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A Bosonic Analog of a Topological Dirac Semi-Metal: Effective Theory, Neighboring Phases, and Wire Construction

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(Dated: November 3, 2016)

We construct a bosonic analog of a two-dimensional topological Dirac Semi-Metal (DSM). The low-energy description of the most basic 2D DSM model consists of two Dirac cones at positions $\pm \mathbf{k}_0$ in momentum space. The local stability of the Dirac cones is guaranteed by a composite symmetry $Z_2^{\mathcal{TI}}$, where $\mathcal{T}$ is time-reversal and $\mathcal{I}$ is inversion. This model also exhibits interesting time-reversal and inversion symmetry breaking electromagnetic responses. In this work we construct a bosonic version by replacing each Dirac cone with a copy of the $O(4)$ Nonlinear Sigma Model (NLSM) with topological theta term and theta angle $\theta = \pm\pi$. One copy of this NLSM also describes the gapless surface termination of the 3D Bosonic Topological Insulator (BTI). We compute the time-reversal and inversion symmetry breaking electromagnetic responses for our model and show that they are twice the value one gets in the DSM case matching what one might expect from, for example, a bosonic Chern insulator. We also investigate the stability of the BSM model and find that the composite $Z_2^{\mathcal{TI}}$ symmetry again plays an important role. Along the way we clarify many aspects of the surface theory of the BTI including the electromagnetic response, the charges and statistics of vortex excitations, and the stability to symmetry-allowed perturbations. We briefly comment on the relation between the various descriptions of the $O(4)$ NLSM with $\theta = \pi$ used in this paper (a dual vortex description and a description in terms of four massless fermions) and the recently proposed dual description of the BTI surface in terms of $2+1$ dimensional Quantum Electrodynamics with two flavors of fermion ($N = 2$ QED$_3$). In a set of four Appendixes we review some of the tools used in the paper, and also derive some of the more technical results.

I. INTRODUCTION

Massless 2+1-d Dirac fermions are one of the most well-studied systems in condensed matter physics. Such fermions often appear in relativistic field theories[1], but more importantly are known to be the low-energy description of the electronic structure some 2D materials, e.g., graphene[2], and as the effective theory of the surface states of time-reversal invariant 3+1-d topological insulators[3]. In fact, in the latter two contexts alone, there have been thousands of articles in the past decade that discuss the properties of this fermion system.

The impetus for the intense focus on 2+1-d Dirac fermions was the experimental discovery of graphene[2]. Years earlier[4,5] it had been theoretically predicted that the electronic band structure of graphene near the Fermi-level would be linear dispersing, gapless cones, i.e., massless Dirac fermions. Indeed, the unique signature of the Dirac fermions was quickly confirmed in quantum Hall measurements on graphene[2]. Graphene itself has four Dirac cones, two more than the minimum of two required to satisfy the Fermion doubling theorem in 2+1-d systems with time-reversal invariance. This theorem implies that a 2+1-d material with time-reversal symmetry cannot harbor an odd number of gapless Dirac cones. Hence, the system will have a semi-metallic nature with an even number of point-like Fermi surfaces, and is often referred to as a topological Dirac semi-metal (DSM). Remarkably, this 2+1-d (semi-)metal is relatively stable upon the requirement of some additional constraints: (i) inter-cone scattering across the Brillouin zone is suppressed (translation symmetry is sufficient for non-interacting fermions), (ii) intra-cone gapping terms are forbidden (minimally we need the composite symmetry of time-reversal combined with inversion), and (iii) the

system does not form a superconductor (we need to preserve $U(1)_c$). With these conditions the 2+1-d DSM forms a robust topological semi-metal phase. Interestingly, if we relax condition (ii) then the system will form a gapped insulator, but will typically have an unusual electromagnetic response (e.g., a quantum anomalous Hall effect[6] or a charge polarization[7]).

We can find examples of systems with an odd number of massless Dirac cones as well. If we do not require time-reversal symmetry then there exist 2+1-d lattice models which have an odd number of Dirac cones, e.g., a Chern insulator model tuned to the topological critical point represents such a system[6]. On the other hand, there is another way to avoid the Fermion doubling theorem while maintaining time-reversal ($\mathcal{T}$). However, this requires something more drastic, i.e., we can produce an odd number of 2+1-d Dirac cones, and maintain $\mathcal{T}$, by considering the surface of a 3+1-d $\mathcal{T}$-invariant (electron) topological insulator (TI). The non-trivial Z_2 3+1-d topological phase is known to have an odd number of massless Dirac cones on its surface with a characteristic spin-momentum locking feature of the states on the Fermi surface. Additionally, there must be at least one massless Dirac cone located at a time-reversal invariant momentum in the Brillouin zone. This is unlike the generic 2+1-d DSM for which the Dirac cones can exist at arbitrary points in the Brillouin zone[8] It is well-known that theories with an odd number of 2+1-d massless Dirac cones typically exhibit the parity anomaly[1], and there are usually subtle features that must be carefully examined when considering the properties of such systems.

More recently there have been rapid developments in understanding symmetry-protected topological (SPT) phases with interactions[9–26]. One development in which we are

particularly interested is the prediction that there could be bosonic analogs of the electron topological insulators. Some examples are the Bosonic Integer Quantum Hall Effect (BIQHE)[15,16,27–33] and the 3D $\mathcal{T}$-invariant Bosonic Topological Insulator (BTI) [17,18,22,34]. The former is characterized by its quantized Hall conductance, which must come in integer multiples of $2e^2/h$, while the latter is characterized by a quantized magneto-electric polarizability with a Θ-angle of 2π instead of the usual value of π for the non-trivial phase of the electron topological insulator[35]. These bosonic phases are not topologically ordered, but they are SPTs that require interactions to exist; at zeroth order the interactions serve to prevent the system of bosons from forming a trivial Bose condensate.

Consider, for a moment, the 3+1-d BTI. In analogy to the electron TI we expect the surface states to exhibit unusual properties. Indeed, for one example, the surface theory can exhibit an effectively 2+1-d $\mathcal{T}$-breaking phase with a Hall conductance of $\pm e^2/h$ which is forbidden for a purely 2+1-d BIQHE phase. To understand the properties of this exotic surface state several equivalent representations of the surface theory have been given in the literature: (i) a network model of quasi-1D strips that are arrayed to form a surface and coupled, (ii) a dual description of the surface bosonic theory in terms of dual vortices, and (iii) an effective field-theory description in terms of the $O(4)$ Nonlinear Sigma Model (NLSM) with a topological theta term with coefficient $\theta = \pi$. All three of these representations of the surface were discussed in Ref. 17. The description in terms of the $O(4)$ NLSM with $\theta = \pi$ was also discussed in Ref. 36. Very recently, inspired by new developments in the description of the electron TI surface[37–39], a new dual description of the BTI surface in terms of $2+1$-d Quantum Electrodynamics with two fermion flavors ($N = 2$ QED$_3$) was proposed[40]. This new dual description was then derived in a coupled wires construction in Ref. 41. When any one of these theories is tuned to criticality it represents a surface state in a symmetry-preserving gapless phase.

In this article our goal is to develop a thorough understanding of the surface of the 3+1d BTI, and then to subsequently combine multiple copies of the theory to form a symmetry preserving bosonic semi-metal state that can exist *intrinsically* in 2+1-d without breaking some requisite symmetries. This type of semi-metal represents the bosonic analog of a 2+1-d DSM. We will present an effective theory for the bosonic semi-metal and explore in detail the requirements for its stability, the resulting electromagnetic responses, and possibilities for neighboring gapped phases with and without intrinsic topological order. We then provide an explicit coupled wires construction of this semi- metal model.

Our article is organized as follows: in Sec. II we give an overview of our main results, and in Sec. III we review the properties of the 2+1-d fermion Dirac semi-metal. Next, in Sec. IV, we review some properties of the surface theory of the 3+1-d $\mathcal{T}$-invariant BTI and provide new results and a synthesis of previous work. In Sec. V we discuss our effective theory for the 2+1-d bosonic semi-metal built from multiple copies of the bosonic TI surface states, including the quasi-topological electromagnetic response, and the stability/instabilities of this critical state. In Sec. VI we derive a criterion for identifying

a gapless semi-metal phase from the value of its polarization response. Finally, in Sec. VII we provide the details of the appropriate wire bundles and couplings to generate the bosonic semi-metal using a coupled-wire array. Following the conclusions we have a set of detailed Appendixes that review some of the technical tools used in the paper, and also contain explicit derivations of some of our more technical results.

II. MOTIVATION AND OVERVIEW OF RESULTS

In this section we provide additional background motivation, describe the logic behind our construction of a bosonic analog of a topological DSM, and give an overview of our results. Henceforth, we call such a system a Bosonic Semi-Metal (BSM). Readers interested in the technical details of the paper can refer to the specific sections for more information. As mentioned above, the main goal of this paper is to construct a model of gapless bosons in 2+1-d which shares many of the properties of the minimal two-node DSM of free fermions studied, for example, in Ref. 7. The main properties we will be interested in are: (1) the electromagnetic response of the system to perturbations which break time-reversal or inversion symmetry, and (2) the perturbative stability of the gapless, low-energy effective theory. As for any topological semi-metal, translation symmetry is an important ingredient as it prevents any scattering processes between the different Dirac or bosonic "cones" (which are generically located at different points in momentum space). Indeed, in our BSM effective theory, translation symmetry will forbid perturbations which could drive the system into a gapped state with only a trivial electromagnetic response.

Before we begin let us make a note about units. In this paper we consider systems constructed from fermions or bosons which all carry a single unit of electric charge e. For most of the paper we work in units where $e = 1$, but will restore the charge e in all final response formulas. We also take $\hbar = 1$, which means that the conductance quantum $\frac{e^2}{h} = \frac{1}{2\pi}$ in our units. We always express Hall conductances in units of $\frac{e^2}{h}$.

We start out in Sec. III by reviewing the continuum description of the two-node DSM. We focus our review on the time-reversal and inversion symmetry breaking electromagnetic responses of the DSM, and also the local (in momentum space) stability of the Dirac cones in the DSM. If the DSM is perturbed by gap-inducing terms that break $\mathcal{T}$ or $\mathcal{I}$ then the respective electromagnetic responses of the DSM take the forms

$$\mathcal{L}_{\mathcal{T}} = \frac{e^2}{4\pi}\epsilon^{\mu\nu\lambda}A_\mu\partial_\nu A_\lambda \tag{2.1a}$$

$$\mathcal{L}_{\mathcal{I}} = \frac{e}{2\pi}\epsilon^{\mu\nu\lambda}B_\mu\partial_\nu A_\lambda , \tag{2.1b}$$

where A_μ is the potential for the external electromagnetic field, and B_μ is another 3-vector field whose meaning is as follows. The two Dirac cones of the DSM are located at different points in the Brillouin zone with a wave-vector difference of $2B_i$ (for simplicity we choose their locations to be at $\mathbf{k}_\pm = \pm(B_x, B_y)$), and the two cones are separated in energy

by an amount $2B_t$. The effective Lagrangian $\mathcal{L}_{\mathcal{T}}$ represents a 2+1-d quantum Hall response with a Hall conductance of 1, while $\mathcal{L}_{\mathcal{I}}$ represents a charge polarization and orbital magnetization response whose precise meaning was discussed in Ref. 7, and which we review in Sec. VII. We note that we have suppressed a sign in these terms that tracks the nature of the inversion or time-reversal breaking.

Typically point-node semi-metals are unstable in 2+1-d unless extra symmetries are imposed. The stability of the DSM is due in part to the translation symmetry of the system. This symmetry prevents scattering processes between Dirac cones at different locations in the Brillouin zone. The local stability (in momentum space) of each Dirac cone in the DSM (at the level of free fermions) is then guaranteed by $U(1)_c$ charge conservation symmetry and a composite symmetry $Z_2^{\mathcal{TI}}$, consisting of a time-reversal transformation combined with an inversion transformation. These two symmetries forbid translation invariant terms which could gap out a single Dirac cone independently of any of the others.

Having reviewed the DSM, we can make the following observation about a minimal, two-cone DSM which directly informs our construction of a BSM model. Since a single Dirac cone is the surface theory for the 3+1-d Electron Topological Insulator (ETI)[3] the degrees of freedom in the two-node DSM can be viewed as being constructed from two copies of the ETI surface theory, but with the two copies separated in momentum space. We are therefore motivated to construct a model for a BSM from two copies of the surface theory for the 3D Bosonic Topological Insulator (BTI), but with those two copies also separated in momentum space. According to Ref. 17, one representation of the surface theory of the BTI is the $O(4)$ NLSM with theta term and $\theta = \pi$, and it is this theory which we discuss next.

In Sec. IV we give a lengthy review of the properties of the $O(4)$ NLSM with $\theta = \pi$ as it appears on the surface of the BTI. There are two reasons for giving an extended discussion of the BTI surface theory: (1) understanding just one copy of this theory is a prerequisite for understanding our BSM effective theory, which consists of two copies of the surface theory, and (2) we provide alternate derivations (and also proofs in Appendixes B and C) for some of the properties of this model. These discussions, and some additional new results, lend further support to many of the claims about this model that have already appeared in the literature. In particular we provide an extended discussion on the stability of the gapless nature of the $O(4)$ surface theory that we will require for our discussion of the BSM theory.

To begin, we recall that the $O(4)$ NLSM can be equivalently formulated in terms of an $SU(2)$ matrix field

$$U = \begin{pmatrix} b_1 & -b_2^* \\ b_2 & b_1^* \end{pmatrix} , \qquad (2.2)$$

where the components b_1 and b_2 are interpreted as representing physical bosons on the surface of the BTI. As such, they transform under the physical $U(1)_c$ charge conservation symmetry as $b_I \to e^{i\chi} b_I$ for $I = 1, 2$ (in units where the boson charge $e = 1$). This theory also has a time-reversal symmetry $Z_2^{\mathcal{T}}$ under which b_1 and b_2 are separately invariant. The ac-

tion for this model includes the conventional NLSM "kinetic energy" term, and the topological theta term,

$$S_\theta[U] = \frac{1}{24\pi^2} \int d^3x \, \epsilon^{\mu\nu\lambda} \text{tr}[(U^\dagger \partial_\mu U)(U^\dagger \partial_\nu U)(U^\dagger \partial_\lambda U)] , \qquad (2.3)$$

where $\text{tr}[\cdots]$ is the usual trace operation. In the action, the theta term is multiplied by a parameter θ, which is an angular variable defined modulo 2π. For the surface theory of the BTI we have $\theta = \pi$[17]. In Sec. IV we review the calculation of the time-reversal breaking electromagnetic response of this theory via its dual vortex description (developed in Refs. 17 and 42) and also discuss an alternate method for calculating this response that confirms this result. We also comment on the relation between the descriptions of the BTI surface used in this paper and the recently proposed dual description of the BTI surface in terms of $N = 2$ QED$_3$[40,41]. We then go on to give a careful discussion of the effects that perturbations allowed by the $U(1)_c$ and $Z_2^{\mathcal{T}}$ symmetries have on the surface theory. These perturbations were only discussed briefly in Ref. 17. Finally, we review the construction of the symmetry-preserving $\mathbb{Z}_2$ topologically ordered surface phase of the BTI which was first derived in Ref. 17.

After all of this setup we are ready to introduce an effective theory of the BSM. In Sec. V we introduce a system with two copies of the $O(4)$ NLSM with theta term. One copy has $\theta = \pi$ and the other copy has $\theta = -\pi$, and just as in the case of the fermionic DSM, the two copies of the $O(4)$ NLSM are located at positions $\mathbf{k}_\pm = \pm(B_x, B_y)$ in momentum space. In our description of the effective theory we discern how charge conservation, translation, time-reversal, and inversion symmetries act on the fields in the model, and then compute the time-reversal and inversion breaking electromagnetic responses analogous to those found in the fermion DSM. We find that these responses also exist in the BSM case, and have exactly twice the value of the responses in Eq. (2.1) for the free fermion DSM. This doubling of the response for bosonic vs. fermionic systems is similar to what happens for the case of the ETI and BTI in 3D, and also the integer quantum Hall effects for fermions and bosons in 2D[15,17].

We then go on to give a partial discussion of the stability of our theory. We argue that the translation symmetry prevents us from coupling one copy of the theory to the other copy in order to drive the system into a trivial insulating state, and that the combined $Z_2^{\mathcal{TI}}$ symmetry ensures the stability of each individual $O(4)$ NLSM. Finally, we discuss some 2D topologically ordered phases which can be accessed from our BSM model by condensing suitable bound states of the vortices in the theory. In particular, we find a phase with $\mathbb{Z}_2 \times \mathbb{Z}_2$ topological order which breaks either the time-reversal or the inversion symmetry of the original BSM model. This phase is essentially two copies of the $\mathbb{Z}_2$ topologically ordered phase found in Ref. 17, but in which the time-reversal and the inversion symmetry of the BSM exchange the two copies. We also discuss phases with $\mathbb{Z}_2$ topological order which break either the inversion or the time-reversal symmetry of the BSM model.

In Sec. VI we give a different perspective on the stability of *any* semi-metal phase by relating the gaplessness of the semi-

metal to its polarization response. In particular, we consider three broad classes of 2D gapped phases with translation symmetry which can have a polarization response, and we show that for these classes of gapped phases the polarization in the x or y direction (in the presence of inversion symmetry) is always of the form $r\frac{e}{2a_0}$, where $r \in \mathbb{Q}$ is a *rational* number and a_0 is the lattice spacing. In addition, for each class of gapped phase we are able to relate the number r to simple measurable properties of that phase. A crucial point is that the three classes of gapped phases that we consider are representative of all gapped 2D phases with translation symmetry which could be expected to exhibit a polarization response. Our result then implies that a generic (i.e., non-rational) value of the polarization in a system with translation symmetry, and in particular a *continuously tunable* polarization, indicates a gapless semi-metal phase. This shows that the gaplessness of the semi-metal is directly related to its physically measurable polarization response. Since this response is expected to be reasonably robust, this also provides additional evidence for the stability of the semi-metal phase itself.

