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# Topological spin ordering via Chern-Simons superconductivity

Tigran A. Sedrakyan,[1,2,3] Victor M. Galitski,[4,5] and Alex Kamenev[1]

[1] *William I. Fine Theoretical Physics Institute, University of Minnesota, Minneapolis, Minnesota 55455, USA*
[2] *Physics Frontier Center at the Joint Quantum Institute, University of Maryland, College Park, Maryland 20742, USA*
[3] *Physics Department, University of Massachusetts Amherst, Amherst, Massachusetts 01003, USA*
[4] *Joint Quantum Institute and Condensed Matter Theory Center, Department of Physics, University of Maryland, College Park, Maryland 20742-4111, USA*
[5] *School of Physics and Astronomy, Monash University, Melbourne, Victoria 3800, Australia*

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We use the Chern-Simons (CS) fermion representation of $s = 1/2$ spin operators to construct topological, long-range magnetically ordered states of interacting two-dimensional (2D) quantum spin models. We show that the fermion-fermion interactions mediated by the dynamic CS flux attachment may give rise to Cooper pairing of the fermions. Specifically, in an XY model on the honeycomb lattice, this construction leads to a "CS superconductor," which belongs to a topologically non-trivial in 2D symmetry class DIII, with particle-hole and time-reversal symmetries. It is shown that in the original spin language, this state corresponds to a symmetry protected topological state, which coexists with a magnetic long-range order. We discuss physical manifestations of the topological character of the corresponding state.

## I. INTRODUCTION

Two-dimensional quantum spin models is a fascinating subject, which continue to attract attention of theoreticians and experimentalists alike[1–11]. What makes it particularly challenging from the theory standpoint is the absence of a simple weakly interacting picture and controlled theoretical tools to describe the plethora of possible ground states where strong quantum fluctuations abound. Much of the earlier theoretical work in quantum magnetism has focused on long-range-ordered magnetic phases, usually well-described in terms of the Schwinger boson representation of the spin operator, with subsequent employment of a mean-field theory or other methods (e.g., large-$N$ approaches and variational analyses)[1,4,5,9].

Another prominent class of ground states are spin liquids, which have received much attention since the early nineties, boosted by the discovery of high-temperature superconductivity and some of its exotic scenarios[12–15]. A hallmark of most spin liquids is a lack of a long-range order and a local order parameter. The theoretical description of these states often involves fractionalization - where the spin operators (or equivalently the operators of hardcore bosons) are represented as a product of two fermions ("partons"), which can "fall" into various mean-field states. This construction often leads to gauge theories, non-locality, and topological order of the underlying quantum liquid[16–29]. By now, these kinds of spin liquids have been thoroughly classified, and there is a promising experimental evidence for their actual existence in solid-state materials[30,31].

Very recently, there has been a tremendous progress in identifying and classifying symmetry-protected topological (SPT) phases of interacting fermionic[29,32–37] and bosonic[29,38–41] systems. The SPT phases have some properties of short-range entangled trivial phases, but are also distinct from those, e.g., by exhibiting edge modes. Hence, they in effect represent a third class of possible ground states of strongly-correlated systems, including quantum magnets. In this paper, we propose a microscopic technical construction that appears to give rise to exotic states of this latter type (and their "gauged" versions) in interacting lattice spin models.

A particularly simple example of an SPT spin phase was proposed by Levin and Gu[39], who considered the Ising paramagnet on a triangular lattice with the deceptively simple Hamiltonian $\hat{H}_{LG} = -\sum_{\mathbf{r}\in\triangle} \hat{S}^x_{\mathbf{r}}$, where the spin operators are either Pauli matrices $\hat{S}^x_{\mathbf{r}} = \frac{1}{2}\hat{\sigma}^x_{\mathbf{r}}$ (which indeed makes the corresponding phase a trivial Ising paramagnet) or $\hat{S}^x_{\mathbf{r}} \equiv \hat{B}^x_{\mathbf{r}} = \frac{1}{2}\hat{\sigma}^x_{\mathbf{r}} \prod_{(\mathbf{r}'\mathbf{r}'')} \exp\left[\frac{i}{4}\left(1-\hat{\sigma}^z_{\mathbf{r}'}\hat{\sigma}^z_{\mathbf{r}''}\right)\right]$, where we use Levin-Gu notations for "twisted" spin operators with $\mathbf{r}'$ and $\mathbf{r}''$ running over the six triangles containing site $\mathbf{r}$. These operators satisfy the usual $su(2)$ algebra's commutation relations and give rise to a distinct SPT phase, with non-trivial edge physics. To motivate the central question of this paper, we note that the Levin-Gu topological Ising model may appear in an interacting quantum spin model where the symmetry is broken either "externally" or spontaneously, e.g., a twisted $XY$-model on a triangular lattice, $\hat{H}_{TXY} = -\sum_{\langle\mathbf{r}\mathbf{r}'\rangle\in\triangle} \hat{B}^+_{\mathbf{r}}\hat{B}^-_{\mathbf{r}'}$. Indeed, the mean-field ordered state (e.g., with the mean-field magnetization along the $x$-direction) essentially reproduces the SPT Ising model above $\hat{H}_{TXY} = -\langle\hat{B}^x\rangle \sum_{\mathbf{r}\in\triangle} \hat{B}^x_{\mathbf{r}}$. Note that edge excitations and fluctuation effects in the "$TXY$-model" may lead to qualitative changes in the nature of the mean-field topological (SPT-like) phase, but the above simple construction does suggest that there exist *topologically non-trivial long-range-ordered states* of interacting quantum magnets.

