{U} = \left(\begin{smallmatrix} 0 & \mathbf{1} \\ \eta_{\mathcal{S}}\mathbf{1} & 0 \end{smallmatrix}\right), \; H = \begin{pmatrix} H_{-q} & 0 \\ 0 & H_{+q} \end{pmatrix}, \tag{11}$$

where $\eta_{\mathcal{S}} = \pm 1$ depends on the projective representation $\mathcal{S}^2 = \eta_{\mathcal{S}}$. But no matter what the value of $\eta_{\mathcal{S}}$, Eq. (10) only establishes a connection between H_{+q} and H_{-q}, i.e. $H_q^* = H_{-q}$, which imposes no further restriction on H_q itself. So for $q \neq 0$, $H_q \in \mathbb{C}$ is a complex Hermitian matrix, which should exhibit GUE level statistics in the ETH phase.[92] - These predictions are confirmed by numerical studies of these spectra, and displayed in Tab. II(a).

IV. SYMMETRY CLASS CII

Now we turn to the MBL-SPT states in symmetry class CII, which are protected by $(\mathrm{U}(1) \rtimes Z_2^{\mathcal{C}}) \times Z_2^{\mathcal{S}}$ symmetry. The $\mathbb{Z}$ classification of the non-interacting 1D SPT phases in this class reduces to $\mathbb{Z}_2$ in the presence of interactions[93]. The symmetry action on the boundary is understood most easily if we embed the $\mathrm{U}(1) \rtimes Z_2^{\mathcal{C}}$ subgroup into the SU(2) group (although the SU(2) symmetry is not necessary[94] to protect this SPT phase). Therefore we consider the boundary degrees of freedom to be spin-1/2 Fermions $f_a = (f_{a\uparrow}, f_{a\downarrow})^T$, where $a = 1, 2, \cdots, N_f$. The SU(2) generators are defined as

$$\vec{\boldsymbol{S}} = \frac{1}{2}\sum_{a=1}^{N_f} f_a^\dagger \vec{\boldsymbol{\sigma}} f_a, \tag{12}$$

where $\boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ are Pauli matrices. The U(1) symmetry in this representation of symmetry class CII corresponds to conservation of S_z, with the U(1) charge operator

$$Q = 2S_z = \sum_{a=1}^{N_f}\sum_{\sigma=1,2}(-1)^\sigma f_{a\sigma}^\dagger f_{a\sigma}. \tag{13}$$

"Charge"-conjugation corresponds to spin-rotation by angle π about the S_y-axis

$$\mathcal{C} = e^{\mathrm{i}\pi S_y}, \tag{14}$$

so that $\mathcal{C}f_{a\uparrow}\mathcal{C}^\dagger = f_{a\downarrow}$, $\mathcal{C}f_{a\downarrow}\mathcal{C}^\dagger = -f_{a\uparrow}$ and $\mathcal{C}Q\mathcal{C}^\dagger = -Q$ (which makes $\mathcal{C}$ consistent with its physical meaning of charge conjugation). The chiral symmetry acts as $\mathcal{S}f_{a\sigma}\mathcal{S}^{-1} = f_{a\sigma}^\dagger$, which also flips the spin $\mathcal{S}\vec{\boldsymbol{S}}\mathcal{S}^{-1} = -\vec{\boldsymbol{S}}$, and in particular the "charge" $Q = 2S_z$. To implement the chiral symmetry operation, we write $\mathcal{S} = \mathcal{U}\mathcal{K}$ where $\mathcal{K}$ denotes complex conjugation and $\mathcal{U}$ a unitary operator in the many-body Fermion Fock space. In complete analogy with the AIII case discussed above, one first chooses again a real representation of the canonical Fermion operator $f_{a\sigma}$ and $f_{a\sigma}^\dagger$ using a Jordan-Wigner type transformation and "qubit operators", as was done in the paragraph above Eq. (8), so that $\mathcal{K}f_{a\sigma}\mathcal{K}^{-1} = f_{a\sigma}$, and $\mathcal{K}f_{a\sigma}^\dagger\mathcal{K}^{-1} = f_{a\sigma}^\dagger$. As before, one immediately verifies that the action of the chiral symmetry transformation $\mathcal{S}$ on the Fermion operators is reproduced by setting

$$\mathcal{U} = \prod_{a=1}^{N_f}\prod_{\sigma=\uparrow,\downarrow}\xi_{a\sigma}, \quad \text{where} \quad \xi_{a\sigma} \equiv \mathrm{i}(f_{a\sigma}^\dagger - f_{a\sigma}), \tag{15}$$

where again $\mathcal{K}\xi_{a\sigma}\mathcal{K}^{-1} = -\xi_{a\sigma}$. One easily verifies

$$\mathcal{S}^2 = \mathcal{U}\mathcal{U}^* = (-1)^{N_f} \quad \text{and} \quad \mathcal{C}^2 = (-1)^Q, \tag{16}$$

where the 2nd equation follows directly from the action of $\mathcal{C}$ on the Fermion operators. One also immediately verifies the algebraic relations $\mathcal{S}e^{\mathrm{i}\theta Q} = e^{\mathrm{i}\theta Q}\mathcal{S}$ and $\mathcal{S}\mathcal{C} = \mathcal{C}\mathcal{S}$ (such that $Z_2^{\mathcal{S}}$ commutes with $\mathrm{U}(1) \rtimes Z^{\mathcal{C}}$). As in the case of AIII, note that since these relations, in particular Eq. (16), are invariant under a change of basis of the many-body Fock space, they hold true in any such basis (even though it was initially derived in a representation in which both f_a and $f_a^\dagger$ are real).