Finally, in Sec. VII we give an explicit construction of the BSM model using an array of coupled 1D wires. This construction is motivated by the fact that 2+1-d fermion DSMs can be constructed out of arrays of coupled wires[7]. A building block for the simplest two-node DSM of fermions is a wire with a single 1+1-d massless Dirac fermion. When coupled, arrays of these wires may exhibit three related phases: (i) if an array of these wires is dominated by an intra-wire topological tunneling term then the system becomes a 2+1-d weak topological insulator that exhibits a charge polarization parallel to the wires, (ii) if the array is dominated by an inter-wire topological tunneling term then the array forms a Chern insulator phase with an integer Hall conductivity of $\pm e^2/h$, or (iii) if there is significant competition between an intra-wire and inter-wire tunneling there can be a parent critical *phase*, i.e., a DSM, which is unstable to the formation of phase (ii) if time-reversal is broken, and unstable to phase (i) if the Dirac nodes meet at the boundary of the Brillouin zone and annihilate.

A key observation of this construction is that a wire with a single 1+1-d Dirac fermion can be thought of as a narrow strip of a $\sigma_{xy} = e^2/h$ integer quantum Hall system. Hence, by analogy, we can immediately propose a 1+1-d bosonic wire model to serve as the building block for a coupled-wire construction of our 2+1-d BSM state: a narrow strip of the BIQHE, which will contain the degrees of freedom from both edges. An edge of the BIQHE can be described by an $SU(2)_1$ Wess-Zumino-Witten (WZW) conformal field theory (CFT)[15,17], therefore our 1+1-d bosonic wires will consist of two (time-reversed) copies of an $SU(2)_1$ WZW theory. The fields in each wire consist of bosons which carry charge 1 under the $U(1)_c$ symmetry.

It has been known for some time that one copy of the $O(4)$ NLSM with $\theta = \pi$ can be obtained from an array of coupled wires in which each wire contains a *single* $SU(2)_1$ WZW theory[17,42,43]. After giving a brief review of this result, we then show how our BSM model can be derived starting with 1+1-d bosonic wires containing two $SU(2)_1$ WZW theories.

We construct inter-wire tunneling terms which not only give the desired $O(4)$ NLSM's with theta angles π and $-\pi$, but also shift the two copies of the $O(4)$ NLSM's to the locations $\mathbf{k}_\pm = \pm(B_x, B_y)$ in momentum space (our specific construction gives the case with $B_x = 0$). We then show how to assign transformations under time-reversal and inversion symmetry to the fields in the coupled wires model so that the transformations of the fields in the BSM model are recovered in the continuum limit. We conclude Sec. VII with a discussion of the different physical interpretations of the coupled wire constructions of the DSM and BSM models, and we also indicate how inversion and time-reversal breaking perturbations of the BSM model can be explored within its quasi-1D coupled wire description.

In Appendix A we review the canonical quantization of the $O(4)$ NLSM, and also work out the commutators for this theory when expressed in terms of the constrained bosonic variables b_1 and b_2. This information is used in Sec. IV to investigate the effects of symmetry-allowed perturbations on the BTI surface theory, and also in Sec. V to discuss the stability of the BSM model to symmetry-allowed perturbations. In Appendix B we study a family of exact, finite energy vortex solutions to the NLSM equations of motion, and we compute the quantum numbers carried by global excitations in the background of a single vortex. In particular, we are able to prove the result, first argued for in Ref. 17, that the main effect of the theta term in the $O(4)$ NLSM is to attach a charge $\frac{\theta}{2\pi}$ of the boson b_1 to vortices in the phase of b_2, and vice-versa. In Appendix C we discuss the role of the theta term of the $O(4)$ NLSM in the Minkowski spacetime path integral of the theory. Finally, in Appendix D we resolve an apparent paradox associated with our alternative calculation, via auxiliary fermions, of the time-reversal breaking electromagnetic response of the BTI surface theory.

III. REVIEW OF THE FREE-FERMION DIRAC SEMI-METAL

Before going into the details of our construction of a bosonic analog of a Dirac semi-metal (DSM), we first give a review of the free fermion DSM for the simple case of two Dirac points in the Brillouin zone. Our review closely follows the discussion in Ref. 7, in which the electromagnetic responses of various topological semi-metals were derived. Specifically, we discuss a square lattice model of a DSM, its symmetry requirements, its low-energy description, the time-reversal and inversion symmetry breaking electromagnetic responses of the system, and finally the *local* stability of the Dirac nodes. A particularly important point is that the local stability of the DSM, which means the stability against perturbations that can gap out individual Dirac cones, is guaranteed by enforcing a composite symmetry $Z_2^{\mathcal{TI}}$, whose action consists of a time-reversal transformation composed with an inversion transformation. This composite symmetry will also play an important role in our bosonic semi-metal model.

We also note here that the surface theory of the 3D Electron Topological Insulator (ETI) is a single Dirac fermion. We

may therefore view the simple two cone DSM as a theory constructed from similar degrees of freedom as two copies of the surface theory of the 3D ETI (but with the two copies of the theory having opposite helicity). This observation is the motivation for our construction of a bosonic semi-metal from two copies of the surface theory of the 3D Bosonic Topological Insulator (BTI), which is an $O(4)$ NLSM with theta term and theta angle $\theta = \pm \pi$. We return to this point in later sections.

A. Lattice model of a DSM

We now describe a lattice model, discussed in more detail in Ref. 7, which realizes a DSM phase for a certain range of parameters. The model consists of two species/orbitals of spinless fermions at half-filling on the square lattice. We therefore have a two-component complex fermion operator $\vec{c}_{\mathbf{n}}$ at each site $\mathbf{n} = (n_x, n_y)$ of the square lattice. We take the lattice spacing $a_0 = 1$. A number of symmetries play an important role in this system. They are discrete translation symmetry, $U(1)_c$ charge conservation symmetry, $Z_2^{\mathcal{T}}$ time-reversal symmetry, and $Z_2^{\mathcal{I}}$ inversion symmetry. The fermions carry charge 1, so they transform under $U(1)_c$ as

$$U(1)_c : \vec{c}_{\mathbf{n}} \rightarrow e^{i\chi} \vec{c}_{\mathbf{n}} , \qquad (3.1)$$

where χ is a constant phase. The action of the time-reversal and inversion symmetries on the fermions is given in terms of the anti-unitary operator $\mathcal{T}$ and the unitary operator $\mathcal{I}$, respectively. We will specify the action of these operators on the complex fermions after introducing the DSM model.

The Bloch Hamiltonian of this model takes the form

$$\mathcal{H}_{2D}(\mathbf{k}) = \sin(k_x)\sigma^x + (1 - m - \cos(k_x) - t_y \cos(k_y))\sigma^z , \qquad (3.2)$$

where σ^a, $a = x, y, z$ are the Pauli matrices acting on the orbital space. The $\sin(k_x)$ term represents a complex hopping for fermions in the x direction, while the terms multiplying σ^z represent a mass term as well as real hopping terms in the x and y directions. This system has time-reversal and inversion symmetry where $\mathcal{T}$ and $\mathcal{I}$ act on the fermions as

$$\mathcal{T}\vec{c}_{\mathbf{n}}\mathcal{T}^{-1} = \sigma^z \vec{c}_{\mathbf{n}} , \qquad (3.3)$$

and

$$\mathcal{I}\vec{c}_{\mathbf{n}}\mathcal{I}^{-1} = \sigma^z \vec{c}_{-\mathbf{n}} . \qquad (3.4)$$

We see that both time-reversal and inversion symmetry act with opposite signs on the two species of fermion. We note that the inversion symmetry also negates the spatial coordinate, and $\mathcal{T}$ is anti-unitary.

The energies of the two bands of this model as a function of $\mathbf{k}$ are given by

$$E(\mathbf{k})_{\pm} = \pm \sqrt{\sin^2(k_x) + \left(1 - m - \cos(k_x) - t_y \cos(k_y)\right)^2} . \qquad (3.5)$$

When $t_y = 0$ and $m = 0$, this model has a band touching at $k_x = 0$ for any value of k_y. However, for non-zero m

and t_y the bands touch only at isolated points along the line $k_x = 0$ in the Brillouin zone. The location of these points is determined by the ratio of m and t_y. We focus our attention on the regime of $m > 0$ but $m \ll 2$ (at $m = 2$ the band touching moves to $k_x = \pi$). In this regime the low-energy physics of this system is completely described by two continuum Dirac Hamiltonians obtained by linearizing around the two band touchings. These band touchings are located at the points $\mathbf{k}_{\pm} = (0, \pm B_y)$ in the Brillouin zone, where B_y is the positive solution, in the first Brillouin zone, to the equation $m + t_y \cos(B_y) = 0$.

Performing a $k \cdot P$ expansion around $\mathbf{k}_{\pm}$ we find the low-energy Dirac Hamiltonians $\mathcal{H}_{\pm}(\mathbf{k})$

$$\mathcal{H}_{\pm}(\mathbf{k}) = k_x \sigma^x \pm t_y \sin(B_y)(k_y \mp B_y)\sigma^z . \qquad (3.6)$$

We emphasize that these two low-energy Hamiltonians have opposite signs on their k_y terms. This means that the two Dirac fermions which emerge at low-energy in this model have opposite *helicity*, i.e., the Berry phase for electrons on the two Fermi surfaces when the chemical potential is tuned away from the Dirac points have opposite signs. This form of the low-energy Hamiltonian for each Dirac point leads us directly to the continuum description of the DSM.

B. Continuum description of the DSM

For the continuum description of the DSM, we take as our starting point an effective Hamiltonian for two continuum Dirac fermions ψ_A and ψ_B with opposite helicity (which is exactly what we found in the linearized Bloch Hamiltonian for the DSM model). We then shift the zero in momentum in the k_y direction by B_y. This leads to the Hamiltonian

$$\mathcal{H}_{DSM}(\mathbf{k}) = k_x \mathbb{I} \otimes \sigma^x + k_y \sigma^z \otimes \sigma^z - B_y \mathbb{I} \otimes \sigma^z , \quad (3.7)$$

where for simplicity we have taken $t_y \sin(B_y) = 1$ to make the dispersion of the Dirac cones isotropic. In position space the original lattice fermions $\vec{c}_{\mathbf{n}}$ may be written at low energies in terms of the two continuum Dirac fermions ψ_A and ψ_B as

$$\frac{\vec{c}_{\mathbf{n}}}{a_0} \sim \psi_A(\mathbf{x})e^{iB_y y} + \psi_B(\mathbf{x})e^{-iB_y y} , \qquad (3.8)$$

where $\mathbf{x} = (x, y) = (n_x a_0, n_y a_0)$, and we have temporarily restored the lattice spacing. Now we define the multi- component fermion operator $\Psi = (\psi_A, \psi_B)^T$. One can show that $\mathcal{I}$ and $\mathcal{T}$ act on Ψ as

$$\mathcal{I}\Psi(\mathbf{x})\mathcal{I}^{-1} = \sigma^x \otimes \sigma^z \Psi(-\mathbf{x}) , \qquad (3.9)$$

and

$$\mathcal{T}\Psi\mathcal{T}^{-1} = \sigma^x \otimes \sigma^z \Psi . \qquad (3.10)$$

In particular, these operators exchange ψ_A and ψ_B, i.e., they each map fermions from one Dirac cone to the other.

At this point we may go ahead and generically allow for an offset B_x to the k_x location of the Dirac points, as well as an

offset B_t between the energies of the two Dirac points. This leads to the effective Hamiltonian

$$\mathcal{H}_{DSM}(\mathbf{k}) = k_x \mathbb{I} \otimes \sigma^x - B_x \sigma^z \otimes \sigma^x + k_y \sigma^z \otimes \sigma^z \\ - B_y \mathbb{I} \otimes \sigma^z + B_t \sigma^z \otimes \mathbb{I}. \quad (3.11)$$

To more clearly see the final structure we can pass to a Lagrangian formulation of this system. The Lagrangian has the form

$$\mathcal{L} = \bar{\Psi} \left(i \slashed{\partial} + \slashed{A} + (\sigma^z \otimes \mathbb{I}) \slashed{B} \right) \Psi, \quad (3.12)$$

where we define the gamma matrices $\gamma^0 = \mathbb{I} \otimes \sigma^y, \gamma^1 = -i\mathbb{I} \otimes \sigma^z$, and $\gamma^2 = i\sigma^z \otimes \sigma^x$, and where $\bar{\Psi} = \Psi^\dagger \gamma^0$. We have also employed the Feynman slash notation $\slashed{\partial} = \gamma^\mu \partial_\mu$, etc., and included minimal coupling to the external electromagnetic field A_μ connected to the $U(1)_c$ symmetry.

C. Electromagnetic response of the DSM

We now briefly review the electromagnetic response of the DSM to time-reversal and inversion symmetry breaking perturbations. The two mass terms

$$\Sigma_\mathcal{I} = \mathbb{I} \otimes \sigma^y \quad (3.13)$$
$$\Sigma_\mathcal{T} = \sigma^z \otimes \sigma^y, \quad (3.14)$$

are the only matrices that anti-commute with the kinetic energy terms of the DSM Hamiltonian in Eq. (3.11), *and* preserve translation invariance (i.e., they do not couple ψ_A to ψ_B, which are located at different points in momentum space). The first term $\Sigma_\mathcal{I}$ breaks inversion symmetry, while the second term $\Sigma_\mathcal{T}$ breaks time-reversal symmetry.

As shown in Ref. 7, perturbing the system with a term $-m\Sigma_\mathcal{T}$ leads to the 2D electromagnetic response $\mathcal{L}_\mathcal{T}$ from Eq. (2.1), i.e., it induces a quantum anomalous Hall effect with Chern number/Hall conductance[6] $\sigma_{xy} = \pm e^2/h$. On the other hand, perturbing the system with $-m\Sigma_\mathcal{I}$ leads to the quasi-1D electromagnetic response $\mathcal{L}_\mathcal{I}$ from Eq. (2.1). This response indicates that, when starting from the gapped, inversion breaking phase and taking $m \to 0$, the DSM limit will have a charge polarization and/or orbital magnetization. In fact, the response does not depend on the magnitude of m at all, and the dependence on m enters only as a global sign $\text{sgn}(m)$ multiplying the response formula. Interestingly, this second response term depends crucially on the properties of the Dirac nodes, i.e., their relative positions in momentum and energy.

D. Combined $\mathcal{TI}$ symmetry ensures local stability of the Dirac cones

We end this section with a quick comment about the stability of the DSM. We saw in the previous section that the only mass terms that we can add to Eq. (3.11) which are allowed by translation symmetry are the terms $\Sigma_\mathcal{T}$ and $\Sigma_\mathcal{I}$. If we add only one of these mass terms to the system then it will gap out both of the Dirac cones. However, suppose we tried to add a linear combination of these two mass terms. The two possible linear combinations are

$$\Sigma_\pm = \frac{1}{2}(\Sigma_\mathcal{I} \pm \Sigma_\mathcal{T}). \quad (3.15)$$

Adding just one of these terms would gap out either ψ_A (add Σ_+) or ψ_B (add Σ_-). However, both of these terms are forbidden by the composite symmetry $\mathcal{TI}$. Therefore, the local stability of the Dirac cones is guaranteed by the combined time-reversal times inversion symmetry $\mathcal{TI}$.[44] If we enforce this symmetry, then it is impossible to gap out one of the Dirac cones independently of the other cone, and hence they can only be removed if they are perturbed enough to collide with each other in momentum space. This means that with translation, $\mathcal{TI}$, and $U(1)_c$ preserved the DSM is a (perturbatively) stable 2+1-d semi-metal phase.

IV. THE SURFACE THEORY OF THE BOSONIC TOPOLOGICAL INSULATOR

In this section we review, and also clarify some aspects of, the surface theory of the 3+1-d BTI. Since our BSM model is constructed from two copies of the surface theory of the BTI, it is essential that we discuss this theory in detail. The surface theory of the BTI was first derived in Ref. 17, where it was obtained from a network model constructed from coupled edge theories of the BIQH state (we briefly discuss this network model in Sec. VII). The authors of Ref. 17 then used this theory, as well as a dual vortex description of the theory, to derive many possible surface phases for the BTI. These different possible surface phases were further investigated and clarified in Ref. 18 which utilized monopole configurations of the external gauge field to probe the properties of the various phases.

In this section we provide a detailed account of the surface theory of the BTI, which is equivalent to an $O(4)$ NLSM with theta term and theta angle $\theta = \pi$. We first discuss the basic properties of this theory, and also the transformations of the $O(4)$ field under the physical symmetry group of the BTI. We then give a summary of the dual description of the theory, but from a different point of view than the one given in Ref. 17. We then show how the time-reversal symmetry breaking electromagnetic response of the BTI surface can be obtained from the dual description. We also describe an alternative method for calculating the electromagnetic response of the theory. This method uses a well-known formula derived by Abanov and Wiegmann in Ref. 45, which allows one to write the original $O(4)$ NLSM as a path integral over a set of auxiliary fermions which couple to the $O(4)$ field. Since the $O(4)$ NLSM is such a difficult system to study, having two different methods for calculating the response which give the same answer is strong corroborating evidence. We next discuss the stability of the gapless theory. In particular, we carefully study the effects of symmetry-allowed perturbations, some of which were briefly discussed in Ref. 17. Finally, we end the section with a brief review of the symmetry-preserving, topologically

ordered surface phase for the BTI proposed in Ref. 17. After all of this is complete we will be ready to discuss the properties of the bosonic semi-metal state.

A. The $O(4)$ NLSM with theta term

In this subsection we review the description of the surface of the Bosonic Topological Insulator (BTI) in terms of one 2+1-d $O(4)$ Nonlinear Sigma Model (NLSM) with a topological theta term having $\theta = \pi$. The $O(4)$ NLSM field $\mathbf{N} = (N^1, N^2, N^3, N^4)$ is a real-valued unit vector field (i.e., $\mathbf{N} \cdot \mathbf{N} = 1$). The action for this theory with a general theta angle takes the form

$$ S = \int d^3x \, \frac{1}{g}(\partial^\mu N^a)(\partial_\mu N^a) - \theta S_\theta[\mathbf{N}] \,, \qquad (4.1) $$

where we sum over all repeated indices ($\mu = t, x, y$ and $a = 1, 2, 3, 4$), and the theta term is

$$ S_\theta[\mathbf{N}] = \frac{1}{12\pi^2} \int d^3x \, \epsilon^{\mu\nu\lambda}\epsilon_{abcd}N^a\partial_\mu N^b\partial_\nu N^c\partial_\lambda N^d \,. \qquad (4.2) $$

The coefficient g is a positive coupling constant. Small g favors an ordered phase in which N^a is constant everywhere in spacetime, while large g favors a disordered phase. The theta term only plays a role in the disordered phase, so we assume that we are working in the large g regime.