## II. CHERN-SIMONS FERMIONIZATION

This paper provides an example and effective description of a topological long-ranged-ordered state of a quantum spin model. We will focus on a specific Hamiltonian – see, Eq. (3) below – but the general method we use works for a wide class of lattice models and is based on the Chern-Simons (CS) flux attachment[25,26,42–49] - the Jordan-Wigner-type transform that "converts" hardcore bosons/spins into fermions via attaching a string to each particle:

$$\hat{S}^{\pm}_{\mathbf{r}} = \hat{f}^{\pm}_{\mathbf{r}}\hat{\mathcal{U}}^{\pm}_{\mathbf{r}}, \quad \hat{\mathcal{U}}^{+}_{\mathbf{r}} = \exp\left[ie\sum_{\mathbf{r}'\neq\mathbf{r}}\arg(\mathbf{r}-\mathbf{r}')\hat{n}_{\mathbf{r}'}\right]. \quad (1)$$

Here $\hat{S}^{\pm}_{\mathbf{r}}$ are the spin-1/2 raising/lowering operators on a lattice cite $\mathbf{r}$, $\hat{n}_{\mathbf{r}} = \hat{S}^{z}_{\mathbf{r}}+1/2 = \hat{f}^{+}_{\mathbf{r}}\hat{f}_{\mathbf{r}}$, the sum runs over all lattice sites except $\mathbf{r}$, and $e$ is an odd integer CS charge, which makes $\hat{f}^{\pm}_{\mathbf{r}}$ into the fermion creation/annihilation operators. The resulting theory depends on a Hamiltonian and a lattice of course, and generally takes the form that is not amenable to an exact treatment. However, the theory - fermions coupled to the CS gauge field resulting from transformation (1),

$$i\hat{\mathcal{U}}^{+}_{\mathbf{r}}\partial_\mu\hat{\mathcal{U}}^{-}_{\mathbf{r}} \to A_\mu(\mathbf{r}) = \varepsilon_{\mu\nu}\sum_{\mathbf{r}'\neq\mathbf{r}}\frac{(\mathbf{r}-\mathbf{r}')_\nu}{|\mathbf{r}-\mathbf{r}'|^2}n_{\mathbf{r}'} \quad (2)$$

(with $\mu,\nu = 1,2$, and $\varepsilon_{\mu\nu}$ being an antisymmetric tensor) provides a convenient field-theoretic platform to formulate an effective description of various stable phases of quantum magnets.

These constructions usually proceed as follows. The CS gauge potential is represented in terms of a mean-field part (assumed static in the Lagrangian formulation) and fluctuations around the mean-field, $\mathbf{A} = \langle\mathbf{A}\rangle + \delta\mathbf{A}$. The fermions are integrated out on the background of the mean-field configuration (to be determined a posteriori via a variational analysis). Notice that in this construction the CS fermions are assumed to simply fill up the single-particle bands (albeit with a non-trivial Hofstadter-type energy landscape) without undergroing a phase transition. The remaining low-energy theory - an expansion in the CS fluctuations, $\delta\mathbf{A}$ - provides a field-theoretical description of the underlying mean-field. This way one can obtain various states - both exotic and ordered ones. For example, an integer quantum Hall state of fermions generates a CS term, which can either add up to the statistical Chern-Simons field originating from transform (1) (this corresponds to a chiral spin liquid) or cancel it with the remaining Maxwell term representing a gapless phonon[45] (this corresponds to an ordered state).

This elegant approach is not without its downsides. Just about any mean-field Ansatz for $\langle\mathbf{A}\rangle$ "accidentally" breaks physical symmetries that one may want to preserve. Furthermore, the CS fermions are actually not free, but rather represent strongly interacting entities. These interactions may lead to instabilities and hence new underlying spin phases. In this paper, we propose such an alternative construction of a topological long-range-ordered spin state via the CS flux attachment, where instead of assuming a specific mean-field for the CS gauge field, we treat it non-perturbatively as an interaction between the fermions that are shown to become unstable against pairing.

## III. THE MODEL

The specific model we use as our starting point is the bulk spin-1/2 antiferromagnetic Hamiltonian on the honeycomb lattice with nearest-neighbor couplings:

$$\hat{H} = J\sum_{\langle\mathbf{r}\mathbf{r}'\rangle\in\hexagon}\left[(1+\gamma)\hat{S}^{x}_{\mathbf{r}}\hat{S}^{x}_{\mathbf{r}'} + (1-\gamma)\hat{S}^{y}_{\mathbf{r}}\hat{S}^{y}_{\mathbf{r}'}\right], \quad (3)$$

We emphasize that the purpose of our theory below is not to "solve" the particular model (in the sense of finding its lowest energy ground state, whose properties in the conventional setting are well known), but to illustrate that the appearance of topologically non-trivial long-range-ordered states is possible in a class of models. The ease and naturalness with which the calculation goes through strongly suggests that this approach is generic in bipartite lattices (a similar calculation for a different model on the square lattice will be presented in a subsequent publication). Eq. (3) describes a 2D anisotropic XY-type model, whose bulk supports an antiferromagnetic ($J > 0$) ground state. At $\gamma > 0$ it corresponds to a doubly degenerate gapped phase with Néel oder parameter $\langle\hat{S}^{x}_{\mathbf{r}}\rangle$ with $\mathbb{Z}_2$ Ising symmetry which in this case is equivalent to reflection.

