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Dual view on sliding phases in U(1) symmetric systems
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The proposal of sliding phases (SP) is revisited from the perspective of duality. A generic argument is formulated as essentially a no go theorem for SP in translationally invariant non-frustrated systems with short range interactions – classical or quantum. Its validity is demonstrated on an asymmetric bilayer and its multilayer variation models where the duality allows obtaining asymptotically exact analytical solution. This solution is in drastic contrast with the perturbative renormalization group prediction and is strongly supported by Monte Carlo simulations. An alternative path toward finding SP is suggested. Its key ingredient is a long range gauge-type interactions suppressing the inter-layer Josephson coupling.

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I. INTRODUCTION

The idea of the sliding phases SP has been emerging in several different contexts – liquid crystals, superconductors, 1D quantum systems, correlated disorder and spin liquids – within a general theme of a possible dimensional decoupling (reduction) when a D-dimensional system breaks into a stack of systems of essentially lower dimensionality. It can be traced back to the suggestion of vanishing shear modulus in layered liquid crystals where each layer is a quasi-solid positionally decoupled from its neighbors. This mechanism has been further explored in Refs.\textsuperscript{2,3}. In layered superconductors magnetic field parallel to the layers has been proposed to suppress the inter-layer Josephson coupling\textsuperscript{4}. However, the frustration due to magnetic field turned out to be insufficient to fully suppress the coupling as shown in Ref.\textsuperscript{5}.

In the context of quantum 1D chains the possibility of the decoupling between chains has been explored as a pathway toward non-Fermi liquid high $T_c$ materials\textsuperscript{6,7}. The main argument for such a decoupling is based on using the scaling dimensions of the Josephson coupling determined with respect to the Luttinger liquid parameter in each chain: if it is larger than 2, the coupling should become irrelevant\textsuperscript{7}. These proposals have been criticized in Refs.\textsuperscript{8,9} where it was shown that the inter-chain tunneling is always relevant. In Refs.\textsuperscript{10} it has been shown that the dimensional reduction is not possible due to pair tunneling in quantum wires. This analysis is based on RG developed for bosonized models with non-zero conformal spin (see in Ref.\textsuperscript{11}).

The results\textsuperscript{1-3} refer to non-compact variables – translation of one layer against its neighbors. That SP can occur in the case of compact XY-variables has been proposed in Ref.\textsuperscript{12} where the inter-layer gradient couplings between classical XY variables in each layer have been considered as a "knob" controlling scaling dimensions of the Josephson coupling and of the vortex fugacity in each layer. The SP would occur if the first one is irrelevant above some temperature $T_d$, while vortices in each layer are still bound into neutral pairs. This approach was also developed for the case of quantum 1D Luttinger liquids coupled by both the Josephson and the gradient terms\textsuperscript{13-16} (which are the analog of the Andreev-Bashkin drag effect\textsuperscript{17} in neutral superfluids or Biot-Savart interactions in superconductors\textsuperscript{18}). More recently, the dimensional reduction was considered in the context of layered disorder\textsuperscript{19,20} and non-Fermi liquids in the spin-liquid regime\textsuperscript{21}.

It is important to note that the proposal of SP is based on applying the Renormalization Group (RG) logic to compact variables characterized by global U(1) symmetry. While these early suggestions were more of a purely academic interest, expanding capabilities of ultra-cold-atoms techniques in recent years emphasize the importance of these suggestions especially in the context of possible new phases in composite lattices\textsuperscript{22} and in the presence of disorder\textsuperscript{19,20}. In more general terms, the question is if it is possible to realize a phase transition, rather than a crossover, from a low- to higher-dimensional behavior.

Here we propose an alternative approach to the problem of SP. It is based on the dual formulation of a field model of compact variables in terms of positive defined statistics of random closed loops of integer currents obeying Kirchhoff’s conservation law\textsuperscript{23}. In this language spontaneous symmetry breaking is equivalent to the formation of a “soup” of fully disordered macroscopic loops. Accordingly, the SP implies that, while proliferating along the layers, such loops do not proliferate perpendicular to them. This immediately leads to the generic requirement for the SP to exist: The energy cost $E_L$ for a loop element to invade a neighboring layer must be macroscopically large with respect to the layer size $L$ because otherwise the entropy for such an invasion will dominate and will cause simultaneous proliferation of the loops along and perpendicular to the layers.

In order to illustrate the above general statement, we will consider a classical XY layered system characterized by gradient inter-layer interactions and the Josephson coupling $u$. The gradient terms are chosen in such a way
that the SP is supposed to exist in some range in accordance with the RG prediction for zero conformal spin. We will present results of the large scale Monte Carlo simulations of this system in its dual representation – in terms of the closed loops. The main finding is that, in accordance with the generic argument, no SP state exists in such a system. As a comparison, the standard asymmetric XY layered model where no SP are expected to occur will be analyzed too. As will be seen, both models demonstrate essentially the same behavior. Furthermore, using dual representation, it becomes possible to find exact analytical solution for the renormalized Josephson coupling \( u_\tau \) in the asymptotic limit \( u \to 0 \). The validity of this solution will be demonstrated numerically for both models. Thus, the main conclusion is that, rather than following the RG prediction, the model of SP demonstrates 3D behavior.

Our paper is organized as follows. In Sec.II we introduce the bilayer model and provide the RG solution for SP. Then, we construct the dual representation in Sec. II B. The asymptotic analytical solution for the renormalized Josephson coupling \( u_\tau \) as well as the numerical results will be discussed in Sec. II C. Then, in Sec.III the stack of bilayers will be discussed. Finally, in Sec. IV we discuss the implications of our analytical and numerical results and also provide an alternative model for the SP.

II. BILAYER MODEL OF SP

The purpose of the following analysis is to introduce a simplest model which admits the RG solution predicting sliding phases. This result will then be tested numerically and analytically in the asymptotic limit of vanishingly small Josephson coupling.

Consider two classical asymmetric parallel layers, each being a square lattice of linear size \( L \) (in terms of the inter-site shortest distance). These layers host two U(1) fields \( \psi_1 = \exp(i\phi_1) \) and \( \psi_2 = \exp(i\phi_2) \) on the layers \( z = 1, 2 \), respectively. The continuous (low energy) action

\[
H_\phi = \int d^2x \left[ \frac{1}{2} K_{zz'} \nabla \phi_z \nabla \phi_{z'} - u \cos(\phi_2 - \phi_1) \right],
\]

features the gradient interaction represented by the (Luttinger parameter) matrix \( K_{zz'} \) as well as by the inter-layer Josephson term \( \sim u \). Here \( \nabla \phi_z \) refers to the \( x, y \) derivatives along the planar directions. The summation over the repeated indexes is used here and hereafter. Stability of the system is guaranteed if \( \det(K_{zz'}) > 0 \), that is,

\[
K_{11}K_{22} - K_{12}^2 > 0.
\]

In the partition function

\[
Z = \int D\phi_1 D\phi_2 \exp(-H_\phi)
\]

the measure of the functional integration must account for the compactness of the phases \( \phi_z \). This can be achieved by using the discrete lattice formulation described in the Appendix A and further discussed in Sec.II B. Let’s first, however, discuss the RG approach to the system.

A. Scaling dimensional analysis for the bilayer

Here we will present the analysis to the system (1,3) based on RG in line with the approach suggested in Refs.\(^ {12} \). It is important that in this analysis the compact nature of the "angles" \( \phi_{1,2} \) is ignored.

Derivation of the RG equations and their solutions for the bilayer are presented in the Appendix B along the line as described in Refs.\(^ {24,25} \). Despite the asymmetric nature of the system, the resulting RG flow equations \((B11,B12)\) for the Josephson coupling and \((B17,B18)\) for the vortex fugacity turn out to be identical to the standard RG equations for the Sine-Gordon model (see in Ref.\(^ {24,25} \)). Thus, in order to identify the critical points (see in Fig.1) \( T_d \) of the SP and \( T_{(q_1,q_2)} \) of the Berezinskii-Kosterlitz-Thouless (BKT) transition, it is enough to evaluate the scaling dimensions (see in Ref.\(^ {26} \)) of the inter-layer Josephson and vortex fugacity, and find the range \( T_d < T < T_{(q_1,q_2)} \) of parameters were both are irrelevant. We begin with the first critical point \( T = T_d \) of the transition from the phase where the interlayer Josephson coupling is relevant (called "Josephson" in Fig.1) to the SP.