For the description of the surface of the BTI, it is more convenient to use a formulation of the $O(4)$ NLSM in terms of an $SU(2)$ matrix U which is related to the unit vector $\mathbf{N}$ via $U = N^4\mathbb{I} + \sum_{a=1}^{3} iN^a\sigma^a$. In terms of U the action takes the form

$$ S = \int d^3x \, \frac{1}{2g}\text{tr}[\partial^\mu U^\dagger \partial_\mu U] - \theta S_\theta[U] \,, \qquad (4.3) $$

where now

$$ S_\theta[U] = \frac{1}{24\pi^2} \int d^3x \, \epsilon^{\mu\nu\lambda}\text{tr}[(U^\dagger\partial_\mu U)(U^\dagger\partial_\nu U)(U^\dagger\partial_\lambda U)] \,, \qquad (4.4) $$

and tr[...] denotes the usual trace operation for matrices. In this form, the $O(4)$ NLSM is also known as the $SU(2)$ Principal Chiral Nonlinear Sigma Model (PCNLSM).

The renormalization group (RG) flows of general $SU(N)$ PCNLSM's in the (g, θ) plane were studied qualitatively in Ref. 46. In that paper the authors argued that the theory with $\theta = \pi$ could either be gapless or have a degenerate ground state. In the gapless case they predicted an RG fixed point at $\theta = \pi$ and $g = g^*$ for some finite g^*, while for the degenerate case they predicted that g flows off to positive infinity. In this paper we focus only on the first possibility of a gapless theory. We might also suspect that the $O(4)$ NLSM at $\theta = \pi$ is gapless on the grounds that its lower dimensional cousin, the $O(3)$ NLSM with $\theta = \pi$, was also shown to be gapless in Ref. 47.

For the description of the BTI surface, one writes U in terms of two complex fields b_1 and b_2 as

$$ U = \begin{pmatrix} b_1 & -b_2^* \\ b_2 & b_1^* \end{pmatrix} \,, \qquad (4.5) $$

where b_1 and b_2 are subject to the constraint $\sum_{I=1}^{2} |b_I|^2 = 1$, which is equivalent to the original constraint $\mathbf{N} \cdot \mathbf{N} = 1$ of the $O(4)$ NLSM. We should think of b_1 and b_2 as representing the physical bosonic degrees of freedom on the surface of the BTI, and so we will refer to b_I, $I = 1, 2$, as "bosonic fields" for the rest of the article. Using these fields we see that the $O(4)$ NLSM can be viewed as being essentially a theory of two complex scalar fields b_1 and b_2, however, these fields interact with each other due to (i) the constraint $\sum_I |b_I|^2 = 1$, and (ii) the theta term $S_\theta[U]$.

The BTI is a gapped bosonic phase of matter protected by $U(1)_c$ charge conservation symmetry and $Z_2^{\mathcal{T}}$ time-reversal symmetry. Under these symmetries the bosonic fields b_I transform as

$$ U(1)_c : b_I \to e^{i\chi}b_I \qquad (4.6) $$
$$ Z_2^{\mathcal{T}} : b_I(t, \mathbf{x}) \to b_I(-t, \mathbf{x}) \,, \qquad (4.7) $$

for $I = 1, 2$, where $\mathbf{x} = (x, y)$ denotes the spatial coordinates. These transformations give the total symmetry group the structure $U(1)_c \rtimes Z_2^{\mathcal{T}}$, where the semi-direct product "$\rtimes$" indicates that the $U(1)_c$ and $Z_2^{\mathcal{T}}$ transformations do not commute with each other. As we explain in the next few paragraphs, the $O(4)$ NLSM theory with a theta term only possesses this time-reversal symmetry when θ is an integer multiple of π.

To see why the only time-reversal symmetric values of θ are $\theta = n\pi$, $n \in \mathbb{Z}$, we first make a transformation to Euclidean spacetime. Euclidean time is defined by $\tau = it$, and the theta term in Euclidean spacetime has the form

$$ S_{\theta,E}[\mathbf{N}] = -\frac{i}{12\pi^2} \int d^3x_E \, \epsilon^{\mu\nu\lambda}\epsilon_{abcd}N^a\partial_\mu N^b\partial_\nu N^c\partial_\lambda N^d \,, \qquad (4.8) $$

where $d^3x_E = d\tau d^2\mathbf{x}$ is the integration measure for Euclidean spacetime, and now $\mu, \nu, \lambda = \tau, x, y$. The theta term is now imaginary, which means that $e^{-\theta S_{\theta,E}[\mathbf{N}]}$ appears as a phase factor in the Euclidean path integral. Under a time-reversal transformation we send $t \to -t$, $i \to -i$ (since this symmetry is anti-unitary), and $b_I(t, \mathbf{x}) \to b_I(-t, \mathbf{x})$. Since $\tau = it$, τ is invariant under this transformation. Therefore we find that under time-reversal $S_{\theta,E}[\mathbf{N}] \to -S_{\theta,E}[\mathbf{N}]$[48].

If we impose boundary conditions on $\mathbf{N}$ such that $\mathbf{N}$ tends to a fixed configuration $\mathbf{N}_0$ at infinity in all directions of Euclidean spacetime, then we may identify Euclidean spacetime with the sphere S^3. The sphere S^3 is also the configuration space for the $O(4)$ NLSM, so in this situation the theta term becomes quantized,

$$ \frac{1}{12\pi^2} \int d^3x_E \, \epsilon^{\mu\nu\lambda}\epsilon_{abcd}N^a\partial_\mu N^b\partial_\nu N^c\partial_\lambda N^d = n_I \in \mathbb{Z} \,, \qquad (4.9) $$

where n_I is the *instanton* number of the field configuration $\mathbf{N}$. The quantization of this integral follows from the homotopy group $\pi_3(S^3) = \mathbb{Z}$. In fact, the theta term is just the pull-back to spacetime of the volume form on S^3. Since Euclidean spacetime (with the boundary conditions discussed above) is just another copy of S^3, the integral is required to be an integer, which just counts the number of times that the spacetime

S^3 wraps around the configuration space (also S^3) of the $O(4)$ NLSM field $\mathbf{N}$.

In the Euclidean path integral, the theta term appears in an exponential, $e^{-\theta S_{\theta,E}[\mathbf{N}]} = e^{i\theta n_I}$, which shows that the parameter θ is only defined modulo 2π. We have already seen that a time-reversal transformation sends $\theta \to -\theta$. It is then immediate to see that the only time-reversal symmetric values of θ are $\theta = n\pi$, $n \in \mathbb{Z}$, since it is only these values of θ which satisfy $\theta \equiv -\theta$ mod 2π. It was shown in Ref. 17 that the gapless surface termination of the BTI is described by the $O(4)$ NLSM with $\theta = \pi$, and hence preserves time-reversal symmetry.

Another comment can be made about the interpretation of the theta term in Euclidean spacetime. It was shown in Ref. 42 that the one-instanton configuration of the $O(4)$ field $\mathbf{N}$ can be re-interpreted in terms of vortex configurations of the bosonic fields b_1 and b_2. Recall that a *vortex* of the field b_I is a point in space around which the phase of b_I winds by 2π. In 2+1-d the spacetime trajectory, or *worldline*, of a vortex is just a line (or curve) in spacetime. In Euclidean spacetime (compactified to the sphere S^3 via appropriate boundary conditions), the worldlines of vortices become closed loops. In Ref. 42 it was shown that the one-instanton configuration of the field $\mathbf{N}$ is equivalent to a linking configuration in which the worldline of a vortex in the phase of b_1 links exactly once with a worldline of a vortex in the phase of b_2. Since this configuration contributes a phase of $e^{i\theta}$ to the Euclidean path integral, the authors of Ref. 42 interpreted this to mean that a vortex in b_1 and a vortex in b_2 have a mutual statistical angle of θ. This means that a braiding process in which a vortex in b_1 makes a complete circuit around a vortex in b_2 should result in an overall phase factor $e^{i\theta}$ for the wave functional of the quantum field theory. This result was then used in Ref. 17 to deduce a topologically ordered surface phase for the BTI. We will review this topologically ordered phase at the end of this section. We remark in passing that similar arguments were also used in Ref. 49 to deduce the braiding statistics of particle and loop-like excitations in gauged SPT phases from their description in terms of NLSM's with theta term.

It is clear from the discussion in the preceding paragraphs that the theta term plays an important role in the physics of the $O(4)$ NLSM. However, in this section we relied extensively on the interpretation of the theta term in Euclidean spacetime to understand its special properties. To better understand the quantum mechanics of the $O(4)$ NLSM with theta term, it is desirable to understand the role the theta term plays in the Minkowksi spacetime path integral. In Appendix C we explain the precise interpretation of the theta term in Minkowksi spacetime, and show that the theta term does indeed contribute a phase $e^{i\theta}$ to the path integral for configurations of the $O(4)$ field in which a vortex in the boson b_2 makes a complete circuit around a vortex in b_1. This result confirms the interpretation of the theta term given by Senthil and Fisher in Ref. 42, which was based on an analysis of the theory in Euclidean spacetime. In addition, following an argument from Ref. 50, this result implies that a bound state of a vortex in b_1 and a vortex in b_2 carries intrinsic angular momentum $J = \frac{\theta}{2\pi}$. At $\theta = \pi$ we have $J = \frac{1}{2}$, which means that the vortex bound state is a fermion, as was discussed in Ref. 17.

B. Time-reversal breaking response

In this section we discuss the calculation of the time-reversal breaking electromagnetic response of the $O(4)$ NLSM with $\theta = \pi$. This response is in principle obtained by coupling the NLSM to the external electromagnetic field A_μ, turning on a small time-reversal breaking perturbation, integrating out the matter fields b_1 and b_2, and then setting the time-reversal breaking perturbation to zero. In practice, however, it is very difficult to integrate out the NLSM field directly, and so we make use of two alternative and completely different methods for calculating the time-reversal breaking response of the theory. The fact that these two methods give the same answer strongly suggests that the answer is the correct one, even though it has not, as yet, been checked with a direct calculation in the $O(4)$ NLSM.

1. Method 1: Dual Vortex Description

The first method for calculating the time-reversal breaking response of the BTI surface is to use the dual vortex description of the $O(4)$ NLSM which becomes possible at the special value of $\theta = \pi$. This dual vortex description was first obtained in Ref. 42 using a lattice formulation of the theory. The continuum version of this dual vortex theory was then used extensively in Ref. 17 to study the possible surface phases of the BTI. In this section we give a review of this dual description from an alternative perspective that is complementary to that given in Refs. 17 and 42.

We have already explained how the $O(4)$ NLSM can be regarded as a theory of two complex scalar fields b_1 and b_2 subject to the constraint $\sum_I |b_I|^2 = 1$. This constraint has a strong effect on the physics of vortices in the fields b_1 and b_2. Recall that a vortex in the field b_1 is a point in space around which the phase of b_1 winds by 2π. At such a point the phase of b_1 is undefined, and so the amplitude of b_1 must vanish at that point. However, since the fields b_1 and b_2 are subject to the constraint discussed above, this means that in the core of a vortex in b_1 we have $|b_2| = 1$. This indicates that vortices in b_1 can trap charge of b_2 and vice-versa. In fact, in Minkowski spacetime the main effect of the theta term is to attach charge $\frac{\theta}{2\pi}$ of boson b_1 to vortices in b_2 and vice-versa. A heuristic argument for this effect was given in Ref. 17. In Appendix B we prove this result explicitly by computing the charges of global excitations on the background of certain exact vortex solutions of the NLSM equations of motion.

We first give a short review of the dual vortex description of the theory of an ordinary charged scalar field in 2+1-d, and refer the reader to Ref. 51 for a more detailed description of this technique. Consider first an ordinary complex scalar field b, with a Lagrangian of the form

$$\mathcal{L} = |(\partial_\mu - iA_\mu)b|^2 - \frac{\mu}{2}|b|^2 - \frac{\lambda}{4}|b|^4 + \dots \quad (4.10)$$

For later convenience we write b in a density phase representation as $b = \rho e^{i\vartheta}$. When $\mu < 0$ this system has a symmetry-broken ground state in which $\rho = \bar{\rho} = \sqrt{-\frac{\mu}{\lambda}}$ and the phase of b is locked to a particular value (thus spontaneously breaking the original $U(1)_c$ symmetry under $b \to e^{i\chi} b$). The low-energy excitations about this ground state are the gapless fluctuations of the phase ϑ of b (the Goldstone modes), which are described by

$$\mathcal{L} = \bar{\rho}^2 (\partial_\mu \vartheta - A_\mu)^2 + \dots . \quad (4.11)$$

The fluctuations ϑ consist of two parts, $\vartheta = \vartheta^s + \vartheta^v$. The smooth part ϑ^s consists of small fluctuations around the fixed vacuum value of ϑ. The second part ϑ^v consists of vortices in which the phase winds by some multiple of 2π around the vacuum manifold (i.e., the circle defined by $|b| = \bar{\rho}$).

In the usual boson-vortex duality a sequence of transformations is now applied to the Lagrangian Eq. (4.11) (more precisely, these transformations are applied to the path integral) to obtain a final Lagrangian of the form

$$\mathcal{L} = |(\partial_\mu - i\alpha_\mu)\phi|^2 - \frac{\tilde{\mu}}{2}|\phi|^2 - \frac{\tilde{\lambda}}{4}|\phi|^4 + \dots$$
$$- \frac{1}{4\bar{\rho}^2}\left(\frac{1}{2\pi}\epsilon^{\mu\nu\lambda}\partial_\nu \alpha_\lambda\right)^2 - \frac{1}{2\pi}\epsilon^{\mu\nu\lambda}A_\mu \partial_\nu \alpha_\lambda . \quad (4.12)$$

This expression features two new fields: the gauge field α_μ and the complex scalar field ϕ. The field α_μ is a non-compact gauge field which is introduced to represent the conserved number current J^μ of the original bosons b via the equation $J^\mu = \frac{1}{2\pi}\epsilon^{\mu\nu\lambda}\partial_\nu \alpha_\lambda$. Non-compactness of α_μ is just the statement that $\epsilon^{\mu\nu\lambda}\partial_\mu \partial_\nu \alpha_\lambda = 0$, which guarantees the conservation of J^μ. The excitations of the new complex scalar field ϕ represent vortices in the phase of the original boson b. The vortex current of b, defined by $K^\mu = \frac{1}{2\pi}\epsilon^{\mu\nu\lambda}\partial_\nu \partial_\lambda \vartheta^v$, is given in this representation by the number current of ϕ as $K^\mu = i(\phi\,\partial^\mu\phi^* - \phi^*\partial^\mu\phi)$ (in other words, the $U(1)$ charge of ϕ is the vortex number). We have also included a number of potential energy terms which could appear in the action for the vortex field ϕ.

We now apply this technique to the boson b_2 in the $O(4)$ NLSM while leaving b_1 untransformed (a nearly identical discussion can be had if one chooses to dualize b_1 and leave b_2 fixed instead). We therefore define a new complex scalar field $\phi_{2,+}$ which creates a vortex in the phase of b_2. From the discussion earlier in this section, and the results of Appendix B, $\phi_{2,+}$ carries charge $\frac{\theta}{2\pi}$ under the $U(1)_c$ symmetry. We represent the conserved number current J_2^μ of b_2 using the non-compact gauge field $\alpha_{2,\mu}$.

At this point, the dual vortex description of the $O(4)$ NLSM with general angle θ takes the form

$$\mathcal{L} = \frac{1}{g}|(\partial_\mu - iA_\mu)b_1|^2 + |(\partial_\mu - i\alpha_{2,\mu} - i\frac{\theta}{2\pi}A_\mu)\phi_{2,+}|^2 + \dots$$
$$- \frac{1}{\kappa_2}\left(\frac{1}{2\pi}\epsilon^{\mu\nu\lambda}\partial_\nu \alpha_{2,\lambda}\right)^2 - \frac{1}{2\pi}\epsilon^{\mu\nu\lambda}A_\mu \partial_\nu \alpha_{2,\lambda} , \quad (4.13)$$

where the ellipses stand for possible potential energy terms. The field $\phi_{2,+}$ carries charge of both the dual gauge field $\alpha_{2,\mu}$

and the external field A_μ. The theta term is entirely responsible for the coupling of $\phi_{2,+}$ to A_μ. Finally, the constant κ_2 is given by $\kappa_2 = \frac{g}{4\bar{\rho}_2^2}$, where $\bar{\rho}_2$ is the absolute value of b_2 in the condensed phase.

Interestingly, exactly at the special value $\theta = \pi$, it becomes possible to replace our description of the theory in terms of b_1 and $\phi_{2,+}$ with a much more symmetric dual description in terms of two types of vortices $\phi_{2,+}$ and $\phi_{2,-}$, as we now explain. At $\theta = \pi$, the vortex $\phi_{2,+}$ carries charge $\frac{1}{2}$ of boson b_1. In this case the composite field

$$\phi_{2,-} = \phi_{2,+}b_1^* , \quad (4.14)$$

carries charge $-\frac{1}{2}$ of boson b_1 (note that we are using * to represent anti-particles). The field $\phi_{2,-}$ can be understood as a vortex-anti-boson bound state, and at $\theta = \pi$ it is a natural object to consider because of the fact that it carries the same magnitude of charge as the original vortex $\phi_{2,+}$ (note that we can always define $\phi_{2,-}$ in this way for *any* value of θ, but this field only transforms nicely under the symmetries of the theory when $\theta = \pi$).

Further justification for the introduction of the field $\phi_{2,-}$ can be obtained by recalling that at the special value $\theta = \pi$, the time-reversal symmetry of the $O(4)$ NLSM is restored. It follows that the vortex $\phi_{2,+}$ should have a well-defined transformation under time-reversal when $\theta = \pi$. Vortices should transform into anti-vortices under the action of time-reversal, since time-reversal is an anti-unitary symmetry. On the other hand, the time-reversal partner of $\phi_{2,+}$ should have the same $U(1)_c$ charge as $\phi_{2,+}$ in order to preserve the structure of the symmetry group of the BTI. It turns out that $\phi_{2,-}^*$ has just the right properties to be the partner of $\phi_{2,+}$ under the time-reversal operation.