In the absence of a net magnetization, CS fermionization yields a half filled fermionic system. The Fermi level of fermions on the honeycomb lattice consists of two Dirac points conventionally denoted by $K$ and $K'$. Using the fermion representation of Eq. (3), and upon expansion in the vicinity of these Dirac points (below, we present calculation details for $\gamma = 0$; for a finite $\gamma$ the calculation is essentially similar) a gauge transformation generates the covariant derivative $\partial_\mu - ieA_\mu(\mathbf{r})$ (and kinetic momentum). The CS gauge field $A_\mu$, that enters into the kinetic term, is bilinear in fermion operators and thus generates a two-particle interaction vertex. This brings the following momentum space representation of the Hamiltonian (3): $\hat{H} = \hat{H}_0 + \hat{H}_{int}$, where

$$\begin{aligned}\hat{H}_0 &= v_F\sum_{\mathbf{k}}\left[\hat{f}^{+}_{\mathbf{k},\alpha}\mathbf{k}\cdot\boldsymbol{\sigma}_{\alpha\beta}\hat{f}_{\mathbf{k},\beta} - \hat{\bar{f}}^{+}_{\mathbf{k},\alpha}\mathbf{k}\cdot\boldsymbol{\sigma}^{T}_{\alpha\beta}\hat{\bar{f}}_{\mathbf{k},\beta}\right] \quad (4)\\ \hat{H}_{int} &= -\sum_{\mathbf{k},\mathbf{k}',\mathbf{q}}V^{\alpha\alpha',\beta\beta'}_{\mathbf{q}}\hat{f}^{+}_{\mathbf{k},\alpha}\hat{\bar{f}}^{+}_{\mathbf{k}'+\mathbf{q},\alpha'}\hat{\bar{f}}_{\mathbf{k}',\beta}\hat{f}_{\mathbf{k}+\mathbf{q},\beta'}.\end{aligned}$$

Here $v_F = \frac{\sqrt{3}J\varepsilon}{2}$ is the velocity at the Fermi level, $\varepsilon$ is lattice constant of the two triangular sub-lattices, $\hat{f}^{\pm}_{\mathbf{k},\alpha}$ and

$\hat{\bar{f}}^{\pm}_{\mathbf{k},\alpha}$ are low energy fermions with momenta measured from $K$ and $K'$ points respectively, and spinor indices correspond to the sub-lattices $\alpha = A, B$. The interaction vertex $V$ in Eq. (4) reads

$$V^{\alpha\alpha',\beta\beta'}_{\mathbf{q}} = 2\pi i e v_F \epsilon_{\mu\nu} \left(\sigma^{\mu}_{\alpha\beta}\delta_{\alpha'\beta'} + \delta_{\alpha\beta}[\sigma^{\mu}]^T_{\alpha'\beta'}\right) A^{\nu}_{\mathbf{q}}, \quad (5)$$

where $\mathbf{A_q} = \mathbf{q}/|\mathbf{q}|^2$ is the Fourier image of the vector potential of the vortex gauge field, $\delta_{\alpha\beta}$ is the Kronecker delta symbol, and the summation over repeating indices is implied.

It is worth noting here that our fermionic Hamiltonian (4) in momentum representation consists of graphene-like kinetic energy term $\hat{H}_0$ and non-local two-particle interactions that arise from integrating out the vortex operators in the fermionized representation. Aside from expanding in the vicinity of $K$ and $K'$ points of the Brillouin zone, the above procedure is formally exact.

## IV. COOPER PAIRING OF CHERN-SIMONS FERMIONS

To proceed further we make use of the Hubbard-Stratonovich transformation based on Cooper pair operators $\hat{\bar{f}}_{-\mathbf{k},\alpha}\hat{f}_{\mathbf{k},\alpha'}$ and $\hat{f}^{+}_{\mathbf{k},\alpha}\hat{\bar{f}}^{+}_{-\mathbf{k},\alpha'}$ to decouple four fermion interaction term in the Hamiltonian (4). By introducing fluctuating superconducting order parameter fields $\Delta^{\alpha\beta}_{\mathbf{k}}$ we obtain

$$\begin{aligned} H &= H_0 + \sum_{\mathbf{k}} \hat{f}^{+}_{\mathbf{k},\alpha}\hat{\bar{f}}^{+}_{-\mathbf{k},\alpha'}\Delta^{*\alpha\alpha'}_{\mathbf{k}} + \Delta^{\alpha\alpha'}_{\mathbf{k}}\hat{\bar{f}}_{-\mathbf{k},\alpha}\hat{f}_{\mathbf{k},\alpha}] \\ &+ \sum_{\mathbf{k},\mathbf{k}'} \Delta^{*\alpha\alpha'}_{\mathbf{k}}\left[V^{-1}\right]^{\alpha\alpha',\beta\beta'}_{\mathbf{k}-\mathbf{k}'}\Delta^{\beta\beta'}_{\mathbf{k}'}, \end{aligned} \quad (6)$$

where $V^{-1}$ is the inverse of the interaction vertex (5). Integrating out the fermionic degrees of freedom in Eq. (6) define an effective action $W(\Delta^{\alpha\beta}_{\mathbf{k}})$ for the superconducting order parameter $\Delta^{\alpha\beta}_{\mathbf{k}}$. We treat the latter in the stationary field approximation, similarly to the standard BCS theory of Cooper pairing. The corresponding saddle point equations, $\delta W/\delta\Delta^{\alpha\beta}_{\mathbf{k}} = 0$, lead to:

$$\Delta^{\alpha\alpha'}_{\mathbf{k}} = \sum_{\beta\beta'\mathbf{k}'} V^{\alpha\alpha',\beta\beta'}_{\mathbf{k}-\mathbf{k}'}\langle\hat{\bar{f}}_{-\mathbf{k}',\beta}\hat{f}_{\mathbf{k}',\beta'}\rangle. \quad (7)$$