The boundary Hamiltonian contains all $(\mathrm{U}(1) \rtimes Z_2^{\mathcal{C}}) \times Z_2^{\mathcal{S}}$ symmetric random interactions. One may think of generating such a Hamiltonian from the Hamiltonian in symmetry class BDI appearing in Eq. (5) containing an even number $N_\chi = 4N_f$ of Majorana Fermion species paired up so as to define the action of an U(1) symmetry, by projecting the latter onto each U(1) charge sector and by then symmetrizing with respect to the $Z_2^{\mathcal{C}}$ group. (I.e., here we think of expressing the complex Fermions $f_{a\sigma}$ and $f_{a\sigma}^\dagger$ in terms of their real and imaginary parts Majorana Fermions, which are those appearing in the corresponding BDI Hamiltonian. Recall the presence of the extra spin index, $N_c = 2N_f$, when comparing to class AIII.) We collect the level statistics numerically in each charge sector labeled by the eigenvalue q of the operator Q. It turns out that the GOE and GSE level statistics appear alternatively with respect to the parity of both the topological number N_f and the charge quantum number q, as summarized in Tab. II(b).

Again, these numerically obtained results for the level statistics can be understood by analyzing the nature of the representations of the $(\mathrm{U}(1) \rtimes Z_2^{\mathcal{C}}) \times Z_2^{\mathcal{S}}$ symmetry (which protects the SPT order). In the charge neutral ($q = 0$) sector, charge conjugation $\mathcal{C}$ is effectively an identity operator. So the analysis is the same as the AIII case, which explains the GOE (or GSE) level statistics at $N_f (\mathrm{mod}\, 2) = 0$ (or 1). For $q \neq 0$, opposite charge sectors $\pm q$ must again be put together for consideration since they are connected by the action of $\mathcal{C}$ and $\mathcal{S}$. In the block-diagonal basis of Q, we have

$$Q = \left(\begin{smallmatrix} -q & 0 \\ 0 & +q \end{smallmatrix}\right),\ \mathcal{C} = \left(\begin{smallmatrix} 0 & \mathbf{1} \\ \eta_{\mathcal{C}}\mathbf{1} & 0 \end{smallmatrix}\right),\ H = \begin{pmatrix} H_{-q} & 0 \\ 0 & H_{+q} \end{pmatrix}. \tag{17}$$

The form of $\mathcal{C}$ is determined by the relation $\mathcal{C}Q = -Q\mathcal{C}$, and $\eta_{\mathcal{C}} = \mathcal{C}^2 = (-1)^q$. To respect the $Z_2^{\mathcal{C}}$ symmetry, we require the Hamiltonian to satisfy $\mathcal{C}H = H\mathcal{C}$, which implies $H_{+q} = H_{-q}$. In the present basis (block-diagonal in Q), the relations $\mathcal{S}e^{\mathrm{i}\theta Q} = e^{\mathrm{i}\theta Q}\mathcal{S}$ and $\mathcal{SC} = \mathcal{CS}$ translate into $\mathcal{U}Q = -Q\mathcal{U}$ and $\mathcal{UC} = \mathcal{CU}$, so that $\mathcal{U}$ must take the form of

$$\mathcal{U} = \left(\begin{smallmatrix} 0 & J \\ \eta_{\mathcal{C}} J & 0 \end{smallmatrix}\right), \tag{18}$$

where J is a real matrix to be determined. Upon substituting Eq. (18) into $\mathcal{UU}^* = \mathcal{S}^2 = (-1)^{N_f} \equiv \eta_{\mathcal{S}}$, it is found that $J^2 = \eta_{\mathcal{C}}\eta_{\mathcal{S}} = (-1)^{q+N_f}$. In order to respect the chiral symmetry ($\mathcal{S}H\mathcal{S}^{-1} = H$), we must have

$$JH_q^* J^{-1} = H_{-q} = H_q. \tag{19}$$

When $(q + N_f)$ is even (odd), $J^2 = +\mathbf{1}$ $(-\mathbf{1})$, then Eq. (19) implies that H_q is a real symmetric (quaternion Hermitian) matrix which leads to the GOE (GSE) level statistics. This result in combination with the analysis in the $q = 0$ sector, thus explains the numerical results displayed in Tab. II(b). One may extend the $(\mathrm{U}(1) \rtimes Z_2^{\mathcal{C}}) \times Z_2^{\mathcal{S}}$ symmetry to an $\mathrm{SU}(2) \times Z_2^{\mathcal{S}}$ symmetry; the SPT classification and the level statistics remain the unchanged. With full SU(2) symmetry, the level statistics is to be considered in each spin-s sector, where the spin quantum number s is determined by $\vec{\boldsymbol{S}}^2 = s(s+1)$. The even (odd) charge q in Tab. II(b) should then be replaced by an integer (half-integer) spin s.