We see then that at the special value $\theta = \pi$, the dual description of the $O(4)$ NLSM is given most naturally in terms of the two-component vortex field $\Phi_2 = (\phi_{2,+}, \phi_{2,-})^T$, which transforms under the $U(1)_c$ and $Z_2^{\mathcal{T}}$ symmetries according to

$$U(1)_c : \Phi_2 \to e^{i\frac{\chi}{2}\sigma^z}\Phi_2 \quad (4.15)$$
$$Z_2^{\mathcal{T}} : \Phi_2(t, \mathbf{x}) \to \sigma^x \Phi_2^*(-t, \mathbf{x}) . \quad (4.16)$$

In terms of the pair of vortex fields making up Φ_2, the final dual action takes the form

$$\mathcal{L} = \sum_{s=\pm} |(\partial_\mu - i\alpha_{2,\mu} - i\frac{s}{2}A_\mu)\phi_{2,s}|^2 + \dots \quad (4.17)$$
$$- \frac{1}{\kappa_2}\left(\frac{1}{2\pi}\epsilon^{\mu\nu\lambda}\partial_\nu \alpha_{2,\lambda}\right)^2 - \frac{1}{2\pi}\epsilon^{\mu\nu\lambda}A_\mu \partial_\nu \alpha_{2,\lambda} ,$$

where again the ellipses stand for possible potential energy terms. Note that the two species of vortex carry the same charge of $\alpha_{2,\mu}$, but opposite charge of A_μ.

As stated in Ref. 17, the original boson field b_1 is now represented approximately by

$$b_1 = \phi_{2,+}\phi_{2,-}^* , \quad (4.18)$$

i.e., it is a bound state of a vortex ($\phi_{2,+}$) and an anti-vortex ($\phi_{2,-}^*$). We should, however, take a moment to consider this

equation carefully. Interestingly, the two sides of this equation do not have the same dimensions. The field b_1 is dimensionless, while the complex scalar fields $\phi_{2,+}$ and $\phi_{2,-}$ carry dimensions of $(\text{length})^{-\frac{1}{2}}$ (this is true because the vortex current K^μ has dimensions of $(\text{length})^{-2}$). A more precise version of this equation would be to write

$$b_1 \sim g\, \phi_{2,+}\phi_{2,-}^* \,, \qquad (4.19)$$

where g is the NLSM coupling which has units of length, and where an arbitrary dimensionless constant could be included on the right-hand side of this equation.

The time-reversal breaking response of the $O(4)$ NLSM at $\theta = \pi$ can now be explored using the dual description in Eq. (4.17). A gapped, time-reversal breaking phase is realized when, for example, $\phi_{2,+}$ condenses and $\phi_{2,-}$ becomes gapped, or vice-versa. In order to induce this phase, one needs to include in Eq. (4.17) a potential energy of the form

$$V(\Phi_2) = \mu\Phi_2^\dagger\sigma^z\Phi + \lambda_+|\phi_{2,+}|^4 + \lambda_-|\phi_{2,-}|^4 \,, \qquad (4.20)$$

where $\lambda_\pm$ are both positive. The choice of which vortex condenses and which is gapped depends on the sign of μ. Note that the term $\Phi_2^\dagger\sigma^z\Phi$ explicitly breaks time-reversal symmetry.

When $\phi_{2,+}$ condenses and $\phi_{2,-}$ is gapped, we may (at low energies) set $\phi_{2,-} = 0$ and $\phi_{2,+} = \text{const.}$ to find that the minimum energy configuration is realized when $\alpha_{2,\mu} = -\frac{1}{2}A_\mu$, which yields the response

$$\mathcal{L}_{eff} = \frac{e^2}{4\pi}e^{\mu\nu\lambda}A_\mu\partial_\nu A_\lambda \,, \qquad (4.21)$$

where we have restored the charge e. This response yields a "half" bosonic quantum Hall effect with $\sigma_{xy} = \frac{1}{2}\frac{2e^2}{h}$. If we had instead condensed $\phi_{2,-}$ and gapped out $\phi_{2,+}$, we would have found the same response but with the opposite sign.

2. Method 2: Abanov-Wiegmann integration over fermions

The second method for calculating the time-reversal breaking response of the BTI surface uses a formula due to Abanov and Wiegmann[45] which allows one to express the $O(4)$ NLSM with theta term as a path integral over a set of auxiliary fermions. The fermions in this construction must also carry charge under the physical $U(1)_c$ symmetry, so we can directly couple the fermions to the $U(1)_c$ gauge field A_μ and then integrate out the fermions to deduce the electromagnetic response of the system. A similar approach was used recently in Ref. 52 to calculate the electromagnetic response of a Bosonic Integer Quantum Hall state in 4+1-d.

The starting point for this construction is a multicomponent fermionic field $\Psi = (\psi_1, \psi_2, \psi_3, \psi_4)^T$, where each of ψ_a, $a = 1, 2, 3, 4$, is a two-component Dirac fermion in 2+1-d. In what follows we use tensor product notation in order to treat spinor and "isospace" indices on equal footing. All indices are traced over in the evaluation of the fermion path integral. The rightmost 2×2 matrix in the tensor products acts on the spinor indices of ψ_a, while the left and middle matrices in the tensor products act on the isospace indices.

We define two sets of gamma matrices γ^μ and Γ^a by

$$\gamma^0 = \mathbb{I} \otimes \mathbb{I} \otimes \sigma^y \qquad (4.22\text{a})$$
$$\gamma^1 = -i\mathbb{I} \otimes \mathbb{I} \otimes \sigma^z \qquad (4.22\text{b})$$
$$\gamma^2 = i\mathbb{I} \otimes \mathbb{I} \otimes \sigma^x \,, \qquad (4.22\text{c})$$

and

$$\Gamma^1 = \sigma^x \otimes \sigma^x \otimes \mathbb{I} \qquad (4.23\text{a})$$
$$\Gamma^2 = \sigma^y \otimes \sigma^x \otimes \mathbb{I} \qquad (4.23\text{b})$$
$$\Gamma^3 = \sigma^z \otimes \sigma^x \otimes \mathbb{I} \qquad (4.23\text{c})$$
$$\Gamma^4 = \mathbb{I} \otimes \sigma^y \otimes \mathbb{I} \,, \qquad (4.23\text{d})$$

where $\mathbb{I}$ is the 2×2 identity matrix. In this case we can also define a fifth matrix for the second set,

$$\Gamma^5 = \mathbb{I} \otimes \sigma^z \otimes \mathbb{I} \,. \qquad (4.24)$$

The first set of gamma matrices obey a Clifford Algebra in Lorentz signature, $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}\mathbb{I}_{8\times 8}$, and are used to construct the derivative operator for the Dirac action. The second set obeys a Euclidean Clifford Algebra, $\{\Gamma^a, \Gamma^b\} = 2\delta^{ab}\mathbb{I}_{8\times 8}$, and is used to construct the mass terms which couple Ψ to the $O(4)$ field $\mathbf{N}$.

According to Ref. 45, a fermionic action of the form

$$\mathcal{L}_f = \bar{\Psi}\left(i\slashed{\partial} - \cos(\nu)M\Gamma^5 - \sin(\nu)M\sum_{a=1}^{4}N^a\Gamma^a\right)\Psi \,, \qquad (4.25)$$

with large mass $M > 0$ will produce, after integration over the fermions, an $O(4)$ NLSM of the form of Eq. (4.1) with the theta angle given by

$$\theta = \pi\left(1 - \frac{9}{8}\cos(\nu) + \frac{1}{8}\cos(3\nu)\right) \,. \qquad (4.26)$$

Taking $\nu = \frac{\pi}{2}$ gives $\theta = \pi$. The evaluation of this fermion path integral is not completely straightforward, and so we refer the reader to Ref. 45 as well as Ref. 53 for explanations of this calculation.

If we set $\nu = \frac{\pi}{2} - \delta$ for small δ, then the action takes the form

$$\mathcal{L}_f = \bar{\Psi}\left(i\slashed{\partial} - (M\delta)\Gamma^5 - M\sum_{a=1}^{4}N^a\Gamma^a\right)\Psi \,. \qquad (4.27)$$

Since the only time-reversal invariant values of θ are multiples of π, this corresponds to adding a small time-reversal breaking perturbation to the action (we would now get $\theta \approx \pi(1 - \frac{3}{2}\delta)$ after integrating out the fermions). We now calculate the response of the theory in the presence of this perturbation, and then take the limit $\delta \to 0$.

Before we proceed with the calculation, we mention the following puzzle. The calculation in Ref. 45 is controlled by an expansion in powers of M^{-1}, so we must take M to be large

for this expansion to make sense. On the other hand, the coupling constant g of the $O(4)$ NLSM is related to M via a formula of the form $M \propto 1/g$. For M large we seem to obtain an $O(4)$ NLSM in the ordered (small g) phase, whereas we are interested in studying the disordered (large g) phase. It is therefore not immediately clear why the calculation in this subsection agrees with the response calculation of Ref. 17 using the dual vortex theory, which we reviewed in the previous subsection. We resolve this puzzle in Appendix D, where we use the Abanov-Wiegmann formula to argue that the theory $S_f = \int d^3x\, i\bar{\Psi}\slashed{\partial}\Psi$ of four massless fermions ψ_a must possess exactly the same topological response as the original $O(4)$ NLSM at $\theta = \pi$. In the rest of this section we will therefore calculate the response of the fermions ψ_a to the time-reversal breaking mass term $-(M\delta)\bar{\Psi}\Gamma^5\Psi$. According to the arguments in Appendix D, this response (or at least its topological part), should be identical to the response of the $O(4)$ NLSM at $\theta = \pi$.

Before we can do this we need to determine the charges q_a of the four Dirac fermions ψ_a. These charges should be chosen so that the coupling term $\sum_{a=1}^{4} N^a \bar{\Psi}\Gamma^a\Psi$ is invariant under the $U(1)_c$ symmetry. Each fermion ψ_a is assumed to transform as

$$U(1)_c : \psi_a \to e^{iq_a\chi}\psi_a . \qquad (4.28)$$

The transformation of the $O(4)$ field under the $U(1)_c$ symmetry was described in Eq. (4.6). Using the relation

$$b_1 = N^4 + iN^3 \qquad (4.29)$$
$$b_2 = -N^2 + iN^1 , \qquad (4.30)$$

and the explicit form of the matrices Γ^a, we find that in order for the term $\sum_{a=1}^{4} N^a \bar{\Psi}\Gamma^a\Psi$ to be invariant under $U(1)_c$, the charges q_a must satisfy the matrix equation

$$\begin{pmatrix} -1 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ -1 \\ 1 \end{pmatrix} . \qquad (4.31)$$

This matrix has a null vector $(1,1,1,1)^T$, so the solution of the system is not unique. One possible way to parameterize a general solution is

$$\begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{pmatrix} = \begin{pmatrix} \bar{q} \\ -1+\bar{q} \\ \bar{q} \\ 1+\bar{q} \end{pmatrix} , \qquad (4.32)$$

where the parameter $\bar{q}$ is completely arbitrary because of the non-uniqueness of the solution. In what follows, we keep $\bar{q}$ to be some arbitrary number. Importantly, when we calculate a physical quantity pertaining to the $O(4)$ NLSM we will see that the answer is independent of $\bar{q}$.

We now define the diagonal matrix of charges $Q = \mathrm{diag}(q_1, q_2, q_3, q_4) \otimes \mathbb{I}$, given explicitly by,

$$Q = \bar{q}\mathbb{I} \otimes \mathbb{I} \otimes \mathbb{I} + \frac{1}{2}(\sigma^z \otimes \sigma^z \otimes \mathbb{I} - \sigma^z \otimes \mathbb{I} \otimes \mathbb{I}) , \qquad (4.33)$$

(note that it acts as the identity on the spinor indices of the fermions) and then use this matrix to couple Ψ to A_μ to obtain the action

$$\mathcal{L}_{f,gauge} = \bar{\Psi}\left(i\slashed{\partial} - (M\delta)\Gamma^5 \right. \qquad (4.34)$$
$$\left. - M\sum_{a=1}^{4} N^a\Gamma^a + Q\slashed{A} \right)\Psi .$$

We now integrate out the fermions and collect the lowest order terms in derivatives involving only A_μ, because those terms will give the dominant contribution to the electromagnetic response. For completeness we give a basic outline of this calculation below.

Since we are currently only interested in the electromagnetic response of the fermions, we set $N^a = 0$ for the response calculation. Integrating out Ψ then gives

$$S_{eff}[A_\mu] = -i\ln\det\left(i\slashed{\partial} - (M\delta)\Gamma^5 + Q\slashed{A} \right)$$
$$= -i\mathrm{Tr}\ln\left(i\slashed{\partial} - (M\delta)\Gamma^5 + Q\slashed{A} \right) , \qquad (4.35)$$

where $\mathrm{Tr}[\dots]$ indicates a trace over spacetime, spinor, and isospace indices. We now write

$$S_{eff}[A_\mu] = -i\mathrm{Tr}\ln(i\slashed{\partial} - (M\delta)\Gamma^5) \qquad (4.36)$$
$$- i\mathrm{Tr}\ln\left[1 + (i\slashed{\partial} - (M\delta)\Gamma^5)^{-1}(Q\slashed{A}) \right] ,$$

and expand the second term using $\ln(1 + x) = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^n}{n}$.

Here is a technical point. The effective action we wrote down is divergent for $\delta \to 0$. Therefore a procedure is needed to define the effective action for $\delta = 0$. Let us indicate the dependence of the effective action on δ by writing it as $S_{eff}[A_\mu, \delta]$. We follow Ref. 1 and define the renormalized effective action at $\delta = 0$ by

$$S^R_{eff}[A_\mu, 0] = S_{eff}[A_\mu, 0] - \lim_{\delta\to\infty} S_{eff}[A_\mu, \delta] . \qquad (4.37)$$

The second term in this expression also has a divergent term which cancels the divergence from the first term. Now as $\delta \to \infty$, the second term gives a finite contribution, which is a Chern-Simons term. We find that

$$S^R_{eff}[A_\mu, 0] = \frac{1}{2}\mathrm{tr}_I[Q^2\Gamma^5]\frac{\mathrm{sgn}(\delta)}{4\pi}\int d^3x\, \epsilon^{\mu\nu\lambda}A_\mu\partial_\nu A_\lambda , \qquad (4.38)$$

where $\mathrm{tr}_I[\dots]$ denotes a trace over isospace indices only (the trace over spacetime and spinor indices has already been performed). Since $\frac{1}{2}\mathrm{tr}_I[Q^2\Gamma^5] = -1$, the final response is given by

$$\mathcal{L}^R_{eff} = -\mathrm{sgn}(\delta)\frac{e^2}{4\pi}\epsilon^{\mu\nu\lambda}A_\mu\partial_\nu A_\lambda , \qquad (4.39)$$

where we have restored the charge e. The answer depends on $\mathrm{sgn}(\delta)$ and not $\mathrm{sgn}(\delta M)$ because the mass M is assumed positive in the Abanov-Wiegmann method. Note that the result is independent of $\bar{q}$ (the arbitrary offset to the charges of the four fermions ψ_a), which is expected because we have calculated a physical quantity related to the $O(4)$ NLSM at $\theta = \pi$.

C. Connection to the dual description of the BTI surface in terms of $N = 2$ QED$_3$

In this section we briefly comment on the relationship between the descriptions of the BTI surface theory discussed above: (i) the dual vortex description, (ii) the description in terms of Abanov-Wiegmann fermions, and (iii) the recently proposed dual description of the BTI surface in terms of $2 + 1$-d Quantum Electrodynamics with two flavors of Dirac fermion, also known as $N = 2$ QED$_3$[40] (a quasi-1D derivation of this dual description was later given in Ref. 41).

Before writing down the dual description of Ref. 40, we first remind the reader that in their original study of the BTI in Ref. 17, Vishwanath and Senthil assigned an additional "pseudo-spin" quantum number to the bosons b_1 and b_2 for convenience, with b_1 carrying spin 1 and b_2 carrying spin -1. We refer to the $U(1)$ symmetry associated with pseudo-spin conservation as $U(1)_s$. Under this symmetry the bosons transform as

$$U(1)_s : b_1 \to e^{i\xi} b_1, \qquad (4.40a)$$
$$b_2 \to e^{-i\xi} b_2 . \qquad (4.40b)$$

The fermions in the $N = 2$ QED$_3$ description of the BTI surface are charged under this $U(1)_s$ symmetry.

The $N = 2$ QED$_3$ description of the BTI surface consists of two flavors of Dirac fermions, χ_1 and χ_2, which can be combined into one multi-component spinor $X = (\chi_1, \chi_2)^T$. These fermions do not carry any $U(1)_c$ charge, but χ_1 carries spin 1 while χ_2 carries spin -1. The time-reversal symmetry of the BTI acts on X like a particle-hole transformation. Both fermions also carry charge 1 of a dual non-compact gauge field α_μ, whose curl represents the total number current J_{tot}^μ of the bosons on the BTI surface via $J_{tot}^\mu = \frac{1}{2\pi} \epsilon^{\mu\nu\lambda} \partial_\nu \alpha_\lambda$. The dual Lagrangian takes the form

$$\mathcal{L} = \bar{X}(i(\mathbb{I} \otimes \gamma^\mu)\partial_\mu + (\sigma^z \otimes \gamma^\mu)A_\mu^s + (\mathbb{I} \otimes \gamma^\mu)\alpha_\mu)X$$
$$- \frac{1}{2\pi}\epsilon^{\mu\nu\lambda}A_\mu^c \partial_\nu \alpha_\lambda , \qquad (4.41)$$

where γ^μ, are the usual 2×2 gamma matrices for $2 + 1$ dimensional Dirac fermions (e.g. the matrices from Eq. (4.22) without the additional identity matrices in the tensor product), $\bar{X} = X^\dagger(\mathbb{I} \otimes \gamma^0)$, A_μ^c is the external electromagnetic field (denoted simply by "A_μ" in the other sections of this paper), and A_μ^s is a new external $U(1)$ gauge field which probes the $U(1)_s$ symmetry.