Since the vertex function $V^{\alpha\alpha',\beta\beta'}_{\mathbf{k}-\mathbf{k}'}$ in this expression is sharply momentum dependent, the order parameter $\Delta^{\alpha\alpha'}_{\mathbf{k}}$ also turned out to be momentum dependent: $\Delta^{11}_{\mathbf{k}} = \Delta^{22}_{\mathbf{k}} = \Delta_{3\mathbf{k}}$, $\Delta^{12}_{\mathbf{k}} = -\Delta^{21}_{\mathbf{k}} = \Delta^{x}_{0\mathbf{k}} - i\Delta^{y}_{0\mathbf{k}}$. Here we have a vector order parameter $\mathbf{\Delta}_{0\mathbf{k}} = (\Delta^{x}_{0\mathbf{k}}, \Delta^{y}_{0\mathbf{k}})$ and a rotation scalar $\Delta_{3\mathbf{k}}$. The latter corresponds to the pairing of fermions residing on the same sublattice, in contrast to the conventional BCS pairing, where particles having the same spin (here, instead of spin we have a pseudospin degree of freedom associated with two sublattices) do not get paired. The lowest energy solution corresponds to $p$-wave pairing[51–53]; namely we look for the momentum dependence of $\mathbf{\Delta}_{0\mathbf{k}}$ in the form $\mathbf{\Delta}_{0\mathbf{k}} = \Delta_{0\mathbf{k}}\mathbf{k}/k$, where $\Delta_{0\mathbf{k}} = |\mathbf{\Delta}_{0\mathbf{k}}|$ . Substituting the solution (7) into Eq. (6) we obtain the Bogoliubov-de Gennes (BdG) Hamiltonian. In the basis of 4-spinors $\psi_{\mathbf{k}} = (\hat{f}^{A}_{\mathbf{k}}, \hat{f}^{B}_{\mathbf{k}}, \hat{\bar{f}}^{A+}_{-\mathbf{k}}, \hat{\bar{f}}^{B+}_{-\mathbf{k}})$, it acquires the form

$$H_{BdG} = \begin{pmatrix} v_F\mathbf{k}\boldsymbol{\sigma} & \hat{\Delta}_{\mathbf{k}} \\ \hat{\Delta}^{\dagger}_{\mathbf{k}} & -v_F\mathbf{k}\boldsymbol{\sigma} \end{pmatrix}, \quad (8)$$

where $\hat{\Delta}_{\mathbf{k}} = \Delta_{3\mathbf{k}}\mathbb{1} + \mathbf{\Delta}_{0\mathbf{k}} \times \boldsymbol{\sigma}$ and $\mathbb{1}$ is the identity matrix. $H_{BdG}$ gives a 4-band gapped spectrum $\pm E^{(a)}_{\mathbf{k}}$, with the quasiparticle energy $E^{(a)}_{\mathbf{k}} = \sqrt{|av_F\mathbf{k} + \mathbf{\Delta}_{0\mathbf{k}}|^2 + |\Delta_{3\mathbf{k}}|^2}$ and $a = \pm$ distinguishing between two upper/lower bands. As we see the spectrum is $U(1)$ rotationally invariant.

To proceed, one needs to solve the self-consistency Eqs. (7) for the order parameters. Replacing the sum over 2D momenta by an integral and performing the angular integration one arrives at (for the details see the supplementary material)

$$\begin{aligned} \Delta_{0\mathbf{k}} &= \frac{ev_F}{2}\sum_{a=\pm}\int_0^k dk'\frac{k'\Delta_{3k'}}{kE^{(a)}_{\mathbf{k}'}}, \\ \Delta_{3\mathbf{k}} &= \frac{ev_F}{2}\sum_{a=\pm}\int_k^{\Lambda} dk'\frac{\Delta_{0k'} + av_Fk'}{E^{(a)}_{\mathbf{k}'}}, \end{aligned} \quad (9)$$

where we have introduced a cutoff parameter $\Lambda$ around $K$(or $K'$) points defined by the area of the half of Brillouin zone (BZ). Solutions of Eqs. (9) in both halfs of BZ should be glued with each other on the boundary to recover periodicity of the spectrum (see Fig.1).

The solution of Eq. (9) depends on the CS charge, which in the case of Hamiltonian (3) must be an odd integer $e = 1, 3, 5\cdots$. Remarkably, the simplest choice of $e = 1$ yields only a trivial solution, with zero order parameter. However, the states with $e \geq 3$ (mathematically a solution exists for any $e > e_c$, where the critical $e_c = 2/3^{1/2}$; see the supplementary material) give rise to a nontrivial, gapped solution to Eq. (9), indicating that the CS fermions are unstable against pairing. The CS charge, $e$, is determined by energetics of a particular model and we found that $e = 3$ yields the lowest energy (interestingly, an analysis of nearest-neighbor spin-spin correlators in our exotic state is quite close to those in the actual ground state of the conventional $XY$ model[50]).

Numerically found gap-functions for the CS superconductor are plotted in Fig. 1. We see, that $\Delta_{0,\mathbf{k}}$ is linear at $k < J/v_F$: $\Delta_{0,\mathbf{k}} \simeq \frac{3v_Fk}{2}$, as follows from the first of Eqs. (9). This asymptote corresponds to the solution of the gap equation in one half of BZ, (e.g., around the $K$ point, i.e. on the segment $(0, \Lambda)$ of the momentum axis). By flipping signs of $\Delta_{0,\mathbf{k}}$ and $\Delta_{3,\mathbf{k}}$ in gap equations (9), we generate a solution with opposite chirality. This is the solution in the vicinity of $K'$.