An equivalent way of reading the above result arises from using the (many-body) time-reversal operator, $\mathcal{T} = \mathcal{SC} = \mathcal{CS}$, whose square becomes $\mathcal{T}^2 = \mathcal{S}^2\mathcal{C}^2 = (-1)^{N_f}\ (-1)^q$. Since the "charge" q defines the corresponding Fermion number parity operator $(-1)^F = (-1)^q$, the square of the many-body time reversal operator is of the form $\mathcal{T}^2 = \gamma_{\mathrm{mb}}\ (-1)^F$, where $\gamma_{\mathrm{mb}} \equiv (-1)^{N_f}$ is a "many-body" phase that may always appear when considering the time-reversal operator on the Fermionic (many-body) Fock space. For a general discussion see Eq. (S17), and the corresponding text in the Appendix.[95]

V. SUMMARY

In conclusion, we have investigated the many-body level statistics of the SYK model for the three symmetry classes BDI, AIII and CII whose SPT phases in 1D are $\mathbb{Z}$ classified in the absence of interactions. The level statistics varies among the three different Wigner-Dyson random matrix ensembles periodically with the Fermion flavor number, which also corresponds to the topological index characterizing the interacting 1D SPT phases in these symmetry classes. There is an interesting interplay between level statistics and symmetry quantum numbers, as summarized in Tab. I and Tab. II. The patterns of level statistics can be understood from the global quantum anomalies which are known to characterize the 1D bulk SPT phases, by considering the SYK models as effective theories for the thermalized boundaries of 1D Fermionic MBL-SPT states.

Acknowledgments

As our work was completed, we became aware of results by Fu and Sachdev[77] who also discover the $\mathbb{Z}_4$ periodicity of the SY model (symmetry class AIII) in the level degeneracy. We are grateful to Wenbo Fu for sharing his unpublished results with us at that point. We also acknowledge helpful discussions with Andrew Potter, Tarun Grover, Xiao-Liang Qi and Marcos Rigol. YZY and CX are supported by the David and Lucile Packard foundation and NSF Grant No. DMR-1151208. AWWL is supported by NSF Grant No. DMR-1309667.

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[71] The meaning of the square of the time-reversal operator, $\mathcal{T}^2$, when acting on the many-body Fock space, is described in the Appendix VII. Here we just note that, as shown in the Appendix, the time-reversal operator may acquire an additional phase, which can only be a fourth root of unity, when acting on the Fock space.

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[83] We thank Andrew Potter for drawing our attention to this fact.

[84] If the spectrum is not separated by symmetry sectors, the level statistics will typically be Poissonian even if the system is in the ETH phase, because there is no level repulsion between different symmetry sectors.

[85] N_χ must be sufficiently large ($N_\chi > 4$), in order to observe the universal Wigner-Dyson level statistics. If N_χ is too small, the system is not thermalized, and hence no Wigner-Dyson statistics is observed.

[86] ν and $-\nu$ may be further distinguished by the projective representation of the time-reversal and Fermion parity combined symmetry $P\mathcal{T}$. For more details, see the Appendix VI.

[87] $\mathcal{C}\ell_{p,q}$ denotes the real Clifford algebra with p symmetric generators and q antisymmetric generators.

[88] Following the arguments in Ref. 68, the symmetry group in these two classes admit, owing to the presence of charge conjugation symmetry, irreducible representations of dimensions larger than one, which turn out to imply an extensive number of local degeneracies for any SPT-MBL state which are invariant under the protecting symmetry group. This extensive degeneracy makes the system unstable in the presence of interactions. This leads to either a spontaneous breaking of the symmetry protecting the SPT order, or thermalization. All these scenarios destroy the boundary modes that would otherwise be a necessary consequence of SPT order.

[89] When defined on a 1D lattice, one needs to include a factor (-1) on the right hand side of this equation, when acting on Fermion operators on one of the two sublattices, and a $(+1)$ sign, when acting on the other sublattice. Since this equation however refers only to one lattice site (at the boundary) we can choose a $(+1)$ without loss of generality.

[90] One may think of expressing the complex Fermions c_a and $c_a^\dagger$ in terms of their Majorana "real"- and "imaginary"-parts, which are those appearing in the BDI Hamiltonian. In this way, we can define the action of a U(1) on the BDI system, we have $N_\chi = 2N_c$, and project that latter onto each U(1) charge sector.

[91] A general discussion of the relationship between the action of the square of anti-unitary operators such as chiral symmetry or time-reversal symmetry on the many-body Fock space, and the action the same symmetry operations on the single-particle Hilbert space, is provided in the Appendix. Let us briefly summarize the results for the chiral symmetry operation. The action of these operators on the single-particle Hilbert space is determined by its action on the canonical Fermion operators. For the chiral symmetry operator we can always make the choice that $\mathcal{S}^2 c_a \mathcal{S}^{-2} = c_a$, $\mathcal{S}^2 c_a^\dagger \mathcal{S}^{-2} = c_a^\dagger$. I.e., at the single-particle level the chiral symmetry operation can always chosen to square to the identity (by a choice of phase). It is shown in the Appendix that the square of the anti-unitary operator $\mathcal{S}$ representing the chiral symmetry operation on the many-body Fermion Fock space, can nevertheless have two possibilities, $\mathcal{S}^2 = \pm\mathbf{1}$, depending on the system. The result presented in Eq. (9) is a particular example of the phenomenon discussed in the Appendix.