We now speculate on the relation between the fermions χ_1 and χ_2, the vortices $\phi_{1,\pm}$ and $\phi_{2,\pm}$ from the dual vortex description of the BTI surface, and the four Abanov-Wiegmann fermions ψ_a. Out of the four vortices $\phi_{1,\pm}$ and $\phi_{2,\pm}$, we may form four composite vortices $\phi_{1,\pm}\phi_{2,\pm}$ by taking every possible combination of "+" and "−" vortices of species 1 and 2. As discussed in Ref. 17, and as we show in Appendix C, when $\theta = \pi$ a bound state of a vortex in b_1 and a vortex in b_2 is a fermion. This means that the four composite vortices $\phi_{1,\pm}\phi_{2,\pm}$ are in fact fermions. The charges and spins of these four composite vortices can be easily determined and they are shown in Table I. The spins s_a of the four Abanov-Wiegmann

	$\phi_{1,+}\phi_{2,+}$	$\phi_{1,+}\phi_{2,-}$	$\phi_{1,-}\phi_{2,+}$	$\phi_{1,-}\phi_{2,-}$
q	1	0	0	-1
s	0	-1	1	0

TABLE I. Charges and spins of the composite vortices of the form $\phi_{1,\pm}\phi_{2,\pm}$. As was discussed in Ref. 17, and as we show in Appendix C, a bound state of a vortex in b_1 and a vortex in b_2 is a fermion.

	ψ_1	ψ_2	ψ_3	ψ_4
q	0	-1	0	1
s	1	0	-1	0

TABLE II. Charges and spins of the fermions ψ_a used in the Abanov-Wiegmann formula for the $O(4)$ NLSM (ignoring the arbitrary offsets $\bar{q}$ and $\bar{s}$ discussed in the main text).

fermions ψ_a can also be calculated, using the same method used to determine their charges q_a (just like the charges, the spins are also determined only up to an arbitrary offset $\bar{s}$, which we ignore here). The charges and spins of the four Abanov-Wiegmann fermions are shown in Table II. Interestingly, each composite vortex has precisely the same charge and spin as one of the Abanov-Wiegmann fermions.

Since the composite vortices have precisely the same charges and spins as the Abanov-Wiegmann fermions, and since the composite vortices in the $O(4)$ NLSM at $\theta = \pi$ are known to be fermions, we conjecture that these objects should be identified with each other. Furthermore, we propose that the fermions χ_1 and χ_2 from the $N = 2$ QED$_3$ description can be identified with ψ_1 and ψ_3, respectively, which in turn correspond to the composite vortices $\phi_{1,-}\phi_{2,+}$ and $\phi_{1,+}\phi_{2,-}$. The particle-hole-like transformation of χ_1 and χ_2 under time-reversal then follows immediately from the transformations of the vortices under time-reversal. Also, since the individual vortices $\phi_{I,\pm}$ couple to the non-compact gauge fields $\alpha_{I,\mu}$, where the conserved current of boson b_I is given by $J_I^\mu = \frac{1}{2\pi} \epsilon^{\mu\nu\lambda} \partial_\nu \alpha_{I,\lambda}$, the composite vortices are coupled to the total gauge field $\alpha_\mu = \alpha_{1,\mu} + \alpha_{2,\mu}$, whose curl represents the total boson number current. This is the exact same gauge field which χ_1 and χ_2 couple to in the $N = 2$ QED$_3$ description. It would be an interesting challenge for future investigations to provide a derivation of the $N = 2$ QED$_3$ description of the BTI surface directly from the description in terms of an $O(4)$ NLSM at $\theta = \pi$. Such a derivation would provide the details necessary to support the picture we have presented here.

D. Symmetry-breaking phases accessible from the dual theory

In this section we will complete our discussion, following Ref. 17, of the symmetry-breaking phases of the surface of the BTI which are accessible from the dual vortex description of the $O(4)$ NLSM at $\theta = \pi$. We have already seen that condensing just one vortex, say $\phi_{2,+}$, and gapping out the other one leads to a phase which breaks time-reversal symmetry. In that

case it was necessary to add the time-reversal breaking term $\Phi_2^\dagger \sigma^z \Phi_2$ to the Lagrangian to simultaneously gap out one vortex and force the other vortex to condense.

There are two other basic options for condensing and/or gapping out the vortices in the dual theory. These options are: (i) condense both vortices, and (ii) gap both vortices. Both options lead to a superfluid phase which can be understood as a phase in which one of the original fields b_1 or b_2 condenses. To identify which boson is condensing in each case, it is convenient to separately gauge the $U(1)$ symmetries corresponding to $b_1 \rightarrow e^{i\chi} b_1$ and $b_2 \rightarrow e^{i\chi} b_2$. We couple b_1 to the external field $A_{1,\mu}$ and b_2 to the external field $A_{2,\mu}$. In this case the dual theory takes the form (recall that $\phi_{2,\pm}$ carry charge $\pm\frac{1}{2}$ of the boson b_1)

$$\mathcal{L} = \sum_{s=\pm} |(\partial_\mu - i\alpha_{2,\mu} - i\frac{s}{2}A_{1,\mu})\phi_{2,s}|^2 + \ldots$$
$$- \frac{1}{\kappa_2}(\epsilon^{\mu\nu\lambda}\partial_\nu \alpha_{2,\lambda})^2 - \frac{1}{2\pi}\epsilon^{\mu\nu\lambda}A_{2,\mu}\partial_\nu \alpha_{2,\lambda} . \quad (4.42)$$

Consider first the phase obtained by condensing both $\phi_{2,+}$ and $\phi_{2,-}$. To be precise, we consider the condensation $\langle\phi_{2,+}\rangle = \langle\phi_{2,-}\rangle^* = v$, which does not break $Z_2^{\mathcal{T}}$. In this case we get a Higgs term for the gauge field $\alpha_{2,\mu}$ and the external field $A_{1,\mu}$. The gauge field $\alpha_{2,\mu}$, which represents the Goldstone boson of a condensate of b_2, is therefore gapped and can be safely integrated out. The resulting action contains only a Higgs term for $A_{1,\mu}$, and so the phase where both $\phi_{2,+}$ and $\phi_{2,-}$ condense can be identified with the phase where b_1 condenses.

Next consider the second case in which $\phi_{2,+}$ and $\phi_{2,-}$ are both gapped. We can then set $\phi_{2,+}$ and $\phi_{2,-}$ equal to zero to study the low energy properties of this phase. At this point the gauge field $\alpha_{2,\mu}$ can be integrated out to give a Higgs term for $A_{2,\mu}$, and so the phase where both $\phi_{2,+}$ and $\phi_{2,-}$ are gapped can be identified with the phase in which b_2 condenses.

Finally, we note that the dual vortex theory can be driven into either of these two phases by a potential that does not break the $U(1)_c$ or $Z_2^{\mathcal{T}}$ symmetries, which means that the superfluid phase of the BTI surface spontaneously breaks the $U(1)_c$ symmetry (and it does not break the time-reversal symmetry).

E. Symmetry-allowed perturbations

In this section we carefully investigate the effects of symmetry-allowed perturbations on the BTI surface. This is important as we want to understand the stability of the gapless phase of the surface, and hence the related 2+1-d semimetal, as explicitly as possible. In Ref. 17 Vishwanath and Senthil initially studied the $O(4)$ NLSM at $\theta = \pi$ assuming a larger symmetry group consisting not only of $U(1)_c$ charge conservation and $Z_2^{\mathcal{T}}$ time-reversal, but also an additional $U(1)_s$ "pseudo-spin" conservation symmetry and a Z_2^s "spin-flip" symmetry. The action of the $U(1)_s$ symmetry on the bosons b_I was already given in Eq. (4.40). The Z_2^s spin-

flip symmetry acts as

$$Z_2^s : b_1 \leftrightarrow b_2 . \quad (4.43)$$

In the presence of these additional symmetries, interspecies tunneling terms of the form $b_1^* b_2 + b_2^* b_1$, as well as chemical potential terms of the form $\mu_1 |b_1|^2 + \mu_2 |b_2|^2$ (with $\mu_1 \neq \mu_2$), cannot be added to the Lagrangian. However, the BTI is supposed to be protected by $U(1)_c$ and $Z_2^{\mathcal{T}}$ symmetry alone. It is therefore essential to understand the effects that such terms can have on the $O(4)$ NLSM at $\theta = \pi$, since we are allowed to add these terms to the Lagrangian in the generic case when the extra $U(1)_s$ and Z_2^s symmetries are broken.

Interspecies tunneling and chemical potential terms can have a drastic effect on the physics of the $O(4)$ NLSM with theta term. However, we will show that these terms always drive the system into a symmetry-breaking phase. To show this we make use of the commutation relations of the $O(4)$ NLSM fields in the canonical formalism. Because of the constraint between the bosonic fields b_I, these commutation relations must be derived using the Dirac Bracket formalism, and we review their derivation in Appendix A.

There is a simple way to understand why interspecies tunneling and chemical potential terms can have a strong effect on the physics of the $O(4)$ NLSM with theta term. When these terms are strong, they can drive the fields into a configuration in which the theta term vanishes identically. This is easiest to see when the theta term is written in Hopf coordinates on the the sphere S^3. In Hopf coordinates the fields b_1 and b_2 are parameterized as $b_1 = \sin(\eta)e^{i\vartheta_1}$, $b_2 = \cos(\eta)e^{i\vartheta_2}$ with $\eta \in [0, \pi/2]$, and $\vartheta_1, \vartheta_2 \in [0, 2\pi)$, and the theta term takes the form

$$S_\theta[U] = \frac{1}{2\pi^2} \int d^3x \ \cos(\eta)\sin(\eta)\epsilon^{\mu\nu\lambda}\partial_\mu\eta\partial_\nu\vartheta_1\partial_\lambda\vartheta_2 . \quad (4.44)$$

The interspecies tunneling and chemical potential terms take the form

$$b_1^* b_2 + b_2^* b_1 = 2\cos(\eta)\sin(\eta)\cos(\vartheta_1 - \vartheta_2) , \quad (4.45)$$

and

$$\mu_1 |b_1|^2 + \mu_2 |b_2|^2 = \mu_1 \cos^2(\eta) + \mu_2 \sin^2(\eta)$$
$$= \mu_1 + (\mu_2 - \mu_1)\sin^2(\eta) . \quad (4.46)$$

Consider the interspecies tunneling term. When it is strong, the lowest energy configurations of the $O(4)$ field are those configurations which have $\vartheta_1 = \vartheta_2 + n\pi$ for some integer n which is even or odd depending on the sign of the coefficient of this term. It is easy to see that the theta term vanishes identically on this kind of field configuration. The analysis of the chemical potential term is even simpler. Depending on the sign of $\mu_2 - \mu_1$, the lowest energy configurations are those with $\sin(\eta) = 0$ or $\sin(\eta) = 1$. In either case η is a constant and so the theta term completely vanishes. This analysis makes it clear that a more thorough understanding of the effects of symmetry-allowed perturbations on the BTI surface is needed.

1. Interspecies tunneling

We now show that interspecies tunneling terms such as $b_1^* b_2 + b_2^* b_1$, and even interaction terms such as $(b_1^* b_2)^n + (b_2^* b_1)^n$ for $n \geq 1$, do not condense (i.e., have zero expectation value) in *any* time-reversal invariant state $|\Psi\rangle$. This means that interspecies tunneling terms can only condense in the ground state of the system if that ground state breaks time-reversal symmetry. It also means that weak interspecies tunneling terms should have a negligible effect on the gapless time-reversal invariant ground state of the $O(4)$ NLSM with $\theta = \pi$.

To show that these expectation values vanish, we canonically quantize the theory and study the (equal-time) commutation relations of the *operators* $b_I(\mathbf{x})$, their hermitian conjugates $b_I^\dagger(\mathbf{x})$, and their canonically conjugate momenta. We discuss the canonical quantization of this system in Appendix A. The only commutation relation we will need here is

$$[b_I(\mathbf{x}), \pi_J(\mathbf{y})] = i\left(\delta_{IJ} - \frac{1}{2}b_I(\mathbf{x})b_J^\dagger(\mathbf{y})\right)\delta^{(2)}(\mathbf{x} - \mathbf{y}) , \tag{4.47}$$

where $\pi_I = \partial\mathcal{L}/\partial(\partial_t b_I)$ is the momentum conjugate to b_I. Consider this commutation relation first in the case where $I = J$, say for $I = J = 1$. We have

$$[b_1(\mathbf{x}), \pi_1(\mathbf{y})] = i\left(1 - \frac{1}{2}b_1(\mathbf{x})b_1^\dagger(\mathbf{y})\right)\delta^{(2)}(\mathbf{x} - \mathbf{y}) . \tag{4.48}$$

In the Hilbert space the action of the time-reversal symmetry $Z_2^\mathcal{T}$ is represented by an anti-unitary operator $\mathcal{T}$, obeying $\mathcal{T}^2 = 1$, which acts on the boson operators $b_I(\mathbf{x})$ as

$$\mathcal{T}b_I(\mathbf{x})\mathcal{T}^{-1} = b_I(\mathbf{x}) . \tag{4.49}$$

Then we must have

$$\mathcal{T}\pi_I(\mathbf{x})\mathcal{T}^{-1} = -\pi_I(\mathbf{x}) , \tag{4.50}$$

in order for the commutation relations to be invariant under conjugation by $\mathcal{T}$. Now suppose we have a state $|\Psi\rangle$ of the system which is time-reversal invariant, i.e., $\mathcal{T}|\Psi\rangle = |\Psi\rangle$. Then the expectation value $\langle\Psi|\mathcal{O}|\Psi\rangle$ of any operator $\mathcal{O}$ which is odd under time-reversal, $\mathcal{T}\mathcal{O}\mathcal{T}^{-1} = -\mathcal{O}$, must vanish.

We now apply this reasoning to the off-diagonal commutation relation

$$[b_1(\mathbf{x}), \pi_2(\mathbf{y})] = -i\frac{1}{2}b_1(\mathbf{x})b_2^\dagger(\mathbf{y})\delta^{(2)}(\mathbf{x} - \mathbf{y}) . \tag{4.51}$$

If we take the expectation value of both sides of this equation in the state $|\Psi\rangle$, then the expectation value of the left-hand side vanishes since all operators on the left-hand side are odd under the action of $\mathcal{T}$. We are left with the equation

$$0 = -i\frac{1}{2}\langle\Psi|b_1(\mathbf{x})b_2^\dagger(\mathbf{y})|\Psi\rangle\delta^{(2)}(\mathbf{x} - \mathbf{y}) , \tag{4.52}$$

and integrating both sides of this equation over $\mathbf{y}$ yields the final result

$$\langle\Psi|b_1(\mathbf{x})b_2^\dagger(\mathbf{x})|\Psi\rangle = 0 . \tag{4.53}$$

So we find that the operator $b_1(\mathbf{x})b_2^\dagger(\mathbf{x})$ has zero expectation value in any time-reversal invariant state $|\Psi\rangle$. Going further, we may first multiply both sides of Eq. (4.51) by *any* time-reversal invariant operator $\tilde{\mathcal{O}}(\mathbf{x})$, and then take an expectation value in $|\Psi\rangle$ to find that

$$\langle\Psi|\tilde{\mathcal{O}}(\mathbf{x})b_1(\mathbf{x})b_2^\dagger(\mathbf{x})|\Psi\rangle = 0 . \tag{4.54}$$

For example we could take $\tilde{\mathcal{O}}(\mathbf{x}) = (b_1(\mathbf{x})b_2^\dagger(\mathbf{x}))^{n-1}$ to find that the expectation value of $(b_1(\mathbf{x})b_2^\dagger(\mathbf{x}))^n$ vanishes for any $n \geq 1$. We can conclude from this analysis that if interspecies tunneling and interaction terms of the form $(b_1^* b_2)^n + (b_2^* b_1)^n$ do condense in the ground state of the system, then that ground state must break time-reversal symmetry. However, our analysis is not limited to just these terms, since there are many more possible choices for the form of the operator $\tilde{\mathcal{O}}(\mathbf{x})$ which we are allowed to insert.

2. Chemical potential

We now discuss the effects of the chemical potential term on the quantum theory. In a theory of two independent complex scalar fields, a chemical potential term, combined with suitable quartic terms in the potential, can have many possible effects on the fields in the theory. For example, both scalar fields could become gapped, or they could both condense, or one scalar field could become gapped and the other scalar field could condense. But the $O(4)$ NLSM is not a theory of two *independent* complex scalar fields. Instead, the fields b_1 and b_2 obey the very important constraint $\sum_I |b_I|^2 = 1$. In fact, with the help of the constraint, any chemical potential term can be re-written as

$$\mu_1|b_1|^2 + \mu_2|b_2|^2 = \frac{1}{2}(\mu_1 + \mu_2) + \frac{1}{2}(\mu_1 - \mu_2)(|b_1|^2 - |b_2|^2) . \tag{4.55}$$

This result indicates that for the $O(4)$ NLSM, the effect of a general quartic potential of the form

$$V(b_1, b_2) = \mu_1|b_1|^2 + \mu_2|b_2|^2 + \lambda_1|b_1|^4 + \lambda_2|b_2|^4 , \tag{4.56}$$

is to cause one of the fields b_1 or b_2 to condense and to cause the other field to become gapped. The choice of which of b_1 or b_2 is condensed and which is gapped depends only on the sign of $\mu_1 - \mu_2$ (assuming that λ_1 and λ_2 are positive). In particular, it seems that it is impossible to write down any potential which could cause both b_1 and b_2 to condense. Further evidence for this conclusion can be obtained from an analysis of the commutation relations of the theory, as we now show.

Consider a state $|\Phi\rangle$ which represents a superfluid ground state of the $O(4)$ NLSM for the boson b_1. In such a state the $U(1)$ symmetry $b_1 \to e^{i\chi}b_1$ is spontaneously broken, and $\langle\Phi|b_1|\Phi\rangle \neq 0$. In general, the state $|\Phi\rangle$ is not an eigenstate of the operator b_1, or even of the phase of b_1 (this can be seen from the form of the symmetry broken ground state for the phase excitations of an ordinary complex scalar field shown in Chapter 11 of Ref. 54, for example). Below we show that in the special case where $|\Phi\rangle$ *is* an eigenstate of b_1, it is possible

to prove that $\langle\Phi|b_2|\Phi\rangle = 0$. For the general case where $|\Phi\rangle$ is not an eigenstate of b_1, we must instead rely on the qualitative argument presented above, and another argument which we present below which is based on the expression for b_2 in terms of the vortices $\phi_{1,\pm}$ in b_1 (Eq. (4.19) with the indices 1 and 2 swapped).