If the vortex fugacity is irrelevant, the compact nature of the phases is usually ignored. Then, introducing the variables \( \phi_+ = \phi_1 + \phi_2 \) and \( \phi_- = \phi_2 - \phi_1 \) in Eqs.(1,3) and, then, integrating out \( \phi_+ \), the resulting partition function becomes

\[
Z_- = \int D\phi_- e^{-H_-},
\]

\[
H_- = \int d^2x \left[ \frac{K}{2} (\nabla \phi_-)^2 - u \cos \phi_- \right].
\]

FIG. 1: (Color online) Two options for phases in the bilayer model: (a) with the SP according to RG; (b) Without SP.
where the notation
\[ K = \frac{K_{11}K_{22} - K_{12}^2}{K_{11} + K_{22} + 2K_{12}} \]  
(5)
is introduced. Eqs. (4,5) represent the standard Sine-Gordon model in 2D. The scaling dimension of the operator \( \sim u \) is \( \Delta_u = 1/(4\pi K) \). Thus, the Josephson term becomes irrelevant if \( \Delta_u > 2 \), that is, at \( K < K_2 = 1/(8\pi) \), so that the renormalized \( u \) should flow to zero as \( u_r \sim uL^K \rightarrow 0 \), \( b = 2(1 - K_d/K) < 0 \). Such a behavior is supposed to occur together with the persistence of the algebraic order along the planes. Without loss of generality let’s assume \( K_{11} < K_{22} \) and introduce the notations:
\[ T = 1/K_{22} \text{ as a measure of temperature, and} \]
\[ Y = K_{22}/K_{11} > 1, \]
\[ X = K_{12}/K_{11}. \]
Then, the condition \( K < 1/(8\pi) \) for \( \text{SP} \) becomes
\[ T > T_d = \frac{8\pi(Y - X^2)}{1 + Y + 2X}. \]  
(6)

In order to guarantee the algebraic order in each layer no BKT transition should occur in the layers. In other words, all backscattering harmonics \( V_{q_1,q_2} \) in the action (B1) must be irrelevant below some temperature \( T_{(q_1,q_2)} \) exceeding \( T_d \) in Eq.(6). In order to determine possible types of vortices responsible for the transition, we examine the form (1) in the limit \( u = 0 \) using the Kosterlitz-Thouless argument for the BKT transition. Specifically, a composite vortex \( (q_1,q_2) \) induced by the drag term \( \sim K_{12}^{12.27,28} \) with circulations \( q_1 \) \( q_2 \) in the layers 1 and 2, respectively, is introduced at the same position \( x,y \) along the layers. The free energy of such a composite vortex is
\[ F_v = [\pi[(q_1 + Xq_2)^2 + (Y - X^2)q_2^2] - 2T] \ln L. \]  
(7)

Then, the stability against the BKT transition is guaranteed by the positivity of \( F_v \) or
\[ T < T_{(q_1,q_2)} = \frac{\pi}{2} [(q_1 + Xq_2)^2 + (Y - X^2)q_2^2], \]  
(8)
where the minimization with respect to \( q_1,q_2 \) must be performed. This condition corresponds to the requirement that the scaling dimension \( \Delta_{q_1,q_2} = \pi \sum_{a,b} \tilde{K}_{ab}q_aq_b \) of the most dangerous backscattering amplitude \( V_{q_1,q_2} \) in Eq.(B13) is above 2.

Proliferation of simple vortices \( q_1 = \pm 1, q_2 = 0 \) or \( q_1 = 0, q_2 = \pm 1 \) corresponds to \( T_{(1,0)} = \pi/2 \) and \( T_{(0,1)} = \pi Y/2 > T_{(1,0)} \), respectively. The minimal solution with composite vortex can exist only as along \( X \neq 0 \), that is, when \( K_{12} \neq 0 \).

Solutions for Eqs.(6,8) exist for integer values of \( X \geq 3 \). Introducing \( \delta = Y - X^2 > 0 \) (due to the stability requirement (2)), one should distinguish cases \( \delta > 1 \) and \( \delta < 1 \). In the first case the dominant vortex is \((1,0)\) and the solution for \( T_d < T_{(1,0)} \) exists if \( 0 < \delta < (1 + X)^2/15 \).

If \( 0 < \delta < 1 \), the dominant vortex is composite \(-X,1\) and the conditions (6,8) become
\[ \frac{8\pi\delta}{\delta + (1 + X)^2} < T < \frac{\pi}{2}\delta. \]  
(9)

For \( X >> 1 \), \( T_d \rightarrow 0 \) while \( T_{(q_1,q_2)} \rightarrow (\pi/2)\min(1,\delta) \) as long as \( \delta \) is kept constant. Such a limit corresponds to the largest range of \( T \) where \( \text{SP} \) are to be anticipated for the two-layer model. However, for practical purposes of simulations using too large \( X \) leads to slower convergence. Thus, we choose \( X = 5, Y = 25.5 \) corresponding to a reasonably wide range where \( \text{SP} \) is anticipated to exist. Then, Eq.(9) becomes \( 8\pi/73 < T < \pi/4 \) or \( 0.344 < T < 0.785 \). [The simulations discussed below have been conducted at \( T \) in the middle of the interval (9), that is, \( T \approx 0.565 \). More specifically, \( K_{11} = 1/T = 1.77, K_{22} = 25.5K_{11}, K_{12} = 5K_{11} \)].

Proliferation of the composite vortex pairs with vorticities \( (q_1,q_2) \) corresponds to disordering of the original fields \( \exp(i(\phi_1,\phi_2)) \). At the same time the composite field \( \Psi = \exp(i(q_1\phi_1 + q_2\phi_2)) \) remains (algebraically) ordered. This mechanism constitutes the formation of thermally induced bound phases (or using the language of superfluidity – Thermally Paired Superfluid29). For the values chosen above this composite field \( \Psi = \exp(i(\phi_1 + X\phi_2)) \). Since \( X > 1 \) we call such a composite phase as thermally bound superfluid (TBS). This effect does not require that \( X \) is necessarily integer. If \( X \) is non-integer, its closest integer part will determine the power of \( \psi_2 \). In Fig.1 the TBS exists in the range \( T_{(q_1,q_2)} < T < T_n \). Full symmetry is restored above \( T_n \) that is, no algebraic order exists in any composite or original fields.

Concluding this section, the presented analysis based on the RG finds the range of temperatures where the sequence of phases is as presented in Fig. 1 in the panel (a): at \( T < T_d \) the Josephson coupling is relevant. At \( T_d < T < T_{(q_1,q_2)} \) there is the SP where the symmetry \( U(1) \) is promoted to \( U(1) \times U(1) \). In the range \( T_{(q_1,q_2)} < T < T_n \), the TBS phase is characterized by the composite field \( \Psi \). Thus, the broken symmetry is partially restored through the subgroup \( Z_N \), where \( N = 1 + q_2 \). At higher temperatures, \( T > T_n \), the composite field \( \Psi \) becomes disordered too. In what follows we will show that the actual sequence of phases is correctly depicted in the panel (b) rather than in (a), Fig. 1.

### B. Dual representation

As described in detail in the Appendix A, the bilayer model (1,3) can be reformulated in terms of the dual variables which account for the compact nature of the variables \( \phi_{1,2} \). [The logic behind this transformation is along the line of the J-current model, Ref.23]. The partition function (3) is now represented as
\[ Z = \sum_{\{J_{z,i}\}} e^{-H_J}, \]  
(10)
with the action
\[ H_J = \sum_{i,j} \frac{1}{2}(K^{-1})_{z,i}J_{z,i}J_{z,i} + \sum_{i} \sum_{z} \frac{1}{2w_V}J_{z,i}^2, \]  
(11)
where \((K^{-1})_{zz'}\) is the matrix inverse to \(K_{zz'}\) introduced in Eq.(1) and \(uv\) is the Villain value of the Josephson coupling \(u\). Since we are interested in the limit \(u \ll 1\), it is \(uv \approx 1/(2 \ln(2/u))^{30,31}\) (for more details see the Appendix A). The summation runs over the integer bond currents \(J_{z,ij} = -J_{z',ji}, z = 1, 2\) defined between neighboring sites \(i\) and \(j\) and oriented from site \(i\) to site \(j\) within each corresponding layer as well as over the integer currents \(J_{z,i}\) oriented along the bond connecting the site \(i\) in the layer 1 to the site \(i\) in the layer 2. All the configurations are restricted by the Kirchhoff’s current conservation rule – the total of all \(J\)-currents incoming to any site must be equal to the total of all outgoing currents from the same site.

The resulting configurational space consists of closed loops of the bond currents as schematically depicted in Fig. 2. Further simulations can be effectively performed by the Worm Algorithm\(^{32}\). As will be shown, in addition to being very effective in numerics, the language of loops also allows obtaining analytic expression for the renormalized Josephson coupling \(u_r\), which is exact in the asymptotic limit \(u \to 0\) while strong algebraic order persists along the layers.