[92] The above approach by analyzing the projective representations of the time-reversal symmetry can be applied to the BDI case as well. For more details, see the Appendixl VI.

[93] This class of Fermionic SPT states turns out to be also related to bosonic SPT states (Haldane chains)[69,70,78] which have the same $\mathbb{Z}_2$ classification.

[94] For $N_f = 1$ (the "root state" of the CII class), there is an accidental SU(2) symmetry that emerges from $\mathrm{U}(1) \rtimes \mathbb{Z}_2^{\mathcal{C}}$. However for generic N_f (with the large N_f limit in mind), no SU(2) symmetry is required for the CII class. The introduction of SU(2) group here is merely a trick to help with the presentation.

[95] We end by commenting that we could have made a canonical transformation by making a particle-hole transformation on the $\sigma =\downarrow$ Fermions (and not on the $\sigma =\uparrow$ Fermions): $F_{a\uparrow} \equiv f_{a\uparrow}$, $F_{a\downarrow} \equiv f_{a\downarrow}$. Instead of Eq. (13), we would have obtained the usual expression $Q = \sum_{a,\sigma} F_{a\sigma}^\dagger F_{a\sigma}$, while charge conjugation and the action of the chiral symmetry would also have acquired the familiar forms $\mathcal{C} F_{a\uparrow} \mathcal{C}^\dagger = F_{a\downarrow}^\dagger$, and $\mathcal{C} F_{a\downarrow}^\dagger \mathcal{C}^\dagger = -F_{a\uparrow}$. All the previous statements would of course have been entirely identical after this canonical transformation.

Appendix

VI. PROJECTIVE REPRESENTATION ANALYSIS FOR SYMMETRY CLASS BDI

In the main text, we have shown that for symmetry classes AIII and CII, the (projective) symmetry action on the boundary restricts the boundary Hamiltonian H to either real, complex or quaternion Hermitian matrices, and hence exhibiting the three classes of Wigner-Dyson level statistics. In this appendix, we will show that the same kind of argument can be applied to symmetry class BDI as well, which will provide another perspective to understand the level statistics apart from the Clifford algebra argument given in the main text.

The projective symmetry representation on the many-body Hilbert space at the boundary of a 1D Fermion system in symmetry class BDI case has been thoroughly studied by Fidkowski and Kitaev in their pioneering work Ref. [6]. Here we will briefly review some results of Ref. [6], and then discuss the their implications on the level statisics. For the Fermion chain in symmetry class BDI, the full symmetry group in consideration is $Z_2^P \times Z_2^T$, where Z_2^P is the Fermion parity symmetry and Z_2^T is the time-reversal symmetry. The many-body state of the boundary Majorana modes form a projective representation of this symmetry group.

In terms of the Majorana operators χ_a ($a = 1, 2, \cdots, N_\chi$) on the boundary, the Fermion parity operator P can be written as

$$P = \begin{cases} (-\mathrm{i}\chi_1\chi_2)(-\mathrm{i}\chi_3\chi_4)\cdots(-\mathrm{i}\chi_{N_\chi-1}\chi_{N_\chi}) & N_\chi \in \text{even}, \\ (-\mathrm{i}\chi_1\chi_2)(-\mathrm{i}\chi_3\chi_4)\cdots(-\mathrm{i}\chi_{N_\chi}\chi_\infty) & N_\chi \in \text{odd}, \end{cases} \tag{S1}$$

such that the Fermion parity operator anti-commutes with all Majorana Fermion operators, i.e. $\forall a : \chi_a P = -P\chi_a$, as expected. Note that for odd N_χ, an extra Majorana mode χ_∞ at infinity is added to complete the physical Hilbert space, and also to make P operator itself an even-Fermion-parity operator. As shown in Ref. [6], there is no non-trivial projective representation associated to P^2, meaning that one can always make $P^2 = 1$ by gauge fixing, and such a gauge choice has been made in Eq. (S1).

For odd N_χ, Ref. [6] also introduces an useful operator Z by "factoring out" the extra Majorana Fermion χ_∞ from the Fermion parity operator P, so that $P = \mathrm{i}\chi_\infty Z$. One can see that Z is similar to P but does not involve χ_∞,

$$Z = (-\mathrm{i})^{(N_\chi-1)/2} \prod_{a=1}^{N_\chi} \chi_a \quad (N_\chi \in \text{odd}). \tag{S2}$$

Z squares to one (i.e. $Z^2 = 1$) and anti-commutes with P (i.e. $ZP = -PZ$). Importantly, Z commutes with all Fermion interaction terms (which are sum of products of four χ_a operators), and thus Z also commutes with the boundary Hamiltonian H. So Z is an additional symmetry of the Hamiltonian H in the case of odd N_χ.