For now we assume that $|\Phi\rangle$ is an eigenstate of the operators $b_1(\mathbf{x})$ and $b_1^\dagger(\mathbf{x})$, and that $b_1(\mathbf{x})|\Phi\rangle = \alpha|\Phi\rangle$ and $b_1^\dagger(\mathbf{x})|\Phi\rangle = \beta|\Phi\rangle$, where α and β are complex numbers which do not depend on $\mathbf{x}$. The relation $\langle\Phi|b_1(\mathbf{x})|\Phi\rangle = \langle\Phi|b_1^\dagger(\mathbf{x})|\Phi\rangle^*$ implies that $\beta = \alpha^*$. Now assume that $\alpha \neq 0$, and take the expectation value of Eq. (4.51) in the state $|\Phi\rangle$. The left-hand side vanishes and we find

$$0 = -i\frac{\alpha^*}{2}\langle\Phi|b_2^\dagger(\mathbf{y})|\Phi\rangle\delta^{(2)}(\mathbf{x}-\mathbf{y}) . \qquad (4.57)$$

Since we assumed that $\alpha \neq 0$, we are forced to conclude that $\langle\Phi|b_2^\dagger(\mathbf{x})|\Phi\rangle = 0$, which shows that b_2 cannot condense in an eigenstate of b_1 (which we have argued is a representative ground state of the superfluid phase of b_1). Similarly, we can show that b_1 cannot condense in an eigenstate of b_2.

Another intuitive way of seeing that b_2 cannot condense in a superfluid ground state of b_1 is to recall that b_2 can be expressed in terms of the two kinds of vortices in b_1 as $b_2 \sim \phi_{1,+}\phi_{1,-}^*$. In a superfluid ground state of b_1 we expect that the vortices $\phi_{1,\pm}$ in b_1 are gapped (i.e., not condensed), which means that we should also have $\langle b_2\rangle = 0$ in such a state.

We conclude that the main effect of a chemical potential term (combined with suitable quartic terms) is to spontaneously break the $U(1)_c$ symmetry, since this term will condense one of b_1 or b_2 and gap out the other one. Our analysis of the commutation relations confirms that the vacuum expectation value of one boson always vanishes in a state which represents a superfluid ground state of the other boson.

F. Symmetry-preserving state with topological order

In Ref. 17 Vishwanath and Senthil showed that it was possible for the surface phase of the BTI to retain the full $U(1)_c \rtimes Z_2^{\mathcal{T}}$ symmetry while gapped, but at the expense of having intrinsic topological order, and they went on to derive a specific topologically ordered state for the BTI surface. That same topologically ordered state was constructed in Ref. 55 using a coupled wires construction consisting of Bosonic Integer Quantum Hall effect edge modes decorated with Toric Code/$\mathbb{Z}_2$ topological order (Abelian Chern-Simons theory with K-matrix given by $\pm 2\sigma^x$) edge modes. In this section we briefly review the construction of this topologically ordered state via vortex condensation in the $O(4)$ NLSM with $\theta = \pi$ as shown in Ref. 17.

Recall the interpretation of the theta term that was derived in Ref. 42 (see also our Appendix C). According to Ref. 42, in the $O(4)$ NLSM with $\theta = \pi$, a braiding process in which a vortex in the phase of b_1 makes a full circuit around a vortex in the phase of b_2 results in an overall phase of $e^{i\pi}$ in the path integral for the theory. In other words, the vortex in b_1 and the

vortex in b_2 can be regarded as anyons with a mutual statistical angle of π.

In the $O(4)$ NLSM at $\theta = \pi$, all quasi-particles with non-trivial statistics can be built just from the fundamental vortices $\phi_{1,+}$ and $\phi_{2,+}$. Indeed, recall that the other two vortices $\phi_{1,-}$ and $\phi_{2,-}$ should really be understood as bound states of a vortex and a boson: $\phi_{1,-} \sim \phi_{1,+}b_2^*$ and $\phi_{2,-} \sim \phi_{2,+}b_1^*$, and are hence not *topologically* distinguishable from $\phi_{1,+}$ and $\phi_{2,+}$. This means that the two vortices $\phi_{1,+}$ and $\phi_{2,+}$ should be sufficient building blocks to describe any topologically ordered states derived from this system. According to the arguments given in the previous paragraph, these two vortices have a mutual statistical angle of π, i.e., they are mutual semions. Also, a composite of a vortex with itself, such as $(\phi_{1,+})^2$, should be regarded as trivial (topologically equivalent to the vacuum quasi-particle), since that object has the exact same quantum numbers as the boson b_2 and braids trivially with all other quasi-particles. This property is partly responsible for the $\mathbb{Z}_2$ structure of the topological order discussed below.

We can now consider condensing some field $\mathcal{O}$ which is a composite of the vortices. Standard reasoning then tells us that any quasi-particles that have trivial mutual statistics with $\mathcal{O}$ will survive as anyons in the state obtained by condensing $\mathcal{O}$. In Ref. 17 Vishwanath and Senthil construct a topologically ordered phase for the surface of the BTI by choosing to condense $\mathcal{O} = \phi_{1,+}\phi_{1,-}$ in such a way that $\langle\mathcal{O}\rangle$ is real. Since $Z_2^{\mathcal{T}}$ maps $\mathcal{O} \to \mathcal{O}^*$, this condensation does not break time-reversal symmetry. In addition, $\mathcal{O}$ is invariant under $U(1)_c$, so the resulting phase actually retains the full $U(1)_c \rtimes Z_2^{\mathcal{T}}$ symmetry of the BTI.

We see that both $\phi_{1,+}$ and $\phi_{2,+}$ have trivial mutual statistics with $\mathcal{O}$, so these vortices both survive as quasi-particles in the condensed state. The condensed state therefore has quasi-particle content (recall that fusing a vortex with itself gives a trivial excitation)

$$\{1, \phi_{1,+}, \phi_{2,+}, \phi_{1,+}\phi_{2,+}\} , \qquad (4.58)$$

where "1" is the trivial (vacuum) quasi-particle and $\phi_{1,+}\phi_{2,+}$ is the composite of the two vortices $\phi_{1,+}$ and $\phi_{2,+}$. The composite vortex $\phi_{1,+}\phi_{2,+}$ is actually a fermion. The exchange and mutual statistics of these quasi-particles is shown in Table III. These quasi-particles with the braiding statistics shown in Table III form a $\mathbb{Z}_2$ topological order which is characterized by a K-matrix $K = 2\sigma^x$ and charge vector $\vec{t} = (1,1)^T$. In a purely 2D system which admits an edge to the vacuum, such a system would exhibit a charge Hall conductance of $\vec{t}\cdot(K^{-1}\vec{t}) = 1$ (it is non-zero and therefore breaks time-reversal symmetry), which hints that the topologically-ordered state on the surface of the BTI realizes time-reversal symmetry in a way which is forbidden in a real 2D system[17].

V. THE BOSONIC SEMI-METAL MODEL: TWO $O(4)$ NLSM'S WITH $\theta = \pm\pi$

In this section we introduce our Bosonic Semi-Metal (BSM) model. The model is constructed from two copies of the $O(4)$ NLSM with theta term, and we take one copy to

	$\phi_{1,+}$	$\phi_{2,+}$	$\phi_{1,+}\phi_{2,+}$
$\phi_{1,+}$	0	π	π
$\phi_{2,+}$	π	0	π
$\phi_{1,+}\phi_{2,+}$	π	π	π

TABLE III. Exchange and mutual statistics for the quasi-particles contained in the topologically ordered surface phase of the BTI which is accessed by condensing the composite field $\mathcal{O} = \phi_{1,+}\phi_{1,-}$. The diagonal entries in the table (*exchange* statistics) are the phase for a process in which two identical particles exchange positions (so a phase of π represents fermions), while the off-diagonal entries (*mutual* statistics) are the phase picked up when a particle of one type makes a complete circuit around a particle of a different type.

have $\theta = \pi$ and the other copy to have $\theta = -\pi$. The intuition behind the construction of our model is as follows. Recall that the surface theory of the 3D ETI is a single massless 2+1-d Dirac fermion. The two-cone DSM phase in 2+1-d can then be viewed as being constructed from two copies of the surface theory of the 3D ETI, with the two copies separated in momentum space and having opposite helicity. For our BSM model we instead take two copies of the $O(4)$ NLSM with $|\theta| = \pi$, (i.e., two copies of the surface theory of the BTI), but we take the two copies to have opposite signs of θ, which is the bosonic analog of the helicity of the 2+1-d Dirac fermion. One way to see this is in the construction by Abanov and Wiegmann in Ref. 45, where the helicity of the auxiliary fermions directly determines the sign of the theta angle in the resulting $O(4)$ NLSM.

This section is broken up into several subsections as follows. We first define our BSM model and the transformations of the fields in the model under $U(1)_c$ charge conservation symmetry, $U(1)_t$ "translation" symmetry (to be defined), $Z_2^{\mathcal{T}}$ time-reversal symmetry, and $Z_2^{\mathcal{I}}$ inversion symmetry. We then discuss the dual description of our BSM model and derive the action of the different symmetries on the vortex fields in the dual theory. Finally, we calculate the time-reversal and inversion breaking electromagnetic responses of our BSM model (again using two different methods), and compare the result with that of the 2+1-d DSM model discussed in Ref. 7, and reviewed in Sec. III. We then discuss the stability of the model and find that the composite $Z_2^{\mathcal{T}\mathcal{I}}$ symmetry again plays an important role. Finally, we close the section with a discussion of phases with $\mathbb{Z}_2$ and $\mathbb{Z}_2 \times \mathbb{Z}_2$ topological order which can be accessed from our BSM model by condensing a composite of the vortices appearing in the dual description of the model. We show that these phases break either the time-reversal or the inversion symmetry of the BSM model. This is interesting because the gapped phases which do not have topological order also must break one of these two symmetries.

A. BSM model and symmetries

Our BSM model consists of two copies of an $O(4)$ NLSM with theta term, called "A" and "B" copies, with the theta angles for the two copies being $\theta_A = \pi$ and $\theta_B = -\pi$. We write

the model in terms of $SU(2)$ matrices U_A and U_B, which are each expressed in terms of bosonic fields $b_{I,A}$ and $b_{I,B}$, $I = 1, 2$, as in Eq. (4.5). The action for the system is

$$S = \int d^3x \left[\frac{1}{2g} \mathrm{tr}[\partial^\mu U_A^\dagger \partial_\mu U_A + (A \to B)] \right.$$
$$\left. - \pi S_\theta[U_A] + \pi S_\theta[U_B] \right] , \quad (5.1)$$

where the explicit form of the theta term was given in Eq. (4.4).

The fields in the BSM model transform under $U(1)_c$ charge conservation symmetry, $U(1)_t$ "translation" symmetry, $Z_2^{\mathcal{T}}$ time-reversal symmetry, and $Z_2^{\mathcal{I}}$ inversion symmetry. In this section we explain the action of each of these symmetries on the fields $b_{I,A}$ and $b_{I,B}$ in the model. Just as for the DSM, the composite symmetry $Z_2^{\mathcal{T}\mathcal{I}}$, consisting of a time-reversal transformation followed by an inversion transformation, will be important for guaranteeing the stability of the gapless phase of our model.

The fields transform under the $U(1)_c$ symmetry as

$$U(1)_c : b_{I,A/B} \to e^{i\chi} b_{I,A/B} . \quad (5.2)$$

This just indicates that each bosonic field $b_{I,A/B}$ carries charge 1 of the external gauge field A_μ. Under the "translation" $U(1)$ symmetry, $U(1)_t$, the fields transform as

$$U(1)_t : b_{I,A} \to e^{i\xi} b_{I,A} \quad (5.3a)$$
$$b_{I,B} \to e^{-i\xi} b_{I,B} . \quad (5.3b)$$

To explain the physical meaning of this $U(1)$ translation symmetry, we need to imagine that our BSM model has been obtained in the low-energy continuum limit of a bosonic lattice model, as we now explain. Let us assume that the fields $b_{I,A}$ and $b_{I,B}$ arise from the low-energy continuum limit of a bosonic lattice model, and that they are related to the lattice boson operators in a way similar to the DSM case illustrated in Eq. (3.8). In other words, the combinations $e^{i\mathbf{k}_+}b_{I,A}$ and $e^{i\mathbf{k}_-}b_{I,B}$ appear in the expression for the lattice boson operator, indicating that the continuum fields $b_{I,A}$ and $b_{I,B}$ are located at positions $\mathbf{k}_\pm = \pm(B_x, B_y)$ in momentum space. We provide an explicit example of such a model using a coupled-wire construction in the next section.

To model this momentum shift, the kinetic term in the Lagrangian for the fields $b_{I,A}$ and $b_{I,B}$ will feature a minimal coupling to the vector field B_μ (and just as in the DSM case, we also allow for a relative energy offset given by B_t). The translation properties of the fields $b_{I,A}$ and $b_{I,B}$ can then be thought of in terms of carrying charges 1 and -1, respectively, of the field B_μ, and the action is invariant under the $U(1)_t$ gauge transformation in which the bosonic fields transform according to Eq. (5.3) while $B_\mu \to B_\mu + \partial_\mu \xi$. This is the physical origin of the $U(1)_t$ symmetry[56].

We now discuss the discrete symmetries $Z_2^{\mathcal{I}}$, $Z_2^{\mathcal{T}}$ and $Z_2^{\mathcal{T}\mathcal{I}}$. We take $Z_2^{\mathcal{I}}$, and $Z_2^{\mathcal{T}}$ to act on the bosonic fields as

$$Z_2^{\mathcal{I}} : b_{I,A}(t, \mathbf{x}) \to b_{I,B}(t, -\mathbf{x}) \quad (5.4)$$
$$Z_2^{\mathcal{T}} : b_{I,A}(t, \mathbf{x}) \to b_{I,B}(-t, \mathbf{x}) , \quad (5.5)$$

and vice-versa. The composite symmetry $Z_2^{\mathcal{T}\mathcal{I}}$ then acts as

$$Z_2^{\mathcal{T}\mathcal{I}} : b_{I,A}(t, \mathbf{x}) \to b_{I,A}(-t, -\mathbf{x}) , \qquad (5.6)$$

with an identical transformation for $b_{I,B}(t, \mathbf{x})$. In the canonical formalism, the action of time-reversal is represented by the anti-unitary operator $\mathcal{T}$, and the action of inversion is represented by the unitary operator $\mathcal{I}$. From the symmetry transformations defined above we can see that these operators satisfy the identities $\mathcal{T}^2 = 1$, $\mathcal{I}^2 = 1$, and $[\mathcal{T}, \mathcal{I}] = 0$, which implies that $(\mathcal{T}\mathcal{I})^2 = 1$ as well.

Just as in the fermionic DSM case, the composite $Z_2^{\mathcal{T}\mathcal{I}}$ symmetry is important for ensuring the local (in momentum space) stability of each $O(4)$ NLSM copy in our BSM model. We will have more to say on this subject later in this section, but for now we note the following important property of the $Z_2^{\mathcal{T}\mathcal{I}}$ symmetry for the BSM model. In the BSM model it is actually the $Z_2^{\mathcal{T}\mathcal{I}}$ symmetry which fixes the theta angles θ_A and θ_B to be multiples of π, just as the $Z_2^{\mathcal{T}}$ symmetry guaranteed this property for the BTI surface theory. Therefore, from this general argument, the gaplessness of the BSM model (which can occur only when the theta angles are odd multiples of π) depends crucially on this symmetry.

B. Dual vortex description of the BSM model

We now turn to the dual vortex description of our BSM model, using the dual description of one $O(4)$ model which we reviewed in Sec. IV. We choose to employ the dual vortex description in terms of vortices in $b_{2,A}$ and $b_{2,B}$, although a description starting in terms of vortices in $b_{1,A}$ and $b_{1,B}$ is also possible. For the "A" NLSM, vortices in $b_{2,A}$ are represented by the two-component field $\Phi_2^{(A)} = (\phi_{2,+}^{(A)}, \phi_{2,-}^{(A)})^T$. For the "B" copy of the NLSM, vortices in $b_{2,B}$ are represented by the two-component field $\Phi_2^{(B)} = (\phi_{2,+}^{(B)}, \phi_{2,-}^{(B)})^T$.