If \(u = 0\), there are two sorts of loops – one in each layer. Thus, each configuration is characterized by definite values of the windings \(W_{z,\alpha}\) in the \(z\)th layer along the \(\alpha = \hat{x}, \hat{y}\) directions of the planes. These quantities are defined as a total of all \(J\)-current crossing any line perpendicular to the direction \(\alpha\). [The Kirchhoff’s rule guarantees that windings are independent of the choice of the line]. It is straightforward to show that statistics of these windings determine the renormalized values \(\tilde{K}_{zz'}\) of the matrix \(K_{zz'}\) along the line of the approach\(^{33}\). More specifically

\[
\tilde{K}_{zz'} = \frac{1}{2} \sum_{\alpha = \hat{x}, \hat{y}} \langle W_{z,\alpha} W_{z',\alpha} \rangle.
\]  

(12)

This expressions are valid for periodic boundary conditions (PBC). It is important to note that \(\tilde{K}_{zz'}\) represents an exact linear response with respect to the Thouless phase twists. In other words, if there are externally imposed infinitesimal constant gradients \(\nabla_\alpha \phi_{1,2} \to 0\) (violating the PBC) of the phases \(\phi_{1,2}\), the free energy acquires the contribution \(\delta F = \frac{1}{2} L^2 \sum_{z,z',\alpha} \tilde{K}_{zz'} \nabla_\alpha \phi_z \nabla_\alpha \phi_{z'}\). On the other hand, in the presence of the gradient the integrand of the partition function gets the factor \(\exp(iL \sum_{z,\alpha} W_{z,\alpha} \nabla_\alpha \phi_z)\). Comparing both expressions leads to the relation (12).

As a test of consistency, we have checked numerically that in the regime where the SP state is supposed to exist (that is, \(X = 5, Y = 25.5, T \approx 0.565\)), the deviations of \(\tilde{K}_{zz'}\) from the bare values \(K_{zz'}\) are within the statistical error less than 1\% for all tested sizes of the layers \(10 \leq L \leq 1000\). Significant deviations are observed only as the system approaches fully disordered state – that is, \(T \to T_n\), Fig.1, where the fields \(\psi_{1,2}\) as well as the composite one \(\Psi\) become disordered. In this case, \(\tilde{K}_{zz'}\) flow to zero as \(L\) increases. The deviations remain small (about 2-3\%) even in the regime where \(\Psi\) is the only ordered field. The emergence of the TBS is detected by observing that windings \(W_{z,\alpha}\) in the layers 1 and 2 are changing exactly by the increment \(\Delta W_1 = 1, \Delta W_2 = X\) (plus or minus), respectively.

At finite values of \(u\) the loops belong to both layers so that no separate windings can be introduced. However, the sums \(W_\alpha = W_{1,\alpha} + W_{2,\alpha}\) remain well defined and can be used to evaluate the rigidity \(\rho_\alpha\) of the fields along the layers. In a general case of \(N_z\) layers \(\rho = \rho_x = \rho_y\) :

\[
\rho = \frac{1}{2N_z} \sum_{\alpha} \langle W^2_\alpha \rangle
\]  

(13)

\[
W_\alpha = \frac{1}{L} \sum_{\langle ij \rangle, a=1,2,...N_z} J_{a,ij},
\]  

(14)

where for a given \(\alpha = \hat{x}, \hat{y}\) in (14) the bond \(\langle ij \rangle\) (as well as \(J_{a,ij}\)) is oriented along the direction \(\alpha\).

Our focus here on the renormalized value \(u_r\) of the Josephson coupling \(u\) in the SP regime. In the case of \(N_z\) layers, if the periodic boundary conditions are also imposed perpendicular to the layers (along \(z\)-direction), the inter-layer response \(u_r\) is given by windings \(W_z\) along \(z\)-direction:

\[
u_r = \frac{N_z}{T^2} \langle W^2_z \rangle, \quad W_z = \frac{1}{N_z} \sum_i J_{z,i},
\]  

(15)

where the summation \(\sum_i\) of the currents \(J_{z,i}\) (oriented along \(z\)-direction) is performed over all sites of all layers. Similarly to the cases (12) and (13), Eq.(15) represents the full linear response at zero momentum – that is, the renormalized value \(u_r\) of the Josephson coupling \(u\).

At this point, we should comment on how to interpret the PBC for two layers, \(N_z = 2\). While in the cases \(N_z \geq 3\) it is a natural procedure to link the \(z\) = \(N_z\)th

![FIG. 2: (Color online) A J-current configuration characterized by \(W_z = 1, W_x = 0, W_y = 0\). Horizontal oriented arrows show \(J\)-currents along planes, with \(|J_{z,ij}| = 1\). The vertical ones indicate \(J\)-currents between the planes, with \(J_{z,i} = 1\), with the dashed lines showing currents which are completing periodic boundary conditions.](image)
layer to the first one, \( z = 1 \), by the Josephson term, the case \( N_z = 2 \) needs an auxiliary construction because the layers 1 and 2 are coupled already directly. The formal procedure, then, consists of adding a third layer, \( z = 3 \), with no rigidity along \( x, y \) directions and coupled by the Josephson term to both layers, \( z = 1, 2 \). If the coupling \( u_{13} \) between the layers 1 and 3 and the coupling \( u_{23} \) between the layers 2 and 3 add up as \( 1/u_{13} + 1/u_{23} = 1/uv \), in the dual action (11) the sum in the last term can be extended to the layers \( z = 1, 2, 3 \) in the periodic manner. The key to this procedure is the Kirchhoff’s rule: the J-current from a site \( (x, y) \) along \( z \)-direction from the layer 2 to the layer 3 must be exactly the same as the current from the site \( (x, y) \) in the layer 3 to the layer 1. Then, in the form (11) the same value \( uv \) can be used for the currents from the layer 1 to the layer 2 directly or through the layer 3 as shown in the sketch, Fig. 2.

C. Asymptotic expression for \( u_r \)

As mentioned above, the dual representation allows obtaining analytically asymptotic solution for \( u_r \). Let’s begin with the simplest case of zero stiffnesses \( K_{zz'} \) and arbitrary number of layers, \( N_z = 2, 3, 4 \). The action in this case in the field representation becomes

\[
H_A = \frac{N_z}{2uv} \sum_i J_{z,i}, \quad J_{z,i} = 0, \pm 1, \pm 2, \ldots
\]

(16)

where the summation runs over all sites \( i \) of only one layer, say, \( z = 1 \). In this expression the Kirchhoff rule dictates that the current \( J_{z,i} \) at a given site along \( z \)-direction must be the same for all values of \( z \). Thus, such a current with \( J_{z,i} \) constitutes one closed loop characterized by the winding \( W = J_{z,i} \). This allows constructing the partition function exactly as

\[
Z_A = \left[ \sum_{W=0, \pm 1, \pm 2, \ldots} \exp \left( -\frac{N_z W^2}{2uv} \right) \right]^{L^2}
\]

(17)

where \( L^2 \) is the number of sites in one layer. The stiffness (15) can be found by taking into account that the total winding along \( z \)-direction is \( W_z = \sum_i J_{z,i} \), where the summation runs over \( L^2 \) sites of only one layer. Then, using statistical independence of different sites we find

\[
u_r = \frac{2N_z \sum_{W=1,2,\ldots} W^2 \exp \left( -\frac{N_z W^2}{2uv} \right)}{1 + 2 \sum_{W=1,2,\ldots} \exp \left( -\frac{N_z W^2}{2uv} \right)}.
\]

(18)

This expression shows that, as long as \( N_z \) is finite, the Josephson coupling remains relevant even if there is no in-plane order. In the limit \( uv \ll 1 \) only the term \( W = 1 \) is important, so that Eq.(18) becomes

\[
u_r = \frac{2N_z}{2 + \exp(N_z/2uv)} \sim 2N_z \exp(-N_z/2uv).
\]

(19)

The exponential decay vs \( N_z \) in Eq.(19) is a direct consequence of the absence of the stiffness along the layers, that is, \( \rho = 0 \) in Eq.(14), so that the shortest loop is “vertical” with the number \( M = N_z \) of the vertical currents \( J_{z=1,i} = J_{z=2,i} = \ldots = J_{z=M,i} \).

Thus, it is natural to anticipate, that the dimensional decoupling in a strong sense, when \( u_r \) scales to zero as some negative power of \( L \) as prescribed by RG, should not occur even in the absence of the algebraic order along the planes, when \( \rho \to 0 \). The stiffness along \( z \)-direction remains finite in the limit \( L \to \infty \) as long as \( N_z \) is finite.

This example indicates that short-range inter-plane correlations rather than long-range intra-plane coherences are responsible for finite inter-plane Josephson coupling. In terms of the original variables \( \phi_z \), the result (19) implies that \( u_r \sim u^{N_z} \) (because \( uv \approx 1/(2 \ln(2/u)) \) in the limit \( u \to 0 \)). This can be viewed as the perturbative result of \( N_z \)th order with respect to \( u \).

If there is a finite strong stiffness \( \rho >> 1 \), Eq.(19) can also be used, with \( N_z \) substituted by some effective value \( M = 1, 2, 3, \ldots \), that is,

\[
u_r = \frac{2M}{2 + \exp(M/2uv)} \to 2M \exp(-M/2uv).
\]

(20)

The value of \( M \) is determined by the length of a ”cheapest” string of \( J \)-currents along \( z \)-directions. The loop proliferation can be viewed from the perspective of the Worm Algorithm\(^{32}\) where one open end of a string of \( J \)-currents walks randomly until it meets another open end so that a closed loop is formed. Thus, the path of the spiral is residing in a layer with only occasional jumps between neighboring layers (in the limit \( uv \to 0 \)). Such an elementary jump has the probability \( \sim \exp(-M/2uv) \) so that all higher values \( M \) are exponentially suppressed. In other words, the situation is reminiscent of the ”ideal gas” of rare fluctuations of the \( J \)-currents of length \( M \) in \( z \)-direction.