## V. MAGNETIC LONG-RANGE ORDER

Now we turn to the discussion of the properties of the Chern-Simons superconducting state. First, we prove that the corresponding magnetic state has off-diagonal long-range order and an associated local order parameter. The most natural correlation function to look at is the spin-spin 2-point correlation function, which in the fermion language takes the form: $\left\langle \hat{S}^{+}_{\mathbf{r}} \hat{S}^{-}_{\mathbf{0}} \right\rangle = \left\langle \hat{f}^{\dagger}_{\mathbf{r}} \hat{f}_{\mathbf{0}} e^{i\Phi_{\mathbf{r}}} \right\rangle$, where $\Phi_{\mathbf{r}} = e \sum_{\mathbf{r}'} \left[\arg \mathbf{r}' - \arg\left(\mathbf{r}-\mathbf{r}'\right)\right] \hat{n}_{\mathbf{r}'}$. The existence of the non-local string makes the calculation of the correlator complicated (we have not been able to evaluate it). However, one can construct high-order correlators, where the string effectively disappears and the spin and fermion correlators are one-to-one related. Note that the CS transform is one of infinitely many Jordan-Wigner-type fermionization transforms, that attach strings in different ways through the lattice. The observables (such as spin-spin correlators) must not depend on the "gauge" choice of a specific Jordan-Wigner string. One can check that there exists a choice of the string such that the four-point spin correlator - see Eq. (10) - is identically equal to the corresponding fermion one. On the other hand, the fermion correlation functions corresponding to two different Jordan-Wigner choices differ only by a phase and hence we arrive at the following relation

$$C^{(4)}(\mathbf{r}-\mathbf{r}') = \langle \hat{S}^{+}_{\mathbf{r}} \hat{S}^{+}_{\mathbf{r}+\mathbf{e}} \hat{S}^{-}_{\mathbf{r}'} \hat{S}^{-}_{\mathbf{r}'+\mathbf{e}} \rangle \sim \langle \hat{f}^{+}_{\mathbf{r}} \hat{\hat{f}}^{+}_{\mathbf{r}+\mathbf{e}} \hat{f}^{-}_{\mathbf{r}'} \hat{\hat{f}}^{-}_{\mathbf{r}'+\mathbf{e}} \rangle,$$

where the $\sim$ symbol implies that the two are equal modulo a phase. The fermion 4-point correlator is calculated using the Wick's decoupling and approaches a constant as $|\mathbf{r}-\mathbf{r}'| \to \infty$. Therefore, $C^{(4)}(\infty) = \text{const}$ and we have an ordered state. This proves the existence of a spin-nematic-type long-range order, but does not prove (or rule out) the existence of a "stronger" magnetic order, which would require calculation of 2-point correlators.

## VI. SYMMETRIES AND TOPOLOGY OF THE CHERN-SIMONS SUPERCONDUCTOR

We now discuss symmetries of the BdG Hamiltonian (8). Importantly, it fulfills simultaneously *(i)* particle-hole (PH) symmetry, $\sigma_2 H^{*}_{BdG} \sigma_2 = H_{BdG}$, physical meaning of which is conservation of pseudo-spin linked with Fermi statistics; and *(ii)* time reversal (TR) symmetry $\tau_1 H^{T}_{BdG} \tau_1 = -H_{BdG}$. Here, the Pauli matrix $\tau_1$ acts in the space of $K$, $K'$ blocks, which is also locked with the Nambu space. The combination of the effective PH and TR symmetries forms a chiral symmetry $\tau_1\sigma_2 H_{BdG} \tau_1 \sigma_2 = -H_{BdG}$, which defines the symmetry class $DIII$ for the Hamiltonian Eq. (8) according to Altland-Zirnbauer classification [32,35,36].

The presence of an effective chiral symmetry implies that one can choose a different basis, where the BdG Hamiltonian decouples into two models having opposite chiralities. Using a unitary transformation $\phi_{\mathbf{k}} = (\phi^{L}_{\mathbf{k}}, \phi^{R}_{\mathbf{k}}) = U\psi_{\mathbf{k}}$ we can reduce $H$ to a simpler form by introducing two copies of two component "left" (L) and "right" (R) fermions with masses of opposite signs, $\pm\Delta_{3,\mathbf{k}}$:

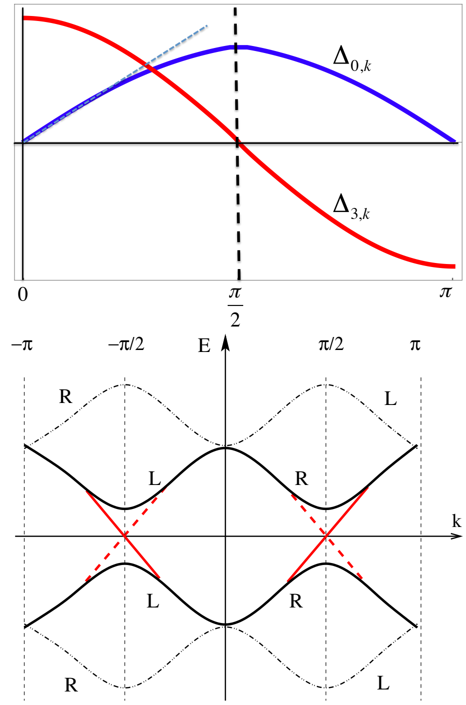


FIG. 1: (Color online) Upper figure: Numerical solution of gap equations (9) for e=3 shown in arbitrary units. Cutoff parameter is $\Lambda \sim \pi/(2\varepsilon)$. Lower figure: Thick black lines represent the low lying part of the bulk spectrum, dotted black lines represent the higher branches. R and L mark the branches corresponding to R/L states, see Eq. (10). Full red lines show R and L brunches of edge state energies.

$$U^{+}HU = \begin{pmatrix} H^{L} & 0 \\ 0 & H^{R} \end{pmatrix}, \quad U = \frac{1}{\sqrt{2}} \begin{pmatrix} \sigma_3 & 1 \\ -1 & \sigma_3 \end{pmatrix}, \quad (10)$$

where $H^{R/L} = \pm\Delta_{3\mathbf{k}}/2 + (\Delta_{0,\mathbf{k}}/k \mp v_F)\mathbf{k}\cdot\boldsymbol{\sigma}$. Figure 1 depicts the spectrum of the lower energy branch of R/L states. Eq. (10) describes a superconducting state, which supports a pair of couter-propagating massless edge states, $R$ and $L$, that are protected by the chiral (combination of TR and PH) symmetry. Note that the edge states are present not only in the theory with $\gamma = 0$ (an isotropic spin model in the original language), but also at a finite $\gamma$. In fact, the existence of a CS superconducting state in the latter is more natural, in that it does not require spontaneous symmetry breaking. Since the gauge symmetry is broken from the outset by the non-zero $\gamma$, there aren't Goldstone excitation and the corresponding *bulk* ground-state is truly gapped.

