This is the accepted manuscript made available via CHORUS. The article has been published as:

Symmetry enrichment in three-dimensional topological phases

Shang-Qiang Ning, Zheng-Xin Liu, and Peng Ye

Phys. Rev. B **94**, 245120 — Published 15 December 2016

DOI: 10.1103/PhysRevB.94.245120

Symmetry Enrichment in Three-Dimensional Topological Phases

Shang-Qiang Ning,[1] Zheng-Xin Liu,[2] and Peng Ye[3,4,*]

[1] *Institute for Advanced Study, Tsinghua University, Beijing, China, 100084*
[2] *Department of Physics, Renmin University of China, Beijing, China, 100872*
[3] *Department of Physics, University of Illinois at Urbana-Champaign, IL 61801, USA*
[4] *Institute for Condensed Matter Theory, University of Illinois at Urbana-Champaign, IL 61801, USA*

While two-dimensional symmetry-enriched topological phases (SETs) have been studied intensively and systematically, three-dimensional ones are still open issues. We propose an algorithmic approach of imposing global symmetry G_s on gauge theories (denoted by GT) with gauge group G_g. The resulting symmetric gauge theories are dubbed "symmetry-enriched gauge theories" (SEG), which may be served as low-energy effective theories of three-dimensional symmetric topological quantum spin liquids. We focus on SEGs with gauge group $G_g = \mathbb{Z}_{N_1} \times \mathbb{Z}_{N_2} \times \cdots$ and on-site unitary symmetry group $G_s = \mathbb{Z}_{K_1} \times \mathbb{Z}_{K_2} \times \cdots$ or $G_s = \mathrm{U}(1) \times \mathbb{Z}_{K_1} \times \cdots$. Each SEG$(G_g, G_s)$ is described in the path integral formalism associated with certain symmetry assignment. From the path-integral expression, we propose how to physically diagnose the ground state properties (i.e., SET orders) of SEGs in experiments of charge-loop braidings (patterns of symmetry fractionalization) and the *mixed* multi-loop braidings among deconfined loop excitations and confined symmetry fluxes. From these symmetry-enriched properties, one can obtain the map from SEGs to SETs. By giving full dynamics to background gauge fields, SEGs may be eventually promoted to a set of new gauge theories (denoted by GT*). Based on their gauge groups, GT*s may be further regrouped into different classes each of which is labeled by a gauge group G_g^*. Finally, a web of gauge theories involving GT, SEG, SET and GT* is achieved. We demonstrate the above symmetry-enrichment physics and the web of gauge theories through many concrete examples.

I. INTRODUCTION

Recently, the field of gapped phases with symmetry has been drawing a lot of attentions in condensed matter physics. There are two kinds of symmetric gapped phases: symmetry-protected topological phases (SPT) and symmetry-enriched topological phases (SET). SPT phases are short-range entangled [1] with a global symmetry and have been studied intensively in strongly-correlated bosonic systems [1–35]. Much progress has also been made in two-dimensional (2D) SETs [36–47], which are partially driven by tremendous efforts in quantum spin liquids (QSL) [36, 48] that respect a certain global symmetry (e.g., spatial reflection, time-reversal, Ising $\mathbb{Z}_2$, U(1) and SU(2) spin rotations, etc.). In contrast to SPTs, SETs are long-range entangled [1] and support emergent excitations, such as anyons in 2D systems. Furthermore, quantum numbers carried by emergent excitations may be fractionalized. Experimentally, it is of interest to detect patterns of such symmetry fractionalization, which may help us characterize QSLs [48]. In addition to the usual global symmetry, there are also SETs enriched by a new kind of symmetry dubbed "topological (anyonic)" symmetry [43, 49–60]. This symmetry denotes an automorphism of the topological data (braiding statistics, quantum dimensions, etc.). A typical example is that $\mathbb{Z}_2$ topological order in two dimensions is invariant under e-m exchange operation, namely, an electric-magnetic duality in discrete gauge theories [49, 50].

Despite much success in 2D SETs, three-dimensional (3D) SET physics, especially the underlying general framework, is still poorly understood so far, partially due to the presence of spatially extended loop excitations [61]. In physical literatures, some attempts have been made, including 3D U(1) QSLs and $\mathbb{Z}_2$ QSLs with symmetry, e.g., in Ref. [62–65]. Field theories of 3D SETs with either time-reversal or 180° spin rotation about y-axis were studied where the dynamical axion electromagnetic action term is considered [18, 66]. The boundary anomaly of some 3D SETs was viewed as 2D anomalous SETs with anyonic symmetry [67]. In Ref. [68, 69], a dimension reduction point of view was proposed to demonstrate how symmetry is fractionalized on loop excitations. In Ref. [70], the notion of "2D anyonic symmetry" was generalized to 3D "charge-loop excitation symmetry" (Charles) which is a permutation operation among particle excitations and among loop excitations. As typical examples of 3D SETs with U(1) and time-reversal, fractional topological insulators were constructed via a parton construction with gauge confinement [70].

In this paper, we study 3D SETs with Abelian topological orders [71] that are encoded by deconfined discrete Abelian gauge theories [72]. We focus on discrete Abelian gauge group $G_g = \mathbb{Z}_{N_1} \times \mathbb{Z}_{N_2} \times \cdots$ and on-site unitary Abelian symmetry group $G_s = \mathbb{Z}_{K_1} \times \mathbb{Z}_{K_2} \times \cdots$ or $G_s = \mathrm{U}(1) \times \mathbb{Z}_{K_1} \times \cdots$. Physically, these 3D SETs can be viewed as 3D gapped QSLs that are enriched by unbroken on-site symmetry G_s. Given a gauge group G_g, there are usually many topologically distinct gauge theories (denoted by GT) including one untwisted and several twisted ones [23, 73], as shown in Fig. 1. After imposing global symmetry group, the resulting gauge

* Corresponding author: yphysics@illinois.edu

field theory is called "symmetry-enriched gauge theory" (SEG). Quantitatively, an SEG is defined through two key ingredients:

1. an action that consists of topological terms (of one-form or two-form Abelian gauge fields) only;
2. symmetry assignment via a specific minimal coupling to background gauge fields (denoted by $\{A^i\}$ with $i = 1, 2, \cdots$, where A^i externally imposes symmetry fluxes in $\mathbb{Z}_{K_i}$ symmetry subgroup).

We also stress that an anomaly-free SEG must *simultaneously* satisfy the following two stringent conditions [74]:

1. global symmetry is preserved;
2. gauge invariance is guaranteed on a closed spacetime manifold.

We use the notation $\mathsf{SEG}(G_g, G_s)$ to denote such an SEG. Then we try to provide answers to the following questions:

1. What is the path-integral formalism of an SEG? And what is the "parent" GT of each SEG?
2. What is the relation between SEG and SET? How can we probe symmetry-enriched properties in experiments?
3. What is the resulting new gauge theory (denoted by GT^*) after giving full dynamics [75] to $\{A^i\}$?

To answer the first question is nothing but to look for anomaly-free SEGs that meet the above definition and conditions. Following the 5-step general procedure (Sec. II C), the path-integral formalism of each SEG can be constructed, which is efficient for the practical purpose. Each SEG can be identified as a descendant of some GT (i.e., "parent"). Many concrete examples, including the simplest case $\mathsf{SEG}(\mathbb{Z}_2, \mathbb{Z}_2)$, are calculated explicitly in this paper. The method we will provide is doable for more general cases, some of which are collected in Appendix.

In the second question, a complete description of an SET order requires the information of both topological orders and symmetry enrichment. In this sense, the total number of SEGs is generically larger than that of distinct SET orders. For example, two anomaly-free SEGs, may possibly give rise to the same SET order. If two SEGs have the same topological order, a practical way to probe symmetry enrichment is to insert symmetry fluxes into the 3D bulk and perform Aharonov-Bohm experiments between symmetry fluxes (flux loop formed by A^i) and bosons that are charged in the symmetry group. In addition, one should also perform the *mixed* version of three-loop braiding experiment [26, 76] among symmetry fluxes and gauge fluxes (i.e., loop excitations). Through these thought experiments, one may find the relations between different SEGs. If two SEGs share the same bulk topological order data as well the same symmetry-enriched properties, they belong to the same SET ordered phase. Otherwise, they belong to two different SET phases (see Fig. 1).

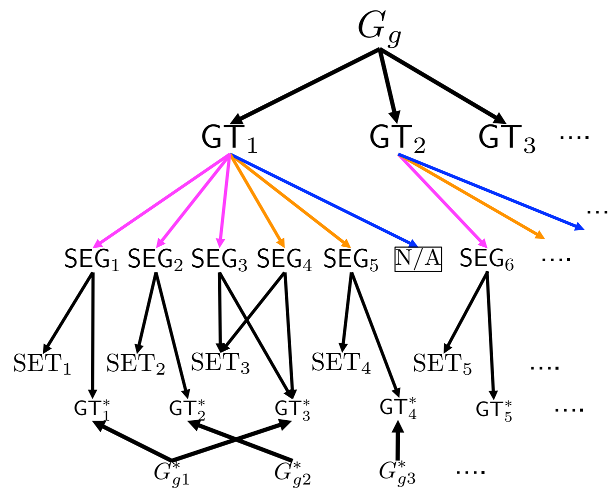


FIG. 1. (Color online) Schematic representation of a web of gauge theories with global symmetry. We start with a discrete gauge group G_g that generates several topologically distinct gauge theories (GTs) one of which is untwisted. We then assign symmetry charge of symmetry group G_s to topological currents of gauge theories through coupling to background gauge fields. There are usually many different ways of symmetry assignment, each of which is represented by a colored arrow. Within each specific symmetry assignment, we obtain many SEGs. For example, SEG_1, SEG_2, and SEG_3 belong to the same symmetry assignment (marked by magenta arrows) in GT_1. It is generically possible that some of symmetry assignment do not provide SEG descendants for GT, which we mark as "N/A". To identify SETs, we need to further study ground state properties of SEGs. We may externally insert symmetry fluxes into the system and perform Aharonov-Bohm experiments and the *mixed* version of three-loop braiding experiments. Two SEGs (e.g., SEG_3 and SEG_4) may possibly describe the same SET phase. Further, by giving full dynamics to the background gauge fields, the resulting new gauge theories (denoted by GT^*) are generated. Since basis transformations are allowed, there should be in general many-to-one correspondence between SEGs and GT^*s. Finally, all GT^*s may be regrouped via identifying their gauge groups (denoted by G_g^*).

For the third question, we note that in the action of an SEG, $\{A^I\}$ is a set of non-dynamical background gauge fields. Symmetry fluxes formed by them are confined loop objects that are externally imposed into the bulk. These loop objects are fundamentally different from the gauge fluxes that are deconfined bulk loop excitations. Therefore, the usual basis transformations (mathematically represented by unimodular matrices of a general linear group) on gauge field variables are strictly prohibited [8] *if* the transformations mix gauge fluxes and symmetry fluxes. However, if we give full dynamics to $\{A^I\}$ [75], then, the action actually represents a new gauge theory (denoted by GT^*) and does not describe a SEG any more.

In GT^*s, symmetry fluxes are legitimate deconfined bulk loop excitations and arbitrary basis transformations are allowed. As a result, it is possible that the actions of two SEGs may be rigorously mapped to each other via basis transformations, both of which lead to the same GT^*. This set of gauge theories "GT^*_1, GT^*_2, $\cdots$" may be further regrouped by identifying their gauge groups (denoted by $G^*_{g1}, G^*_{g2}, \cdots$). Finally, a web of gauge theories is obtained, as schematically shown in Fig. 1.

The remainder of the paper is organized as follows. Sec. II is devoted to general discussions on GTs, topological interactions and global symmetry. Especially, in Sec. II C, the 5-step general procedure is introduced in detail. Some calculation details in Sec. II D,II E,II F will be useful for quantitatively understanding the remaining sections, especially, Sec. III. For readers who are only interested in the final results, these details may be either skipped or gone through quickly. In Sec. III, many simple examples are studied in details, including $\mathsf{SEG}(\mathbb{Z}_2,\mathbb{Z}_K)$, $\mathsf{SEG}(\mathbb{Z}_2\times\mathbb{Z}_2,\mathbb{Z}_2)$, and $\mathsf{SEG}(\mathbb{Z}_2\times\mathbb{Z}_2,\mathrm{U}(1))$. In Sec. IV, physical characterization of SEGs is studied, including symmetry fractionalization and mixed three-loop braiding statistics among gauge fluxes and symmetry fluxes. In this way, we may achieve the map from SEG to SET as schematically shown in Fig. 1. Simple examples are given, including $\mathsf{SEG}(\mathbb{Z}_2,\mathbb{Z}_K)$ with $K\in\mathbb{Z}_{\text{even}}$ and $K\in\mathbb{Z}_{\text{odd}}$. In Sec. V, full dynamics is given to the background gauge field, which promotes SEGs to GT^*s. Again, the discussions are followed by some simple examples including $\mathsf{SEG}(\mathbb{Z}_2,\mathbb{Z}_2)$, $\mathsf{SEG}(\mathbb{Z}_2,\mathbb{Z}_3)$ and $\mathsf{SEG}(\mathbb{Z}_2\times\mathbb{Z}_2,\mathbb{Z}_2)$. Summary and outlook are made in Sec. VI. More technical details and concrete examples are collected in Appendix.

II. GAUGE THEORIES, TOPOLOGICAL INTERACTIONS, AND GLOBAL SYMMETRY

A. Inter-"layer" topological interactions and addition of "trivial" layers

In the continuum limit, gauge theories with discrete gauge groups can be written in terms of the following multi-component topological BF term [77]:

$$S = i\sum_{I,J}\frac{\Lambda^{IJ}}{2\pi}\int_{\mathcal{M}^4} b^I\wedge da^J\,, \tag{1}$$

where $\{b^I\}$ and $\{a^I\}$ are two sets of 2-form and 1-form U(1) gauge fields respectively. $I=1,2,\cdots,n$. Λ^{IJ} is some $n\times n$ integer matrix, which may not be symmetric but the determinant of Λ must be nonzero: $\mathrm{Det}\Lambda\neq 0$ [78]. In comparison to Horowitz's action term [77], here we do not consider $b^I\wedge b^J$. $\mathcal{M}^4$ is the 4D closed spacetime (with imaginary time) manifold where our topological phases are defined. In the following, the notation $\mathcal{M}^4$ will be neglected from the action for the sake of simplicity.

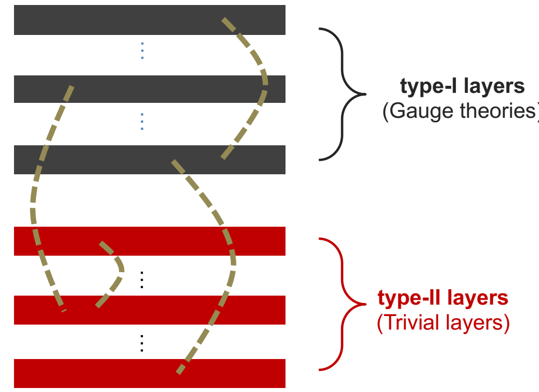


FIG. 2. (Color online) A schematic representation of "layers" (Sec. II A) and the general procedure (Sec. II C). Each "layer" denotes a 3D system. It should be noted that all layers are stacked together in the *same* 3D spatial region although they are not so in this figure. GT before imposing symmetry resides in type-I layers. Type-II layers are described by level-1 BF terms before imposing symmetry. By "trivial", we mean that these layers do not carry gauge groups. The dashed curves represent topological interactions between layers. Actually, three-layer and four-layer topological interactions should also be considered.

There are two *independent* general linear transformations represented by two unimodular matrices $W,\Omega\in\mathbb{GL}(n,\mathbb{Z})$ that "rotate" loop lattice and charge lattice respectively. Therefore, Λ can always be sent into its *canonical form* via:

$$W\Lambda\Omega^T = \mathrm{diag}(N_1,N_2,\cdots,N_I,\cdots,N_n)\,, \tag{2}$$

where $\{N_I\}$ are a set of *positive* integers. The superscript "T" denotes "transpose". It is in sharp contrast to the multi-component Chern-Simons theory [71] where $W=\Omega$ and the above diagonalized basis usually doesn't exist. In the remaining parts of this paper, we work in this new basis unless otherwise specified. In this new basis, each I labels a "layer system" as schematically shown in the "type-I layers" in Fig. 2 (N.B., the word "layer" actually denotes a 3D spatial region). N_I is the *level* of the BF term in the I-th layer.

$\{b^I\}$ and $\{a^I\}$, as two sets of gauge fields, are subject to the following Dirac quantization conditions:

$$\frac{1}{2\pi}\int_{\mathcal{M}^3} db^I\in\mathbb{Z}\,, \tag{3}$$

$$\frac{1}{2\pi}\int_{\mathcal{M}^2} da^I\in\mathbb{Z}\,, \tag{4}$$

where $\mathcal{M}^3$ and $\mathcal{M}^2$ denote 3D and 2D closed manifolds embedded in $\mathcal{M}^4$ respectively. These two equations will play important roles in the following discussions.

The BF term in the canonical form is a field theory of *untwisted* $G_g=\mathbb{Z}_{N_1}\times\mathbb{Z}_{N_2}\times\cdots\times\mathbb{Z}_{N_n}$ gauge theory where layers are decoupled to each other. However, there are topological interactions that can couple them together:

$$S = i\sum_I \frac{N_I}{2\pi}\int b^I\wedge da^I + i\sum_{IJK}\frac{q^{IJK}}{4\pi^2}\int a^I\wedge a^J\wedge da^K + i\sum_{IJKL}\frac{t^{IJKL}}{8\pi^3}\int a^I\wedge a^J\wedge a^K\wedge a^L\,, \tag{5}$$

where $\{q^{IJK}\}$ and $\{t^{IJKL}\}$ are two sets of coefficients. These newly introduced action terms are topological since their expressions are wedge products of differential forms. Recently a lot of progress has been made based on these topological terms in gauge theories as well as SPT phases [13, 23, 79–81]. The presence of interlayer topological interactions leads to *twisted* G_g gauge theories. Since these new topological terms are explicitly not gauge invariant (even in a closed manifold) alone, the definitions of usual gauge transformations on $\{b^I\}$ must be properly modified [to appear in Eq. (10)]. To be a legitimate GT action, $\{q^{IJK}\}$ and $\{t^{IJKL}\}$ are expected to be quantized and compact (i.e., periodic), which eventually leads to finite number of distinct GTs *before* global symmetry is imposed. All of them are classified by the fourth group cohomology with U(1) coefficient: $\mathcal{H}^4(\mathbb{Z}_{N_1}\times\mathbb{Z}_{N_2}\cdots,\mathrm{U}(1)) = \prod_{I<J}(\mathbb{Z}_{N_{IJ}})^2\times\prod_{I<J<K}(\mathbb{Z}_{N_{IJK}})^2\times\prod_{I<J<K<L}\mathbb{Z}_{N_{IJKL}}$, where $N_{IJ,\dots}$ is the greatest common divisor of $N_I, N_J,\cdots$. Technical details are shown in Sec. II D.