Thus, generically, it is expected that \( u_r \propto u^M \) in the limit \( u \to 0 \) because then \( uv \approx 1/(2 \ln(2/u)) \).\(^{30,31}\) Below we will show that for the model we consider \( M = 2 \) and, thus, \( u_r \propto u^2 \).

In the standard XY model (with no drag effect and no asymmetry between the layers) in its \( J \)-current representation\(^{23}\), characterized by finite in-plane stiffness \( \rho \) and small inter-layer coupling \( u_r \), the ”cheapest” string in \( z \)-direction has \( M = 1 \) in Eq.(20). The standard XY model and its comparison with the multi-layer extension of the bilayer model will be discussed in more detail in the Sec. IIIID. Below we will show that \( M = 2 \) in Eq.(20) for the bilayer in the SP regime and will present the numerical support for this. In other words, contrary to the RG prediction, the Josephson inter-layer coupling \( u_r \) remains finite in the limit \( L \to \infty \).
D. Numerical results for \( N_z = 2 \)

Here we discuss the results of Monte Carlo simulations of the bilayer in the regime of SP. The action (11) can be represented in the notations \( T,X,Y,\delta \) (introduced below Eq.(5)) as

\[
H_J = \sum_{(ij)} \left[ \frac{T}{2} J_{1,ij}^2 + \frac{T}{2\delta} (J_{2,ij} - X J_{1,ij})^2 \right] + \\
+ \sum_i J_{2,i}^2 / 2u_V.
\]

(21)

where the values of the parameters have been discussed at the end of Sec.II A: \( X = 5,\delta = 1/2, T = (T_u + T_{(X,-1)}) / 2 \approx 0.565 \).

The structure of the loops is determined by the energy of creating a \( J \)-current element along a given direction. A typical energy of this combined element is \( \delta E \) (introduced below Eq.(5)) as \( \sim \exp(-1), \) and no entropy contribution (due to 4 optional directions along the plane) can compensate for such a low value. This implies that no large isolated loops can exist in the plane 2. In contrast, the energy to create an isolated element in the plane 1 (with no \( J \) currents along the same bond in the layer 2) costs much larger energy: \( \delta E_1 \approx T(1 + X^2/\delta)/2 \approx 15 \). Accordingly, the probability to create such an element is exponentially suppressed as \( \sim \exp(-1), \) and no entropy contribution (due to 4 optional directions along the plane) can compensate for such a low value. This implies that no large isolated loops can exist in the layer 1. The only option to create a large loop in the layer 1 is if each element \( J_{1,ij} \) is mirrored by the current \( J_{2,ij} = X J_{1,ij} \) along the same bonds in the layer 2. A typical energy of this combined element is \( \delta E_{12} \approx \approx T/2 \approx 0.25 \). This strong asymmetry between the layers has immediate implication for the windings along \( z \)-direction — the minimal length \( M \) of the element \( J_{z,i} \) must be \( M = 2 \) in Eq.(20). Thus, the stiffness \( u_r \) in the limit \( u << 1 \) becomes

\[
u_r = \frac{4}{2 + \exp(1/u_V)} \approx 4e^{-1/u_V} = u^2,
\]

(22)

where the asymptotic expression \( u_V = 1/2(2/u) \) has been used. Accordingly, for the simple XY model (with no drag interaction) the corresponding dependence is \( u_r \propto u^2 \). This will be discussed below.

As discussed above, the power \( u^2 \) stems from the value \( M = 2 \) in Eq.(20). Formally speaking, Eq. (22) appears to be as though the weak layer \( (z = 1) \) is incoherent and, thus, is eliminated in second order of perturbation with respect to \( u \) — very much alike to the situation discussed in Sec.II C. It is, however, important to note that the weak layer is coherent and the application of the perturbative approach in terms of the original variables — the phases \( \phi_z \) — is not that apparent. In contrast, the dual representation leading to the picture of the “ideal gas” of the vertical currents gives the result \( u_r \sim u^2 \) quite naturally.

![Graph](image)

FIG. 3: (Color online) Monte Carlo results for the inter-layer stiffness \( u_r \) vs its bare value \( u_V \) for the bilayer for various layer sizes (shown in the legend). Error bars are shown, and for the majority of the data points these are smaller than symbols. The fit line is the solution (22).

The results of the simulations is shown in Fig.3. The first striking feature to notice is that \( u_r \), while changing over 7 orders of magnitude, does not depend on the layers size \( L \). Second, \( u_r \) vs \( u_V \) follows the analytical result (22) with high accuracy — even for values \( u_V \sim 1 \). Both features are in the striking conflict with the RG prediction stating that \( u_r \) should scale as \( \propto L^{2(1-T/T_u)} \approx L^{-1.28} \rightarrow 0 \) in the SP regime. It should be also noted that the stiffness along the layers (13) remains finite and much larger than \( u_r \), that is, \( \rho = 32.3 \pm 0.1 \) for all simulated sizes from \( L = 8 \) to \( L = 960 \). This justifies the validity of Eq.(22) even in the case \( u_V \sim 1 \).

III. STACK OF BILAYERS

As it became evident from the previous analysis, no SP can occur in the double layer. Referring to the sketch of the possibilities, Fig. 1, the option (b) is realized instead of (a). Here we will address a possibility of SP in a \( N_z \)-layers setup. In other words, we will be looking for a behavior where the renormalized Josephson stiffness \( u_r \) decays as a function of \( N_z \) in the limit \( L \rightarrow \infty \), while the stiffness along planes remains finite. This would be a “weaker” version of the SP in a clean system. [cf. the SP in the layered disorder case20].

We consider the PBC setup: the odd \( z = 1,3,5,7,... \) and the even \( z = 2,4,6,... \) layers are characterized by the inplane stiffnesses \( K_{11} \) and \( K_{22} > K_{11} \), respectively, with the nearest layers coupled by the current-current term \( \propto K_{12} \) (the same for all pairs of layers) as well as by the Josephson coupling \(-u \sum_{x,y,z} \cos(\phi_{z+1} - \phi_z) \) , where \( \psi_z(x,y) = \exp(i\phi_z(x,y)) \) is the XY variable defined on a site \((x,y)\) belonging to the layer \( z \).

In the linearized with respect to the gradients of \( \phi_z \)
approximation analogous to Eq.(1) the model becomes

$$H_N = \sum_{z=1,3,5,\ldots} \{H_z - \int d^2xz [\cos(\phi_{z+1} - \phi_z) + \cos(\phi_{z-1} - \phi_z)]\}$$  \hspace{1cm} (23)

where the summation runs over odd values of \(z\) and the notation

$$H_z = \int d^2x [K_{11} \frac{1}{2} (\nabla \phi_z)^2 + K_{22} \frac{1}{2} (\nabla \phi_{z+1})^2$$

$$+ K_{12} \nabla \phi_z (\nabla \phi_{z+1} + \nabla \phi_{z-1})]$$  \hspace{1cm} (24)

is used. The gaussian part of the action can be diagonalized by using Fourier representation along \(z\)-direction with doubled unit cell. Then, the matrix \(K_{zz'}\) becomes dependent on the wave vector \(q_z = \frac{4\pi n_z}{N_z}\), \(n_z = 0,1,2,\ldots, N_z/2 - 1\) along \(z\)-axis. The corresponding partition function becomes

$$Z = \int D\phi_z \exp(-H_N),$$  \hspace{1cm} (25)

where the measure of functional integration \(D\phi_z\) must explicitly account for the definition of the phases \(\phi_z(x,y)\) modulo \(2\pi\).

A. RG solution

The corresponding RG equation for \(u_r\) (B19) is analogous to that discussed in the Appendix B for the bilayer. Then the critical temperature of the dimensional decoupling becomes

$$T_d^{-1} = \frac{1}{4\pi N_z} \sum_{m=0}^{(N_z/2) - 1} \frac{1 + Y + 4X \cos^2 q_m}{Y - 4X^2 \cos^2 q_m}.$$  \hspace{1cm} (26)

where the wavevectors along \(z\) take values dictated by the periodic boundary conditions \(q_m = 4\pi m/N_z, m = 0,1,2,\ldots,(N_z/2) - 1\). Here we use the same notations

\(T = 1/K_{11}, X = K_{12}/K_{11}, Y = K_{22}/K_{11}\) introduced in Sec.II.A. Thus, RG predicts irrelevance of \(u_r\) at \(T > T_d\).