In this work we showed that the CS superconductor is free from a $Z_2$ topological order. The reason for this is that we employed the direct flux attachment procedure Eq. (1) to fermionize spin-1/2 operators by introducing single fermionic degree of freedom on a honeycomb lattice with nonlocal interactions and no additional gauge fields. In the action formulation, the model can be equivalently formulated as as a theory fermions coupled to a $U(1)$ CS field with $K$-matrix being just unity (for details see e.g. Refs. 26,45). The fact that the CS mediated superconductivity can be free from topological order, is one of the novelties of the present work.

In principle, one can obtain an excited, $Z_2$ topologically ordered state in the model (3) by applying a fermionization transformation on different sublattices (A and B) separately. This procedure, upon superconducting mean-field approximation, will result in a mutual $U(1)\times U(1)$ CS theory[54–56] – a theory of a $Z_2$ topological order (including also Kitaev's toric code model[57]).This is a theory with an effective CS Lagrangian containing two mutual $U(1)$ gauge fields, $(\mathcal{A}_0^a; \mathcal{A}_\mu^a)$, $a = 1, 2;\ \mu = 1, 2,$ and the $K$-matrix given by $K = (\{0, 2\}; \{2, 0\})$[54–56].

## VII. RELATION TO SPT PHASES

The existence of two counter-propagating Majorana edge modes in the Bogoliubov-de Gennes mean-field of the CS superconductor should correspond to some low energy edge excitations in the original lattice spin model with a boundary. Specifically, the edge theory should be equivalent to the 1D critical quantum spin-1/2 Ising chain[51–53,58] – an example of 1+1D conformal field theory with central charge $c = (1/2, 1/2)$. The choice of spin-1/2 operators in the original Hamiltonian (3) as $\hat{\mathbf{S}}_{\mathbf{r}} = \frac{\hbar}{2}\hat{\boldsymbol{\sigma}}_{\mathbf{r}}$, where $\hat{\sigma}^j_{\mathbf{r}}$, $j = x; y; z$, are the Pauli matrices, yields a topologically trivial, conventional XY magnet, $H = H_{XY}(\gamma)$, with the well-known antiferromagnetic Néel ground state. It does not exhibit any edge states. One can show that the reason is that the microscopic edge Hamiltonian for would-be gapless modes of such a model, breaks the chiral symmetry and thus the corresponding edge-states are actually gaped.

Nevertheless this suggests that CS $p$-wave superconductor corresponds to the spin operators being *not* the Pauli matrices, but "twisted spin operators" – B-operators of Levin and Gu, mentioned in the introduction. Indeed, the Levin-Gu construction hinges on the existence of non-trivial element(s) of the group $\mathcal{H}^3[\mathbb{G}, U(1)]$ in Wen's[29,38,40] classification of SPT states. In the case of an Ising model on the triangular lattice, the group $\mathbb{G}$ is the Ising $Z_2$ symmetry, giving rise to two kind of phases, which can be associated with two types of spin operators - the Pauli matrices (yielding the trivial phase) and B-operators (yielding the non-trivial SPT phase). Similarly in our case of two copies of Levin-Gu-type models for each triangular sub-lattice, there is non-trivial element of $\mathcal{H}^3$ indicating the B-operators representation. The corresponding edge Hamiltonian complies with the bulk symmetries, keeping edge-modes gapless.

Our construction of $p$-wave CS superconductor thus corresponds to such new kind of a XY model that can be dubbed twisted $XY$ or $TXY$ model. We notice that this SPT state coexists with the long-ranged nematic (and possibly Neel) order, spontaneously breaking the $U(1)$ symmetry of the parent Hamiltonian (3) at $\gamma = 0$. One attribute the $TXY$ model must have is that it should be possible to gap out its counter-propagating edge states on two sublattices and turn it into a conventional $XY$ magnet by breaking the "protecting" symmetry, reproducing a topologically trivial state at the mean-field level.

In conclusion, we note that gauging the Chern-Simons superconductor gives rise to the topologically ordered state (in the sense that it allows anyon excitations in the bulk). We also note that the corresponding state is different from the Moore-Read state (which corresponds to a gauged $p+ip$ superconductor, while our parent fermionic state preserves time-reversal symmetry). We defer a detailed discussion of this kind of topological order to future studies.