In addition, one may always add arbitrary number of "trivial layers" into the action S in Eq. (5):

$$S\to S + i\frac{1}{2\pi}\int b^{n+1}\wedge da^{n+1} + i\frac{1}{2\pi}\int b^{n+2}\wedge da^{n+2} + \cdots\,. \tag{6}$$

These trivial layers do not introduce additional gauge structures. However, as we will see, adding trivial layers will be very useful and sometimes necessary when global symmetry G_s is imposed.

B. Symmetry assignment

Now, let us consider how to impose global symmetry group $G_s = \mathbb{Z}_{K_1}\times\mathbb{Z}_{K_2}\times\cdots\times\mathbb{Z}_{K_m}$. In topological quantum field theory, there is a 1-form topological current J^I for each I: $*J^I = \frac{1}{2\pi}db^I$, where $*$ denotes the Hodge dual operation. It is conserved automatically since $d^2=0$. The fact that the total particle number is integral is nicely guaranteed by Dirac quantization condition (3). Therefore, a natural definition of global symmetry is to enforce that the symmetry charge is carried by this topological current. This is the so-called hydrodynamical approach that was applied successfully in the fractional quantum Hall effect with the multi-component Chern-Simons theory description [71]. This is also a key step of the topological quantum field theory description of SPTs [13].

In order to identify global symmetry, a background gauge field A^i is turned on. Mathematically, a minimal coupling term between background gauge fields and topological currents is introduced into the action (6): $S_{sym.} = i\sum_I^n\sum_i^m L^{Ii}\int J^I\wedge *A^i$, where L^{Ii} is an $n\times m$ integer matrix. By noting that the total symmetry group $G_s=\mathbb{Z}_{K_1}\times\mathbb{Z}_{K_2}\times\cdots$, the background 1-form U(1) gauge field A^i is subject to the following constraints:

$$\frac{K_i}{2\pi}\int_{\mathcal{M}^1} A^i\in\mathbb{Z}\ \text{ for } \mathbb{Z}_{K_i}\text{ symmetry subgroup}\,, \tag{7}$$

where $\mathcal{M}^1$ denotes a closed spacetime loop. As mentioned in Sec. II A, trivial layers in Eq. (6) may be taken into consideration once symmetry is imposed. Therefore, the index I in $S_{sym.}$ is allowed to be larger than n. Once the topological current carries symmetry charge, a new set of stringent constraints on the coefficients $\{q^{IJK}\}$ and $\{t^{IJKL}\}$ will be imposed such that global symmetry is compatible with gauge invariance principle, the quantization and periodicity of $\{q^{IJK}\}$ and $\{t^{IJKL}\}$ may be changed dramatically after global symmetry is imposed. It means that, one GT may generate many distinct SEG descendants after symmetry is imposed, which manifestly shows patterns of symmetry enrichment (see Fig. 1). If symmetry is not imposed, those distinct SEGs become indistinguishable and reduce back to the same parent GT.

C. Summary of the 5-step general procedure

Based on the preparation done in Sec. II A and II B, in this part, we summarize the general procedure for obtaining SEGs and connecting them to their parent GTs. There are five main steps.

Step-1. Add trivial layers (i.e., type-II in Fig. 2). Mathematically, trivial layers are described by Eq. (6).

Step-2. Assign symmetry via the minimal coupling terms ($\sim J\wedge *A$). Symmetry assignment can be either made purely inside type-I or purely inside type-II or both [82].

Step-3. Add all possible topological interactions among layers via the topological terms with coefficients $\{q^{IJK}\}$ and $\{t^{IJKL}\}$ in Eq. (5) and the indices $I,J,K,\cdots$ are extended to all layers including trivial layers. In Fig. 2, only two-layer interactions (denoted by dashed lines) are drawn for simplicity. However, generic three-layer and four-layer interactions should also be taken into considerations.

Step-4. Consider all consistent conditions and determine the quantization and periodicity of coefficients of topological interactions. These consistent conditions are *(i)* Dirac quantization conditions; *(ii)* "small" gauge transformations; *(iii)* "large" gauge transformations; *(iv)* shift operation of coefficients that leads to coefficient periodicity; *(v)*. total symmetry charge for $\mathbb{Z}_{K_i}$ subgroup is conserved mod K_i. Once the above four steps are done, the path-integral expressions and symmetry assignment for SEGs are obtained. Definitions and quantitative studies of these consistent conditions will be provided in Sec. II D,II E,II F, and Appendix A.

Step-5. Regroup all SEGs obtained above into distinct GTs in Fig. 1. For example, in Fig. 1, $\mathsf{SEG}_{1,\cdots,5}$ are SEG descendants of GT_1, while, SEG_6 is a SEG descendant of GT_2. If gauge group is $G_g = \mathbb{Z}_N$ that will be calculated in Sec. III A, this step can be skipped for the reason that there is only one $\mathbb{Z}_N$ GT, i.e., the untwisted GT. If gauge group contains more than one $\mathbb{Z}_N$s, e.g., $G_g = \mathbb{Z}_{N_1}\times\mathbb{Z}_{N_2}$, usually gauge theories have twisted versions. Under the circumstances, the role of Step-5 becomes critical. We will discuss pertinent details in Sec. III B.

D. General calculation on $G_g = \mathbb{Z}_{N_1}\times\mathbb{Z}_{N_2}$ with no symmetry

In the following, we present some useful calculation details on gauge theories with $G_g = \mathbb{Z}_{N_1}\times\mathbb{Z}_{N_2}$ and demonstrate, especially, what the consistent conditions listed in Step-4 of Sec. II C are, at quantitative level. Several mathematical notations are introduced and will be frequently used in the remaining parts of this paper. All other calculation details are present in Appendix A.

Consider the following two-layer BF theories with inter-layer topological couplings in the form of "$aada$":

$$S = \sum_{I=1}^{2}\frac{iN_I}{2\pi}\int b^I\wedge da^I + i\frac{q}{4\pi^2}\int a^1\wedge a^2\wedge da^2 + i\frac{\bar{q}}{4\pi^2}\int a^2\wedge a^1\wedge da^1\,, \tag{8}$$

where $q\equiv q^{122}$ and $\bar{q}\equiv q^{211}$. Since $a^1a^2da^2$ and $a^2a^1da^1$ are linearly independent, we may study them separately. First consider $\bar{q}=0$. The action is invariant under the following gauge transformations parametrized by scalars $\{\chi^I\}$ and vectors $\{V^I\}$:

$$a^I \longrightarrow a^I + d\chi^I\,, \tag{9}$$

$$b^I \longrightarrow b^I + dV^I - \frac{q}{2\pi N^I}\epsilon^{IJ3}\chi^J\wedge da^2\,, \tag{10}$$

where $\epsilon^{123} = -\epsilon^{213} = 1$. It is clear that the usual gauge transformations of b^I [77] are modified through adding a q-dependent term in Eq. (10). As usual, the gauge parameters χ^I and V^I satisfy the following conditions:

$$\frac{1}{2\pi}\int_{\mathcal{M}^1} d\chi^I\in\mathbb{Z},\quad \frac{1}{2\pi}\int_{\mathcal{M}^2} dV^I\in\mathbb{Z}\,. \tag{11}$$

Once the integers on the r.h.s. are nonzero, the associated gauge transformations are said to be "large". Let us investigate the integral $\frac{1}{2\pi}\int_{\mathcal{M}^3} db^I$.

Under the above modified gauge transformations (10), the integral will be changed by the amount below (for $I=1$, $\mathcal{M}^3=\mathcal{M}^1\times\mathcal{M}^2$ is considered):

$$\begin{aligned}\frac{1}{2\pi}\int_{\mathcal{M}^3} db^1 \longrightarrow &\frac{1}{2\pi}\int_{\mathcal{M}^3} db^1 - \frac{q}{4\pi^2N_1}\int_{S^1}d\chi^2\int_{M^2}da^2\\ =&\frac{1}{2\pi}\int_{\mathcal{M}^3} db^1 - \frac{q}{4\pi^2N_1}\times 2\pi\ell\times 2\pi\ell'\,,\end{aligned} \tag{12}$$

where $\ell\,,\,\ell'\in\mathbb{Z}$, and, the Dirac quantization condition (4) and gauge parameter condition (11) are applied. In order to be consistent with the Dirac quantization condition (3), the change amount must be integral, namely, q must be divisible by N_1. Similarly, q is also divisible by N_2 due to:

$$\begin{aligned}\frac{1}{2\pi}\int_{\mathcal{M}^3} db^2 \longrightarrow &\frac{1}{2\pi}\int_{\mathcal{M}^3} db^2 + \frac{q}{4\pi^2N_2}\int_{S^1}d\chi^1\int_{M^2}da^2\\ =&\frac{1}{2\pi}\int_{\mathcal{M}^3} db^2 + \frac{q}{4\pi^2N_2}\times 2\pi\ell''\times 2\pi\ell''',\end{aligned} \tag{13}$$

where $\ell''\,,\,\ell'''\in\mathbb{Z}$. Hence, $q=\frac{kN_1N_2}{N_{12}}, k\in\mathbb{Z}$, where the symbol "$N_{12}$" denotes the greatest common divisor of N_1 and N_2.

Below, we will show that k has a periodicity N_{12} and thereby q is compactified: $q\sim q+N_1N_2$. Let us consider the following redundancy due to shift operations:

$$\frac{1}{2\pi}\int db^1 \longrightarrow \frac{1}{2\pi}\int db^1 - \frac{N_2\tilde{K}_1}{4\pi^2N_{12}}\int a^2\wedge da^2\,, \tag{14}$$

$$\frac{1}{2\pi}\int db^2 \longrightarrow \frac{1}{2\pi}\int db^2 + \frac{N_1\tilde{K}_2}{4\pi^2N_{12}}\int a^1\wedge da^2\,, \tag{15}$$

$$k\longrightarrow k+\tilde{K}_1+\tilde{K}_2\,. \tag{16}$$

Under the above shift operation, the total action (8) is invariant. Again, in order to be consistent with Dirac quantization (3), the change amount of the integral $\frac{1}{2\pi}\int_{\mathcal{M}^3} db^I$ should be integral, namely:

$$\frac{N_2\tilde{K}_1}{4\pi^2N_{12}}\int_{\mathcal{M}^3} a^2\wedge da^2\in\mathbb{Z}\,, \tag{17}$$

$$\frac{N_1\tilde{K}_2}{4\pi^2N_{12}}\int_{\mathcal{M}^3} a^1\wedge da^2\in\mathbb{Z}\,. \tag{18}$$

We may apply the Dirac quantization condition (4) and the quantized Wilson loop $\frac{N_I}{2\pi}\int_{\mathcal{M}^1}a^I\in\mathbb{Z}$ that is obtained via equations of motion of b^I. As a result, two constraints are achieved: $\tilde{K}_1/N_{12}\in\mathbb{Z}$, $\tilde{K}_2/N_{12}\in\mathbb{Z}$. By using Bezout's lemma, the minimal periodicity of k is given by the greatest common divisor (GCD) of N_{12} and N_{12}, which is still N_{12}. As a result, we obtain the conditions on q *if* symmetry is not taken into consideration.

$$q = k\frac{N_1N_2}{N_{12}} \text{ mod } N_1N_2\,,\quad k\in\mathbb{Z}_{N_{12}}. \tag{19}$$

Similarly, for $\frac{\bar{q}}{4\pi^2}a^2\wedge a^1\wedge da^1$ term, we also have the same quantization and the same periodicity:

$$\bar{q} = k\frac{N_1N_2}{N_{12}} \text{ mod } N_1N_2\,,\quad k\in\mathbb{Z}_{N_{12}}. \tag{20}$$

In conclusion, we have $(\mathbb{Z}_{N_{12}})^2$ different kinds of gauge theories with $G_g = \mathbb{Z}_{N_1} \times \mathbb{Z}_{N_2}$.

E. General calculation on $G_g = \mathbb{Z}_{N_1} \times \mathbb{Z}_{N_2}$ with $G_s = \mathbb{Z}_{K_1} \times \mathbb{Z}_{K_2}$-(I)

To impose the symmetry, we add the following coupling term into S in Eq. (8) (again, we consider $\bar{q} = 0$ only):

$$\sum_i^2 \frac{i}{2\pi} \int A^i \wedge db^i \,, \tag{21}$$

where A^i is subject to the constraints in Eq (7). This coupling term simply means that the first layer carries $\mathbb{Z}_{K_1}$ symmetry while the second layer carries $\mathbb{Z}_{K_2}$ symmetry. The total symmetry group $G_s = \mathbb{Z}_{K_1} \times \mathbb{Z}_{K_2}$.

Our goal is to determine all legitimate values of q in the presence of global symmetry. And we expect that the period of q is in general larger than the original gauge theory with no symmetry, which leads to a set of SEGs. The key observation is that the change amounts of the integral $\frac{1}{2\pi}\int_{\mathcal{M}^3} db^I$ in both gauge transformations and shift operations should not only be integral [in order to be consistent with the Dirac quantization condition (3)] but also be multiple of K_i such that the coupling term (21) is gauge invariant modular 2π. Physically, it can be understood via the definition of the integral. This integral is nothing but the total symmetry charge of the associated symmetry group. Since the total symmetry charge of $\mathbb{Z}_{K_i}$ is allowed to be changed by K_i while still respecting symmetry. This is a peculiar feature of cyclic symmetry group, compared to U(1) symmetry.

More quantitatively, with symmetry taken into account, from Eqs. (12, 13), we may obtain the quantization of q: $q = \frac{kN_1N_2K_1K_2}{\mathrm{GCD}(N_1K_1,N_2K_2)}$ with $k \in \mathbb{Z}$ such that the change amounts are multiple of K_i. Then, with these new quantized values, the shift operations (14,15) are changed to:

$$\frac{1}{2\pi}\int db^1 \longrightarrow \frac{1}{2\pi}\int db^1 - \frac{\tilde{K}_1 N_2 K_1 K_2 \int a^2 \wedge da^2}{4\pi^2 \mathrm{GCD}(N_1K_1, N_2K_2)} \,, \tag{22}$$

$$\frac{1}{2\pi}\int db^2 \longrightarrow \frac{1}{2\pi}\int db^2 + \frac{\tilde{K}_2 N_1 K_1 K_2 \int a^1 \wedge da^2}{4\pi^2 \mathrm{GCD}(N_1K_1, N_2K_2)} \,. \tag{23}$$

The change amounts should be quantized at K^1 in Eq. (22) and K^2 in Eq. (23), respectively, such that symmetry is kept. We may apply the Dirac quantization condition (4) and the quantized Wilson loop $\frac{N_I K_I}{2\pi}\int_{\mathcal{M}^1} a^I \in \mathbb{Z}$ that is obtained via equations of motion of b^I in the presence of A^I background. As a result, two necessary and sufficient constraints are achieved: $\frac{\tilde{K}_1}{\mathrm{GCD}(N_1K_1,N_2K_2)} \in \mathbb{Z}$, $\frac{\tilde{K}_2}{\mathrm{GCD}(N_1K_1,N_2K_2)} \in \mathbb{Z}$. By using Bezout's lemma, the minimal periodicity of k is given by GCD of $\mathrm{GCD}(N_1K_1, N_2K_2)$ and $\mathrm{GCD}(N_1K_1, N_2K_2)$, which is still $\mathrm{GCD}(N_1K_1, N_2K_2)$. Therefore, once symmetry is imposed, q is changed from Eq. (19) to:

$$\begin{aligned} q =& k\frac{N_1N_2K_1K_2}{\mathrm{GCD}(N_1K_1, N_2K_2)} \bmod N_1N_2K_1K_2 \,, \\ &\text{with } k \in \mathbb{Z}_{\mathrm{GCD}(N_1K_1,N_2K_2)} \end{aligned} \tag{24}$$

which gives $\mathrm{GCD}(N_1K_1, N_2K_2)$ SEGs. In other words, the allowed values of q are enriched by symmetry. For $\bar{q}$ term, the conditions are completely the same as q, which leads to another $\mathrm{GCD}(N_1K_1, N_2K_2)$ SEGs.

$$\begin{aligned} \bar{q} =& k\frac{N_1N_2K_1K_2}{\mathrm{GCD}(N_1K_1, N_2K_2)} \bmod N_1N_2K_1K_2 \,, \\ &\text{with } k \in \mathbb{Z}_{\mathrm{GCD}(N_1K_1,N_2K_2)} \,. \end{aligned} \tag{25}$$

In short, before imposing symmetry, according to Eqs. (19,20), there are $(N_{12})^2$ distinct GTs with gauge group $G_g = \mathbb{Z}_{N_1} \times \mathbb{Z}_{N_2}$. After imposing symmetry group $G_s = \mathbb{Z}_{K_1} \times \mathbb{Z}_{K_2}$, according to Eqs. (24,25), there are $[\mathrm{GCD}(N_1K_1, N_2K_2)]^2$ distinct SEGs *if* the symmetry assignment is given by Eq. (21). Likewise, one can consider that $\mathbb{Z}_{K_1}$ and $\mathbb{Z}_{K_2}$ symmetry charges are carried by the second layer and the first layer respectively, i.e., Eq. (21) is changed to:

$$\frac{i}{2\pi}\int (A^1 \wedge db^2 + A^2 \wedge db^1) \,. \tag{26}$$

Then, there will be $[\mathrm{GCD}(N_1K_2, N_2K_1)]^2$ new SEGs.