The upper limit on \(T\) is determined by the algebraic order along the layers. \([\text{At } u_r = 0 \text{ there should be no 3D vortices }].\) Clearly, if \(T\) is as high as \(> \pi Y/2 >> 1\), all layers will become disordered. Less drastic situation occurs when only weak layers (odd) are disordered \(\pi/2 < T < \pi Y/2\). In this case the Josephson coupling between even layers will be supported by the proximity effect. We, however, will be considering the situation \(T < \pi/2\) which implies algebraic order in all layers.

We considered also a possibility of proliferation of the composite vortices. One option is a composite vortex characterized by phase windings \(q_1 = 1\) and \(q_2 = X > 1\) in odd and even layers, respectively, forming a string of length \(N_z\) perpendicular to the layers. In this case the vortex energy will have a factor \(~N_z\) which makes such vortices too energetically costly to play any role in the limit \(N_z >> 1\), provided the system is not too close to the instability (when the matrix of the gradient interactions acquires zero eigenvalue, that is, \(Y - 4X^2 = 0\)). In our simulations we have been avoiding this region. Thus, such “infinite” vertical vortices are excluded. Another option, is when composite vortices occur as finite length vertical strings – say, of length 2 (along \(z\)-axis) with \(q_1 = \pm 1\) in an odd layer and \(q_2 = -[2X]q_1\) in the neighboring (even) layer. However, a simple analysis shows that energy of such (and longer) composite vortices turns out to be higher than that of the simple vortex with \(q_1 \pm 1, q_2 = 0\) destroying order in the odd layers. Thus, we impose the requirement \(T_d < \pi/2\) in order to have a finite range \(T_d < T < \pi/2\) for the SP to exist within the RG approximation. This implies

$$\frac{1}{N_z} \sum_{m=0}^{N_z/2-1} \frac{1 + Y + 4X \cos^2 q_m}{Y - 4X^2 \cos^2 q_m} > 8.$$  \hspace{1cm} (27)

It can surely be achieved for large enough \(X\) in the limit \(N_z >> 1\). Replacing the summation by integration in this limit and considering \(\delta << 1\), Eq.(27) gives \(\delta < X^2/64\). For the simulations we have chosen \(\delta = 0.3\) and \(X = 6\), which gives \(T_d \approx 0.983\) with \(T = 1.28\) chosen in the middle of the interval between \(T_{(1,0)} = \pi/2 \approx 1.57\) and \(T_d\). The chosen value of \(T_d\) corresponds to the limit \(N_z \rightarrow \infty\), and for any finite \(N_z\), the actual \(T_d\) from Eq.(26) is below this value.

It is worth reminding that, according to RG, \(u_r\) should exhibit suppression as some power of \(L \rightarrow \infty\) in the range \(T_d < T < T_{(1,0)}\). However, as shown below analytically and then numerically, there is no such a suppression in the asymptotic limit \(u_V << 1\).

B. Dual formulation

The dual formulation of the model (23,24,25) in terms of the closed loops of integer J-currents (along bonds in and between the layers) can be achieved similarly to the case \(N_z = 2\). Using Villain approach (see Appendix A) to the discrete gradients : \(\nabla \phi_z \rightarrow (\nabla \phi_z + 2\pi m_{z,ij})\) along the planes and \(-u \cos(\phi_{z+1} - \phi_z) \rightarrow (u/2)(\phi_{z+1} - \phi_z + 2\pi m_{z,ij})\) for the Josephson terms, where \(m_{z,ij}\) refers to integer defined on the bond \(ij\) belonging to the plane \(z\) and \(m_{z,ij}\) stands for an integer on a bond connecting site \(i\) in the plane \(z\) to the same site in the plane \(z + 1\), the partition function (25) follows as a result of explicit integration over all \(\phi_z(i)\) and summations over all bond integers.

The J-currents enter through the Poisson identity

$$\sum_{m=0,1,\ldots} f(m) = \sum_{m=0,\pm 1,\pm 2,\ldots} \int dx \exp(2\pi i Jx) f(x)$$

applied to each bond integer. This allows explicit integration over all phases \(\phi_z\) as well as over the bond integers \(m_{z,ij}, m_{z,ij}\). There are two types of J-currents: inplane \(J_{z,ij}^{(a)}\), \(a = 1,2\)
within each “elementary cell” (along $z$) and between the planes $J_{i,z}$. The label $a = 1, 2$ refers to $J$-current defined on the bond $ij$ belonging to a plane with odd and even $z$, respectively. $J_{z,i}$ denotes the current from the site $i$ from the plane $z$ to the plane $z + 1$. The integration over phases $\phi$ generates the Kirchhoff constraint — similarly to the bilayer case.

Finally the $J$-current ensemble can be represented as

$$
Z = \sum_{(J)} \exp(-H_J),
$$

$$
H_J = \frac{1}{2} \sum_{ij,z,z'} V_{ab}(z - z') J_{z,i}^{(a)} J_{z',ij}^{(b)} + \frac{1}{2uV} \sum_{i,z} J_{z,i}^2,
$$

(28)

where the matrix $V_{ab}(z - z')$ is defined in terms of the matrix $K_{z,z'}$. It reflects the asymmetry between odd and even layers. Explicitly, $V_{11}(z) = YV_{22}(z)$, for $z = z - z'$ being even, describes the interaction between odd layers, and $V_{22}(z)$ is defined between even layers; $V_{12}(z) = -X[V_{22}(z + 1) + V_{22}(z - 1)]$ refers to the interaction between odd and even layers (that is, $z$ is odd), and

$$
V_{22}(z) = \frac{2T}{N_z} \sum_{q_m} \frac{\cos(q_m z)}{\sqrt{Y - 4X^2 \cos^2(q_m z)}},
$$

(29)

with $z = 0, \pm 2, \pm 4, \ldots$ and the summation running over $q_m = 4\pi m/N_z$, $m = 0, 1, \ldots, N/2 - 1$.

C. Asymptotic solution

Analogously to the case of the bilayer, the dual representation allows constructing the asymptotic solution for $u_r$ for arbitrary $N_z$. We begin by finding the renormalized Josephson coupling in the asymptotic limit $u \to 0$, $L \to \infty$ with $N_z$ kept fixed. The dual formulation (28) for $N_z$ layers allows obtaining the asymptotic expression for $u_r$ within the same logic used for deriving Eq.(22). We will repeat it here. The loop formation can be viewed as a process of random walks of two ends of a broken loop — exactly along the line of the Worm Algorithm. Such a walk at each end is controlled by energetics of creating one bond element $|J| = 1$ in a randomly chosen direction — either along a given plane or perpendicular to it. Similarly to the case of the two layers, the energy to create such an element alone along an odd layer costs energy $\gg T \sim 1$, while the same element along an even layer costs energy $\sim 1$. The only option for creating a loop in an odd layer is if its energy is compensated by parallel elements in the even plane. This feature is caused by the strong current-current interaction $\sim X$. Thus, if the walk occurs along $z$-direction from some even layer $z$ toward the neighboring odd layer $z + 1$, the subsequent move along the odd layer will be too energetically costly so that the walker would either move further toward $z + 2$ layer or will go back to the original layer $z$. Thus, the inter-layer elements are characterized by either $J_{i,z} = J_{i,z+1} = \pm 1$ or $J_{z,z+1} = J_{z+1,z} = 0$. The weight of such a process is either $\exp(-1/uV)$ or 1, respectively. Even if the walker makes a step or two along the layer $z + 1$ (which is a highly improbable event) and then chooses to go toward the layer $z + 2$, the contribution to the partition function will be further reduced exponentially by the energy of the element $J$ along the odd plane. Thus, such processes can be ignored, and we arrive at the conclusion that $u_r$ given by Eq.(22) must be valid for arbitrary $N_z$ in the asymptotic limit. The validity of this solution will be verified numerically as explained below.

It is instructive to discuss the dependence on $N_z$ in the situation when $L \gg 1$ is fixed and $uV \to 0$. In this situation the renormalized Josephson stiffness $u_r$ does exhibit the SP-like behavior $u_r \sim \exp(-...N_z)$ (which, however, transforms into the solution (22) as $L \to \infty$). The reason for this, however, is of a purely geometrical nature (which has nothing to do with the drag interactions). Indeed, for any finite $N_z \gg 1$, the system becomes essentially of (quasi-) 1D nature as long as $uV \to 0$. In this case, there is such a value $u^*$ of $uV$ below which there is essentially only one macroscopic vertical loop with $W_z = \pm 1$ for a given area $L^2$, with higher ones exponentially suppressed. This situation corresponds to the contribution of zero modes to the stiffness along $z$-direction. These modes are characterized by $V_{zx} \phi_z(x,y) = 0$ which leads to the effective Hamiltonian $H_0 = \sum_z u_r L^2 \cos(\phi_{z+1} - \phi_z)$, with $u_r \approx 2\exp(-1/uV)$ being the renormalized mesoscopic stiffness. Zero modes become dominant excitations as long as $u_r L^2 \ll K_{11}$.