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### Appendix: The Threshold for Chern-Simons Superconductivity

In this Appendix, we present details on *(i)* the derivation of gap equations (9) of the main text for superconducting order parameters $\Delta_{0\mathbf{k}}$ and $\Delta_{3\mathbf{k}}$ and *(ii)* their asymptotic solution. We start with the saddle point equations Eq. (7) that follow from minimization of action:

$$\Delta_{\mathbf{k}}^{\alpha\alpha'} = \sum_{\beta\beta'\mathbf{k}'} V_{\mathbf{k}-\mathbf{k}'}^{\alpha\alpha',\beta\beta'} \langle \hat{\bar{f}}_{-\mathbf{k}',\beta} \hat{f}_{\mathbf{k}',\beta'} \rangle. \tag{A.1}$$

The vacuum expectation value of Cooper pair annihilation operator $\langle \hat{\bar{f}}_{-\mathbf{k}',\beta} \hat{f}_{\mathbf{k}',\beta'} \rangle$ can be represented as the derivative of the effective fermionized action $W(\{\Delta_{\mathbf{k}}^{\alpha\alpha'}\}) = \sum_{\mathbf{k}, a=\pm} E_{\mathbf{k}}^{(a)}$ with respect to Hubbard-Stratonovich fields $\Delta_{\mathbf{k}}^{\beta\beta'}$ as

$$\langle \hat{\bar{f}}_{-\mathbf{k}',\beta} \hat{f}_{\mathbf{k}',\beta'} \rangle = \frac{\delta W(\{\Delta_{\mathbf{k}}^{\alpha\alpha'}\})}{2\delta\Delta_{\mathbf{k}}^{\beta\beta'}}. \tag{A.2}$$

Substituting Eq.(A.2) into (A.1), one obtains

$$\Delta_{\mathbf{k}}^{\alpha\alpha'} = \frac{1}{2}\sum_{\beta\beta'\mathbf{k}'} V_{\mathbf{k}-\mathbf{k}'}^{\alpha\alpha',\beta\beta'} \frac{\delta W(\{\Delta_{\mathbf{k}}^{\alpha\alpha'}\})}{\delta\Delta_{\mathbf{k}}^{\beta\beta'}}. \tag{A.3}$$

Following the reasoning of the main text, we introduce the following convenient notations

$$\begin{aligned} \Delta^{12}_{\mathbf{k}} &= -\Delta^{21}_{\mathbf{k}} = \Delta^{x}_{\mathbf{k}} - i\Delta^{y}_{\mathbf{k}} = \Delta_{0\mathbf{k}} k^{-}/k \\ \Delta^{11}_{\mathbf{k}} &= -\Delta^{22}_{\mathbf{k}} = \Delta_{3\mathbf{k}}, \end{aligned} \tag{A.4}$$

where $k^{\pm} = k_x \pm i k_y$. Then, using the dispersion relation $E^{(a)}_{\mathbf{k}} = \sqrt{|a v_F \mathbf{k} + \mathbf{\Delta}_{0\mathbf{k}}|^2 + |\Delta_{3\mathbf{k}}|^2}$, it is straightforward to take the variation of the action $W(\{\Delta^{\alpha\alpha'}_{\mathbf{k}}\})$ with respect to the Hubbard-Stratonovich fields. Finally, we use explicit expression (5) of momentum dependent interaction vertices $V^{\alpha\alpha',\beta\beta'}_{\mathbf{k}-\mathbf{k}'}$,

$$\begin{aligned} V^{12,11}_{\mathbf{k}-\mathbf{k}'} &= 2\pi e v_F A^{-}, \;\; V^{21,11}_{\mathbf{k}-\mathbf{k}'} = -2\pi e v_F A^{+} \\ V^{12,22}_{\mathbf{k}-\mathbf{k}'} &= 2\pi e v_F A^{-}, \;\; V^{21,22}_{\mathbf{k}-\mathbf{k}'} = -2\pi e v_F A^{+} \\ V^{22,12}_{\mathbf{k}-\mathbf{k}'} &= -2\pi e v_F A^{+}, \;\; V^{11,12}_{\mathbf{k}-\mathbf{k}'} = -2\pi e v_F A^{+} \\ V^{11,21}_{\mathbf{k}-\mathbf{k}'} &= 2\pi e v_F A^{-}, \;\; V^{22,21}_{\mathbf{k}-\mathbf{k}'} = 2\pi e v_F A^{-}, \end{aligned} \tag{A.5}$$

to rewrite self-consistent gap equations (A.3) in the following form:

$$\begin{aligned} \Delta_{0\mathbf{k}} \frac{k^{-}}{k} &= -\pi e v_F \sum_{\mathbf{k}', a=\pm} A^{-}_{\mathbf{k}-\mathbf{k}'} \frac{\Delta_{3\mathbf{k}'}}{E^{(a)}_{\mathbf{k}'}}, \\ \Delta_{3\mathbf{k}} &= \frac{\pi e v_F}{2} \sum_{\mathbf{k}', a=\pm} \frac{A^{-}_{\mathbf{k}-\mathbf{k}'} k'^{+} + A^{+}_{\mathbf{k}-\mathbf{k}'} k'^{-}}{k'} \\ &\times \frac{\Delta_{0k'} + a v_F k'}{E^{(a)}_{\mathbf{k}'}}. \end{aligned} \tag{A.6}$$

Here, $\mathbf{A}_{\mathbf{q}} = \mathbf{q}/|\mathbf{q}|^2$ is the Fourier image of the vector potential of the vortex gauge field $\mathbf{A}(\mathbf{r})$ , and $A^{\pm}_{\mathbf{q}} = A^{x}_{\mathbf{q}} \pm iA^{y}_{\mathbf{q}}$.