F. General calculation on $G_g = \mathbb{Z}_{N_1} \times \mathbb{Z}_{N_2}$ with $G_s = \mathbb{Z}_{K_1} \times \mathbb{Z}_{K_2}$-(II)

In the following, we alter the definition of symmetry assignment and still consider $a^1a^2da^2$ first. The coupling term in Eq. (21) is now changed to:

$$\frac{i}{2\pi}\int (A^1 + A^2) \wedge db^1 \tag{27}$$

which means that both $\mathbb{Z}_{N_1}$ and $\mathbb{Z}_{N_2}$ symmetry charges are carried by the first layer. We will show that (LCM stands for "least common multiple"):

$$\begin{aligned} q =& k\, \mathrm{LCM}(N_1K_1, N_1K_2, N_2) \bmod N_1N_2\, \mathrm{LCM}(K_1, K_2) \,, \\ &\text{with } k \in \mathbb{Z}_{\frac{N_1N_2\,\mathrm{LCM}(K_1,K_2)}{\mathrm{LCM}(N_1K_1,N_1K_2,N_2)}} \end{aligned} \tag{28}$$

meaning that the total number of SEGs are $\frac{N_1N_2\,\mathrm{LCM}(K_1,K_2)}{\mathrm{LCM}(N_1K_1,N_1K_2,N_2)}$ if (i) both symmetry charges are carried by the first layer shown in Eq. (27) and (ii) $a^1a^2da^2$ is considered (i.e., $\bar{q} = 0$). As a side note, by exchanging $1 \leftrightarrow 2$, the above result directly implies that the total number of SEGs are $\frac{N_1N_2\,\mathrm{LCM}(K_1,K_2)}{\mathrm{LCM}(N_2K_1,N_2K_2,N_1)}$ if (i) both symmetry charges are carried by the second layer [replacing b^1 in Eq. (27) by b^2] and (ii) $a^2a^1da^1$ is considered (i.e., $q=0$):

$$\bar{q} = k\,\mathrm{LCM}(N_2K_1, N_2K_2, N_1) \bmod N_1N_2\,\mathrm{LCM}(K_1,K_2)\,,\quad \text{with } k\in\mathbb{Z}_{\frac{N_1N_2\,\mathrm{LCM}(K_1,K_2)}{\mathrm{LCM}(N_2K_1,N_2K_2,N_1)}}\,. \tag{29}$$

Let us present several key steps towards Eq. (28) below. The change amount in Eq. (12) should be divisible *simultaneously* by K_1 and K_2 such that symmetry is kept. Meanwhile, the change amount in Eq. (13) should be integral in order to be consistent with Dirac quantization condition (3). Therefore, q should be quantized as: $q = k\,\mathrm{LCM}(N_1K_1, N_1K_2, N_2)$ with $k\in\mathbb{Z}$. Then, with these new quantized values, the shift operations (14,15) are changed to:

$$\frac{1}{2\pi}\int db^1 \longrightarrow \frac{1}{2\pi}db^1 + \frac{1}{4\pi^2N_1}\tilde{K}_1\,\mathrm{LCM}(N_1K_1,N_1K_2,N_2)\int a^2\wedge da^2\,, \tag{30}$$

$$\frac{1}{2\pi}\int db^2 \longrightarrow \frac{1}{2\pi}db^2 - \frac{1}{4\pi^2N_2}\tilde{K}_2\,\mathrm{LCM}(N_1K_1,N_1K_2,N_2)\int a^1\wedge da^2\,. \tag{31}$$

Again, the change amount in Eq. (30) should be divisible simultaneously by K_1 and K_2 such that symmetry is kept. The change amount in Eq. (31) should be integral such that Dirac quantization condition (3) is satisfied. Before evaluating the integral, the Wilson loop of a^1 may be obtained via equation of motion of b^1:

$$\frac{N_1K_1K_2}{2\pi\,\mathrm{GCD}(K_1,K_2)}\int_{\mathcal{M}^1} a^1\in\mathbb{Z}\,, \tag{32}$$

where Eq. (7) and Bezout's lemma are applied. The Wilson loop of a^2 may be obtained via equation of motion of b^2:

$$\frac{N_2}{2\pi}\int_{\mathcal{M}^1} a^2\in\mathbb{Z}\,. \tag{33}$$

With this preparation, we may calculate the change amounts in Eqs. (30,31) and obtain the conditions on $\tilde{K}_1$ and $\tilde{K}_2$:

$$\frac{\mathrm{LCM}(N_1K_1,N_1K_2,N_2)}{N_1N_2\,\mathrm{LCM}(K_1,K_2)}\tilde{K}_1\in\mathbb{Z}\,, \tag{34}$$

$$\frac{\mathrm{LCM}(N_1K_1,N_1K_2,N_2)}{N_1N_2\,\mathrm{LCM}(K_1,K_2)}\tilde{K}_2\in\mathbb{Z}\,. \tag{35}$$

Therefore, by using Bezout's lemma, the minimal periodicity of k can be fixed and k is thus compactified: $k\in\mathbb{Z}_{\frac{N_1N_2\,\mathrm{LCM}(K_1,K_2)}{\mathrm{LCM}(N_1K_1,N_1K_2,N_2)}}$.

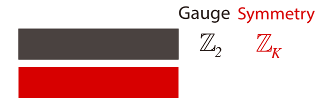
Following the similar procedure, we may obtain the results for the remaining two cases: (i). $a^2a^1da^1$ (labeled by $\bar{q}$) and both symmetry charges are in the first layer; (ii). $a^1a^2da^2$ (labeled by q) and both symmetry charges are in the second layer. For (a), $\bar{q}$ is given by:

$$\bar{q} = k\,\mathrm{LCM}(N_1K_1, N_1K_2, N_2) \bmod N_1N_2\,\mathrm{LCM}(K_1,K_2)\,,\quad \text{with } k\in\mathbb{Z}_{\frac{N_1N_2\,\mathrm{LCM}(K_1,K_2)}{\mathrm{LCM}(N_1K_1,N_1K_2,N_2)}}\,. \tag{36}$$

For (b), q is given by:

$$q = k\,\mathrm{LCM}(N_2K_1, N_2K_2, N_1) \bmod N_1N_2\,\mathrm{LCM}(K_1,K_2)\,,\quad \text{with } k\in\mathbb{Z}_{\frac{N_1N_2\,\mathrm{LCM}(K_1,K_2)}{\mathrm{LCM}(N_2K_1,N_2K_2,N_1)}}\,. \tag{37}$$

TABLE I. $\mathsf{SEG}(\mathbb{Z}_2,\,\mathbb{Z}_K)$. Both gauge group and symmetry group are in the same layer (the first layer). There is no non-trivial symmetry enrichment but a trivial stacking of symmetry and gauge theory.

Symmetry assignment			
GT	$q/4\pi^2a^1a^2da^2$	$\bar{q}/4\pi^2a^2a^1da^1$	
	0 mod 2	0 mod 2	
SEG	0 mod $2K$	0 mod $2K$	1

III. TYPICAL EXAMPLES OF SYMMETRY-ENRICHED GAUGE THEORIES

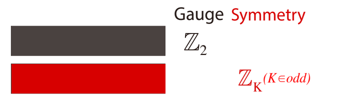
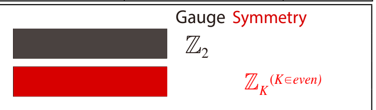
In this section, through a few concrete examples, we apply the general procedure shown in Sec. II C and construct SEGs that satisfy the definition and conditions listed in Sec. I. Useful technical details are present in Sec. II D,II E,II F and Appendix A. More examples are collected in Appendix B.

A. $\mathsf{SEG}(\mathbb{Z}_2,\mathbb{Z}_K)$

We begin with $G_g=\mathbb{Z}_N$ and $G_s=\mathbb{Z}_K$. The common features of this class are that: *(i)* there is only one gauge theory before imposing global symmetry; *(ii)* there are two complementary choices of symmetry assignment [82], namely, $\mathbb{Z}_K$ is either in the first layer or in the second layer (trivial layer). More concretely, before imposing global symmetry, there is only one $\mathbb{Z}_N$ gauge theory since all additional topological terms like $aada$, $aaaa$ vanish identically. Despite that, we still formally explicitly add $a^1a^2da^2$ and $a^2a^1da^1$ in all tables in order to see whether or not these topological terms will eventually have chance to be nonvanishing after symmetry is taken into consideration. Since we only have one cyclic symmetry subgroup, i.e., $G_s=\mathbb{Z}_K$, inclusion of two layers (the second one is a trivial layer in a sense that the level of b^2da^2 term is 1) is enough in the current simple cases.

We choose $N=K=2$ which was studied thoroughly in Ref. [69] via a completely different approach. The results are collected in Tables I and II ($K=2$). In Table I, the symmetry charge is carried by the first layer.

TABLE II. $\mathsf{SEG}(\mathbb{Z}_2, \mathbb{Z}_K)$. Gauge group and symmetry group are in different layers. $K \in \mathbb{Z}_{\text{odd}}$ ($\mathbb{Z}_{\text{even}}$) for first (second) sub-table.

Symmetry assignment			
GT	$q/4\pi^2 a^1a^2da^2$	$\bar{q}/4\pi^2a^2a^1da^1$	
	0 mod 2	0 mod 2	
SEG	0 mod $2K$	0 mod $2K$	1
Symmetry assignment			
GT	$q/4\pi^2 a^1a^2da^2$	$\bar{q}/4\pi^2a^2a^1da^1$	
	0 mod 2	0 mod 2	
SEG	0 mod $2K$ K mod $2K$	0 mod $2K$ K mod $2K$	2^2

Before imposing symmetry, we find that both q and $\bar{q}$ are 0 mod 2, indicating that topological interactions between layers are irrelevant. Mathematically, this conclusion can be achieved from Eqs. (19,20) by simply setting $N_1 = 2, N_2 = 1$. Physically, it means that there is only one $\mathbb{Z}_2$ GT which is described by the BF term with level-2: $i\frac{2}{2\pi}\int b\wedge da$. After symmetry is imposed, both q and $\bar{q}$ are 0 mod 4. This conclusion can be easily obtained by setting $N_1 = 2, K_1 = 2, N_2 = K_2 = 1$ in Eq. (24). Physically, after imposing symmetry, for each topological interaction, there is still only one choice of the coefficient but which is always connected to zero via a periodic shift. As a result, the total number of SEGs from this table is just one although the periodicity of both q and $\bar{q}$ is enhanced by symmetry.

In Table II ($K = 2$), the symmetry charge is carried by the second layer that is a trivial layer. In this case, we find that there are 2 distinct choices for both q and $\bar{q}$: either 0 mod 4 or 2 mod 4. Quantitatively, this result can be obtained by simply setting $N_1 = 2, N_2 = 1, K_1 = 1, K_2 = 2$ in Eqs. (24,25). As a result, there are in total 2^2 SEGs from this table. Among them, the SEG with $q = \bar{q} = 2$ mod 4 can be simply regarded as stacking of symmetry enrichments from $(q,\bar{q})$=(2 mod 4, 0 mod 4) and $(q,\bar{q})$=(0 mod 4, 2 mod 4). In other words, both $a^1a^2da^2$ and $a^2a^1da^1$ topological interactions are present in this SEG.

In summary, there are $1 + 2^2 = 5$ SEGs with $G_g = \mathbb{Z}_2$ and $G_s = \mathbb{Z}_2$. One of them, labeled by $(2, 2)$ in Table II can be regarded as stacking of symmetry enrichment patterns of $(0, 2)$ and $(2, 0)$. For generic even K in Tables I and II, there are in total five SEGs, just like $K = 2$ case. For odd K, there are two SEGs only. One is from Table I where symmetry group is in the same layer as gauge group. The other one is from Table II where gauge group and symmetry group are in different layers.

B. $\mathsf{SEG}(\mathbb{Z}_2\times\mathbb{Z}_2, \mathbb{Z}_2)$

The calculation in Sec. III A only involves one gauge group. Therefore, before imposing symmetry group, there is only one gauge theory, i.e., the untwisted one. In the following, we calculate SEGs with $G_g = \mathbb{Z}_2\times\mathbb{Z}_2$ and $G_s = \mathbb{Z}_2$. Before imposing symmetry, there are already four topologically distinct GTs labeled by $(q,\bar{q})$ = (0 mod 4, 0 mod 4), (0 mod 4, 2 mod 4), (2 mod 4, 0 mod 4), and (2 mod 4, 2 mod 4), which can be derived from Eqs. (19,20) by setting $N_1 = N_2 = 2$. Under this circumstances, *Step-5* in Sec. II C cannot be skipped. All SEGs are listed in Table III, where three different ways of symmetry assignment are considered.

Taking the first symmetry assignment ($\mathbb{Z}_2$ symmetry is assigned to the first layer, see the first subtable of Table III) as an example, there are two choices of q after symmetry is imposed: either 0 mod 8 or 4 mod 8. This result can be easily obtained by setting $N_1 = 2, K_1 = 2, N_2 = 2, K_2 = 1$ in Eq. (24). Similarly, there are also two choices of $\bar{q}$. Therefore, totally there are 2^2 SEGs from the first subtable of Table III. However, one may wonder what is the parent gauge theory (GT) for each choice. This line of thinking is the goal of Step-5 in Sec. II C. Interestingly, both choices of q mathematically belong to the sequence "0 mod 4". In other words, 0 mod 8 and 4 mod 8, both of which belong to the sequence 0 mod 4 and thus are indistinguishable before imposing symmetry, become distinguishable after symmetry is imposed. This is nothing but a consequence of symmetry enrichment.

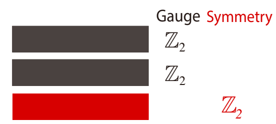
Meanwhile, both choices do not match the sequence "2 mod 4" at all, which is indicated by the mark "N/A" in the table. Similar analysis can be applied to $a^2a^1da^1$. This phenomenon tells us that, $\mathbb{Z}_2\times\mathbb{Z}_2$ GT labeled by $(q,\bar{q})$ = (2 mod 4, 2 mod 4) cannot generate SEG descendants *if symmetry is assigned to either the first layer (the first subtable of Table III) or the second layer (the second subtable of Table III)*. Both layers are of type-I in Fig. 2. One may wonder what will happen if we still enforce G_s on this twisted GT in such kinds of symmetry assignment. Can the gauge group and symmetry group be compatible with each other simultaneously? To answer these questions, recalling the general procedure shown in Sec. II C, there are several conditions (symmetry requirement and gauge invariance) listed in Step-4 that determine $\mathsf{SEG}(G_g, G_s)$. Therefore, if there is a SEG replacing the mark "N/A", it *either* breaks symmetry *or* preserves symmetry but violates gauge invariance principle. The latter case is an anomalous SEG and possibly realizable on the boundary of some (4+1)D system.

In the third subtable of Table III, symmetry is assigned to the third layer, i.e., the type-II layer in Fig. 2. It is clear that there are 8 linearly independent topological interaction terms that can be applied [83]. In this symmetry assignment, each topological interaction term has two choices of its coefficient: either 0 mod 4 or 2 mod 4 (for $a^1a^2da^3$ and $a^2a^3da^1$, the result can be obtained from the

TABLE III. SEG($\mathbb{Z}_2\times\mathbb{Z}_2,\mathbb{Z}_2$). Three different ways of symmetry assignment are considered. Interestingly, all SEGs in first and second ways of symmetry assignment come from the untwisted $\mathbb{Z}_2\times\mathbb{Z}_2$ GT only. "N/A" means that SEGs do not exist. Those states necessarily either break symmetry or violate gauge invariance principle. For the former, the ground states should be discrete-symmetry-breaking phases. The latter may exist on the boundary of some (4+1)D systems.

<table>
<tr><td>Symmetry assignment</td><td colspan="5">
</td></tr>
<tr><td rowspan="2">GT</td><td colspan="2">$\frac{q}{4\pi^2}a^1a^2da^2$</td><td colspan="2">$\frac{\bar q}{4\pi^2}a^2a^1da^1$</td><td></td></tr>
<tr><td>0 mod 4</td><td>2 mod 4</td><td>0 mod 4</td><td>2 mod 4</td><td></td></tr>
<tr><td rowspan="2">SEG</td><td>0 mod 8</td><td rowspan="2">N/A</td><td>0 mod 8</td><td rowspan="2">N/A</td><td rowspan="2">2^2</td></tr>
<tr><td>4 mod 8</td><td>4 mod 8</td></tr>
</table>

<table>
<tr><td>Symmetry assignment</td><td colspan="5">
</td></tr>
<tr><td rowspan="2">GT</td><td colspan="2">$\frac{q}{4\pi^2}a^1a^2da^2$</td><td colspan="2">$\frac{\bar q}{4\pi^2}a^2a^1da^1$</td><td></td></tr>
<tr><td>0 mod 4</td><td>2 mod 4</td><td>0 mod 4</td><td>2 mod 4</td><td></td></tr>
<tr><td rowspan="2">SEG</td><td>0 mod 8</td><td rowspan="2">N/A</td><td>0 mod 8</td><td rowspan="2">N/A</td><td rowspan="2">2^2</td></tr>
<tr><td>4 mod 8</td><td>4 mod 8</td></tr>
</table>

<table>
<tr><td>Symmetry assignment</td><td colspan="11">
</td></tr>
<tr><td rowspan="2">GT</td><td colspan="2">$a^1a^2da^2$</td><td colspan="2">$a^2a^1da^1$</td><td>$a^1a^3da^3$</td><td>$a^3a^1da^1$</td><td>$a^2a^3da^3$</td><td>$a^3a^2da^2$</td><td>$a^1a^2da^3$</td><td>$a^2a^3da^1$</td><td></td></tr>
<tr><td>0 mod 4</td><td>2 mod 4</td><td>0 mod 4</td><td>2 mod 4</td><td>0 mod 2</td><td>0 mod 2</td><td>0 mod 2</td><td>0 mod 2</td><td>0 mod 2</td><td>0 mod 2</td><td></td></tr>
<tr><td rowspan="2">SEG</td><td rowspan="2">0 mod 4</td><td rowspan="2">2 mod 4</td><td rowspan="2">0 mod 4</td><td rowspan="2">2 mod 4</td><td>0 mod 4</td><td>0 mod 4</td><td>0 mod 4</td><td>0 mod 4</td><td>0 mod 4</td><td>0 mod 4</td><td rowspan="2">2^8</td></tr>
<tr><td>2 mod 4</td><td>2 mod 4</td><td>2 mod 4</td><td>2 mod 4</td><td>2 mod 4</td><td>2 mod 4</td></tr>
</table>

general calculation in Appendix A 1 and A 2). Therefore, totally, there are 2^8 SEGs. Interestingly, for those four SEGs with topological interactions $a^1a^2da^2$ and $a^2a^1da^1$ only, they can be simply regarded as stacking of a twisted $\mathbb{Z}_2\times\mathbb{Z}_2$ gauge theory and a direct product state with $\mathbb{Z}_2$ symmetry.