As an anti-unitary operator, the time-reversal operator $\mathcal{T} = \mathcal{U}_T\mathcal{K}$ can be considered as complex conjugation $\mathcal{K}$ followed by a unitary transformation $\mathcal{U}_T$. One needs to specify the meaning of $\mathcal{K}$ (which is basis-dependent) as follows (following Ref. [6]). First we pick a Fermion occupation number basis (Fock basis) by assuming that the complex Fermion annihilation and creation operators c_m and $c_m^\dagger$ (for $m = 1, 2, \cdots$) are defined as

$$c_m = \tfrac{1}{2}(\chi_{2m-1} + \mathrm{i}\chi_{2m}), \quad c_m^\dagger = \tfrac{1}{2}(\chi_{2m-1} - \mathrm{i}\chi_{2m}). \tag{S3}$$

For odd N_χ, we will include χ_∞ to define the last pair of complex Fermion operators. Let $|0\rangle$ be the state annihilated by all the c_m operators. Any Fermion many-body state $|\psi\rangle$ in the boundary Hilbert space can be represented in the Fock basis as

$$|\psi\rangle = \sum_{n_m \in \{0,1\}} C_{n_1 n_2 \cdots} c_1^{\dagger n_1} c_2^{\dagger n_2} \cdots |0\rangle. \tag{S4}$$

Now we define $\mathcal{K}$ to be the complex conjugation operator in this basis of Fock space, which leaves the basis kets $c_1^{\dagger n_1} c_2^{\dagger n_2} \cdots |0\rangle$ invariant and acts by complex conjugating the coefficients $C_{n_1 n_2 \cdots}$. With this definition of complex conjugation, the Majorana Fermion operator χ_a will have an alternating sign under complex conjugation depending on whether the index a is even or odd, i.e. $\mathcal{K}\chi_a\mathcal{K} = -(-1)^a\chi_a$. This alternating sign must be compensated for by the unitary transformation $\mathcal{U}_T$ via $\mathcal{U}_T\chi_a\mathcal{U}_T^{-1} = -(-1)^a\chi_a$, so that the time-reversal transformation $\mathcal{T}\chi_a\mathcal{T}^{-1} = \chi_a$ leaves the Fermion operator χ_a unchanged. A unitary operator satisfying this condition

$$\mathcal{U}_T = P^{\lceil N_\chi/2 \rceil + 1} \prod_{a=1:2:N_\chi} \chi_a, \tag{S5}$$

where $\lceil N_\chi/2 \rceil$ denotes the smallest integer larger than $N_\chi/2$ ("integer ceiling"), and $a = 1 : 2 : N_\chi$ means that a steps from 1 to N_χ with increment 2. When N_χ is odd, there is an ambiguity in the choice of $\mathcal{U}_T$, because the transform of χ_∞ under $\mathcal{T}$ is not specified. In this case we choose $\mathcal{T}\chi_\infty\mathcal{T}^{-1} = \chi_\infty$, which differs from the choice made in Ref. [6] for N_χ =1 and 5. However, with our choice, the time-reversal operator has a unified expression, Eq. (S5), for all N_χ.

Using the explicit representations for P in Eq. (S1), for Z in Eq. (S2), and for $\mathcal{T} = \mathcal{U}_T\mathcal{K}$ in Eq. (S5), their algebraic relations can be explicitly calculated, and the result is summarized in Tab. SI. The projective representations are fully classified by three invariants: $\mathcal{T}^2$, $(P\mathcal{T})^2$ and $(Z\mathcal{T})^2$, where the last one $(Z\mathcal{T})^2$ is only defined for odd N_χ. In particular, $(P\mathcal{T})^2$ distinguishes the topological

TABLE SI: Projective symmetry group invariants that distinguish the $\mathbb{Z}_8$ anomaly.

$N_\chi (\mathrm{mod}\, 8)$	0	1	2	3	4	5	6	7
$\mathcal{T}^2$	+	+	+	−	−	−	−	+
$(P\mathcal{T})^2$	+	−	−	−	−	+	+	+
$(Z\mathcal{T})^2$		+		+		−		−

index ν from $-\nu$ (where $\nu \equiv N_\chi (\mathrm{mod}\, 8)$). So by combining the invariant $(P\mathcal{T})^2$ with the level statistics, one can fully resolve the $\mathbb{Z}_8$ anomaly pattern.

Having determined the algebraic relations between P, Z and $\mathcal{T}$, we seek the matrix representations of the symmetries and the Hamiltonian in the boundary many-body Hilbert space for all $N_\chi (\mathrm{mod}\ 8)$. We can work in the block-diagonal basis of the Fermion parity operator P, so that

$$P = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix}, \text{ and } Z = \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix} \text{ (for odd } N_\chi), \tag{S6}$$

satisfy $P^2 = Z^2 = 1$ and $ZP = -PZ$. In this basis, to meet the requirements of $\mathcal{T}^2$, $(P\mathcal{T})^2$ and $(Z\mathcal{T})^2$ listed in Tab. SI, the representations of the time-reversal operator $\mathcal{T} = \mathcal{U}_T \mathcal{K}$ can be determined, as summarized in Tab. SII. Here, Ω is a real matrix that squares to -1, i.e. $\Omega^2 = -1$. Without loss of generality, we may choose Ω to be $\Omega = \begin{pmatrix} 0 & +\mathbf{1} \\ -\mathbf{1} & 0 \end{pmatrix}$.