Eqs. (A.6) can be simplified further, namely integration over the relative angle $\phi$ between $\mathbf{k}$ and $\mathbf{k}'$ vectors can be performed analytically. This task can be accomplished using the following identities:

$$\begin{aligned} &\int_0^{2\pi} \frac{d\phi}{2\pi} \frac{(\mathbf{k}-\mathbf{k}')\mathbf{k}'}{(\mathbf{k}-\mathbf{k}')^{\mathbf{2}}} = \int_0^{2\pi} \frac{d\phi}{2\pi} \frac{k - k'\cos[\phi]}{k^2 + k'^2 - 2kk'\cos[\phi]} \\ &= \frac{1}{k}\theta[k-k'] \\ &\int_0^{2\pi} \frac{d\phi}{2\pi} \frac{k - k' e^{i\phi}}{k^2 + k'^2 - 2kk'\cos[\phi]} = \frac{1}{k}\theta[k-k']. \end{aligned} \tag{A.7}$$

After performing the angular integration in Eqs. (A.6), one obtains

$$\begin{aligned} \Delta_{0\mathbf{k}} &= \frac{e v_F}{2} \sum_{a=\pm} \int_0^k dk' \frac{k' \Delta_{3k'}}{k E^{(a)}_{\mathbf{k}'}}, \\ \Delta_{3\mathbf{k}} &= \frac{e v_F}{2} \sum_{a=\pm} \int_k^{\Lambda} dk' \frac{\Delta_{0k'} + a v_F k'}{E^{(a)}_{\mathbf{k}'}}. \end{aligned} \tag{A.8}$$

In this way, one reproduces gap equations of the main text. Below we will discuss the asymptotic solution of these equations.

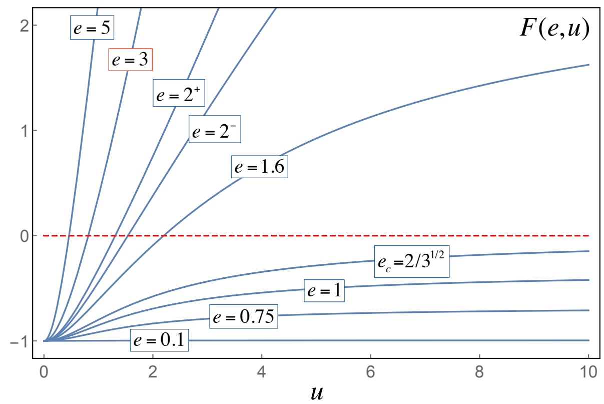


FIG. 2: (Color online) The function $F(e,u)$ is plotted vs $u$ for various values of $e$. At $e > e_c = 2/\sqrt{3}$ the function $F(e,u)$ crosses the dashed line giving raise to a solution to $F(e,u) = 0$ equation. Exactly at the critical point $e = e_c$ the function $F(e_c, u)$ asymptotically approaches the dashed line from below.

In the limit $k \to 0$ it is seen upon inspecting the first Eq. (A.8) that $\Delta_{0k\to 0} \simeq e v_F k/2 + O(k)^3$ and $\Delta_{3k\to 0} \equiv \Delta_3 = const \cdot (v_F \Lambda) + O(k)^2$ is indeed a solution of it. The self-consistency requires however that this asymptotic solution should also satisfy the second Eq. (A.8). Substituting $\Delta_{0k\to 0}$ and $\Delta_3$ into the latter, and using the form of the spectrum $E^{(a)}_{\mathbf{k}}$, one obtains

$$\begin{aligned} \Delta_3 = \frac{e v_F}{2} \int_0^{\Lambda} dk' \Bigg[ &\frac{(\frac{e}{2}+1) v_F k'}{\sqrt{\left(\frac{e}{2}+1\right)^2 (v_F k')^2 + \Delta_3^2}} \\ &+ \frac{(\frac{e}{2}-1) v_F k'}{\sqrt{\left(\frac{e}{2}-1\right)^2 (v_F k')^2 + \Delta_3^2}} \Bigg]. \end{aligned} \tag{A.9}$$

Integration over $k'$ in Eq. (A.9) can be readily performed. Upon introducing a new dimensionless variable $u = (\Lambda v_F)/\Delta_3$, Eq. (A.9) assumes the simple algebraic form:

$$F(e,u) = 0, \tag{A.10}$$

where

$$\begin{aligned} F(e,u) = \frac{e}{2} \Bigg[ &\frac{1}{\frac{e}{2}+1} \left( \sqrt{\left(\frac{e}{2}+1\right)^2 u^2 + 1} - 1 \right) \\ &+ \frac{1}{\frac{e}{2}-1} \left( \sqrt{\left(\frac{e}{2}-1\right)^2 u^2 + 1} - 1 \right) \Bigg] - 1. \end{aligned} \tag{A.11}$$

The function $F(e,u)$ is plotted vs $u$ for various values of $e$ in Fig. 2. We see that the solution to Eq. (A.10) (and thus to self-consistent gap equations Eq. (A.8)) exists only for $e > e_c = 2/\sqrt{3}$. Such a phase transition at $e_c$ can be seen from large and small $u$ asymptotes of function $F$. These are given by

$$F(e, u \gg 1) = \frac{eu}{2}\left[1 + \mathrm{sgn}(e-2)\right] + \frac{3e^2-4}{4-e^2} + O(1/u),$$
$$F(e, u \ll 1) = -1 + \frac{e^2u^2}{4} + O(u)^4. \qquad \text{(A.12)}$$

We see that at $e < e_c = 2/\sqrt{3}$, the monotonically increasing function $F(e,u)$ at $u \to \infty$ asymptotically approaches a negative constant value, $F(e,\infty) = (3e^2-4)/(4-e^2)$, implying that there is no solution to Eq. (A.10) in this region. The large-$u$ behavior of $F(e,u)$ is the same at $2 > e > e_c$, but in this region the constant $F(e,\infty)$ is positive, and thus $F(e,u)$ passes through zero (notice that $F(e,0) = -1$), giving raise to a solution to Eq. (A.10) at some finite $u$. For $e > 2$, the linear large-$u$ asymptote sets in and the condition Eq. (A.10) is being satisfied at even smaller $u$. At $e = 3$, the equation $F(3,u) = 0$ has a solution $u = u_0 = 0.814$. This means that for physical value $e = 3$, the order parameter acquires the asymptotic form $\Delta_{3k\to 0} \simeq 1.23\Lambda v_F + O(k)^2$.


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