C. SEG($\mathbb{Z}_2\times\mathbb{Z}_2,\mathrm{U}(1)$)

In this part, we discuss the gauge theory $\mathbb{Z}_2\times\mathbb{Z}_2$ enriched by the continuous symmetry U(1). The result can be obtained by following the general calculation in Appendix A 3, A 4, and A 5. Similar to the case of SEG($\mathbb{Z}_2\times\mathbb{Z}_2,\mathbb{Z}_2$), we consider 3 ways to assign the symmetry, as shown in Table IV. Considering the first symmetry assignment (U(1) is assigned at the first layer), we find that there is only one SEG($\mathbb{Z}_2\times\mathbb{Z}_2,\mathrm{U}(1)$) for both interaction terms with $q=\bar q=0$. In other words, this SEG is a descendant of the untwisted $\mathbb{Z}_2\times\mathbb{Z}_2$ GT with $q=\bar q=0$, while all other three twisted GTs do not have SEG descendants *in this symmetry assignment*. Similarly, for the second symmetry assignment, there is also only one SEG and it is also a descendant of the untwisted GT.

However, there is one subtle feature that is absent for discrete symmetry group. $q=\bar q=0$ means that q and $\bar q$ are absolutly zero with no periodicity (or periodicity=0 formally) after symmetry is imposed. We note that periodicity is always nonzero in all previous examples with discrete symmetry group. It means that if we start with an untwisted GT but with $q=4$, the resulting gauge theory after imposing U(1) symmetry either breaks symmetry or violates gauge invariance principle. For the latter case, the theory can be regarded as an anomalous SEG which is possibly realizable on the boundary of certain (4+1)D systems.

Now we consider the third symmetry assignment (the last row in Table IV) which is much more complex. There are 8 linearly independent topological interaction terms of *aada* type [83]. We find that there are 2^3 SEG($\mathbb{Z}_2\times\mathbb{Z}_2,\mathrm{U}(1)$). Each coefficient of $a^1a^2da^2$, $a^2a^1da^1$ and $a^1a^2da^3$ has two choices while the coefficient of other *aada* interaction terms vanish identically after periodicity shift, which leads to 2^3 SEGs. For SEGs where only $a^1a^2da^2$ $a^2a^1da^1$ are considered (other topological interaction terms vanish), they can be simply regarded as the stacking of a twisted $\mathbb{Z}_2\times\mathbb{Z}_2$ gauge theory and a direct product state with U(1) symmetry. For SEGs with at least $a^1a^2da^3$ topological interaction term, they are more interesting ones since they induce the nontrivial couplings between type-I layers and type-II layers as shown in Fig. 2.

IV. PROBING SET ORDERS

In Sec. III, we have constructed anomaly-free SEGs in a few concrete examples. In this section, we probe SET orders possessed by the ground states of SEGs. Then, the map from SEGs to SETs in Fig. 1 is achieved. In order to identify SET order in a given SEG, one should know the topological orders and symmetry-enriched properties.

Given a gauge group G_g, the total number of topological orders is generically smaller than that of GTs that are classified by $\mathcal{H}^4(G_g,\mathrm{U}(1))$. Intuitively, the labelings of gauge fluxes / gauge charges probably have redundancy from the aspect of topological orders. For example, if $G_g=\mathbb{Z}_2\times\mathbb{Z}_2$, there are four GTs. However, at least

TABLE IV. $\mathsf{SEG}(\mathbb{Z}_2\times\mathbb{Z}_2,\mathrm{U}(1))$ When the symmetry is assigned at the first and second layer, there is only one SEG:$(q,\bar{q})$=(0,0). The notation N/A denotes that there is no SEG descendant for *the* specific symmetry assignment. 0 means that q or $\bar{q}$ exactly takes zero.

Symmetry assignment	Gauge Symmetry $\mathbb{Z}_2$ U(1) $\mathbb{Z}_2$				
GT	$q/4\pi^2 a^1a^2da^2$		$\bar{q}/4\pi^2 a^2a^2da^1$		
	0 mod 4	2 mod 4	0 mod 4	2 mod 4	
SEG	0	N/A	0	N/A	1

Symmetry assignment	Gauge Symmetry $\mathbb{Z}_2$ $\mathbb{Z}_2$ U(1)				
GT	$q/4\pi^2 a^1a^2da^2$		$\bar{q}/4\pi^2 a^2a^2da^1$		
	0 mod 4	2 mod 4	0 mod 4	2 mod 4	
SEG	0	N/A	0	N/A	1

Symmetry assignment	Gauge Symmetry $\mathbb{Z}_2$ $\mathbb{Z}_2$ U(1)										
GT	$a^1a^2da^2$		$a^2a^1da^1$		$a^1a^3da^3$	$a^3a^1da^1$	$a^2a^3da^3$	$a^3a^2da^2$	$a^1a^2da^3$	$a^2a^3da^1$	
	0 mod 4	2 mod 4	0 mod 4	2 mod 4	0 mod 2	0 mod 2	0 mod 2	0 mod 2	0 mod 2	0 mod 2	
SEG	0 mod 4	2 mod 4	0 mod 4	2 mod 4	0	0	0	0	0 mod 4 2 mod 4	0	2^3

GT with $q = 2 \bmod 4$ and $\bar{q} = 0 \bmod 4$ and GT with $\bar{q} = 2 \bmod 4$ and $q = 0 \bmod 4$ share the same topological order since both are just connected to each other via exchanging superscripts 1 and 2.

For the sake of simplicity, in this section, we will only consider $G_g = \mathbb{Z}_N$ such that both GT and topological order are unique. In these cases, we find that: *(i)* quasiparticles that carry unit gauge charge of the gauge group G_g may carry fractionalized symmetry charge of the symmetry group G_s, which is classified by the second group cohomology with G_g coefficient: $\mathcal{H}^2(G_s, G_g)$; *(ii)* there is an interesting mixed version of three-loop braiding statistics among symmetry fluxes and gauge fluxes. Both features are gauge-invariant and topological, which can be detected in experiments.

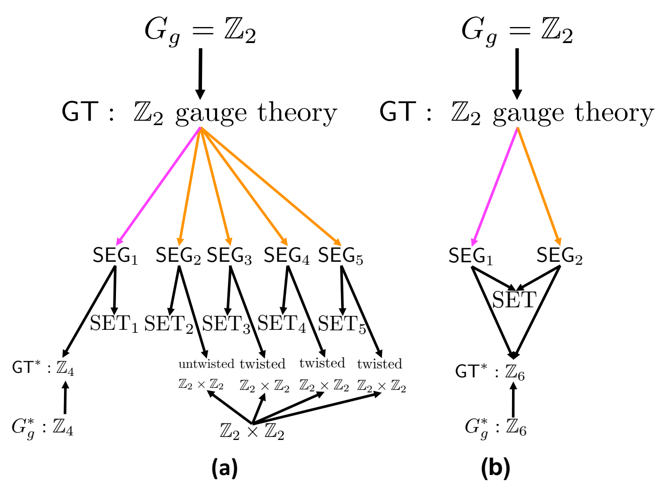


FIG. 3. (Color online) Two concrete examples of webs of gauge theories shown in Fig. 1. $\mathsf{SEG}(\mathbb{Z}_2,\mathbb{Z}_2)$ and $\mathsf{SEG}(\mathbb{Z}_2,\mathbb{Z}_3)$ are shown in (a) and (b), respectively. SEG_1 in (a) can be found in Table I. $\mathsf{SEG}_{2,\cdots,5}$ in (a) can be found in Table II. SEG_1 in (b) can be found in Table I by setting $K=3$. SEG_2 in (b) can be found in the first subtable of Table II by setting $K=3$.

A. SET orders in $\mathsf{SEG}(\mathbb{Z}_2,\mathbb{Z}_K)$ with $K\in\mathbb{Z}_{\text{even}}$

In this part, we probe SET orders with $\mathbb{Z}_2$ gauge group and $\mathbb{Z}_2$ symmetry group in the five SEGs listed in Table I and Table II. General even K is straightforward. When the gauge group G_g only includes one $\mathbb{Z}_N$ subgroup, e.g., $G_g = \mathbb{Z}_2$, there is only one GT, i.e., the untwisted one. The topological order of the GT is dubbed "$\mathbb{Z}_N$ topological order", characterized by the charge-loop braiding statistics data, i.e., the $e^{i2\pi/N}$ phase accumulated by a unit gauge charge moving around a unit gauge flux. For $N=2$, the phase is just $e^{i\pi}=-1$. Due to this simplification, in order to characterize SET orders in these five SEGs, the only remaining task is to diagnose the symmetry-enriched properties. From the following analysis, we obtain five distinct SET orders with $\mathbb{Z}_2$ topological order and $\mathbb{Z}_2$ global symmetry.

1. $\mathsf{SEG}(\mathbb{Z}_2,\mathbb{Z}_K)$ *with* $K\in\mathbb{Z}_{\text{even}}$ *in Table I*

For *the* SEG in Table I, we may consider the following action in the presence of excitation terms ($K=2$ as an example):

$$S = i\frac{2}{2\pi}\int b\wedge da + i\frac{1}{2\pi}\int b\wedge dA + i\int b\wedge *\Sigma + i\int a\wedge *j\,, \tag{38}$$

where the 2-form tensor Σ represents the unit loop excitation current (world-sheet) of the $\mathbb{Z}_2$ gauge theory. The 1-form vector j represents the unit gauge particle current (world-line) of the $\mathbb{Z}_2$ gauge theory. Since only one layer is considered in this case, the superscripts of b^1, a^1 are removed. The background gauge field A is constrained by Eq. (7) with $K_1 = 2$. Next, integrating out b field leads to: $\frac{2}{2\pi} da = -*\Sigma - \frac{1}{2\pi} dA$. Then, a can be formally solved by adding $*d*$ in both sides: $a = -\pi \frac{*d}{\hat{\Delta}}\Sigma - \frac{1}{2}A$, where the Laplacian operator $\hat{\Delta} \equiv *d*d$. Plugging this expression into the last term of Eq. (38), we obtain the following effective action about excitations in the presence of symmetry twist: $-i\frac{1}{2}\int A \wedge *j + i\pi \int j \wedge d^{-1}\Sigma$. In this effective action, the second term characterizes the $\mathbb{Z}_2$ topological order with charge-loop braiding phase $e^{i\pi} = -1$. Mathematically, this is a Hopf term and represents the long-range Aharonov-Bohm statistical interaction between gauge fluxes (i.e., the loop excitations) and particles. The operator $d^{-1} = \frac{d}{\hat{\Delta}}$ is a formal notation defined as the operator inverse of d, whose exact form can be understood in momentum space by Fourier transformations. The first term of this effective action encodes the symmetry-enriched properties of the SEG. It indicates that the unit gauge charge carries 1/2 symmetry charge of symmetry group $G_s = \mathbb{Z}_2$, which corresponds to the second group cohomology classification $\mathcal{H}^2(G_s, G_g) = \mathbb{Z}_2$ (see Appendix C for details).

In summary, for the SEG given by Table I, the $\mathbb{Z}_2$ gauge charged bosons carry half quantized $\mathbb{Z}_2$ symmetry charge. This is the first SET order we identify.

2. SEG$(\mathbb{Z}_2, \mathbb{Z}_K)$ *with* $K \in \mathbb{Z}_{\text{even}}$ *in Table II*

For Table II, we first consider the q-topological interaction term. The action in the presence of A is given by ($K = 2$ as an example):

$$S = i\frac{2}{2\pi}\int b^1 \wedge da^1 + i\frac{1}{2\pi}\int b^2 \wedge da^2 + i\frac{1}{2\pi}\int b^2 \wedge dA + i\frac{q}{4\pi^2}\int a^1 \wedge a^2 \wedge da^2 + i\int b^1 \wedge *\Sigma^1 + i\sum_I^2 \int a^I \wedge *j^I, \quad (39)$$

where Σ^1 and $\{j^I\}$ are loop excitation currents and particle excitation currents of the Ith layer respectively. Σ^2 is not considered for the reason that the second layer is trivial and Σ^2 carries 0 mod 2π fluxes which are not detectable. One may first integrate out $\{b^I\}$, which enforces that the path-integral configurations of $\{a^I\}$ are completely fixed by excitations and the background gauge field: $a^1 = -\frac{2\pi}{2} * d^{-1}\Sigma^1$, $a^2 = -\frac{2\pi}{2} * d^{-1}\sigma$. Here, the symbol d^{-1} has been defined in Sec. IV A 1. The new 2-form variable σ is defined through: $\sigma = *\frac{2}{2\pi}dA$ which represents the number density / current of the π-symmetry twist induced by the background gauge field. Plugging the expressions of $\{a^I\}$ into $\{j^I\}$-dependent terms in Eq. (39), we obtain the following effective action terms: $i\pi \int j^1 \wedge d^{-1}\Sigma^1 + i\int j^2 \wedge *A$, where the first Hopf term indicates that the first layer has a $\mathbb{Z}_2$ topological order. The second term indicates that the quasiparticles in the second layer carry integer symmetry charge. In other words, there doesn't exist symmetry fractionalization.

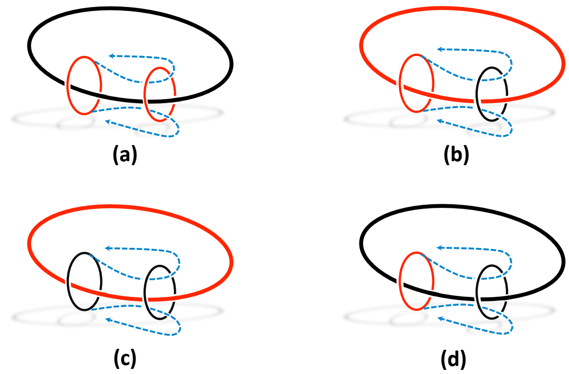


FIG. 4. (Color online) A mixed version of three-loop braiding process among gauge fluxes and symmetry fluxes in SEG$(\mathbb{Z}_2, \mathbb{Z}_2)$ of Table II. Loops in red and black denote symmetry flux loop (σ) and gauge flux loop (Σ^1), respectively. The dashed curves shows the trajectory of one loop that moves around another loop, both of which are linked to the base loop.

Despite that, we will show that there is interesting *mixed* three-loop statistics among symmetry fluxes (σ) and gauge fluxes (Σ^1). For this purpose, plugging the expressions of $\{a^I\}$ into the q-dependent term in Eq. (39), we obtain: $-i\frac{q\pi}{4}\int (*d^{-1}\Sigma^1) \wedge (*d^{-1}\sigma) \wedge (*\sigma) = -i\int (*d^{-1}\Sigma^1) \wedge \frac{\pi q}{4}(*d^{-1}\sigma) \wedge (*\sigma)$ which is the topological invariant that characterizes the *mixed* three-loop statistics among symmetry fluxes and gauge fluxes and provides important symmetry-enriched properties of SEGs. This mixed version of three-loop statistics enriches our previous understandings on three-loop statistics among gauge fluxes [26–30]. Pictorially, the topological invariant corresponds to the three-loop process shown in Fig. 4(a) where the gauge flux Σ^1 is a base loop (a term coined by Wang and Levin [26]). The entire process leads to Berry phase (denoted by $\theta_{\sigma,\sigma;\Sigma^1}$): $\theta_{\sigma,\sigma;\Sigma^1} = 2 \times \frac{q\pi}{4} = \pi$, where $q = 2$ is used and the factor of 2 is due to the fact that the full braiding process accumulates two times of half-braiding (exchange between σ and σ in the presence of the base loop Σ^1). If the base loop is provided by σ instead, the topological invariant gives rise to the full braiding of another σ around a Σ^1 as shown in Fig. 4(b), and the associated Berry phase is given by: $\theta_{\sigma,\Sigma^1;\sigma} = \frac{q\pi}{4} = \frac{\pi}{2}$ mod π, where π phase ambiguity arises from the possibility that $\mathbb{Z}_2$ gauge charge may be attached to σ such that there is π phase contribution from the Aharonov-Bohm phase from the topological invariant $i\pi\int j^1 \wedge d^{-1}\Sigma^1$.

Likewise, the $\bar{q}$ term can also be written in terms of the topological invariant: $-i\frac{\bar{q}\pi}{4}\int (*d^{-1}\sigma) \wedge (*d^{-1}\Sigma^1) \wedge (*\Sigma^1)$. Pictorially, the topological invariant corresponds to the three-loop process shown in Fig. 4(c) where the symmetry flux σ is a base loop. The entire process leads to Berry phase (denoted by $\theta_{\Sigma^1,\Sigma^1;\sigma}$): $\theta_{\Sigma^1,\Sigma^1;\sigma} = 2\times\frac{\bar{q}\pi}{4} =$

π , where $\bar{q}=2$ is used for the SEG labeled by $(0,2)$ in Table II. By choosing Σ^1 as the base loop, we may obtain the Berry phase accumulated by fully braiding Σ^1 around σ with the base loop provided by another Σ^1 [see Fig. 4(d)]: $\theta_{\sigma,\Sigma^1;\Sigma^1}=\frac{\bar{q}\pi}{4}=\frac{\pi}{2}\mod\pi$, where π phase ambiguity arises from the possibility that $\mathbb{Z}_2$ gauge charge may be attached to σ such that there is π phase contribution from the Aharonov-Bohm phase from the topological invariant $i\pi\int j^1\wedge d^{-1}\Sigma^1$.

In summary, for the four SEGs given by Table II, they support four different SET orders. All point-particles are either symmetry-neutral or carry integer $\mathbb{Z}_2$ symmetry charge. In other words, symmetry is not fractionalized and charge-loop braiding data is always trivial. However, they can be experimentally distinguished by the mixed three-loop braiding process. In total, we obtain five distinct SET orders with $\mathbb{Z}_2$ topological order and $\mathbb{Z}_2$ global symmetry. Likewise, for generic even K, there are also five SET orders.

B. SET orders in SEG$(\mathbb{Z}_2,\mathbb{Z}_K)$ with $K\in\mathbb{Z}_{\text{odd}}$

We consider $K=3$ as an example. General odd K is straightforward. In this case, there are two distinct SEGs that are collected in Table I ($K=3$) and the first subtable of Table II ($K=3$) respectively. For the first SEG, the discussion is similar to that of $K=2$ in Table I. We start with the action (38) and the background gauge field A is now constrained by Eq. (7) with $K_1=3$. Integrating out b,a leads to $-i\frac{1}{2}\int A\wedge *j+i\pi\int j\wedge d^{-1}\Sigma$ where the first term indicates that the bosons (denoted by "e") that carry unit $\mathbb{Z}_2$ gauge charge also carry 1/2 symmetry charge of $\mathbb{Z}_3$ group. However, there is no projective representation (with $\mathbb{Z}_2$ coefficient) for $\mathbb{Z}_3$ symmetry group indicated by the trivial second group cohomology: $\mathcal{H}^2(\mathbb{Z}_3,\mathbb{Z}_2)=\mathbb{Z}_1$ (see Appendix C), which means that this half-quantized symmetry charge cannot be detected by symmetry fluxes. The physical effect of this half-quantized symmetry charge is completely identical to that of -1 symmetry charge.