TABLE SII: Representations of $\mathcal{U}_T$ and H that are consistent with all the algebraic relations. Here, Ω can be any real matrix that squares to $\Omega^2 = -1$. $H_\mathbb{R}$, $H_\mathbb{C}$, $H_\mathbb{H}$ stands for real, complex and quaternion Hermitian matrices. A prime on H' indicates H' is in general differed from H. The Clifford algebra $\mathcal{C}\ell_{0,N_\chi-1}$ and level statistics (lev. stat.) in each Fermion number parity sector are also listed.

$N_\chi (\mathrm{mod}\, 8)$	0	1	2	3
$\mathcal{U}_T =$	$\begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix}$	$\begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}$	$\begin{pmatrix} \Omega & 0 \\ 0 & -\Omega \end{pmatrix}$
$H =$	$\begin{pmatrix} H_\mathbb{R} & 0 \\ 0 & H'_\mathbb{R} \end{pmatrix}$	$\begin{pmatrix} H_\mathbb{R} & 0 \\ 0 & H_\mathbb{R} \end{pmatrix}$	$\begin{pmatrix} H_\mathbb{C} & 0 \\ 0 & H^*_\mathbb{C} \end{pmatrix}$	$\begin{pmatrix} H_\mathbb{H} & 0 \\ 0 & H_\mathbb{H} \end{pmatrix}$
$\mathcal{C}\ell_{0,N_\chi-1}$	$\mathbb{R} \oplus \mathbb{R}$	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$
lev. stat.	GOE	GOE	GUE	GSE

$N_\chi (\mathrm{mod}\, 8)$	4	5	6	7
$\mathcal{U}_T =$	$\begin{pmatrix} \Omega & 0 \\ 0 & -\Omega \end{pmatrix}$	$\begin{pmatrix} 0 & \Omega \\ \Omega & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & \mathbf{1} \\ -\mathbf{1} & 0 \end{pmatrix}$	$\begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix}$
$H =$	$\begin{pmatrix} H_\mathbb{H} & 0 \\ 0 & H'_\mathbb{H} \end{pmatrix}$	$\begin{pmatrix} H_\mathbb{H} & 0 \\ 0 & H_\mathbb{H} \end{pmatrix}$	$\begin{pmatrix} H_\mathbb{C} & 0 \\ 0 & H^*_\mathbb{C} \end{pmatrix}$	$\begin{pmatrix} H_\mathbb{R} & 0 \\ 0 & H_\mathbb{R} \end{pmatrix}$
$\mathcal{C}\ell_{0,N_\chi-1}$	$\mathbb{H} \oplus \mathbb{H}$	$\mathbb{H}$	$\mathbb{C}$	$\mathbb{R}$
lev. stat.	GSE	GSE	GUE	GOE

The Hamiltonian H must respect all the symmetries. From the Fermion parity symmetry $PH = HP$, we know H must be block diagonal, and takes the form of

$$H = \begin{pmatrix} H_+ & 0 \\ 0 & H_- \end{pmatrix}, \tag{S7}$$

where H_+ (H_-) is the Hamiltonian that acts in the even (odd) Fermion parity subspace. If the number N_χ of Majorana operators is odd, Z is an additional symmetry of H. Then $ZH = HZ$ further requires $H_+ = H_-$ for odd N_χ. Finally $\mathcal{T}H = H\mathcal{T}$ implies $\mathcal{U}_T H^* \mathcal{U}_T^{-1} = H$. Using the representation of $\mathcal{U}_T$ listed in Tab. SII, we can determine whether H_+ and H_- are matrices with real, complex or quaternion matrix elements, and the result is summarized in Tab. SII. $H_\mathbb{R}$, $H_\mathbb{C}$, and $H_\mathbb{H}$ stand for the set of $n \times n$ Hermitian matrices with real, complex and quaternion matrix elements with some n. The prime on H' indicates that H' is in general differed from H. One can see that the result is consistent with the Clifford algebra analysis (by considering $\mathcal{C}\ell_{0,N_\chi-1}$) discussed in the main text. So we reach at the same conclusion about the level statistics from the analysis of the projective symmetry representations on the many-body Hilbert space at the boundary.

VII. SQUARE OF ANTI-UNITARY SYMMETRIES IN MANY-BODY FOCK SPACE

In this Appendix we discuss in general the action of an anti-unitary operator Θ such as the time-reversal $\Theta = \mathcal{T}$ or the chiral symmetry operation $\Theta = \mathcal{S}$ on the many-body Fock space of a system of Fermions. The square of these operators, as defined by its action on the Fermion creation and annihilation operators (which determine their action on the single-particle Hilbert space) is characterized in the familiar way by a number that we call γ_{sp}, which can take only values $\gamma_{\mathrm{sp}} = \pm 1$ (“single-particle phase”) - see Eq.s (S9,S14,S16). Here we show that the action of the square of the same anti-unitary operators on the many-body Fock space many acquire an additional *many-body phase* γ_{mb}, see Eq. (S17), whose value is related to γ_{sp} in the manner displayed in Eq. (S18). This phase can only be a 4th root on unity. The notion of $\mathcal{T}^2 = \pm \mathrm{i}$ was also discussed in Ref. 79,80.