As discussed in Sec. IV, and as we explicitly prove in Appendix B, a vortex in the phase of one boson binds a charge of $\frac{\theta}{2\pi}$ of the other boson. This result holds for any $U(1)$ symmetry under which the bosons are charged; for example, the $U(1)_c$ and $U(1)_t$ symmetries in our case. This means that under the $U(1)_c$ and $U(1)_t$ symmetries, the field $\Phi_2^{(A)}$ transforms as

$$U(1)_c : \Phi_2^{(A)} \to e^{i\frac{\chi}{2}\sigma^z} \Phi_2^{(A)} \qquad (5.7)$$

$$U(1)_t : \Phi_2^{(A)} \to e^{i\frac{\xi}{2}\sigma^z} \Phi_2^{(A)} . \qquad (5.8)$$

On the other hand, the "B" copy of the $O(4)$ NLSM in our BSM model has theta angle $\theta_B = -\pi$. The elementary vortices $\phi_{1,+}^{(B)}$ and $\phi_{2,+}^{(B)}$ now both carry charges $-\frac{1}{2}$ and $\frac{1}{2}$ under the $U(1)_c$ and $U(1)_t$ symmetries, respectively. The "$-$" vortices must now be defined as $\phi_{2,-}^{(B)} = \phi_{2,+}^{(B)} b_{1,B}$ and $\phi_{1,-}^{(B)} = \phi_{1,+}^{(B)} b_{2,B}$. Unlike for the "A" copy, these relations involve the bosons $b_{I,B}$ and not the anti-bosons $b_{I,B}^*$ since $\theta_B = -\pi$ and not $+\pi$. We then find that under the $U(1)_c$ and

$U(1)_t$ symmetries, the field $\Phi_2^{(B)}$ transforms as

$$U(1)_c : \Phi_2^{(B)} \to e^{-i\frac{\chi}{2}\sigma^z} \Phi_2^{(B)} \qquad (5.9)$$

$$U(1)_t : \Phi_2^{(B)} \to e^{i\frac{\xi}{2}\sigma^z} \Phi_2^{(B)} . \qquad (5.10)$$

In terms of the fields $\Phi_2^{(A)}$ and $\Phi_2^{(B)}$, the dual description of the BSM model has the Lagrangian $\mathcal{L} = \mathcal{L}^{(A)} + \mathcal{L}^{(B)}$, with

$$\mathcal{L}^{(A)} = \sum_{s=\pm} |[\partial_\mu - i\alpha_{2,\mu}^{(A)} - i\frac{s}{2}(A_\mu + B_\mu)]\phi_{2,s}^{(A)}|^2 \qquad (5.11)$$

$$- \frac{1}{\kappa_{2,A}} \left(\frac{1}{2\pi} \epsilon^{\mu\nu\lambda} \partial_\nu \alpha_{2,\lambda}^{(A)} \right)^2 - \frac{1}{2\pi} \epsilon^{\mu\nu\lambda} (A_\mu + B_\mu) \partial_\nu \alpha_{2,\lambda}^{(A)} ,$$

and

$$\mathcal{L}^{(B)} = \sum_{s=\pm} |[\partial_\mu - i\alpha_{2,\mu}^{(B)} + i\frac{s}{2}(A_\mu - B_\mu)]\phi_{2,s}^{(B)}|^2 \qquad (5.12)$$

$$- \frac{1}{\kappa_{2,B}} \left(\frac{1}{2\pi} \epsilon^{\mu\nu\lambda} \partial_\nu \alpha_{2,\lambda}^{(B)} \right)^2 - \frac{1}{2\pi} \epsilon^{\mu\nu\lambda} (A_\mu - B_\mu) \partial_\nu \alpha_{2,\lambda}^{(B)} .$$

In these expressions, $\frac{1}{2\pi}\epsilon^{\mu\nu\lambda}\partial_\nu\alpha_{2,\lambda}^{(A)}$ and $\frac{1}{2\pi}\epsilon^{\mu\nu\lambda}\partial_\nu\alpha_{2,\lambda}^{(B)}$ represent the number currents of the bosons $b_{2,A}$ and $b_{2,B}$, respectively. We have included coupling to the external probe fields A_μ and B_μ associated with the two $U(1)$ symmetries $U(1)_c$ and $U(1)_t$. It is also possible to add various potential energy terms to these dual Lagrangians.

C. Transformation of vortices under $\mathcal{T}$ and $\mathcal{I}$ symmetries

In this section we deduce the transformations of the vortices under the $Z_2^{\mathcal{T}}$ and $Z_2^{\mathcal{I}}$ symmetries. First we note that because of the quantum numbers carried by the vortex fields, we have the approximate relations

$$b_{1,A} \sim \phi_{2,-}^{(A),*} \phi_{2,+}^{(A)} \qquad (5.13)$$

$$b_{1,B} \sim \phi_{2,+}^{(B),*} \phi_{2,-}^{(B)} , \qquad (5.14)$$

which are just Eq. (4.19) written for the two copies of the $O(4)$ NLSM, and taking into account the fact that the "B" copy of the $O(4)$ NLSM has $\theta_B = -\pi$. Also, recall that a dimensionful quantity like g, the NLSM coupling constant, is needed to balance the units in this equation, but we ignore that subtlety here. We now deduce the transformations of the vortices under $Z_2^{\mathcal{T}}$ and $Z_2^{\mathcal{I}}$ by requiring that the transformations of the vortices under these symmetries reproduce the transformations of $b_{1,A}$ and $b_{1,B}$ under the symmetries, and that the action of the symmetries on the vortices is consistent with the general structure of the symmetry group.

Consider first the inversion symmetry. Inversion commutes with the $U(1)_c$ symmetry, whereas it negates the $U(1)_t$ charge. In addition, since inversion is a unitary symmetry, it should take vortices to vortices, not anti-vortices (conjugation by the operator $\mathcal{I}$ does not negate the phase of $b_{I,A/B}$). We have only two options: either $\Phi_2^{(A)}(t, \mathbf{x}) \to \sigma^x \Phi_2^{(B)}(t, -\mathbf{x})$ or

$\Phi_2^{(A)}(t, \mathbf{x}) \to i\sigma^y \Phi_2^{(B)}(t, -\mathbf{x})$. Only the first option is consistent with Eq. (5.4). We hence find that

$$Z_2^{\mathcal{I}} : \Phi_2^{(A)}(t, \mathbf{x}) \to \sigma^x \Phi_2^{(B)}(t, -\mathbf{x}) , \qquad (5.15)$$

and vice-versa.

Next consider time-reversal symmetry. Time-reversal is anti-unitary, so it should take vortices to anti-vortices (conjugation by $\mathcal{T}$ *does* negate the phase of $b_{I,A/B}$). In addition, time-reversal preserves the $U(1)_c$ charge and negates the $U(1)_t$ charge. The only two possibilities are then $\Phi_2^{(A)}(t, \mathbf{x}) \to \Phi_2^{(B),*}(-t, \mathbf{x})$ or $\Phi_2^{(A)}(t, \mathbf{x}) \to \sigma^z \Phi_2^{(B),*}(-t, \mathbf{x})$. Only the first option is consistent with Eq. (5.5), so we find that

$$Z_2^{\mathcal{T}} : \Phi_2^{(A)}(t, \mathbf{x}) \to \Phi_2^{(B),*}(-t, \mathbf{x}) , \qquad (5.16)$$

and vice-versa.

We see that the time-reversal and inversion symmetry continue to commute with each other when acting on the vortices. The combined $Z_2^{\mathcal{T}\mathcal{I}}$ symmetry then acts on the vortices as

$$Z_2^{\mathcal{T}\mathcal{I}} : \Phi_2^{(A)}(t, \mathbf{x}) \to \sigma^x \Phi_2^{(A),*}(-t, -\mathbf{x}) , \qquad (5.17)$$

and similarly for $\Phi_2^{(B)}$.

D. Time-reversal and inversion breaking mass terms , electromagnetic response, and a bosonic Chern insulator

Now that we know how the vortex fields transform under the various symmetries, we can use the dual vortex theory to calculate the responses of our BSM model to time-reversal and inversion breaking perturbations. Analogous to the fermion DSM, we can define a time-reversal breaking mass term for the vortices,

$$\Sigma_{\mathcal{T}} = \Phi_2^{(A),*}\sigma^z\Phi_2^{(A)} - \Phi_2^{(B),*}\sigma^z\Phi_2^{(B)} , \qquad (5.18)$$

and also an inversion breaking mass term

$$\Sigma_{\mathcal{I}} = \Phi_2^{(A),*}\sigma^z\Phi_2^{(A)} + \Phi_2^{(B),*}\sigma^z\Phi_2^{(B)} . \qquad (5.19)$$

The term $\Sigma_{\mathcal{T}}$ is odd under $Z_2^{\mathcal{T}}$ but even under $Z_2^{\mathcal{I}}$. On the other hand, $\Sigma_{\mathcal{I}}$ is even under $Z_2^{\mathcal{T}}$ but odd under $Z_2^{\mathcal{I}}$.

Now let us consider the electromagnetic response in these two gapped phases. Suppose we add the time-reversal breaking mass term $\mu\Sigma_{\mathcal{T}}$ to the vortex potential energy. If $\mu < 0$ (and in the presence of suitable quartic terms in the vortex action), this will cause $\phi_{2,-}^{(A)}$ and $\phi_{2,+}^{(B)}$ to become gapped, and $\phi_{2,+}^{(A)}$ and $\phi_{2,-}^{(B)}$ to condense. In this case we can then integrate out $\phi_{2,-}^{(A)}$ and $\phi_{2,+}^{(B)}$. A mean-field treatment of the remaining terms in the action then gives $\alpha_{2,\mu}^{(A)} = -\frac{1}{2}(A_\mu + B_\mu)$ and $\alpha_{2,\mu}^{(B)} = -\frac{1}{2}(A_\mu - B_\mu)$, which gives the 2D time-reversal breaking response

$$\mathcal{L}_{\mathcal{T}} = \frac{e^2}{2\pi}\epsilon^{\mu\nu\lambda}A_\mu\partial_\nu A_\lambda + \frac{1}{2\pi}\epsilon^{\mu\nu\lambda}B_\mu\partial_\nu B_\lambda . \qquad (5.20)$$

The first term in this expression is a Quantum Hall response with Hall conductivity $\sigma_{xy} = 2e^2/h$, exactly the same as one finds for the Bosonic Integer Quantum Hall effect[15,16]. If we took $\mu > 0$ we would get the same response but with the opposite sign. We note that we cannot add a simple mass term to find a σ_{xy} quantized as an odd multiple of e^2/h. This gapped phase represents a Bosonic Chern insulator.

On the other hand, we can add the inversion breaking mass term $\mu\Sigma_{\mathcal{I}}$ to the vortex potential energy instead. If $\mu < 0$ (and again, assuming suitable quartic terms), this will cause $\phi_{2,-}^{(A)}$ and $\phi_{2,-}^{(B)}$ to become gapped and $\phi_{2,+}^{(A)}$ and $\phi_{2,+}^{(B)}$ to condense. In a mean-field treatment this gives $\alpha_{2,\mu}^{(A)} = -\frac{1}{2}(A_\mu + B_\mu)$ and $\alpha_{2,\mu}^{(B)} = \frac{1}{2}(A_\mu - B_\mu)$, which yields the quasi-1D inversion breaking response

$$\mathcal{L}_{\mathcal{I}} = \frac{e}{\pi}\epsilon^{\mu\nu\lambda}B_\mu\partial_\nu A_\lambda . \qquad (5.21)$$

Again, if we took $\mu > 0$ then we would get the same response but with the opposite sign. This response encodes a charge polarization $P^i = \frac{e}{\pi}\epsilon^{ij}B_j$ ($i, j = x, y$) and an orbital magnetization $M = \frac{e}{\pi}B_t$.

We see that both the time-reversal breaking and inversion breaking electromagnetic responses of the BSM are *twice* as large as the responses for the free fermion DSM shown in Eq. (2.1). Let us now provide an alternate derivation of these responses.

E. Electromagnetic Responses from Abanov-Wiegmann Method

We now briefly show how the time-reversal and inversion breaking responses of our BSM model can be computed using the Abanov-Wiegmann method of integration over auxiliary fermions which we discussed in Sec. IV. We first rewrite our BSM model in terms of two four-component unit vector fields $\mathbf{N}_A$ and $\mathbf{N}_B$. Now introduce the multi-component complex fermion $\Psi = (\psi_{1,A}, .., \psi_{4,A}, \psi_{1,B}, .., \psi_{4,B})^T$, where each of $\psi_{a,A/B}$ is a two-component Dirac fermion in $2 + 1$ dimensions. The fermion Ψ has a total of 16 components. In terms of the two sets of gamma matrices introduced in Eqs. (4.22) and (4.23), our BSM model can be obtained from the fermionic Lagrangian

$$\tilde{\mathcal{L}}_f = \bar{\Psi}\left(i\tilde{\gamma}^\mu\partial_\mu - \frac{M}{2}\sum_{a=1}^4 N_A^a(\mathbb{I} + \sigma^z)\otimes\Gamma^a \right.$$
$$\left. - \frac{M}{2}\sum_{a=1}^4 N_B^a(\mathbb{I} - \sigma^z)\otimes\Gamma^a \right)\Psi , \qquad (5.22)$$

where we have defined

$$\tilde{\gamma}^0 = \mathbb{I} \otimes \gamma^0 \qquad (5.23a)$$
$$\tilde{\gamma}^1 = \mathbb{I} \otimes \gamma^1 \qquad (5.23b)$$
$$\tilde{\gamma}^2 = \sigma^z \otimes \gamma^2 , \qquad (5.23c)$$

and $\bar{\Psi} = \Psi^\dagger\tilde{\gamma}^0$ now. The extra σ^z on $\tilde{\gamma}^2$ means that the fermions $\psi_{a,B}$ have opposite helicity to the fermions $\psi_{a,A}$.

This change directly accounts for the opposite signs of the theta angle for the "A" and "B" copies of the $O(4)$ NLSM that we get when we integrate out Ψ. This is the reason why we stated earlier that the sign of θ is the analogue in the BSM of the helicity of the Dirac fermions in the DSM. Indeed, we see that the helicity of the Abanov-Wiegmann auxiliary fermions directly translates into the sign of θ in the $O(4)$ NLSM.

By the same reasoning used in Sec. IV to deduce the charges of the fermions used to generate one $O(4)$ NLSM on the surface of the BTI, we now find that the field Ψ transforms under the $U(1)_c$ and $U(1)_t$ symmetries as

$$U(1)_c : \Psi \to e^{i\chi \mathbb{I} \otimes Q}\Psi \tag{5.24}$$

$$U(1)_t : \Psi \to e^{i\xi \sigma^z \otimes Q}\Psi , \tag{5.25}$$

where Q is the 8×8 charge matrix introduced in Eq. (4.33). We can now couple Ψ to the background gauge fields A_μ and B_μ and calculate the response of the system to various perturbations.

The time-reversal breaking response is obtained by adding the term $-(M\delta)\bar{\Psi}(\sigma^z \otimes \Gamma^5)\Psi$ to the Lagrangian. According to Eq. (4.26), this will give $\theta_A \approx \pi(1-\frac{3}{2}\delta)$ and $\theta_B \approx -\pi(1+\frac{3}{2}\delta)$, so this breaks $Z_2^{\mathcal{T}}$ (which requires $\theta_A \equiv -\theta_B$ mod 2π). The inversion breaking response is obtained by adding the term $-(M\delta)\bar{\Psi}(\mathbb{I} \otimes \Gamma^5)\Psi$. This will give $\theta_A \approx \pi(1-\frac{3}{2}\delta)$ and $\theta_B \approx -\pi(1-\frac{3}{2}\delta)$, so this breaks $Z_2^{\mathcal{I}}$ (which requires $\theta_A \equiv \theta_B$ mod 2π).

In the limit that $\delta \to 0$, the time-reversal breaking perturbation generates the 2+1-d response

$$\tilde{\mathcal{L}}_{\mathcal{T}} = -\text{sgn}(\delta)\frac{1}{2\pi}\epsilon^{\mu\nu\lambda}\left(e^2 A_\mu \partial_\nu A_\lambda + B_\mu \partial_\nu B_\lambda\right) , \tag{5.26}$$

coming from the contributions of each of the Dirac fermions (and according to their charge), while the inversion breaking perturbation gives the quasi-1D response

$$\tilde{\mathcal{L}}_{\mathcal{I}} = -\text{sgn}(\delta)\frac{e}{\pi}\epsilon^{\mu\nu\lambda}B_\mu \partial_\nu A_\lambda . \tag{5.27}$$

These are the same responses which we derived in the previous subsection using the dual vortex formulation of the BSM model.

F. Stability of the BSM Effective Theory

We have provided an effective theory for a gapless bosonic semi-metal in 2+1-d and we now want to evaluate the perturbative stability of this theory to see under what conditions the semi-metal is a stable phase. In discussing the stability of the BSM model, there are a few expected properties which we would like to verify. First, the translation symmetry of the model should prevent us from trivially gapping out the model by coupling the "A" copy of the $O(4)$ NLSM to the "B" copy (for our purposes, by a trivially gapped phase, we mean a gapped phase which retains all the symmetries of the original gapless system and has no interesting electromagnetic response). And second, the composite $Z_2^{\mathcal{T}\mathcal{I}}$ symmetry should guarantee the *local* stability of the $O(4)$ NLSM's which make

up our BSM model (recall that local stability means that it should be impossible to gap out one copy of the $O(4)$ NLSM independently of the other copy without breaking required symmetries). We claim that analogous to the 2+1-d fermionic DSM, these symmetries are enough to provide perturbative stability to the BSM. However, just as with any symmetry-protected phase (gapped or gapless) it is also important to keep in mind the possibility that even symmetry-allowed perturbations may spontaneously break one or more of the symmetries of the system if those perturbations are strong enough.

Also, as a caveat, the $O(4)$ NLSM with $\theta = \pi$ is a difficult interacting theory to study in general. Since many of its properties are still unknown, it is impossible for us to give a complete characterization of the stability of our BSM effective theory. We do provide a thorough analysis of the the effects of many important perturbations on the BSM model, but there are still many other symmetry-allowed perturbations that we have not been able to completely understand: for example, a quartic coupling of the form $|b_{I,A}|^2|b_{J,B}|^2$ between bosons in the "A" and "B" copies of the $O(4)$ NLSM. Our discussion in this section gives strong evidence for the stability of the semi-metal phase so we will leave a possible discussion of these untreated terms to future work.

Let us begin by addressing the issue of trivially gapping out the system by coupling the "A" copy of the $O(4)$ NLSM to the "B" copy. Since the two copies of the $O(4)$ NLSM have opposite theta angles, an interaction which could enforce $\mathbf{N}_A = \pm\mathbf{N}_B$ would have the effect of canceling the theta terms and leaving us with just a single $O(4)$ NLSM without theta term. According to Ref. 48, an $O(4)$ NLSM with $\theta = 0$ represents a trivial gapped phase of charged bosons in $2+1$ dimensions (i.e., this phase has no topological term in its electromagnetic response). Hence, as one consideration, we should make sure that it is not possible to get $\mathbf{N}_A = \pm\mathbf{N}_B$ in our model. To show that it is impossible to drive our system into a phase where $\mathbf{N}_A = \pm\mathbf{N}_B$, we should examine the term $\mathbf{N}_A \cdot \mathbf{N}_B$, as any interaction which could set $\mathbf{N}_A = \pm\mathbf{N}_B$ should be a function of $\mathbf{N}_A \cdot \mathbf{N}_B$. In terms of U_A and U_B we have

$$\mathbf{N}_A \cdot \mathbf{N}_B = \frac{1}{2}\text{tr}[U_A^\dagger U_B] . \tag{5.28}$$

This term is invariant under the $U(1)_c$ symmetry, but under $U(1)_t$ we have

$$\text{tr}[U_A^\dagger U_B] \to \text{tr}[U_A^\dagger U_B e^{-i2\xi\sigma^z}] , \tag{5.29}$$

where we used the fact that the $U(1)_t$ transformation of the bosons from Eq. (5.3) is equivalent to $U_A \to U_A e^{i\xi\sigma^z}$ and $U_B \to U_B e^{-i\xi\sigma^z}$. Since this term is not invariant under $U(1)_t$, we see that translation symmetry forbids terms which could drive our BSM model into a trivial gapped phase.