More physically, let us perform an Aharonov-Bohm experiment by inserting symmetry fluxes (a loop) with flux $\Phi_A=0,\frac{2\pi}{3},\frac{4\pi}{3}$. The boson e that moves around a symmetry flux with Φ_A will pick up a Berry phase $e^{i\frac{1}{2}\Phi_A}$ where 1/2 is the symmetry charge carried by e. However, during this process, it is possible that a gauge flux ($\Phi_g=0,\pi$) is dynamically excited and eventually attached to the symmetry flux. As a result, an additional Berry phase is accumulated: $e^{i\Phi_g}$, leading to the Berry phase $e^{i\frac{1}{2}\Phi_A+\Phi_g}$. After repeating the experiments for each Φ_A sufficient times, the observer will eventually collect two data for each symmetry flux. If $\Phi_A=0$, the Berry phase is either 0 or $e^{i\pi}$; If $\Phi_A=\frac{2\pi}{3}$, the Berry phase is either $e^{i\frac{\pi}{3}}$ or $e^{i\frac{4\pi}{3}}$; If $\Phi_A=\frac{4\pi}{3}$, the Berry phase is either $e^{i\frac{2\pi}{3}}$ or $e^{i\frac{5\pi}{3}}$. It is clear that these observed data can be exactly obtained by considering the boson that carry unit gauge charge and -1 non-fractionalized symmetry charge whose Berry phase is given by $e^{-i\Phi_A+i\Phi_g}$. In other words, the half-quantized symmetry charge can not be distinguished from -1 symmetry charge. Therefore, for SEG in Table I ($K=3$), there is no symmetry fractionalization.

For the second SEG (the first subtable of Table II with $K=3$), since there doesn't exist nontrivial topological interactions between the two layers, this SEG is nothing but a simple stacking of a $\mathbb{Z}_2$ gauge theory and a direct product state with $\mathbb{Z}_3$ symmetry. By definition, it is still a SEG but it doesn't have interesting symmetry-enriched properties.

In summary, both SEGs support the same SET order as shown schematically in Fig. 3(b). In this SET order, the topological order is $\mathbb{Z}_2$-type. However, the $\mathbb{Z}_3$ symmetry always trivially acts on the topological order due to the absence of both symmetry fractionalization and mixed three-loop braiding statistics. In other words, there is no interesting interplay beween $\mathbb{Z}_2$ topological order and $\mathbb{Z}_3$ symmetry. Likewise, for generic odd K, there is also only one SET order.

V. PROMOTING SEG TO GT*, BASIS TRANSFORMATIONS, AND THE WEB OF GAUGE THEORIES

In the above discussions, we obtained many SEGs, where the background gauge fields $\{A^I\}$ are treated as non-dynamical fields. A caveat is that basis transformations that mix $\{A^I\}$ and dynamical variables $\{a^I\}$ are strictly prohibited. However, one may further give *full* dynamics to the background gauge fields $\{A^I\}$, which leads to the mapping from SEGs to GT* as shown in Fig. 1. In other words, the symmetry twist now becomes dynamical [75]. As a result, arbitrary basis transformations now can be applied. It is legitimate to mix gauge fluxes and symmetry fluxes together to form a flux of a new gauge variable.

A. SEG$(\mathbb{Z}_2,\mathbb{Z}_2)$

Let us consider SEG$(\mathbb{Z}_2,\mathbb{Z}_2)$ in Table I with $K=2$. The associated dynamical gauge theory of b,a,A,B (here, $b=b^1$,$a=a^1$ for this single layer case) can be written as:

$$S=\frac{1}{2\pi}\int\begin{pmatrix}B & b\end{pmatrix}\begin{pmatrix}2&0\\1&2\end{pmatrix}\wedge d\begin{pmatrix}A\\a\end{pmatrix}\,, \tag{40}$$

where the two-form gauge field B is introduced to relax the holonomy of A to U(1)-valued in the path integral measure. According to Eq. (2), one can apply the following two unimodular matrices to send the above theory to its canonical form:

$$W = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}, \quad \Omega = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}, \tag{41}$$

$$W \begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix} \Omega^T = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix} \tag{42}$$

which directly indicates that the resulting new gauge theory GT^* after giving full dynamics to the background gauge field is $\mathbb{Z}_4$ gauge theory (Fig. 3).

Likewise, for Table II, the level matrix of the BF term is given by:

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix} \tag{43}$$

in the basis of (b^1, b^2, B) and (a^1, a^2, A). It can be diagonalized by using the following two unimodular matrices:

$$W = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \Omega = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}, \tag{44}$$

$$W \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix} \Omega^T = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}. \tag{45}$$

As a result, the new 1-form gauge variables are given by the vector $(\tilde{a}^1, \tilde{a}^2, \tilde{A})^T$ where,

$$a^1 = \tilde{a}^1, a^2 = \tilde{a}^2 - \tilde{A}, A = \tilde{A}. \tag{46}$$

From the canonical form (45), it is clear that the resulting theory after giving full dynamics to the background gauge field is $\mathbb{Z}_2 \times \mathbb{Z}_2$ gauge theory. But we should also examine how topological interaction terms transform. Since the second layer in the new basis is a trivial layer (level-1), we may neglect all topological interaction terms that include $\tilde{a}^2$. Keeping this in mind, After the basis transformations, the topological interaction terms $\int \frac{iq}{4\pi^2} a^1 \wedge a^2 \wedge da^2 + \int \frac{i\bar{q}}{4\pi^2} a^2 \wedge a^1 \wedge da^1$ are transformed to:

$$\int \frac{iq}{4\pi^2} \tilde{a}^1 \wedge \tilde{A} \wedge d\tilde{A} - \int \frac{i\bar{q}}{4\pi^2} \tilde{A} \wedge \tilde{a}^1 \wedge d\tilde{a}^1. \tag{47}$$

Therefore, we reach the following conclusions. The resulting theory starting from SEG labeled by $(0,0)$ in Table II is "untwisted" $\mathbb{Z}_2 \times \mathbb{Z}_2$ gauge theory. The remaining SEGs lead to twisted $\mathbb{Z}_2 \times \mathbb{Z}_2$ gauge theory after giving dynamics to the background gauge field (Fig. 3), which is also derived in [69] from a different point of view.

B. $\mathsf{SEG}(\mathbb{Z}_2, \mathbb{Z}_3)$

For SEGs in Table I, GT^* is always $\mathbb{Z}_{2K}$ gauge theory which are "untwisted". For SEGs in Table II, for even K, GT^*s are $\mathbb{Z}_2 \times \mathbb{Z}_K$ gauge theories which have one untwisted version and three twisted versions, in a similar manner to $K = 2$ discussed above. But for odd K, the resulting theory GT^* is still $\mathbb{Z}_{2K}$ gauge theory since the two groups are isomorphic: $\mathbb{Z}_2 \times \mathbb{Z}_K \cong \mathbb{Z}_{2K}$ when $K \in \mathbb{Z}_{\text{odd}}$. For example, for $K = 3$:

$$\begin{pmatrix} -1 & 1 \\ -3 & 2 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 1 & -3 \\ 1 & -2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 6 \end{pmatrix}. \tag{48}$$

Therefore, for odd K, the resulting gauge theory is the same as that in Table I. In other words, after giving full dynamics to the background gauge field A, there is only one output: a $\mathbb{Z}_{2K}$ gauge theory (Fig. 3). From this simple case, we see there is an interesting pattern of many-to-one correspondence between SEGs and GT^*s.

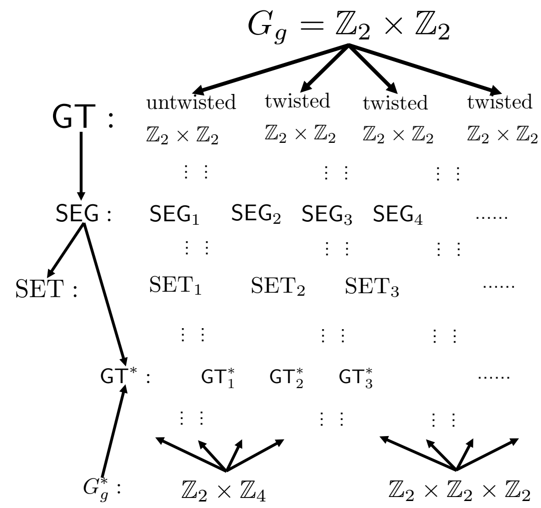


FIG. 5. A skeleton of the web of gauge theories for $\mathsf{SEG}(\mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_2)$.

C. $\mathsf{SEG}(\mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_2)$

For $\mathsf{SEG}(\mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_2)$, all SEGs are collected in Table III. Before imposing symmetry, there are already four distinct gauge theories. Therefore, the resulting web of gauge theories is much more complex. A rough skeleton is shown in Fig. 5 where the resulting GT^* theories can be regrouped into two gauge groups $G^*_{g1} = \mathbb{Z}_2 \times \mathbb{Z}_4$ and $G^*_{g2} = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. The first gauge group arises from the first and second subtables of Table III while the second gauge group arises from the third subtable of Table III. More concretely, let us consider the BF term of the first subtable after the background gauge field becomes fully dynamical:

$$\frac{1}{2\pi} \int \begin{pmatrix} B & b^1 & b^2 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \wedge d \begin{pmatrix} A \\ a^1 \\ a^2 \end{pmatrix}, \tag{49}$$

where the two-form gauge field B is introduced to relax the holonomy of A to U(1)-valued in the path integral measure. According to Eq. (2), one can apply the following two unimodular matrices to send the above theory to its canonical form:

$$W = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \Omega = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{50}$$

$$W \begin{pmatrix} 2 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \Omega^T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 2 \end{pmatrix} \tag{51}$$

which indicates that $G^*_{g1} = \mathbb{Z}_2 \times \mathbb{Z}_4$. Likewise, we have the following matrix calculation for the second subtable:

$$\frac{1}{2\pi} \int \begin{pmatrix} B & b^1 & b^2 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{pmatrix} \wedge d \begin{pmatrix} A \\ a^1 \\ a^2 \end{pmatrix}, \tag{52}$$

and

$$W = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 2 \end{pmatrix}, \quad \Omega = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 2 & 0 & -1 \end{pmatrix}, \tag{53}$$

$$W \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{pmatrix} \Omega^T = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix} \tag{54}$$

which still leads to $G^*_{g1} = \mathbb{Z}_2 \times \mathbb{Z}_4$.

For the third subtable, the BF term is given by:

$$\frac{1}{2\pi} \int \begin{pmatrix} B & b^1 & b^2 & b^3 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} \wedge d \begin{pmatrix} A \\ a^1 \\ a^2 \\ a^3 \end{pmatrix}, \tag{55}$$

where the 4×4 matrix can be diagonalized through:

$$W = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad \Omega = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix}, \tag{56}$$

$$W \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} \Omega^T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}. \tag{57}$$

As a result, $G^*_{g2} = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.

VI. SUMMARY AND OUTLOOK

In this paper, we have studied the symmetry enrichment through topological quantum field theory description of three-dimensional topological phases. All phases constructed in this paper can be viewed as 3D gapped quantum spin liquid candidates enriched by unbroken spin symmetry G_s. Using the 5-step general procedure in Sec. II C, we have efficiently constructed symmetry-enriched gauge theories (SEG) with gauge group $G_g = \mathbb{Z}_{N_1} \times \mathbb{Z}_{N_2} \times \cdots$ and symmetry group $G_s = \mathbb{Z}_{K_1} \times \mathbb{Z}_{K_2} \times \cdots$ as well as $G_s = \mathrm{U}(1) \times \mathbb{Z}_{K_1} \times \cdots$. The relation between SEG and its parent gauge theory GT has been shown. We have also shown how to physically diagnose the ground state properties of SEGs by investigating charge-loop braidings (patterns of symmetry fractionalization) and mixed multi-loop braiding statistics. By means of these physical detections, one can obtain a set of SET orders which represent the phase structures of ground states of SEGs. It is generally possible that two SEGs may give rise to the same SET order. Finally, by providing full dynamics to the background gauge fields [75], the resulting new gauge theories GT*s can be obtained and have been studied, all of which are summarized in a web of gauge theories (Fig. 1). Throughout the paper, many concrete examples have been studied in details. From those examples, we have seen that the general procedure provided in this paper is doable and efficient for the practical purpose of understanding 3D SET physics.

We highlight some questions for future studies. **(i)** Lattice models of SEGs. Dijkgraaf-Witten models [73] and string-net models [84] have been well studied. It is interesting to impose global symmetry (e.g., on-site finite unitary group) on these models in 3D. Then, lattice models can be regarded as an ultra-violet definition of SEGs. Some progress on 2D SETs has been made in Ref. [44, 45]. **(ii)** Material search and the experimental fingerprint of the mixed three-loop braiding statistics. There are several possible experimental realizations of $\mathbb{Z}_2$ spin liquids, such as the so-called Kitaev spin liquid state in the lattices in β- and γ-Li_2IrO_3 [85–91]. By further considering the unbroken $\mathbb{Z}_2$ Ising symmetry, the resulting ground state should exhibit SET orders. As we studied in the paper, the features of these SETs are patterns of symmetry fractionalization and mixed three-loop braiding statistics. It is thus of interest to theoretically propose an experimental fingerprint, especially, for the three-loop braiding statistics. **(iii)** Anomalous SEGs. In our construction, by anomaly, we mean that global symmetry and gauge invariance cannot be compatible with each other. If both are preserved, the resulting SEG is anomaly-free as what we have calculated. As mentioned in Sec. III B, the entries with "N/A" in Table III means that there do not exist SEG descendants for the twisted gauge theory (with both nonzero q and $\bar{q}$) in *the* symmetry assignment (the first and second subtables) such that both global symmetry and gauge invariance are preserved simultaneously. In other words, either symmetry is broken or gauge invariance is violated. For the case in which symmetry is preserved but gauge invariance is violated, we conjecture it can be realized on the boundary of certain (4+1)D systems. More careful studies in the future along anomaly will be meaningful. **(iv)** GT*s originated from SEGs with U(1) symmetry. In Sec. III C and Appendix, some examples of SEGs with U(1) symmetry are studied. After U(1) symmetry group becomes a dynamical gauge group, the resulting theory GT* should admit a mixed phenomenon generated by mixture of discrete gauge group and U(1) gauge group. It will be interesting to study the properties of such a type of gauge theory and eventually build the web (i.e., Fig. 1) of gauge theories for these cases. **(v)** SEGs with Charles symmetry [70]. Charles symmetry, which was introduced in [70], is a 3D analog of 2D anyonic (topological) symmetry. A simple example is $\mathbb{Z}_3$ gauge theory where quasiparticle is permuted to its antiparticle while quasi-loop is permuted to its antiloop. And there is one species of defect-charge-loop composites. This is just one gauge theory by giving a gauge group and a Charles symmetry group. It will be interesting to investigate the possibility that there are more than one gauge theories enriched by Charles.

ACKNOWLEDGEMENTS

We thank Ying Ran, Xie Chen, Zheng-Cheng Gu, Meng Cheng, Chien-Hung Lin, and Michael Levin for enlightening discussions. P.Y. acknowledges Eduardo Fradkin's enrouragement and conversation during the preparation and also thanks Shing-Tung Yau's hospitality at the Center of Mathematical Sciences and Applications at Harvard University where the work was done in part. Part of this work was done in Banff International Research Station, Banff, Calgary, Canada (P.Y.). S.Q.N. and Z.X.L. are supported by NSFC (Grant Nos.11574392), Tsinghua University Initiative Scientific Research Program, and the Fundamental Research Funds for the Central Universities, and the Research Funds of Renmin University of China (No. 15XNLF19). This work was supported in part by the NSF through grant DMR 1408713 at the University of Illinois (P.Y.)

[1] Xie Chen, Zheng-Cheng Gu, Zheng-Xin Liu, and Xiao-Gang Wen, *Symmetry protected topological orders and the group cohomology of their symmetry group*, Phys. Rev. B **87**, 155114 (2013).

[2] Z.-C. Gu and X.-G. Wen, *Tensor-entanglement-filtering renormalization approach and symmetry-protected topological order*, Phys. Rev. B **80**, 155131 (2009).

[3] Frank Pollmann, Ari M. Turner, Erez Berg, and Masaki Oshikawa, *Entanglement spectrum of a topological phase in one dimension*, Phys. Rev. B **81**, 064439 (2010).

[4] X. Chen, Z.-C. Gu, Z.-X. Liu, and X.-G. Wen, *Symmetry-Protected Topological Orders in Interacting Bosonic Systems*, Science **338**, 1604 (2012).

[5] X. Chen, Z.-C. Gu, and X.-G. Wen, *Local unitary transformation, long-range quantum entanglement, wave function renormalization, and topological order*, Phys. Rev. B **82**, 155138 (2010).

[6] Y.-M. Lu and A. Vishwanath, *Theory and classification of interacting integer topological phases in two dimensions: A Chern-Simons approach*, Phys. Rev. **86**, 125119 (2012).

[7] F. J. Burnell, X. Chen, L. Fidkowski, and A. Vishwanath, *Exactly soluble model of a three-dimensional symmetry-protected topological phase of bosons with surface topological order*, Phys. Rev. B **90**, 245122 (2014).

[8] Meng Cheng and Zheng-Cheng Gu, *Topological Response Theory of Abelian Symmetry-Protected Topological Phases in Two Dimensions*, Phys. Rev. Lett. **112**, 141602 (2014).

[9] X.-G. Wen, *Symmetry-protected topological invariants of symmetry-protected topological phases of interacting bosons and fermions*, Phys. Rev. B **89**, 035147 (2014).

[10] C. Xu and T. Senthil, *Wave functions of bosonic symmetry protected topological phases*, Phys. Rev. B **87**, 174412 (2013).

[11] P. Ye and Xiao-Gang Wen, *Projective construction of two-dimensional symmetry-protected topological phases with U(1), SO(3), or SU(2) symmetries*, Phys. Rev. B 87, 195128 (2013).

[12] M. A. Metlitski, C. L. Kane, and M. P. A. Fisher, *Bosonic topological insulator in three dimensions and the statistical Witten effect*, Phys. Rev. B **88**, 035131 (2013).