The dual form of the zero mode action takes a form

$$
\tilde{H}_0 = \frac{N_z}{2L^2 u_r} W_z^2,
$$

(30)

where the duality procedure has been implemented as explained earlier. Calculation of the Josephson stiffness according to Eq.(15) gives the resulting stiffness

$$
u_r^{(0)} \sim \exp\left(-\frac{N_z \exp(1/uV)}{8L^2}\right)
$$

(31)

in the main exponential approximation in the limit $u_r^{(0)} \ll u_r \approx 4e^{-1/uV}$. Thus, if $uV$ is taken to zero, there is such a value $u_\star = u^*$ below which this inequality will be satisfied for fixed $L, N_z \gg 1$. The corresponding value can be obtained from $\frac{N_z \exp(1/uV)}{8L^2} \geq 1$ which gives

$$
\nu_\star \approx \frac{1}{\ln(8L^2/N_z)}
$$

(32)

in the main logarithmic approximation. Thus, for fixed $L, N_z$ the solution (22) is valid as long as $uV > u^*$ and it must cross over to (31) as long as $uV < u^\star$. However, as $L \to \infty$, the crossover value of $uV$, Eq.(32), goes to zero which means the recovery of the asymptotic solution (22) for any finite $uV$. This effect will be seen in
the simulations as discussed below. It is important to mention, though, that such a suppression has nothing to do with SP because the RG solution (discussed above) implies the suppression of \( u_r \to 0 \) in the limit \( L \to \infty \) for fixed \( u_V \), while the asymptotic solution gives finite \( u_r \), Eq.\((22)\), in the same limit.

D. Numerical results for \( N_z > 2 \)

The model \((28)\) has been simulated by the Worm Algorithm\(^{28}\). The renormalized inter-layer stiffness \( u_r \) was found for a range of layer sizes, \( 6 \leq L \leq 640 \) and layer numbers \( 10 \leq N_z \leq 40 \). The resulting data is presented in Figs. 4, 5. As can be seen in Fig. 4, the solution \((22)\) plays the role of the envelop for the family of the curves \( u_r \) vs \( 1/u_V \) for various \( L \) and \( N_z \). We note that the stiffness \( \rho \) along the layers (as determined by Eq.\((13)\)) remains independent of the sizes and much larger \((\rho = 22.6 \pm 0.5)\) than \( u_r \). This justifies the applicability of the asymptotic limit for Eq.\((22)\). We have also controlled that the system is far enough from any possible composite phases\(^{28}\) state by measuring the lowest order correlator \( \exp(i\varphi_{x,y})\exp(-i\varphi_{x',y'}) \) and observing that it exhibits long-range order [in the composite phase state such a correlator is short ranged]. Thus, the system is well in the putative SP state. Its behavior, however, is drastically different from the RG prediction.

At this point we should discuss the deviations of the numerical curves from the analytical result seen in Fig.4. As discussed above, this behavior is a consequence of zero modes. The value of \( u_V = u^* \) below which the suppression begins decreases as

\[ (u^*)^{-1} = \gamma \ln(L^2/N_z), \quad \gamma = 1.00 \pm 0.02 \tag{33} \]

for \( L^2/N_z >> 1 \) in the main logarithmic approximation.

This behavior is demonstrated in Fig.5, where the value \( u^* \) corresponds the offset for \( u_r \) taken at \( 1/10 \) of the value given by the analytical expression \((22)\). The result \((33)\) is consistent with the analytical formula \((32)\).

Clearly, such a quasi-1D suppression (zero modes effect) is also present in the standard XY model (where no SP are anticipated to exist). In order to demonstrate this explicitly we have also simulated a simple XY model given by the system

\[ Z_{XY} = \int D\phi \exp(-H_{XY}), \tag{34} \]
dual representation this system is described by
\[ u \text{ occurs in each layer for } \] with some \( \tilde{K} \) that is, with the slope \( \frac{1}{2} \).

\[ \text{linear fit of this line gives the slope } 1.95 \pm 0.05. \]

\[ H_{XY} = - \sum_{\langle ij \rangle, z} [\tilde{K} \cos(\nabla_{ij} \phi_z) + u \cos(\nabla_{z} \phi_z)], \]

with some \( \tilde{K} \gg 1 \) (guaranteeing that no BKT transition occurs in each layer for \( u = 0 \)), and \( 0 < u << \tilde{K} \). In the dual representation this system is described by

\[ H_{XY} \rightarrow \tilde{H}_{XY} = \sum_{\langle ij \rangle, z} \frac{1}{2\tilde{K}} J^2_{i,j} + \sum_{i,z} \frac{1}{2u_V} J^2_{i,z}, \]

(35)

where \( J_{i,j,z} \) and \( J_{i,z} \) are the same J-currents introduced above for the model (28). The results of the simulations of this model are presented in Fig.6,7. In the asymptotic limit the inter-layer stiffness is described by Eq.(20) with \( M = 1 \). Then, according to the above discussion, the value \( u^{*} \) determining where the deviations from the analytical formula begin is given by \( (u^{*})^{-1} = 2 \ln(L^2/N_z) \), that is, with the slope \( \gamma = 2 \) which should be compared with the numerical value \( \gamma = 1.95 \pm 0.05 \) in Fig.7. Thus, both models demonstrate essentially the same 3D behavior, with the only difference being the slope of the renormalized Josephson coupling in \( u_r \) vs its bare value \( u_V \).

IV. DISCUSSION

The RG approach to 2D systems proves to be very effective in many cases including 2D XY model when it can be mapped on the Sine-Gordon (SG) one34. A successful implementation of the RG analysis to the Josephson coupling was demonstrated in Ref.35 where a single weak link can make one channel Luttinger liquid insulating.

The merit of RG, however, should be taken with caution when applied to the dimensional reduction situations in layered systems hosting compact U(1) variables. In this case there is no exact mapping between XY and SG representations at finite inter-layer Josephson coupling, and the approximation ignoring the compact nature of the variables becomes uncontrolled. As our analysis of one particular layered system shows, no SP exists in such a system despite the RG prediction: the system shows essentially the 3D behavior of the asymmetric XY model. Clearly, the simplest example presented here reveals the flaw in extending RG to the dimensional decoupling situations12–16 when the effective model corresponds to zero conformal spin11. [At non-zero spin, tunneling of pairs can take over10]. As shown in Sec.II C, the interlayer Josephson coupling exists even when there is no intra-layer order – which is consistent with the proximitif effect.

The dual formulation in terms of the closed loops gives a very important insight. Specifically, the SP means that as layer size \( L \rightarrow \infty \), a suppression of the Josephson coupling between layers would require that the number of times elements of closed loops fluctuate between layers must scale slower than \( L^2 \) so that the density of such events is zero in the limit \( L = \infty \). The loops statistics, however, is controlled by local energies of creating finite elements and the entropy due to 6 directions in 3D vs 4 along layers. Thus, as long as there is a finite energy to cross between neighboring layers, the entropy will lead to a finite density of crossings for large enough \( L \). Similar argument can be applied to quantum wires in terms of the quantum to classical mapping where imaginary time is treated as an extra dimension.

The dual approach and the argumentation along the line of the numerical algorithm32, treating closed loops formation as a process of the worm head wondering around and eventually finding its tail, allowed us to expose the actual stages of the renormalization of the Josephson coupling: i) At small scales Josephson coupling is controlled by exponentially suppressed random and independent (in the asymptotic limit) events of crossings between layers. It can be viewed as an ideal gas of J-currents between the layers. This stage leads to the renormalized coupling, in general, represented by Eq.(20) with \( M = 1, 2, 3, \ldots \). ii) If the number of layers \( N_z \) increases, with \( L \) being fixed, quasi 1D fluctuations further suppress the coupling exponentially as demonstrated in Eq.(31).

Here we have discussed a local model characterized by short range interactions between the inter-layer J-current elements. This feature in combination with the low density of such elements justifies the "ideal gas", which in its turn leads to finite values of the renormalized inter-layer Josephson coupling.

The question may be raised if a presence of long-range forces between the inter-plane J-currents \( J_{i,z} \) can change the situation and lead to the SP or its weaker version – where \( u_r \rightarrow 0 \) with the growing number of layers \( N_z \) in the limit \( L = \infty \). In this respect we note that in order to realize this, fluctuations of the difference of the J-
currents with positive and negative orientations must be macroscopically suppressed. In this case the fluctuation of the winding numbers in $z$-direction ($W_z^2$) will scale slower than $L^2$ so that $u_r \sim (W_z^2)/L^2 \to 0$. This may be caused by interactions between the inter-layer J-currents decaying not faster than second power of their separation along planes. More specifically, the following additional repulsive term

$$H_{SP} = \frac{1}{2} \sum_{i,j} U(\vec{x}_i - \vec{x}_j) J_{z,i} J_{z,j}$$

(36)

in the simple XY J-current model (35) with $U(\vec{x})$ having the long range tail $\sim 1/|\vec{x}|^\sigma$ with $\sigma < 2$ will generate the energy contribution $\sim W_z^2 L^{-\sigma}$ in terms of the windings in $z$-direction. Consequently, the renormalized Josephson coupling (15) would scale as $u_r \sim L^{\sigma-2} \to 0$.