Consider first a many-body system defined by a set of Majorana Fermion operators $\chi_j = \chi_j^\dagger$, where $\{\chi_i, \chi_j\} = 2\delta_{ij}$. Let $\mathcal{T}$ be the time-reversal operator. The meaning of the time-reversal operation at the single-particle level is defined by its action on the canonical Majorana Fermion operators,

$$\mathcal{T} \chi_j \mathcal{T}^{-1} = \sum_k W_{jk} \chi_k \tag{S8}$$

where in order to preserve the canonical anti-commutation relations, W_{jk} is an orthogonal matrix. At the single-particle level, the meaning of the square of the time-reversal operator, “$\mathcal{T}^2 = \gamma_{\mathrm{sp}} = \pm 1$”, is defined by its action on the canonical Fermion operators,

$$\mathcal{T}^2 \chi_i \mathcal{T}^{-2} = \gamma_{\mathrm{sp}} \chi_i, \quad (i = 1, 2, ...) \tag{S9}$$

where the sign factor $\gamma_{\mathrm{sp}} = \pm 1$ characterizes the square of the "*single-particle*" time-reversal operator. Here

$$\sum_j W_{ij} W_{jk} = \gamma_{\rm sp} \delta_{ik}. \tag{S10}$$

Consider now the time-reversal operator $\mathcal{T}$ when acting on the many-body Hilbert space (Fock space). As an anti-unitary operator, it takes on general grounds the form

$$\mathcal{T} = \mathcal{U}_T \mathcal{K}, \tag{S11}$$

where $\mathcal{U}_T$ is a unitary operator acting on the many-body Hilbert space, and $\mathcal{K}$ denotes the complex conjugation operator acting on the same space. As a consequence of Eq. (S9) the square of the time-reversal operator acting on the many-body Fock space takes in general the form

$$\mathcal{T}^2 = \mathcal{U}_T \mathcal{K} \mathcal{U}_T \mathcal{K} = \mathcal{U}_T \mathcal{U}_T^* = \gamma_{\rm mb} (\gamma_{\rm sp})^F, \tag{S12}$$

where $\gamma_{\rm sp}^F = (\pm 1)^F$ is the Fermion number parity operator when $\gamma_{\rm sp} = -1$. The point we want to stress in this Appendix is that there can be an *extra phase* $\gamma_{\rm mb}$ ("many-body phase") that cannot be removed, or "gauged away". Eq. (S12), *containing* this additional phase $\gamma_{\rm mb}$, defines the notion of the many-body time-reversal operator $\mathcal{T}^2$ that is used throughout this paper. As a consistency check, one immediately sees that Eq. (S12) is consistent with Eq. (S9), since $\mathcal{T}^2 \chi_i \mathcal{T}^{-2} = \gamma_{\rm mb} \gamma_{\rm sp}^F \chi_i \gamma_{\rm sp}^{-F} \gamma_{\rm mb}^{-1} = \gamma_{\rm sp} \chi_i$. The many-body phase $\gamma_{\rm mb}$ always cancels out in this equation as $\gamma_{\rm mb} \gamma_{\rm mb}^{-1} = 1$.

Moreover, an expression of the form of Eq. (S12) holds true in general for both[17,19] *anti-unitary* operators in Fock space, the time-reversal operator $\mathcal{T}$ as well as the chiral symmetry operator $\mathcal{S}$. The former acts on canonical Fermion creation- and annihilation operators as

$$\mathcal{T} c_j^\dagger \mathcal{T}^{-1} = \sum_k c_k^\dagger \, U_{kj}; \quad \mathcal{T} c_j \mathcal{T}^{-1} = \sum_k (U^\dagger)_{j,k} c_k \tag{S13}$$

where U is a unitary matrix and

$$\mathcal{T}^2 c_j \mathcal{T}^{-2} = \gamma_{\rm sp} c_j, \tag{S14}$$

with $\gamma_{\rm sp} = \pm 1$. The chiral symmetry acts on the same operators as

$$\mathcal{S} c_j^\dagger \mathcal{S}^{-1} = \sum_k c_k \, V_{kj}; \quad \mathcal{S} c_j \mathcal{S}^{-1} \sum_k (V^\dagger)_{j,k} c_k^\dagger \tag{S15}$$

where V is a unitary matrix and

$$\mathcal{S}^2 c_j \mathcal{S}^{-2} = \gamma_{\rm sp} c_j. \tag{S16}$$

Here (for the chiral symmetry) it is always possible[19,81] to choose $\gamma_{\rm sp} = 1$.

If we now denote a general *anti-unitary* operator in the many-body Fock space by Θ, representing either time-reversal, $\Theta = \mathcal{T}$, or chiral symmetry, $\Theta = \mathcal{S}$, then, owing to Eq.s (S14,S16), its square has in general the form

$$\Theta^2 = \gamma_{\rm mb} \; (\gamma_{\rm sp})^F \tag{S17}$$

where $\gamma_{\rm mb}$ is a phase. We will now demonstrate that in this general setting the possible choices for the phase $\gamma_{\rm mb}$ are related to the value of $\gamma_{\rm sp}$ in the following way:

$$\begin{cases} \gamma_{\rm mb} = +1, -1 & \text{if } \gamma_{\rm sp} = +1; \\ \gamma_{\rm mb} = \pm 1, \pm \mathrm{i} & \text{if } \gamma_{\rm sp} = -1. \end{cases} \tag{S18}$$