[13] P. Ye and Z.-C. Gu, *Topological quantum field theory of three-dimensional bosonic Abelian-symmetry-protected topological phases*, Phys. Rev. B **93**, 205157 (2016).

[14] P. Ye and Z.-C. Gu, *Vortex-Line Condensation in Three Dimensions: A Physical Mechanism for Bosonic Topological Insulators*, Phys. Rev. X **5**, 021029 (2015); arXiv:1410.2594.

[15] Z.-X. Liu and X.-G. Wen, *Symmetry-Protected Quantum Spin Hall Phases in Two Dimensions* , Phys. Rev. Lett. **110**, 067205 (2013).

[16] A. Vishwanath and T. Senthil, *Physics of Three-Dimensional Bosonic Topological Insulators: Surface-Deconfined Criticality and Quantized Magnetoelectric Effect*, Phys. Rev. X **3**, 011016 (2013).

[17] Zheng-Xin Liu, Jia-Wei Mei, P. Ye, and Xiao-Gang Wen, *U(1)×U(1) symmetry-protected topological order in Gutzwiller wave functions*, Phys. Rev. B **90**, 235146 (2014).

[18] P. Ye and Juven Wang, *Symmetry-protected topological phases with charge and spin symmetries: Response theory and dynamical gauge theory in two and three dimensions*, Phys. Rev. B **88**, 235109 (2013).

[19] P. Ye and Xiao-Gang Wen, *Constructing symmetric topological phases of bosons in three dimensions via fermionic projective construction and dyon condensation*, Phys. Rev. B **89**, 045127 (2014).

[20] Z. Bi and C. Xu, *Construction and field theory of bosonic-symmetry-protected topological states beyond group cohomology*, Phys. Rev. B **91**, 184404 (2015).

[21] A. Kapustin and R. Thorngren, *Anomalous Discrete Symmetries in Three Dimensions and Group Cohomology*, Phys. Rev. Lett. **112**, 231602 (2014).

[22] A. Kapustin, arXiv:1403.1467; arXiv:1404.6659.

[23] A. Kapustin and R. Thorngren, arXiv:1404.3230.

[24] Z. Bi, A. Rasmussen, and C. Xu, *Line defects in three-dimensional symmetry-protected topological phases*, Phys. Rev. B **89**, 184424 (2014).
[25] Olabode Mayodele Sule, Xiao Chen, and Shinsei Ryu, *Symmetry-protected topological phases and orbifolds: Generalized Laughlin's argument*, Phys. Rev. B **88**, 075125 (2013).
[26] C. Wang and M. Levin, *Braiding Statistics of Loop Excitations in Three Dimensions*, Phys. Rev. Lett. **113**, 080403 (2014).
[27] C. Wang and M. Levin, *Topological invariants for gauge theories and symmetry-protected topological phases*, Phys. Rev. B **91**, 165119 (2015).
[28] C.-H. Lin and M. Levin, *Loop braiding statistics in exactly soluble three-dimensional lattice models*, Phys. Rev. B **92**, 035115 (2015).
[29] Shenghan Jiang, Andrej Mesaros, and Ying Ran, *Generalized Modular Transformations in (3+1)D Topologically Ordered Phases and Triple Linking Invariant of Loop Braiding*, Phys. Rev. X **4**, 031048 (2014).
[30] Juven Wang and Xiao-Gang Wen, *Non-Abelian string and particle braiding in topological order: Modular SL(3,Z) representation and (3+1)-dimensional twisted gauge theory*, Phys. Rev. B **91**, 035134 (2015).
[31] Zheng-Cheng Gu, Juven C. Wang, and Xiao-Gang Wen, *Multikink topological terms and charge-binding domain-wall condensation induced symmetry-protected topological states: Beyond Chern-Simons/BF field theories*, Phys. Rev. B **93**, 115136 (2016).
[32] Juven Wang, Zheng-Cheng Gu, and Xiao-Gang Wen, *Field-Theory Representation of Gauge-Gravity Symmetry-Protected Topological Invariants, Group Cohomology, and Beyond*, Phys. Rev. Lett. **114**, 031601 (2015).
[33] Zheng-Xin Liu, Zheng-Cheng Gu, and Xiao-Gang Wen, *Microscopic Realization of Two-Dimensional Bosonic Topological Insulators*, Phys. Rev. Lett. **113**, 267206 (2014).
[34] H. He, Y. Zheng, and C. v. Keyserlingk, arXiv:1608.05393.
[35] Yidun Wan, Juven C. Wang, and Huan He, *Twisted gauge theory model of topological phases in three dimensions*, Phys. Rev. B **92**, 045101 (2015)
[36] X.-G. Wen, *Quantum orders and symmetric spin liquids*, Phys. Rev. B **65**, 165113 (2002).
[37] M. Levin and A. Stern, *Classification and analysis of two-dimensional Abelian fractional topological insulators*, Phys. Rev. B **86**, 115131 (2012).
[38] A. M. Essin and M. Hermele, *Classifying fractionalization: Symmetry classification of gapped Z_2 spin liquids in two dimensions*, Phys. Rev. B **87**, 104406 (2013).
[39] A. Mesaros and Y. Ran, *Classification of symmetry enriched topological phases with exactly solvable models*, Phys. Rev. B **87**, 155115 (2013).
[40] Yuxiang Gu, Ling-Yan Hung, Yidun Wan, *Unified framework of topological phases with symmetry*, Phys. Rev. B **90**, 245125 (2014).
[41] L.-Y. Hung and X.-G. Wen, *Quantized topological terms in weak-coupling gauge theories with a global symmetry and their connection to symmetry-enriched topological phases*, Phys. Rev. B **87**, 165107 (2013).
[42] Y.-M. Lu and A. Vishwanath, *Classification and properties of symmetry-enriched topological phases: Chern-Simons approach with applications to Z_2 spin liquids*, Phys. Rev. B **93**, 155121 (2016).
[43] M. Barkeshli, P. Bonderson, M. Cheng, and Z. Wang, arXiv:1410.4540.
[44] Meng Cheng, Zheng-Cheng Gu, Shenghan Jiang, and Yang Qi, arXiv:1606.08482.
[45] C. Heinrich, F. Burnell, L. Fidkowski, and M. Levin, arXiv:1606.07816.
[46] Yang Qi and M. Cheng, arXiv:1606.04544.
[47] Y. Qi, M. Cheng, and C. Fang, arXiv:1509.02927.
[48] L. Balents, *Spin liquids in frustrated magnets*, Nature **464**, 199 (2010).
[49] A. Kitaev, *Anyons in an exactly solved model and beyond*, Ann. Phys. **321**, 2 (2006).
[50] H. Bombin, *Topological Order with a Twist: Ising Anyons from an Abelian Model*, Phys. Rev. Lett. **105**, 030403 (2010).
[51] J. C. Teo, T. L. Hughes, and E. Fradkin, *Theory of twist liquids: Gauging an anyonic symmetry*, Ann. Phys. **360**, 349 (2015).
[52] M. N. Khan, J. C. Y. Teo, and T. L. Hughes, *Anyonic symmetries and topological defects in Abelian topological phases: An application to the ADE classification*, Phys. Rev. B **90**, 235149 (2014).
[53] A. Mesaros, Y. B. Kim, and Y. Ran, *Changing topology by topological defects in three-dimensional topologically ordered phases*, Phys. Rev. B **88**, 035141 (2013).
[54] M. Barkeshli and X.-G. Wen, *$U(1) \times U(1) \rtimes Z_2$ Chern-Simons theory and Z_4 parafermion fractional quantum Hall states*, Phys. Rev. B **81**, 045323 (2010).
[55] J. C. Y. Teo, A. Roy, and X. Chen, *Braiding statistics and congruent invariance of twist defects in bosonic bilayer fractional quantum Hall states*, Phys. Rev. B **90**, 155111 (2014).
[56] Y.-Z. You and X.-G. Wen, *Projective non-Abelian statistics of dislocation defects in a Z_N rotor model*, Phys. Rev. B **86**, 161107 (2012).
[57] M. Barkeshli and X.-L. Qi, *Topological Nematic States and Non-Abelian Lattice Dislocations*, Phys. Rev. X **2**, 031013 (2012).
[58] M. Barkeshli, C.-M. Jian, and X.-L. Qi, *Twist defects and projective non-Abelian braiding statistics*, Phys. Rev. B **87**, 045130 (2013).
[59] M. Barkeshli and X.-L. Qi, *Synthetic Topological Qubits in Conventional Bilayer Quantum Hall Systems*, Phys. Rev. X **4**, 041035 (2014).
[60] M. Barkeshli, H.-C. Jiang, R. Thomale, and X.-L. Qi, *Generalized Kitaev Models and Extrinsic Non-Abelian Twist Defects*, Phys. Rev. Lett. **114**, 026401 (2015).
[61] L. Kong and X.-G. Wen, arXiv:1405.5858.
[62] C. Xu, *Three-dimensional Z_2 topological phases enriched by time-reversal symmetry*, Phys. Rev. B **88**, 205137 (2013).
[63] C. Wang and T. Senthil, *Time-Reversal Symmetric U(1) Quantum Spin Liquids*, Phys. Rev. X **6**, 011034 (2016).
[64] Y. P. Huang, G. Chen, and M. Hermele, *Quantum Spin Ices and Topological Phases from Dipolar-Octupolar Doublets on the Pyrochlore Lattice*, Phys. Rev. Lett. **112**, 167203 (2014).
[65] Itamar Kimchi, James G. Analytis, and Ashvin Vishwanath, *Three-dimensional quantum spin liquids in models of harmonic-honeycomb iridates and phase diagram in an infinite-D approximation*, Phys. Rev. B **90**, 205126 (2014).

[66] E. Witten, *Dyons of charge $e\theta/2\pi$*, Phys. Lett. B **86**, 283 (1979).
[67] L. Fidkowski and A. Vishwanath, arXiv:1511.01502.
[68] M. Cheng, arXiv:1511.02563.
[69] X. Chen and M. Hermele, arXiv:1602.00187.
[70] P. Ye, T. L. Hughes, J. Maciejko, and E. Fradkin, *Composite particle theory of three-dimensional gapped fermionic phases: Fractional topological insulators and charge-loop excitation symmetry*, Phys. Rev. B **94**, 115104 (2016).
[71] X.-G. Wen, *Quantum Field Theory of Many-Body Systems - From the Origin of Sound to an Origin of Light and Electrons* (Oxford Univ. Press, Oxford, 2004).
[72] It should be noted that confined phases may also support topological orders. An example is shown in a parton construction in [70] where topological order is given by untwisted $\mathbb{Z}_2 \times \mathbb{Z}_6$ gauge group. Throughout the paper, only deconfined regime of gauge theories will be taken into consideration. In deconfined phases, the gauge fluxes and gauge particles naturally behave as emergent excitations of topological orders.
[73] R. Dijkgraaf and E. Witten, *Topological gauge theories and group cohomology*, Commun. Math. Phys. **129**, 393 (1990).
[74] For an anomaly-free effective field theory in condensed matter sense, these two conditions are necessary but their sufficiency is problematic as the UV definition of a continuum field theory should be justified by lattice models with local interactions. Although GTs have Dijkgraaf-Witten lattice model correspondence, symmetry enriched ones, i.e., SEGs should be more carefully investigated. The future studies of lattice models of SEGs will be interesting. Throughout the paper, we assume sufficiency holds but with the above caveat.
[75] Unless otherwise specified, throughout the paper, we try to avoid to use the terminology like "gauging", "fully gauging", "partially gauging", "weakly gauging", "orbifolding". In literatures, these terms are sometimes mixed with "symmetry twist" (i.e., response theory). By "giving full dynamics to..." in this paper, we really mean that, mathematically, a new path-integral measure $DA^1DA^2\cdots$ is introduced.
[76] In general, mixed version of four-loop braiding process should be considered if the total subgroups of symmetry and gauge are no less than 4. In this paper, mixed three-loop process is already enough for all examples we consider.
[77] G. T. Horowitz, *Exactly soluble diffeomorphism invariant theories*, Commun. Math. Phys. **125**, 417 (1989).
[78] If $\text{Det}\Lambda = 0$, then the zero eigenvalues give rise to gapless photon modes described by the sub-leading Maxwell action term.
[79] X. Chen, A. Tiwari, and S. Ryu, *Bulk-boundary correspondence in (3+1)-dimensional topological phases*, Phys. Rev. B **94**, 045113 (2016).
[80] A. Tiwari, X. Chen, and S. Ryu, arXiv:1603.08429.
[81] Juven Wang, Xiao-Gang Wen, and Shing-Tung Yau, arXiv:1602.05951.
[82] In this paper, one main goal is to present the general methods of constructing path-integral formalism of SEGs and diagnose SET orders, rather than completeness of classification. We only consider the cases where topological currents carry unit symmetry charge while generically they can carry charge-2 or more. Fortunately, for $\mathbb{Z}_2$ symmetry considered in the main text, 1 is sufficient since $2 \sim 0$ in $\mathbb{Z}_2$ symmetry. Furthermore, we also do not discuss the cases in which there exist more than one layers that couple to the same background gauge field. In principle, if all these cases are involved, we can obtain the complete classification based on the current working assumptions, which will be leaved to future work.
[83] As a side note, there are only two linearly independent three-layer topological interaction terms since $a^3a^1da^2$ is $a^1a^2da^3 + a^2a^3da^1$ up to a total derivative.
[84] Michael A. Levin and Xiao-Gang Wen, *String-net condensation: A physical mechanism for topological phases*, Phys. Rev. B **71**, 045110 (2005).
[85] Robert Schaffer, Eric Kin-Ho Lee, Bohm-Jung Yang, and Yong Baek Kim, *Recent progress on correlated electron systems with strong spin-orbit coupling*, Rep. Prog. Phys. **79**, 094504 (2016).
[86] E. K.-H. Lee, R. Schaffer, S. Bhattacharjee, and Y. B. Kim, *Heisenberg-Kitaev model on the hyperhoneycomb lattice*, Phys. Rev. B **89**, 045117 (2014).
[87] Saptarshi Mandal and Naveen Surendran, *Exactly solvable Kitaev model in three dimensions* , Phys. Rev. B **79**, 024426
[88] K. A. Modic, *et.al.*, *Realization of a three-dimensional spin-anisotropic harmonic honeycomb iridate*, Nature Communications, **5**, 4203 (2014).
[89] T. Takayama, *et.al.*, *Hyperhoneycomb Iridate β-Li_2IrO_3 as a Platform for Kitaev Magnetism*, Phys. Rev. Lett. **114**, 077202 (2015).
[90] A. Biffin, *et.al.*, *Un-conventional magnetic order on the hyperhoneycomb Kitaev lattice in β-Li_2IrO_3: Full solution via magnetic resonant x-ray diffraction*, Phys. Rev. B **90**, 205116 (2014).
[91] A. Biffin, *et.al.*, *Noncoplanar and Counterrotating Incommensurate Mag- netic Order Stabilized by Kitaev Interactions in γ-Li_2IrO_3*, Phys. Rev. Lett. **113**, 197201 (2014).

Appendix A: General calculation of gauge theories with global symmetry

1. $G_g = \mathbb{Z}_{N_1} \times \mathbb{Z}_{N_2} \times \mathbb{Z}_{N_3}$ with no symmetry

In this part, we present several details about $G_g = \mathbb{Z}_{N_1} \times \mathbb{Z}_{N_2} \times \mathbb{Z}_{N_3}$ and $G_s = \mathbb{Z}_{K_1} \times \mathbb{Z}_{K_2} \times \mathbb{Z}_{K_3}$. Each layer carries a unique symmetry charge. This case is relevant to those SEGs even with only one gauge group but with two symmetry subgroups (via, e.g., setting $N_2 = N_3 = 1$ and $K_1 = 1$). Most of derivations are similar to the previous cases except some subtle differences in the shift operations. The gauge theory before imposing the global symmetry is given by:

$$S = \sum_{I=1}^{3} \frac{iN_I}{2\pi} \int b^I \wedge da^I + i\frac{\bar{\bar{q}}}{4\pi^2} \int a^1 \wedge a^2 \wedge da^3 \,. \tag{A1}$$

The action is invariant under the following gauge transformations parametrized by scalars $\{\chi^I\}$ and vectors $\{V^I\}$:

$$a^I \longrightarrow a^I + d\chi^I \,, \tag{A2}$$

$$b^I \longrightarrow b^I + dV^I - \frac{\bar{\bar{q}}}{2\pi N^I} \epsilon^{IJ3} \chi^J \wedge da^3 \,. \tag{A3}$$

Let us investigate the integral $\frac{1}{2\pi}\int_{\mathcal{M}^3} db^I$. Under the above modified gauge transformations (A3), the integral will be changed by the amount below (for $I = 1$, $\mathcal{M}^3 = \mathcal{M}^1 \times \mathcal{M}^2$ is considered):

$$\frac{1}{2\pi}\int_{\mathcal{M}^3} db^1 \longrightarrow \frac{1}{2\pi}\int_{\mathcal{M}^3} db^1 - \frac{\bar{\bar{q}}}{4\pi^2 N_1}\int_{S^1} d\chi^2 \int_{M^2} da^3 = \frac{1}{2\pi}\int_{\mathcal{M}^3} db^1 - \frac{\bar{\bar{q}}}{4\pi^2 N_1} \times 2\pi\ell \times 2\pi\ell' \,, \tag{A4}$$

where $\ell\,,\ \ell' \in \mathbb{Z}$, and, the Dirac quantization condition (4) and homotopy mapping condition (11) are applied. In order to be consistent with the Dirac quantization condition (3), the change amount must be integral, namely, $\bar{\bar{q}}$ must be divisible by N_1. Similarly, $\bar{\bar{q}}$ is also divisible by N_2 due to:

$$\frac{1}{2\pi}\int_{\mathcal{M}^3} db^2 \longrightarrow \frac{1}{2\pi}\int_{\mathcal{M}^3} db^2 + \frac{\bar{\bar{q}}}{4\pi^2 N_2}\int_{S^1} d\chi^1 \int_{M^2} da^3 = \frac{1}{2\pi}\int_{\mathcal{M}^3} db^2 + \frac{\bar{\bar{q}}}{4\pi^2 N_2} \times 2\pi\ell'' \times 2\pi\ell''', \tag{A5}$$

where $\ell''\,,\ \ell''' \in \mathbb{Z}$. Hence, $\bar{\bar{q}} = \frac{kN_1N_2}{N_{12}}, k \in \mathbb{Z}$. Below, we want to show that k has a periodicity N_{123} (i.e., GCD of N_1, N_2, N_3) and thereby $\bar{\bar{q}}$ is compactified: $\bar{\bar{q}} \sim \bar{\bar{q}} + \frac{N_{123}N_1N_2}{N_{12}}$. Let us consider the following redundancy due to shift operations:

$$\frac{1}{2\pi}\int db^1 \longrightarrow \frac{1}{2\pi}\int db^1 + \frac{N_2\tilde{K}_1}{4\pi^2 N_{12}}\int a^2 \wedge da^3 \,, \tag{A6}$$

$$\frac{1}{2\pi}\int db^2 \longrightarrow \frac{1}{2\pi}\int db^2 - \frac{N_1\tilde{K}_2}{4\pi^2 N_{12}}\int a^1 \wedge da^3 \,, \tag{A7}$$

$$\frac{1}{2\pi}\int db^3 \longrightarrow \frac{1}{2\pi}\int db^3 + \frac{N_1N_2\tilde{K}_3}{4\pi^2 N_3N_{12}}\int (da^1 \wedge a^2 + a^1 \wedge da^2) \,, \tag{A8}$$

$$k \longrightarrow k + \tilde{K}_1 + \tilde{K}_2 + \tilde{K}_3 \,. \tag{A9}$$

Again, in order to be consistent with Dirac quantization (3), the change amount of the integral $\frac{1}{2\pi}\int_{\mathcal{M}^3} db^I$ should be integral, namely:

$$\frac{N_2\tilde{K}_1}{4\pi^2 N_{12}}\int_{\mathcal{M}^3} a^2 \wedge da^3 \in \mathbb{Z} \,, \tag{A10}$$

$$\frac{N_1\tilde{K}_2}{4\pi^2 N_{12}}\int_{\mathcal{M}^3} a^1 \wedge da^3 \in \mathbb{Z} \,, \tag{A11}$$

$$\frac{N_1N_2\tilde{K}_3}{4\pi^2 N_3N_{12}}\int_{\mathcal{M}^3} (da^1 \wedge a^2 + a^1 \wedge da^2) \in \mathbb{Z} \,. \tag{A12}$$