As one particular example, long-range forces can be introduced in the standard XY model (34) by some effective gauge-type term $-u \cos(\nabla_z \phi - g_z A_z) + (\bar{\nabla} A_z)^2$, where $\bar{\nabla}$ refers to the derivatives along the layers of some soft mode $A_z$, with $g_z$ being a constant. The resulting interaction in the dual form (36) becomes $U \sim g_z^2 \ln(|\vec{x}|)$ and, thus, it suppresses the inter-layer Josephson as $u_r = N_z W_z^2 L^2 \sim 1/(L^2 \ln L)$ in the limit $L \to \infty$ for fixed $N_z$. At the moment we do not comment on how realizable in practice such mechanism is.

Here we have analyzed a clean system and found no SP. The situation is completely different in the presence of layered disorder where the weakly sliding phases occur due to rare fluctuations of disorder resulting in a large stack of insulating layers simply blocking the flow perpendicular to the layers. The number of such layers scales logarithmically with the total number of layers, so that $u_r \sim N_z^{-c}$ with some non-universal exponent $c > 0$. This effect does not need any drag-type interactions and can occur in a simple XY model.

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### Appendix A: Lattice formulation

In order to introduce the cross-gradient term $\sim K_{12}$ in Eq.(1) (cf.3,13,14,16) consistent with the compact nature of the phases, we use an effective gauge-type field $A_{ij}$ defined on bonds of the lattice:

$$H_{A,\phi} = -\sum_{\langle ij \rangle} [t_1 \cos(\nabla_{ij} \phi_1 - A_{ij}) + t_2 \cos(\nabla_{ij} \phi_2 - g_2 A_{ij})] + \frac{1}{2g_1} \langle \nabla_{ij} J_{z}^2 \rangle - \sum_i u \cos(\phi_2(i) - \phi_1(i))$$

(A1)

where $t_1, t_2 > 0, g > 0$ and $g_2$ are parameters; $\langle ij \rangle$ denotes summation over nearest neighbor sites within each layer; $\nabla_{ij} \phi_2 \equiv \phi_2(i) - \phi_2(j)$; $A_{ij}$ is a bond vector field (that is, $A_{ij} = -A_{ji}$) oriented along the bond $\langle ij \rangle$. Accordingly, the partition function (3) should be rewritten as

$$Z = \int DAD\phi_1 D\phi_2 \exp(-H_{A,\phi}),$$

(A2)

where the temperature is absorbed into the the parameters $t_1, t_2, u, g$. Our focus here is on verifying the applicability of the RG analysis to the renormalization of the Josephson coupling $u$. Hence, we will not discuss physical origins of the variables and the parameters.

If the fugacity of the impaline vortexes is irrelevant, the terms $-\cos(\nabla_{ij} \phi_1 - A_{ij})$ and $-\cos(\nabla_{ij} \phi_2 - g_2 A_{ij})$ in Eq.(A1) can be replaced by $\langle \nabla_{ij} \phi_1 - A_{ij} \rangle^2/2$ and $\langle \nabla_{ij} \phi_2 - g_2 A_{ij} \rangle^2/2$, respectively. Then, the gaussian integration over $A_{ij}$ can be carried out explicitly in Eq.(A2), so that (A1) in terms of the phases becomes exactly the expression (1), where the $2 \times 2$ matrix $K_{zz'}$, $z, z' = 1, 2$ is related to the original parameters as

$$K_{11} = \frac{t_1(1 + g_2^2 t_2)}{1 + g(t_1 + g_2^2 t_2)}, \quad K_{22} = \frac{t_2(1 + g t_1)}{1 + g(t_1 + g_2^2 t_2)},$$

$$K_{12} = -\frac{g g_2 t_1 t_2}{1 + g(t_1 + g_2^2 t_2)}.$$

(A3)

[As a matter of taste, we will keep $g_2 < 0$ in order to have $K_{12} > 0$].

The stability requirement (2) is guaranteed by

$$K_{11} K_{22} - K_{12}^2 = \frac{t_1 t_2}{1 + g(t_1 + g_2^2 t_2)} > 0.$$  

(A4)

The condition (9) for the existence of SP for the chosen values $Y = 25.5, X = 5$ in terms of the parameters $t_1, t_2, g_2, g$, implies that $g_2(g_2^2(1 - 5g_2) = 5, g t_1(5.1|g_2| - 1) = 1, t_1 \approx 0.177|g_2|/|(1 - 4|g_2|)(5.1|g_2| - 1)|$ and $10/51 < |g_2| < 1/5$.

The partition function $Z$, Eq. (A2), with the full action (A1) can be evaluated by the high-temperature expansion method (see e.g. in36) in terms of $t_1, t_2, u$ with further explicit integration over the variables. This approach allows obtaining $Z$ in terms of the integer bond variables – powers of the corresponding Taylor series.
We will be utilizing the Villain approximation\textsuperscript{30} for the cosines to obtain the so called J-current version\textsuperscript{23} of Eqs.(A2),(A1):

\[ Z = \sum_{\{m_{a,ij},m_i\}} \int D\phi \int DA e^{-H_V}, \quad (A5) \]

\[ H_V = \sum_{(ij)} \frac{t_1}{2}(\nabla_{ij}\phi_1 - A_{ij} + 2\pi m_{1,ij})^2 \]
\[ + \frac{t_2}{2}(\nabla_{ij}\phi_2 - g_2A_{ij} + 2\pi m_{1,ij})^2 + \frac{1}{2g_1^2}A_{1,ij}^2 \]
\[ + \sum_i \frac{u_V}{2}((\phi_2(i) - \phi_1(i) + 2\pi m_i)^2, \quad (A6) \]

where \( m_{a,ij} = -m_{a,ji} = 0, \pm 1, \pm 2, \ldots (a = 1,2) \) are integer numbers defined along bonds between two nearest sites \( i \) and \( j \) along the planes and \( m_i = 0, \pm 1, \pm 2, \ldots \) is an integer assigned to a site \( i \) and oriented from the layer 1 to the layer 2.

The Villain approximation proves to be very accurate for establishing the transition points as well as in general if the effective constants \( t_1, t_2, u_V \) are properly expressed in terms of the corresponding bare values \( t_1, t_2, u \) (see in Ref.\textsuperscript{31}). The "renormalization"\textsuperscript{6} can be essentially ignored for \( t_1, t_2 \geq 1 \), so that in what follows we will be using \( t_1 = t_2 = t_2 \). Similarly, for the Josephson coupling \( u \sim 1 \) one should take \( u_V = u \) and, if \( u < 1 \), the corresponding relation is \( u_V = 1/(2\ln(2/u))\textsuperscript{30,31} \)

After using the Poisson identity, \( \sum_j f(m) = \int dm f(m) \exp(2\pi imJ) \) for arbitrary function \( f, \) for each integer and performing the integrations over \( \phi_i \) and \( A \), the resulting expression becomes the dual formulation (10,11) of the original system (1,3).

### Appendix B: RG equations for the bilayer

Here we will provide the derivation of the RG equations based on the quantum to classical mapping. In our case it should rather be viewed as classical to quantum mapping. Treating one of the layers direction (say \( y \)) as imaginary time \( \tau \) and using Haldane’s approach\textsuperscript{37} in terms of the phases \( \phi_i \) and the \( \angle \) angles \( \theta_i \) counting particles as mutually conjugated variables, the corresponding action in \( D = 1 + 1 \) becomes\textsuperscript{20,22}

\[ H_Q = \int_0^L dx \int_0^\beta dt \left[ \frac{i}{\pi} \partial_x \theta_x \partial_x \phi_x + \frac{1}{2} K_{zz'} \partial_x \phi_{z} \partial_x \phi_{z'} + \frac{1}{2\pi^2} (K^{-1})_{zz'} \partial_x \theta_{z} \partial_x \theta_{z'} \right] - u \cos(\phi_1 - \phi_2) - \sum_{q_1,q_2} V_{q_1,q_2} \cos(2(q_1 \theta_1 + q_2 \theta_2)), \quad (B1) \]

where \( (K^{-1})_{zz'} \) are the matrix elements of the matrix inverse of \( K_{zz'} \); \( \beta = L \) (that is, the "speed of sound" \( V_s = L/\beta = 1 \)) and the summation over the repeated indexes \( (z,z' = 1,2) \) labeling layers is used here and below. The last summation terms account for the backscattering events with \( q_1,q_2 \) being arbitrary integers (from \(-\infty \) to \(+\infty \)) which represent charges of the instantons (or composite vortices – in the "language" of the original classical layers).