Before proving Eq. (S18) let us list the following examples of this result that apply to systems discussed in this paper: (i): For the time-reversal operator $\Theta = \mathcal{T}$ in symmetry class BDI, for which $\gamma_{\rm sp} = +1$, its square in Fock space is $\mathcal{T}^2 = \gamma_{\rm mb} \mathbf{1}$ with $\gamma_{\rm mb} = +1$ or $\gamma_{\rm mb} = -1$. (See also Table SI of the Appendix.) (ii): For the Chiral symmetry operator $\Theta = \mathcal{S}$ in symmetry class AIII, for which we choose by convention $\gamma_{\rm sp} = +1$, its square in Fock space can be $\mathcal{S}^2 = \gamma_{\rm mb} \mathbf{1}$ with $\gamma_{\rm mb} = +1$ or $\gamma_{\rm mb} = -1$. (See Eq. (9) in the main text.) (iii): For the time-reversal operator $\Theta = \mathcal{T}$ in symmetry class CII, which has $\gamma_{\rm sp} = -1$, its square in Fock space takes on values $\mathcal{T}^2 = \gamma_{\rm mb} (-1)^F$ with $\gamma_{\rm mb} = (-1)^{N_f}$ in the examples given in the main part of this paper.

Let us now proceed to the proof of Eq. (S18). To this end, consider

$$\begin{aligned} \Theta^3 &= \Theta^2 \; \Theta = \Theta \; \Theta^2 = \\ &= \gamma_{\rm mb} \; (\gamma_{\rm sp})^F \; \Theta = \Theta \; \gamma_{\rm mb} \; (\gamma_{\rm sp})^F \end{aligned} \tag{S19}$$

where in the last line use was made of Eq. (S17). The anti-linearity of Θ implies $\Theta \gamma_{\rm mb} = \gamma_{\rm mb}^* \Theta = \gamma_{\rm mb}^{-1} \Theta$ (since $\gamma_{\rm mb}$ is a phase), so that Eq. (S19) leads to

$$\Theta = (\gamma_{\rm mb})^2 \;\; (\gamma_{\rm sp})^F \; \Theta \; (\gamma_{\rm sp})^F \tag{S20}$$

where we have used the fact that $(\gamma_{\rm sp}^F)^2 = (\gamma_{\rm sp}^2)^F = 1$. Now, if $\gamma_{\rm sp} = +1$, Eq. (S20) becomes $\Theta = (\gamma_{\rm mb})^2 \Theta$, implying $(\gamma_{\rm mb})^2 = 1$, and hence $\gamma_{\rm mb}$ can only be ± 1 when $\gamma_{\rm sp} = +1$. On the other hand, if $\gamma_{\rm sp} = -1$, applying Eq. (S20) twice, and making again use of using again $(\gamma_{\rm sp}^F)^2 = (\gamma_{\rm sp}^2)^F = 1$, yields

$$\Theta = (\gamma_{\rm mb})^2 \; (\gamma_{\rm sp})^F \; \Theta \; (\gamma_{\rm sp})^F = (\gamma_{\rm mb})^4 \;\; \Theta. \tag{S21}$$

Therefore we conlude that $\gamma_{\rm mb}^4 = 1$, meaning that $\gamma_{\rm mb}$ must be a 4th root of unity, and hence can only take the values ± 1 and $\pm \mathrm{i}$. Note that the sign of $(\gamma_{\rm mb})^2$ controls the commutation relation between the Fermion number parity operator and Θ. In particular, when (i): $\gamma_{\rm mb} = \pm 1$, Eq. (S20) implies that, for $\gamma_{\rm sp} = -1$, the Fermion number parity operator $(\gamma_{\rm sp})^F = (-1)^F$ *commutes* with Θ. When on the other hand (ii): $\gamma_{\rm mb} = \pm i$, Eq. (S20) implies that the operator $(\gamma_{\rm sp})^F$ must *anti-commute* with Θ. This is only possible when $\gamma_{\rm sp} = -1$, and the action of Θ changes the Fermion number parity. These two cases (i) and (ii) are of course again in line with Eq. (S18). In conclusion, we have demonstrated the dependence of the many-body phase $\gamma_{\rm mb}$ on the single-particle sign $\gamma_{\rm sp}$ (which determines the square of the anti-unitary operator at the single-particle level), which was claimed in Eq. (S18).

We close by noting that our discussion in the Appendix has focused on the properties of the anti-unitary operators Θ when acting on the Hilbert space of a $(0+1)$-dimensional system, i.e. on quantum mechanics at a single point. These considerations generalize to localized excitations described by states $|s\rangle$ in the Hilbert space of a quantum system in spatial dimension larger than one. Our previous analysis in Eq.s (S19),(S20),(S21) still applies except that these equations now need to act on the state $|s\rangle$, and the many-body phase $\gamma_{\rm mb}$ becomes a diagonal operator whose value depends on the state on which it acts. The result of our previous discussion is then that also such a many-body phase of a localized state $|s\rangle$ must always be fourth root of unity. Systems in which the many-body phases $\gamma_{\rm mb} = \pm 1$ and $\gamma_{\rm mb} = \pm i$ occur have been discussed in Ref. 79,80. A careful discussion of the notion of the action of the antiunitary operators Θ on localized states in higher dimensional systems was provided in Ref. 82.