We may apply the Dirac quantization condition (4) and the quantized Wilson loop $\frac{N_I}{2\pi}\int_{\mathcal{M}^1} a^I \in \mathbb{Z}$ that is obtained via equations of motion of b^I. As a result, three constraints are achieved: $\tilde{K}_1/N_{12} \in \mathbb{Z}$, $\tilde{K}_2/N_{12} \in \mathbb{Z}$, $\tilde{K}_3/N_3 \in \mathbb{Z}$. In deriving the result for $\tilde{K}_3$, Bezout's lemma is applied. By using Bezout's lemma again, the minimal periodicity of k is given by GCD of N_{12} and N_3, which is N_{123}. As a result, we obtain the conditions on $\bar{\bar{q}}$ *if* symmetry is not taken into consideration.

$$\bar{\bar{q}} = k\frac{N_1N_2}{N_{12}} \bmod \frac{N_{123}N_1N_2}{N_{12}}\,, \quad k \in \mathbb{Z}_{N_{123}}. \tag{A13}$$

2. $G_g = \mathbb{Z}_{N_1} \times \mathbb{Z}_{N_2} \times \mathbb{Z}_{N_3}$ with $G_s = \mathbb{Z}_{K_1} \times \mathbb{Z}_{K_2} \times \mathbb{Z}_{K_3}$

To impose the symmetry, we add the following coupling term in the action (A1):

$$\sum_i^3 \frac{i}{2\pi}\int A^i \wedge db^i\,. \tag{A14}$$

The change amounts of the integral $\frac{1}{2\pi}\int_{\mathcal{M}^3} db^I$ in Eqs. (A4, A5, A6, A7, A8) should not only be integral [in order to be consistent with the Dirac quantization condition (3)] but also be multiple of K_i such that the coupling term (A14) is gauge invariant modular 2π. More quantitatively, with symmetry taken into account, from Eqs. (A4, A5), we may obtain the quantization of $\bar{\bar{q}}$: $\bar{\bar{q}} = \frac{kN_1N_2K_1K_2}{\mathrm{GCD}(N_1K_1,N_2K_2)}$ with $k \in \mathbb{Z}$ such that the change amounts are multiple of K_i. Then, with these new quantized values, the shift operations (A6, A7, A8) are changed to:

$$\frac{1}{2\pi}\int db^1 \longrightarrow \frac{1}{2\pi}\int db^1 + \tilde{K}_1\frac{N_2K_1K_2}{4\pi^2\mathrm{GCD}(N_1K_1,N_2K_2)}\int a^2\wedge da^3\,, \tag{A15}$$

$$\frac{1}{2\pi}\int db^2 \longrightarrow \frac{1}{2\pi}\int db^2 - \tilde{K}_2\frac{N_1K_1K_2}{4\pi^2\mathrm{GCD}(N_1K_1,N_2K_2)}\int a^1\wedge da^3\,, \tag{A16}$$

$$\frac{1}{2\pi}\int db^3 \longrightarrow \frac{1}{2\pi}\int db^3 + \tilde{K}_3\frac{N_1N_2K_1K_2}{4\pi^2N_3\,\mathrm{GCD}(N_1K_1,N_2K_2)}\cdot\int (da^1\wedge a^2 + a^1\wedge da^2)\,. \tag{A17}$$

After the integration over $\mathcal{M}^3$, the change amounts should be quantized at K_1 in Eq. (A15), K_2 in Eq. (A16), and K_3 in Eq. (A17). We may apply the Dirac quantization condition (4) and the quantized Wilson loop $\frac{N_IK_I}{2\pi}\int_{\mathcal{M}^1} a^I \in \mathbb{Z}$ that is obtained via equations of motion of b^I in the presence of A^I background. As a result, three necessary and sufficient constraints are achieved: $\frac{\tilde{K}_1}{\mathrm{GCD}(N_1K_1,N_2K_2)} \in \mathbb{Z}$, $\frac{\tilde{K}_2}{\mathrm{GCD}(N_1K_1,N_2K_2)} \in \mathbb{Z}$, $\frac{\tilde{K}_3}{N_3K_3} \in \mathbb{Z}$. By using Bezout's lemma, the minimal periodicity of k is given by GCD of $\mathrm{GCD}(N_1K_1,N_2K_2)$, $\mathrm{GCD}(N_1K_1,N_2K_2)$, and N_3K_3, which is $\mathrm{GCD}(N_1K_1,N_2K_2,N_3K_3)$. As a result, once symmetry is imposed, $\bar{\bar{q}}$ is changed from Eq. (A13) to:

$$\bar{\bar{q}} = k\frac{N_1N_2K_1K_2}{\mathrm{GCD}(N_1K_1,N_2K_2)} \bmod \frac{N_1N_2K_1K_2\,\mathrm{GCD}(N_1K_1,N_2K_2,N_3K_3)}{\mathrm{GCD}(N_1K_1,N_2K_2)}\,,\ \text{with } k \in \mathbb{Z}_{\mathrm{GCD}(N_1K_1,N_2K_2,N_3K_3)} \tag{A18}$$

which gives $\mathrm{GCD}(N_1K_1,N_2K_2,N_3K_3)$ SEGs. Since $\mathrm{GCD}(N_1K_1,N_2K_2,N_3K_3) \geq \mathrm{GCD}(N_1,N_2,N_3)$, the allowed values of $\bar{\bar{q}}$ are enriched by symmetry.

3. $G_g = \mathbb{Z}_{N_1} \times \mathbb{Z}_{N_2} \times \mathbb{Z}_{N_3}$ with $G_s = \mathbb{Z}_{K_1} \times \mathbb{Z}_{K_2} \times \mathrm{U}(1)$

In this part, we consider $U(1)$ symmetry. We consider the following symmetry assignment and add it in the action (A1):

$$\frac{i}{2\pi}\sum_i^2\int A^i\wedge db^i + A^{U(1)}\wedge db^3\,. \tag{A19}$$

where the $\mathsf{U}(1)$ Wilson loop

$$\int_{S^1} A_{U(1)} \in \mathbb{R} \tag{A20}$$

meaning that the $\mathsf{U}(1)$ Wilson loop can be any real value. Under the gauge transformation (A3), the change amounts of the integral $\frac{1}{2\pi}\int_{\mathcal{M}^3} db^I$ in Eqs. (A4, A5) should be multiple of K_1 or K_2 such that the coupling terms (A19) is gauge invariant modular 2π. More quantitatively, with symmetry taken into account, from Eqs. (A4, A5), we may obtain the quantization of $\bar{\bar{q}}$: $\bar{\bar{q}} = \frac{kN_1N_2K_1K_2}{\mathrm{GCD}(N_1K_1,N_2K_2)}$ with $k \in \mathbb{Z}$ such that the change amounts are multiple of K_i. To remove the redundancy in the possible value of $\bar{\bar{q}}$, we do the shift operations as that from (A15) to (A17). Similarly to the case above, after the integration over $\mathcal{M}^3$, the change amounts should be quantized at K_1 in Eq. (A15), K_2 in Eq. (A16), and zero in Eq. (A17) due to the fact that the U(1) Wilson loop can be any real value. As a result, three necessary and sufficient constraints are achieved: $\frac{\tilde{K}_1}{\mathrm{GCD}(N_1K_1,N_2K_2)} \in \mathbb{Z}$, $\frac{\tilde{K}_2}{\mathrm{GCD}(N_1K_1,N_2K_2)} \in \mathbb{Z}$, $\tilde{K}_3 = 0$. By using Bezout's lemma, the minimal period of k is $GCD(N_1K_1, N_2K_2)$, i.e.

$$\bar{\bar{q}} = k\frac{N_1N_2K_1K_2}{\mathrm{GCD}(N_1K_1, N_2K_2)} \text{ mod } N_1N_2K_1K_2\,, \text{ with } k \in \mathbb{Z}_{\mathrm{GCD}(N_1K_1,N_2K_2)} \tag{A21}$$

which gives $\mathrm{GCD}(N_1K_1, N_2K_2)$ SEGs.

4. $G_g = \mathbb{Z}_{N_1} \times \mathbb{Z}_{N_2}$ with $G_s = \mathbb{Z}_K \times \mathrm{U}(1)$-(I)

Here we consider the symmetry assignment and add it in the action (8):

$$\frac{i}{2\pi}\int A^K \wedge db^1 + A^{U(1)} \wedge db^2 \tag{A22}$$

which indicates that the first layer carries the discrete symmetry $\mathbb{Z}_K$ while the second layer carries U(1). To determine the possible values of q in the presence of this global symmetry, We observe that the change amounts of the integral $\frac{1}{2\pi}\int_{\mathcal{M}^3} db^1$ in Eq. (12) should be multiple of K such that the first coupling term in Eq. (A22) is gauge invariant modular 2π. But the key observation is that the U(1) Wilson loop (A20) is any real value, therefore, to keep the second coupling term in Eq. (A22) gauge invariant, the change amount $\frac{1}{2\pi}\int_{\mathcal{M}^3} db^2$ in Eq. (13) should be strictly zero, which would be only the case that $q = 0$. Similarly, $\bar{q} = 0$. Therefore, SEG only happens when $q = \bar{q} = 0$.

5. $G_g = \mathbb{Z}_{N_1} \times \mathbb{Z}_{N_2}$ with $G_s = \mathbb{Z}_K \times \mathrm{U}(1)$-(II)

In this part, we consider the whole symmetry group G_s at the same layer and add the following part in the action (8) where we first set $\bar{q} = 0$:

$$\frac{i}{2\pi}\int A^K \wedge db^1 + A^{U(1)} \wedge db^1 \tag{A23}$$

Similar to the case that the symmetry subgroup are assigned at different layers, in order to keep to the second term in (A23) gauge invariant, the change amount of the integral $\frac{1}{2\pi}db^1$ should be strictly zero. Therefore, $q = 0$. For the similar reason, $\bar{q} = 0$. This symmetry assignment also only happens when $q = \bar{q} = 0$.

Appendix B: Several examples

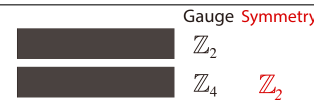
1. SEG($\mathbb{Z}_2 \times \mathbb{Z}_4, \mathbb{Z}_2$)

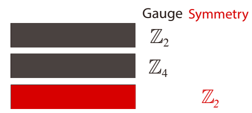
In the main text, we illustrate the example of $\mathbb{Z}_2 \times \mathbb{Z}_2$ gauge with $\mathbb{Z}_2$ symmetry. Here, we calculate another example: $G_g = \mathbb{Z}_2 \times \mathbb{Z}_4$ with $G_s = \mathbb{Z}_2$. Before imposing symmetry, there are 4 gauge theories in total, denoted by $(q, \bar{q})$:(0,0),(0,4),(4,0) and (4,4). In the first subtable of Table S5, the symmetry $\mathbb{Z}_2$ is assigned at the first layer where the $\mathbb{Z}_2$ gauge subgroup lives. From this table, it is clear that both q and $\bar{q}$ have four choices, resulting in 4^2 SEGs. Among these four choices of, say, q, we may further regroup them into two groups: $\{0 \text{ mod } 16, 8 \text{ mod } 16\}$ and $\{4 \text{ mod } 16, 12 \text{ mod } 16\}$. The two choices in the former group are SEG descendants of GT with $q = 0$ mod 8 before imposing symmetry. The two choices in the latter group are SEG descendants of GT with $q = 4$ mod 8 before imposing symmetry. In this sense, this table is sharply different from the first subtable of Table III where some entries are marked by "N/A".

In the second subtable of Table S5, the symmetry is assigned at the second layer where the $\mathbb{Z}_4$ gauge subgroup lives. The results are similar to the second table of Table III, where some entries are marked by "N/A". Totally, there are 2^2 SEGs.

TABLE S5. $\mathsf{SEG}(\mathbb{Z}_2 \times \mathbb{Z}_4, Z_2)$.

Symmetry assignment					
	$q/4\pi^2 a^1 a^2 da^2$		$\bar{q}/4\pi^2 a^1 a^2 da^2$		
GT	0 mod 8	4 mod 8	0 mod 8	4 mod 8	
SEG	0 mod 16 8 mod 16	4 mod 16 12 mod 16	0 mod 16 8 mod 16	4 mod 16 12 mod 16	4^2

Symmetry assignment					
	$q/4\pi^2 a^1 a^2 da^2$		$\bar{q}/4\pi^2 a^1 a^2 da^2$		
GT	0 mod 8	4 mod 8	0 mod 8	4 mod 8	
SEG	0 mod 16 8 mod 16	N/A	0 mod 16 8 mod 16	N/A	2^2

Symmetry assignment											
GT	$a^1 a^2 da^2$		$a^2 a^1 da^1$		$a^1 a^3 da^3$	$a^3 a^1 da^1$	$a^2 a^3 da^3$	$a^3 a^2 da^2$	$a^1 a^2 da^3$	$a^2 a^3 da^1$	
	0 mod 8	4 mod8	0 mod 8	4 mod 8	0 mod 2	0 mod 2	0 mod 4	0 mod 4	0 mod 4	0 mod 4	
SEG	0 mod 8	4 mod 8	0 mod 8	4 mod 8	0 mod 4 2 mod 4	0 mod 4 2 mod 4	0 mod 8 4 mod 8	0 mod 8 4 mod 8	0 mod 8 4 mod 8	0 mod 8 4 mod 8	2^8

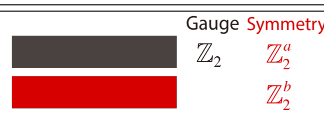
In the third subtable of Table S5, the symmetry is assigned at the third layer where there is no gauge group. This symmetry assignment induces some new nonvanishing topological interactions involving the third layer. There are in total 8 kinds of topological interactions [83]. Each topological interaction contains two choices of coefficients, rendering 2^8 SEGs.

2. SEG($\mathbb{Z}_2, \mathbb{Z}_2 \times \mathbb{Z}_2$)

In this part, we consider SEGs whose symmetry group contains more than one cyclic subgroup. In this case, a lot of new ways of symmetry assignment exist. Specifically, we consider a relatively simple example: $\mathbb{Z}_2$ gauge theory with $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry. In order to differentiate the two subgroups from each other, we introduce superscripts: $G_s = \mathbb{Z}_2^a \times \mathbb{Z}_2^b$.

In Table S6, the two symmetry subgroups are assigned to the first and second layer, respectively. Before imposing symmetry, the coefficients $q, \bar{q}$ can only take value 0 mod 2, so all topological interaction terms identically vanish. This is exactly the fact that there is only one $\mathbb{Z}_2$ gauge theory. After imposing symmetry, however, the periods of both $q, \bar{q}$ are enlarged from 2 to 8. Within one period, they can take either 0 or 4, resulting in 2^2 different SEGs. Another 2^2 SEGs can be obtained by simply exchanging the subscripts a, b.

TABLE S6. SEG $(\mathbb{Z}_2, \mathbb{Z}_2 \times \mathbb{Z}_2)$. The superscripts a and b are added to distinguish the two $\mathbb{Z}_2$ subgroups. The two symmetry subgroups are carried by the two layers respectively. There are two *independent* ways of symmetry assignment obtained by exchanging the auxiliary superscripts $a \longleftrightarrow b$.

Symmetry assignment			
GT	$q/4\pi^2 a^1 a^2 da^2$	$\bar{q}/4\pi^2 a^2 a^1 da^1$	
	0 mod 2	0 mod 2	
SEG	0 mod 8 4 mod 8	0 mod 8 4 mod 8	2^2

In Table S7, we assign the two symmetry subgroups at the second and third layer, both of which are trivial layers. In this case, as there are three layers, we need to consider 8 different topological interactions as collected in the table. As explained also in the main text, there are only two linearly independent three-layer topological interaction terms since $a^3 a^1 da^2$ is $a^1 a^2 da^3 + a^2 a^3 da^1$ up to a total derivative. Again, before imposing symmetry, coefficients of any kinds of topological terms identically vanish. After symmetry is considered, it turns out that these 8 topological interactions generate 2^8 different SEGs. In addition, in Table S8, two ways to assign the two symmetry subgroups in

TABLE S7. SEG $(\mathbb{Z}_2, \mathbb{Z}_2 \times \mathbb{Z}_2)$ The superscripts a and b are added to distinguish the two $\mathbb{Z}_2$ subgroups. The gauge group is carried by the first layer, while the two symmetry subgroups by the second and third layers respectively.