We begin by looking for a solution where all the harmonics amplitudes \( V_{q_1,q_2} \) are irrelevant. In this case the gaussian integration of the \( \theta_i \) variables leads the effective low energy action (1). In this regime the renormalization of \( u \) and \( K_{zz'} \) can be obtained within the standard RG procedure (see, e.g., in Ref.\textsuperscript{25}). It consists of the repeated elimination of the high momenta harmonics from some cutoff \( \Lambda \) to \( \Lambda/(1 + s) \) with \( s \to 0 \) and further rescaling of the unit of length (and time) by the factor \((1 + s) \). More specifically, the variables \( \phi_z \)

\[ \phi_z = \phi_z^{(<)} + \phi_z^{(>)} \quad (B2) \]

are separated into the low energy \( \phi_z^{(<)} \) and the high energy \( \phi_z^{(>)} \) parts, where the latter is to be integrated out from the partition function \( Z = \int D\phi D\theta \exp(-H_Q) \). This (with the rescaling) will generate the effective action \( H_Q^{(<)} \) which depends on the low energy harmonics only and the renormalized values of \( K_{zz'} \) and \( u \). To the lowest order the resulting RG equation for \( u \) can be represented as

\[ \frac{du}{dl} = \left( 2 - \frac{1}{2s} (\langle \phi_1^{(>} - \phi_2^{(>)} \rangle^2)_s \right) u, \quad (B3) \]

where the averaging \( \langle \ldots \rangle_s \) is performed over the harmonics in the narrow shell \( \Lambda < |q| < \Lambda/(1 + s) \) in the gaussian part of the action (1).

The renormalization of \( K_{zz'} \) is determined by the terms \( \sim u^2 \) in the lowest order. The resulting equations are

\[ \frac{dK_{11}}{dl} = \frac{Cu^2}{s} \left( \langle \phi_1^{(>} - \phi_2^{(>)} \rangle^2 \rangle_s \right), \quad (B4) \]
\[ \frac{dK_{22}}{dl} = \frac{Cu^2}{s} \left( \langle \phi_1^{(>} - \phi_2^{(>)} \rangle^2 \rangle_s \right), \quad (B5) \]
where \( C > 0 \) stands for a constant which depends on the cutoff procedure. As usual, this constant can be absorbed into the definition of \( u \), and we choose it as \( C = 1 \).

Using the notations \( K_{22} = K_{11}Y \), \( K_{12} = K_{11}X \) in the gaussian integral \( \langle (\phi_1^{(>)} - \phi_2^{(>)} )^2 \rangle_s \), the above equations become

\[
\frac{du}{dl} = \left( 2 - \frac{1}{4\pi K_{11}} \frac{1 + Y + 2X}{Y - X^2} \right) u, \tag{B7}
\]

\[
\frac{dK_{11}}{dl} = \frac{u^2}{2\pi K_{11}} \frac{1 + Y + 2X}{Y - X^2}, \tag{B8}
\]

\[
\frac{d(K_{11}Y)}{dl} = \frac{u^2}{2\pi K_{11}} \frac{1 + Y + 2X}{Y - X^2}, \tag{B9}
\]

and

\[
\frac{d(K_{11}X)}{dl} = -\frac{u^2}{2\pi K_{11}} \frac{1 + Y + 2X}{Y - X^2}. \tag{B10}
\]

Eqs.(B8,B9,B10) imply \( K_{11} = C_1/(1 + X) \), \( Y = 1 + C_2/(1 + X) \), where \( C_1 > 0, C_2 > 0 \) are constants of integration. Finally, Eqs.(B7,B8) can be expressed in terms of two variables \( u \) and \( K_\theta \equiv K_{11} - C_1/(2 + C_2) \) as

\[
\frac{du}{dl} = \left( 2 - \frac{1}{4\pi K_\theta} \right) u, \tag{B11}
\]

\[
\frac{dV_{q_1,q_2}}{dl} = [2 - \pi K_{ab}q_0q_0] V_{q_1,q_2}, \quad \frac{dV_{q_1,q_2}}{dl} = [2 - \pi K_{11} ((q_1 + Xq_2)^2 + (Y - X^2)q_2^2)] V_{q_1,q_2}. \tag{B15}
\]

As can be immediately seen, this equation features the critical point of the transition into the composite phase described by Eq.(8) (where \( T \equiv 1/K_{11} \)).

The renormalization of the \( K \)-matrix in the second order in the amplitude \( V_{q_1,q_2} \) is given by

\[
\frac{d(K^{-1})_{ab}}{dl} = q_0q_0V_{q_1,q_2}^2 K_{rs}q_0q_0. \tag{B16}
\]

Eqs.(B16) have two integrals. Using the notations \( (K^{-1})_{22} = \hat{Y}(K^{-1})_{11} \) and \( (K^{-1})_{12} = \hat{X}(K^{-1})_{11} \) (which are related to the previously introduced variables as \( \hat{Y} = Y/1 + X \) and \( \hat{X} = -X/Y \)), we find \( Y = q_0^2q_1^{-2} - B_1q_1^{-2}/(K^{-1})_{11} \) and \( X = q_0^2q_1^{-2} - B_2q_1q_2/(K^{-1})_{11} \), where \( B_1, B_2 \) are constants of integration. Using these relations in Eqs.(B15,B16), we find

\[
\frac{dV_{q_1,q_2}}{dl} = \left[ 2 - \frac{\pi q_0^2}{K_\theta} \right] V_{q_1,q_2}. \tag{B17}
\]

and

\[
\frac{dK_\theta}{dl} = \frac{u^2}{2\pi K_\theta}. \tag{B12}
\]

These are the standard RG equations which are fully integrable. The SP phase corresponds to \( K_\theta < 1/(8\pi) \) which is represented by Eq.(6) (with \( T \equiv 1/K_{11} \)). In this phase \( u \) flows to zero and the Luttinger matrix \( K_{zz} \) remains essentially scale independent.

The SP implies that Luttinger liquids in both wires remain gapless. Thus, the condition \( K_\theta < 1/(8\pi) \) should be consistent with the requirement that all the harmonics \( V_{q_1,q_2} \) are irrelevant. In the regime \( K_\theta < 1/(8\pi) \) (where \( u \) is irrelevant), Eq.(B1) can be expressed in terms of the "angles" \( \theta_a \) as

\[
H_\theta = \int_0^L dx \int_0^\beta d\tau \left[ \frac{1}{2\pi^2} (K^{-1})_{ab} \nabla \theta_a \nabla \theta_b \right.
\]

\[
- \sum_{q_1,q_2} V_{q_1,q_2} \cos(2(q_1\theta_1 + q_2\theta_2)). \tag{B13}
\]

The RG equation for the most relevant harmonic can be obtained along the same lines as discussed above. It is

\[
\frac{dV_{q_1,q_2}}{dl} = \left( 2 - \frac{1}{2\pi^2} \langle (q_1\theta_1^{(>)} + q_2\theta_2^{(>)} )^2 \rangle_s \right) V_{q_1,q_2}. \tag{B14}
\]

Evaluation of the correlator \( \langle (q_1\theta_1^{(>)} + q_2\theta_2^{(>)} )^2 \rangle_s \) within the gaussian part of the action (B13) gives

\[
\frac{dV_{q_1,q_2}}{dl} = q_0^2 V_{q_1,q_2}. \tag{B15}
\]

where the notation \( K_\theta = (K^{-1})_{11} - q_0^2 B_2/(2B_2 - B_1) \) is introduced.

Eqs.(B17,B18) are also the standard RG equations. For \( K_\theta < \pi q_0^2/2 \) the most "dangerous" harmonic \( V_{q_1,q_2} \) is irrelevant, that is, the system remains in the superfluid regime with two gapless modes (provided the SP phase exists).

The above analysis implies that the SP phase exists if two conditions hold: \( K_\theta < \pi q_0^2/2 \) and \( K_\theta < 1/(8\pi) \). These conditions are represented by Eqs.(8,6), respectively. As further analysis in the main text, Sec.II.A, has shown Eq.(9) is one of the solutions satisfying both inequalities.
1. RG for arbitrary $N_z$

The equation for $u_r$ in the case of a stack of bilayers, as discussed in Sec.III, can be obtained along the same line as for the bilayer (see also in Ref.\textsuperscript{38} in the context of the bosonic composite phases in a layered system):

$$\frac{du_r}{d\ln l} = (2 - \frac{1}{2s})((\phi_{z+1} - \phi_z)^2)_{x}u_r. \quad (B19)$$

We note that, due to the PBC along $z$-direction, the mean $(\langle (\phi_{z+1} - \phi_z)^2 \rangle)$ does not depend on $z$. Using discrete Fourier representation along $z$ direction with doubled unit cell containing two layers (the odd and the even) with two sorts of phases $\phi_z = \phi_1(z)$ and $\phi_z = \phi_2(z)$ along odd and even layers, respectively, the part $H_z$ in Eq.(23) can be diagonalized and the correlator in Eq.(B19) found. This gives Eq.(B19) rewritten as

$$\frac{du_r}{d\ln l} = 2 \left( 1 - \frac{T}{T_d} \right) u_r, \quad (B20)$$

where $T_d$ is given by Eq.(26).

The flow equations for the matrix $K$ can also be found along the same line as described in Ref.\textsuperscript{38}. In this case the matrix $K_{zz'}$, which now depends on the wavevector $q_z$, remains essentially unrenormalized as long as $T > T_d$.

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