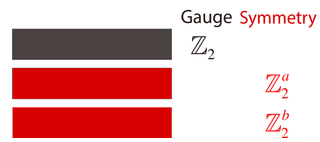
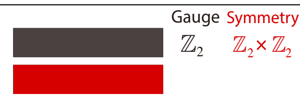
Symmetry assignment									
GT	$a^1a^2da^2$	$a^2a^1da^1$	$a^1a^3da^3$	$a^3a^1da^1$	$a^2a^3da^3$	$a^3a^2da^2$	$a^1a^2da^3$	$a^2a^3da^1$	
	0 mod 2	0 mod 2	0 mod 2	0 mod 2	0 mod 2	0 mod 2	0 mod 2	0 mod 2	
SEG	0 mod 4	0 mod 4	0 mod 4	0 mod 4	0 mod 4	0 mod 4	0 mod 4	0 mod 4	
	2 mod 4	2 mod 4	2 mod 4	2 mod 4	2 mod 4	2 mod 4	2 mod 4	2 mod 4	2^8

TABLE S8. SEG $(\mathbb{Z}_2, \mathbb{Z}_2 \times \mathbb{Z}_2)$ $G_s = \mathbb{Z}_2 \times \mathbb{Z}_2$ is carried entirely by either the first layer (the first subtable) or the second layer (the second subtable).

Symmetry assignment			
GT	$q/4\pi^2 a^1a^2da^2$	$\bar{q}/4\pi^2 a^2a^1da^1$	
	0 mod 2	0 mod 2	
SEG	0 mod 4	0 mod 4	1

Symmetry assignment			
GT	$q/4\pi^2 a^1a^2da^2$	$\bar{q}/4\pi^2 a^2a^1da^1$	
	0 mod 2	0 mod 2	
SEG	0 mod 4 2 mod 4	0 mod 4 2 mod 4	2^2

the same layer are considered. In the first subtable, there is only one SEG. But in the second subtable, the calculation shows that there are 2^2 SEGs.

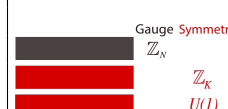
3. SEG$(\mathbb{Z}_N, \mathrm{U}(1))$

To impose the U(1) symmetry to the $\mathbb{Z}_N$ gauge theory, there are two ways, i.e. two symmetry assignments. The first one is to assign the symmetry at the same layer as that where $\mathbb{Z}_N$ gauge lives. For this symmetry assignment, it is equivalent to SET $N_1 = N, N_2 = 1, K = 1$ in the Appendix A 4, so there is only one SEG$(\mathbb{Z}_N, \mathrm{U}(1))$. The other way is to assign it at another layer whose BF term is level-one, which is equivalent to SET $N_1 = N, N_2 = 1, K = 1$ in the Appendix A 5, so there is also only one SEG$(\mathbb{Z}_N, \mathrm{U}(1))$.

4. SEG$(\mathbb{Z}_N, \mathbb{Z}_K \times \mathrm{U}(1))$

For the $\mathbb{Z}_N$ gauge enriched by $\mathbb{Z}_K \times \mathrm{U}(1)$ symmetry, there are five symmetry assignments in Table S9. Four of them only involve two layers which all have only one SEG. The fifth symmetry assignment gives rise to $[\mathrm{GCD}(N,K)]^3$. As we would see below, two roots of $[\mathrm{GCD}(N,K)]^3$ come from the stacking of SEG$(\mathbb{Z}_N, \mathbb{Z}_K)$ and a direct product state with U(1) symmetry (n.b., U(1) SPTin 3D is always trivial). The third root comes from the nontrivial interaction $a^1a^2da^3$ which correlates all layers together. Note that since the layers where the symmetry are assigned are level-one, exchanging the $\mathbb{Z}_K$ and U(1) symmetry does not lead to anything new.

TABLE S9. The five symmetry assignments of $\mathbb{Z}_N$ Gauge with $\mathbb{Z}_K \times \mathrm{U}(1)$ symmetry and the number of corresponding SEG.

	I	II	III	IV	V
Symmetry Assignment					
SEG	1	1	1	1	$[\mathrm{GCD}(N,K)]^3$

We main focus on the symmetry assignment V in Table S9. As there are three layers that we have to take into account, there are in total 8 $aada$ type topological interaction terms. To count the total number of SEGs in this symmetry assignment, we have to determine the period of of the coefficients of these eight topological interaction terms. Below we consider each of them separately because each alone can determine a set of root SEG.

1. For topological interaction $a^1a^2da^2$ or $a^2a^1da^1$, the theory reduces to that of stacking $\mathsf{SEG}(\mathbb{Z}_N,\mathbb{Z}_K)$ and U(1) SPTin three dimensions. From the calculation in Appendix II E by setting $N_1=N, N_2=1, K_1=1, K_2=K$, there are GCD$(N,K)$ different root $\mathsf{SEG}(\mathbb{Z}_N,\mathbb{Z}_K)$s from $a^1a^2da^2$ and another GCD(N,K) root $\mathsf{SEG}(\mathbb{Z}_N,\mathbb{Z}_K)$s from $a^2a^1da^1$. From Ref. [13] there is only one U(1) SPTin three dimensions. Therefore, there are GCD(N,K) different $\mathsf{SEG}(\mathbb{Z}_N,\mathbb{Z}_K\times\mathrm{U}(1))$s from the topological interaction $a^1a^2da^2$ and another GCD(N,K) root $\mathsf{SEG}(\mathbb{Z}_N,\mathbb{Z}_K\times\mathrm{U}(1))$s from $a^2a^1da^1$.

2. For topological interaction $a^1a^3da^3$ or $a^3a^1da^1$, the $\mathsf{SEG}(\mathbb{Z}_N,\mathbb{Z}_K\times\mathrm{U}(1))$ reduces to the stacking of $\mathsf{SEG}(\mathbb{Z}_N,\mathrm{U}(1))$ and $\mathbb{Z}_K$ SPTin three dimensions. From the result in Appendix B 3 (when $\mathbb{Z}_N$ and U(1) are not in the same layer), we know that there is only one $\mathsf{SEG}(\mathbb{Z}_N,\mathrm{U}(1))$ and from Ref. [13], there is only one $\mathbb{Z}_K$ SPT. Therefore, there is only one root $\mathsf{SEG}(\mathbb{Z}_N,\mathbb{Z}_K\times\mathrm{U}(1))$ from $a^1a^3da^3$ and also only one from $a^3a^1da^1$.

3. For topological interaction $a^2a^3da^3$ or $a^3a^2da^2$, the $\mathsf{SEG}(\mathbb{Z}_N,\mathbb{Z}_K\times\mathrm{U}(1))$ reduces to the stacking of $\mathbb{Z}_N$ gauge theory and $\mathbb{Z}_K\times\mathrm{U}(1)$ SPTin three dimension. It is known that there is only one $\mathbb{Z}_N$ gauge theory and from Ref. [13], there is only one $\mathbb{Z}_K\times\mathrm{U}(1)$ SPT. Therefore, there is only one root $\mathsf{SEG}(\mathbb{Z}_N,\mathbb{Z}_K\times\mathrm{U}(1))$ from $a^2a^3da^3$ and also only one from $a^3a^2da^2$.

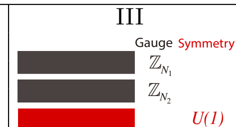
4. For topological interaction $a^1a^2da^3$, the symmetry assignment V is equivalent to SET $N_1=N, N_2=N_3=1$ and $K_1=1, K_2=K$ in Appendix A 3. Therefore, there are GCD(N,K) $\mathsf{SEG}(\mathbb{Z}_N,\mathbb{Z}_K\times\mathrm{U}(1))$s in total. For another three-layer topological interaction $a^2a^3da^1$, it is equivalent to exchange the layer index as $1\longleftrightarrow 3$, $2\longleftrightarrow 1$, $3\longleftrightarrow 2$ in Appendix. A 3. Employing the similar procedure as those for $a^1a^2da^3$, we find that the $q=0$, and so there is only one $\mathsf{SEG}(\mathbb{Z}_N,\mathbb{Z}_K\times\mathrm{U}(1))$.

In summary, for symmetry assignment V, each of $a^1a^3d^3$,$a^3a^1da^1$,$a^2a^3da^3$,$a^3a^2da^2$ and $a^2a^3da^1$ contributes only one root $\mathsf{SEG}(\mathbb{Z}_N,\mathbb{Z}_K\times\mathrm{U}(1))$ and each of $a^1a^2da^2$, $a^2a^1da^1$ and $a^1a^2da^3$ contributes GCD(N,K) root $\mathsf{SEG}(\mathbb{Z}_N,\mathbb{Z}_K\times\mathrm{U}(1))$, so in total there are $[\mathrm{GCD}(N,K)]^3$ $\mathsf{SEG}(\mathbb{Z}_N,\mathbb{Z}_K\times\mathrm{U}(1))$s for the symmetry assignment in Table S9.

5. $\mathsf{SEG}(\mathbb{Z}_{N_1}\times\mathbb{Z}_{N_2},\mathrm{U}(1))$

Without U(1) symmetry, there are in total $(N_{12})^2$ $\mathbb{Z}_{N_1}\times\mathbb{Z}_{N_2}$ gauge theories, where N_{12} is the greatest common divisor of N_1 and N_2. With the U(1) symmetry, there are three symmetry assignments, as shown in Table S10. For the assignment I and II, it is equivalent to SET $K=1$ in Appendix A 4. so there is only one SEG whose parent gauge theory is untwisted $\mathbb{Z}_{N_1}\times\mathbb{Z}_{N_2}$ gauge theory.

TABLE S10. The symmetry assignments of $\mathbb{Z}_{N_1}\times\mathbb{Z}_{N_2}$ gauge with U(1) symmetry and the numbers of corresponding gauge theory and symmetry enriched gauge theory.

	I	II	III
Symmetry assignment			
SEG	1	1	$(N_{12})^3$

For the assignment III, the number of $\mathsf{SEG}(\mathbb{Z}_{N_1}\times\mathbb{Z}_{N_2},\mathrm{U}(1))$ is $(N_{12})^3$ compared to the $(N_{12})^2$ gauge theories. For topological interaction $a^1a^2da^2$ or $a^2a^1da^1$, the root $\mathsf{SEG}(\mathbb{Z}_{N_1}\times\mathbb{Z}_{N_2},\mathrm{U}(1))$ are just stacking the $\mathbb{Z}_{N_1}\times\mathbb{Z}_{N_2}$ root gauge theories and U(1) SPTin three dimension. We know that there are $(N_{12})^2$ $\mathbb{Z}_{N_1}\times\mathbb{Z}_{N_2}$ gauge theories and only one U(1) SPTin three dimensions. Therefore there are N_{12} root $\mathsf{SEG}(\mathbb{Z}_{N_1}\times\mathbb{Z}_{N_2},\mathrm{U}(1))$ from $a^1a^2da^2$ and another N_{12} root $\mathsf{SEG}(\mathbb{Z}_{N_1}\times\mathbb{Z}_{N_2},\mathrm{U}(1))$ from $a^2a^1da^1$.

For the choice of interaction $a^1a^3da^3$, $a^3a^1da^1$, $a^2a^3da^3$ or $a^3a^2da^2$, there is only one $\mathsf{SEG}(\mathbb{Z}_{N_1}\times\mathbb{Z}_{N_2},\mathrm{U}(1))$ for all cases.

For topological interaction $a^1a^2da^3$, it is equivalent to SET $N_3=K_1=K_2=1$ in Appendix A 3, so there are N_{12} $\mathsf{SEG}(\mathbb{Z}_{N_1}\times\mathbb{Z}_{N_2},\mathrm{U}(1))$s. But for $a^2a^3da^1$, it is equivalent to exchange the layer index as $1\longleftrightarrow 3$, $2\longleftrightarrow 1$, $3\longleftrightarrow 2$

in Appendix A 3. Employing the similar procedure as those for $a^1a^2da^3$, we find that $q = 0$ and so there is only one $\mathsf{SEG}(\mathbb{Z}_N, \mathbb{Z}_K \times \mathrm{U}(1))$.

In summary, for symmetry assignment III, each of $a^1a^3d^3$,$a^3a^1da^1$,$a^2a^3da^3$,$a^3a^2da^2$ and $a^2a^3da^1$ contributes only one root $\mathsf{SEG}(\mathbb{Z}_{N_1} \times \mathbb{Z}_{N_2}, \mathrm{U}(1))$ and each of $a^1a^2da^2$, $a^2a^1da^1$ and $a^1a^2da^3$ contributes N_{12} root $\mathsf{SEG}(\mathbb{Z}_{N_1} \times \mathbb{Z}_{N_2}, \mathrm{U}(1))$, so in total there are $(N_{12})^3$ $\mathsf{SEG}(\mathbb{Z}_{N_1} \times \mathbb{Z}_{N_2}, \mathrm{U}(1))$ for the symmetry assignment III in Table S10.

Appendix C: Calculation of $\mathcal{H}^2(\mathbb{Z}_2, \mathbb{Z}_2) = \mathbb{Z}_2$ and $\mathcal{H}^2(\mathbb{Z}_3, \mathbb{Z}_2) = \mathbb{Z}_1$

In this Appendix, we calculate the second group cohomology $\mathcal{H}^2(G_s, G_g)$ which describes topologically distinct patterns of G_s symmetry fractionalization in the charge of G_g (which is abelian) gauge field. Mathematically, $\mathcal{H}^2(G_s, G_g)$ is a set of equivalent classes of 2-cocycles $\omega_2(g_1, g_2)$, where $g_1, g_2 \in G_s$ and $\omega_2(g_1, g_2)$ are G_g valued. The 2-cocycles are solutions of the 2-cocycle equations:

$$\begin{aligned} d\omega_2(g_1, g_2, g_3) &= \omega_2(g_2, g_3)\omega_2^{-1}(g_1g_2, g_3)\omega_2(g_1, g_2g_3)\omega_2^{-1}(g_1, g_2) \\ &= 1. \end{aligned} \tag{C1}$$

If $G_g = \mathbb{Z}_2$, then $\omega_2(g_1, g_2)$ takes value ± 1. Two 2-cocycles $\omega_2'(g_1, g_2)$ and $\omega_2(g_1, g_2)$ are equivalent if they differ by a 2-coboundary $\omega_2'(g_1, g_2) = \omega_2(g_1, g_2)\Omega_2(g_1, g_2)$, with

$$\Omega_2(g_1, g_2) = \frac{\Omega_1(g_1)\Omega_1(g_2)}{\Omega_1(g_1g_2)}, \tag{C2}$$

where $\Omega_1(g)$ are G_g variables. A 2-cocycle is said to be trivial if it is equivalent to $\omega_2(g_1, g_2) = 1$ for all $g_1, g_2 \in G_s$.

In the following we adopting the canonical gauge choice[1] such that $\omega_2(E, g) = \omega_2(g, E) \equiv 1$. To ensure that this is still the case after a gauge transformation, namely, to ensure $\omega_2'(g, E) = \omega_2(g, E)\Omega_2(g, E) = 1$ still holds, $\Omega_1(E) \equiv 1$ is required.

Now we calculate two simple examples $\mathcal{H}^2(\mathbb{Z}_2, \mathbb{Z}_2)$ and $\mathcal{H}^2(\mathbb{Z}_3, \mathbb{Z}_2)$ using above definition.

Cohomology $\mathcal{H}^2(\mathbb{Z}_2, \mathbb{Z}_2)$. If $G_s = \mathbb{Z}_2 = \{E, Q\}$, then there is only one 2-cocycle equation,

$$d\omega_2(Q, Q, Q) = \omega_2(Q, Q)\omega_2^{-1}(E, Q)\omega_2(Q, E)\omega_2^{-1}(Q, Q) = 1.$$

Since $\omega_2(E, Q) = \omega_2(Q, E) = 1$, above equation gives no constraint for the variable $\omega_2(Q, Q)$. Since $G_g = \mathbb{Z}_2$, $\omega_2(Q, Q)$ is a free $\mathbb{Z}_2$ variable and can freely take values ± 1. On the other hand, the 2-coboundary

$$\Omega_2(Q, Q) = \frac{\Omega_1(Q)\Omega_1(Q)}{\Omega_1(E)} = 1$$

is trivial, so there is no gauge degrees of freedom under the canonical gauge condition. This means that $\omega_2(Q, Q) = 1$ and $\omega_2(Q, Q) = -1$ stand for two different classes of 2-cocycles, which yields the result

$$\mathcal{H}^2(\mathbb{Z}_2, \mathbb{Z}_2) = \mathbb{Z}_2.$$

Cohomology $\mathcal{H}^2(\mathbb{Z}_3, \mathbb{Z}_2)$. If $G_s = \mathbb{Z}_3 = \{E, P, P^2\}$, substituting g_1, g_2, g_3 by P, P^2, we obtain eight equations, two of which are independent. The first two equations are

$$\begin{aligned} \omega_2(P, P)\omega_2^{-1}(P^2, P)\omega_2(P, P^2)\omega_2^{-1}(P, P) = 1, \\ \omega_2(P, P^2)\omega_2^{-1}(P^2, P^2)\omega_2(P, E)\omega_2^{-1}(P, P) = 1. \end{aligned}$$

We obtain,

$$\begin{aligned} &\omega_2(P, P^2) = \omega_2(P^2, P), \\ &\omega_2(P, P)\omega_2(P^2, P^2) = \omega_2(P, P^2). \end{aligned}$$

If we let $\omega_2(P, P) = \sigma,\ \omega_2(P^2, P^2) = \eta$, where σ, η are $G_g = \mathbb{Z}_2$ variables, then $\omega_2(P, P^2) = \sigma\eta$.

On the other hand, from equation (C2), we obtain,

$$\begin{aligned} \Omega_2(P, P) &= \frac{\Omega_1(P)\Omega_1(P)}{\Omega_1(P^2)} = \Omega_1(P^2), \\ \Omega_2(P, P^2) &= \Omega_2(P^2, P) = \Omega_1(P)\Omega_1(P^2), \\ \Omega_2(P^2, P^2) &= \frac{\Omega_1(P^2)\Omega_1(P^2)}{\Omega_1(P)} = \Omega_1(P). \end{aligned}$$

If we chose $\Omega_1(P^2) = \sigma$, $\Omega_1(P) = \eta$, then we obtain a new 2-cocyle

$$\omega_2'(g_1, g_2) = \omega_2(g_1, g_2)\Omega_2(g_1, g_2) = 1$$

for all $g_1, g_2 \in \mathbb{Z}_3$. Thus we have shown that these 2-cocyles are trivial, namely,

$$\mathcal{H}^2(\mathbb{Z}_3, \mathbb{Z}_2) = \mathbb{Z}_1.$$