As we mentioned earlier in this subsection, there are symmetry-allowed quartic terms which can couple the two copies of the $O(4)$ NLSM in the BSM model, for example the term $|b_{I,A}|^2|b_{J,B}|^2$. Another possibility would be a current-current interaction of the form $\eta^{\mu\nu}J_{I,A}^\mu J_{J,B}^\nu$ where $J_{I,A}^\mu = \frac{i}{g}(\partial^\mu b_{I,A}^* b_{I,A} - b_{I,A}^* \partial^\mu b_{I,A})$ is the conserved number

current for boson I in the "A" NLSM, similarly for $J^{\nu}_{J,B}$, and $\eta^{\mu\nu} = \text{diag}(1, -1, -1)$ is the Minkowksi metric. A precise analysis of these terms is very difficult and beyond the scope of this paper. To address them what is really needed is the scaling dimension of the $O(4)$ field at the RG fixed point at $\theta = \pi$ discussed in Ref. 46. Despite this, we expect the BSM model to be perturbatively stable to these interactions since, at least when treated in a mean-field limit, these terms do not cause the theta terms for the "A" and "B" copies of the $O(4)$ NLSM to cancel each other.

We see that translation symmetry prevents us from coupling the two NLSM copies (if they are not at the same momentum point), so it remains to discuss the local stability of each NLSM copy . Recall that in the dual description of the BSM model we added mass terms of the form $\Phi^{(A),\dagger}_2 \sigma^z \Phi^{(A)}_2 \pm \Phi^{(B),\dagger}_2 \sigma^z \Phi^{(B)}_2$ to gap out the system and induce an interesting electromagnetic response. Suppose instead that we tried to add just a single term $\Phi^{(A),\dagger}_2 \sigma^z \Phi^{(A)}_2$ or $\Phi^{(B),\dagger}_2 \sigma^z \Phi^{(B)}_2$ to the dual theory in order to gap out just one of the "A" or "B" copies of the model. It turns out that adding one of these terms alone is actually forbidden by the composite $Z^{\mathcal{TI}}_2$ symmetry. Indeed, under $Z^{\mathcal{TI}}_2$ we have $\Phi^{(A),\dagger}_2(t, \mathbf{x}) \sigma^z \Phi^{(A)}_2(t, \mathbf{x}) \rightarrow -\Phi^{(A),\dagger}_2(-t, -\mathbf{x}) \sigma^z \Phi^{(A)}_2(-t, -\mathbf{x})$, and likewise for the "B" copy. Thus, if we require our system to obey $Z^{\mathcal{TI}}_2$ then these terms are forbidden, and some measure of stability is provided for the BSM phase.

While the requirement of $Z^{\mathcal{TI}}_2$ forbids the conventional mass terms listed above, we should also consider the local stability of each $O(4)$ NLSM in the presence of symmetry-*allowed* perturbations. The discussion here closely parallels the discussion in Sec. IV of the effects of symmetry-allowed perturbations on the surface theory of the BTI. We start by considering interspecies tunneling terms of the form $b^*_{1,A} b_{2,A} + \text{c.c.}$ for the "A" copy of the $O(4)$ NLSM. In the canonical formalism the operators $b_{I,A}(\mathbf{x})$ and their conjugate momenta $\pi_{I,A}(\mathbf{x})$ also obey the commutation relation of Eq. (4.47). Since the composite $Z^{\mathcal{TI}}_2$ symmetry acts on the bosons as $(\mathcal{TI}) b_{I,A}(\mathbf{x})(\mathcal{TI})^{-1} = b_{I,A}(-\mathbf{x})$, we deduce from the diagonal commutator that $(\mathcal{TI}) \pi_{I,A}(\mathbf{x})(\mathcal{TI})^{-1} = -\pi_{I,A}(-\mathbf{x})$. Now consider a state $|\Psi\rangle$ which is $Z^{\mathcal{TI}}_2$-symmetric, i.e., $(\mathcal{TI})|\Psi\rangle = |\Psi\rangle$. Then in such a state we find that

$$\langle\Psi|[b_{1,A}(\mathbf{x}), \pi_{2,A}(\mathbf{y})]|\Psi\rangle = -\langle\Psi|[b_{1,A}(-\mathbf{x}), \pi_{2,A}(-\mathbf{y})]|\Psi\rangle . \tag{5.30}$$

If we now plug in for the commutators on both sides of this equation using Eq. (4.47), then we find (again, after integration over the $\mathbf{y}$ coordinate) that

$$\langle\Psi|b_{1,A}(\mathbf{x}) b^{\dagger}_{2,A}(\mathbf{x})|\Psi\rangle = -\langle\Psi|b_{1,A}(-\mathbf{x}) b^{\dagger}_{2,A}(-\mathbf{x})|\Psi\rangle . \tag{5.31}$$

On the other hand, if the state $|\Psi\rangle$ is really invariant under the action of $Z^{\mathcal{TI}}_2$, then we should have

$$\langle\Psi|b_{1,A}(\mathbf{x}) b^{\dagger}_{2,A}(\mathbf{x})|\Psi\rangle = \langle\Psi|b_{1,A}(-\mathbf{x}) b^{\dagger}_{2,A}(-\mathbf{x})|\Psi\rangle . \tag{5.32}$$

Therefore we find that $\langle\Psi|b_{1,A}(\mathbf{x}) b^{\dagger}_{2,A}(\mathbf{x})|\Psi\rangle = 0$ in any state $|\Psi\rangle$ which is invariant under the combined $Z^{\mathcal{TI}}_2$ symmetry.

Just as in Sec. IV, we may conclude that weak interspecies tunneling terms should have a negligible effect on the BSM model (which has $Z^{\mathcal{TI}}_2$ symmetry), but strong interspecies tunneling can drive the system into a phase which spontaneously breaks $Z^{\mathcal{TI}}_2$ symmetry. The same conclusion holds for interspecies tunneling terms in the "B" copy of the $O(4)$ NLSM.

Also, in close analogy to the case in Sec. IV, this result may be generalized to include insertions of any operator $\tilde{\mathcal{O}}(\mathbf{x})$ which transforms nicely under the action of $\mathcal{TI}$ (recall that in Sec. IV the result was generalized to include operators $\tilde{\mathcal{O}}(\mathbf{x})$ invariant under $\mathcal{T}$). Suppose $\tilde{\mathcal{O}}(\mathbf{x})$ transforms under the action of $\mathcal{TI}$ as $(\mathcal{TI})\tilde{\mathcal{O}}(\mathbf{x})(\mathcal{TI})^{-1} = \tilde{\mathcal{O}}(-\mathbf{x})$. Then we find that

$$\langle\Psi|\tilde{\mathcal{O}}(\mathbf{x})[b_{1,A}(\mathbf{x}), \pi_{2,A}(\mathbf{y})]|\Psi\rangle = \\ -\langle\Psi|\tilde{\mathcal{O}}(-\mathbf{x})[b_{1,A}(-\mathbf{x}), \pi_{2,A}(-\mathbf{y})]|\Psi\rangle , \tag{5.33}$$

and following the same steps as above gives the result that $\langle\Psi|\tilde{\mathcal{O}}(\mathbf{x}) b_{1,A}(\mathbf{x}) b^{\dagger}_{2,A}(\mathbf{x})|\Psi\rangle = 0$ in any state $|\Psi\rangle$ which is invariant under $Z^{\mathcal{TI}}_2$. Note that $\tilde{\mathcal{O}}(\mathbf{x})$ could in principle contain operators from both the "A" and "B" copies of the NLSM, as long as it transforms under $\mathcal{TI}$ as specified above.

Finally, we can again consider chemical potential terms; the discussion of these terms is nearly identical to that in Sec. IV since the discussion of the terms in that section did not involve the time-reversal symmetry at all. For the BSM model we can add chemical potential terms of the form $\mu_1|b_{1,A}|^2 + \mu_2|b_{2,A}|^2$ for just one copy of the $O(4)$ NLSM. As in Sec. IV we again find that this term (combined with suitable quartic terms) will in general cause one of $b_{1,A}$ or $b_{2,A}$ to condense and the other to be come gapped (with the choice depending on the sign of $\mu_1 - \mu_2$). The only new feature in this context is that if a boson from one of the $O(4)$ NLSM's were to condense, then both $U(1)_c$ *and* $U(1)_t$ symmetries would be spontaneously broken (i.e., condensing a boson from just one of the $O(4)$ NLSM's also spontaneously breaks translation symmetry).

G. Topologically ordered phases accessible from the BSM theory

In this section we briefly discuss the possibility of generating topologically ordered states from the BSM model by condensing composite vortices. As in Sec. IV, a basis for describing any possible topological orders generated from the BSM model is provided by the "+" vortices $\phi^{(A)}_{1,+}, \phi^{(A)}_{2,+}, \phi^{(B)}_{1,+}$, and $\phi^{(B)}_{2,+}$, since the "−" vortices may be obtained by binding a "+" vortex with a trivial boson excitation.

In exploring different composite vortices to condense, we note first that if we condense a composite vortex of the form $\phi^{(A)}_{I,\pm} \phi^{(B)}_{J,\pm}$, then the only "+" vortices which braid trivially with this object are $\phi^{(A)}_{I,+}$ and $\phi^{(B)}_{J,+}$, and these two vortices braid trivially with each other. The resulting state is therefore trivial. This means that it is impossible to generate any topologically ordered states by condensing a product of one vortex from the "A" NLSM and one vortex from the "B" NLSM. We

must therefore consider composites which have at least two vortices from the same copy of the $O(4)$ NLSM. In this section we discuss one particular phase with $\mathbb{Z}_2 \times \mathbb{Z}_2$ topological order which is generated by condensing two fields which are themselves quadratic in the vortex fields from a single $O(4)$ NLSM. We then show that this same phase can be constructed by condensing a single field which is quartic in the vortex fields. We also show how to construct phases with $\mathbb{Z}_2$ topological order by condensing a composite vortex in one copy of the $O(4)$ NLSM and in the other copy simultaneously condensing a single vortex of one species and gapping out the other one.

We now show how to construct a phase with $\mathbb{Z}_2 \times \mathbb{Z}_2$ topological order by condensing the composite vortices $\mathcal{O}_A = \phi_{1,+}^{(A)}\phi_{1,-}^{(A)}$ and $\mathcal{O}_B = \phi_{1,+}^{(B)}\phi_{1,-}^{(B)}$ in such a way that $\langle \mathcal{O}_A \rangle = \langle \mathcal{O}_B \rangle \equiv \bar{\mathcal{O}}$ with $\bar{\mathcal{O}}$ real. The vortices $\phi_{1,+}^{(A)}, \phi_{2,+}^{(A)}, \phi_{1,+}^{(B)}$, and $\phi_{2,+}^{(B)}$ all braid trivially with $\mathcal{O}_A$ and $\mathcal{O}_B$ and so they survive as quasi-particles in the resulting topologically ordered state. The particular condensation shown here, with $\bar{\mathcal{O}}$ real, appears to respect all symmetries of the system ($U(1)_c$, $U(1)_t$, $Z_2^{\mathcal{T}}$, and $Z_2^{\mathcal{I}}$), however, we show below that this state must break either the time-reversal ($Z_2^{\mathcal{T}}$) or the inversion ($Z_2^{\mathcal{I}}$) symmetry.

Since $\{\phi_{1,+}^{(A)}, \phi_{2,+}^{(A)}\}$ braid trivially with $\{\phi_{1,+}^{(B)}, \phi_{2,+}^{(B)}\}$ the resulting state is nearly identical to two copies of the $\mathbb{Z}_2$ topological order shown in Table III. The first factor of $\mathbb{Z}_2$ is represented exactly by Table III. This part of the topological order is generated by $\{\phi_{1,+}^{(A)}, \phi_{2,+}^{(A)}\}$ and is described in the K-matrix formalism by $K^{(A)} = 2\sigma^x$, $\vec{t}^{(A)} = (1,1)^T$ and $\vec{u}^{(A)} = (1,1)^T$, where $\vec{u}^{(A)}$ is a $U(1)_t$ charge vector which describes the coupling of the vortices to the external field B_μ. Based on this data, the contribution of the "A" vortices to the electromagnetic responses of this state are

$$\mathcal{L}_{\mathcal{T}}^{(A)} = \frac{e^2}{4\pi} \epsilon^{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda \, , \qquad (5.34)$$

and

$$\mathcal{L}_{\mathcal{I}}^{(A)} = \frac{e}{2\pi} \epsilon^{\mu\nu\lambda} B_\mu \partial_\nu A_\lambda \, . \qquad (5.35)$$

The second factor of $\mathbb{Z}_2$ is generated by $\{\phi_{1,+}^{(B)}, \phi_{2,+}^{(B)}\}$. For the "B" copy, since we actually have $\theta_B = -\pi$, it seems that we should choose $K^{(B)} = -2\sigma^x$, however, there is some ambiguity here because a statistical phase of π is equivalent to a phase of $-\pi$. So let us consider both possibilities $K^{(B)} = \pm 2\sigma^x$. On the other hand, there is no ambiguity in the charges of the "B" vortices under the $U(1)_c$ and $U(1)_t$ symmetries: the coupling of $\{\phi_{1,+}^{(B)}, \phi_{2,+}^{(B)}\}$ to A_μ and B_μ is described by the charge vectors $\vec{t}^{(B)} = (-1,-1)^T$ and $\vec{u}^{(B)} = (1,1)^T$, respectively. Based on this, the contribution of the "B" copy to the responses of this state are given by

$$\mathcal{L}_{\mathcal{T}}^{(B)} = \pm\frac{e^2}{4\pi} \epsilon^{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda \, , \qquad (5.36)$$

and

$$\mathcal{L}_{\mathcal{I}}^{(B)} = \mp\frac{e}{2\pi} \epsilon^{\mu\nu\lambda} B_\mu \partial_\nu A_\lambda \, , \qquad (5.37)$$

where the signs out front correspond to the choice of $K^{(B)} = \pm 2\sigma^x$.

We see that if we choose $K^{(B)} = 2\sigma^x$, then the entire system will break time-reversal (we get the full 2D time-reversal breaking response of the BSM), but if we choose $K^{(B)} = -2\sigma^x$, the entire system breaks inversion (we get the full quasi-1D inversion breaking response of the BSM). In particular, it seems like one cannot construct a topological order consisting of the quasi-particles $\phi_{1,+}^{(A)}, \phi_{2,+}^{(A)}, \phi_{1,+}^{(B)}$, and $\phi_{2,+}^{(B)}$, which also preserves all of the symmetries of the BSM model.

The topologically ordered phase which we constructed above can also be accessed by condensing the single quartic vortex field $\mathcal{O}' = \phi_{1,+}^{(A)}\phi_{1,-}^{(A)}\phi_{1,+}^{(B)}\phi_{1,-}^{(B)}$ in such a way that the expectation value $\langle \mathcal{O}' \rangle$ is real. The field $\mathcal{O}'$ does not carry any charge under the $U(1)_c$ or $U(1)_t$ symmetries, is invariant under inversion, and is complex conjugated by time-reversal (so we should take $\langle \mathcal{O}' \rangle$ real in an attempt to preserve time-reversal).

In analyzing the resulting topological order, we first note that all four fundamental vortices $\phi_{1,+}^{(A)}, \phi_{2,+}^{(A)}, \phi_{1,+}^{(B)}$, and $\phi_{2,+}^{(B)}$ braid trivially with $\mathcal{O}'$, so they all survive as quasi-particles in the resulting topologically ordered state. The composite quasi-particles that can be constructed from these four fundamental vortices have the form $(\phi_{1,+}^{(A)})^{n_1}(\phi_{2,+}^{(A)})^{n_2}(\phi_{1,+}^{(B)})^{n_3}(\phi_{2,+}^{(B)})^{n_4}$, where the integers n_j are either 0 or 1 (since the fusion of a vortex with itself is topologically trivial). A total of 16 possible quasi-particles can be constructed by letting all n_j range over their values 0 and 1. To see whether the resulting topologically ordered state actually supports all of these quasi-particles as distinct excitations, we need to check whether any quasi-particle can be obtained from another one by fusing with the condensate $\mathcal{O}'$, which is equivalent to the vacuum (in the phase where $\mathcal{O}'$ is condensed). We find that each of the 16 quasiparticles is topologically distinct and that this set is sufficient to label all of the anyon sectors. Hence, the resulting state is actually identical to the state obtained earlier from simultaneously condensing $\mathcal{O}_A$ and $\mathcal{O}_B$. This result could have been anticipated since $\mathcal{O}' = \mathcal{O}_A\mathcal{O}_B$, and the vortices from the "A" copy of the NLSM braid trivially with the vortices from the "B" copy.

Another way to see that condensing $\mathcal{O}'$ leads to $\mathbb{Z}_2 \times \mathbb{Z}_2$ topological order, and not, for example, $\mathbb{Z}_4$ topological order, is as follows. First, note that $\phi_{1,\pm}^{(A)}$ carry charge 1 of the dual gauge field $\alpha_{1,\mu}^{(A)}$ (whose curl is the number current of $b_{1,A}$), while $\phi_{1,\pm}^{(B)}$ carry charge 1 of the dual gauge field $\alpha_{1,\mu}^{(B)}$ (whose curl is the number current of $b_{1,B}$). So the composite field $\mathcal{O}'$ carries charge 2 of $\alpha_{1,\mu}^{(A)}$ and charge 2 of $\alpha_{1,\mu}^{(B)}$. Therefore, condensing $\mathcal{O}'$ will break the $U(1)$ symmetries associated with $\alpha_{1,\mu}^{(A)}$ and $\alpha_{1,\mu}^{(B)}$ down to a $\mathbb{Z}_2$ subgroup, i.e., the symmetry-breaking associated with this condensation is $U(1) \times U(1) \to \mathbb{Z}_2 \times \mathbb{Z}_2$. If instead it were the case that the four vortices $\phi_{1,\pm}^{(A)}$ and $\phi_{1,\pm}^{(B)}$ all carried charge 1 of the *same* $U(1)$ gauge field, then we would expect the condensation of $\mathcal{O}'$ to break that $U(1)$ symmetry down to a $\mathbb{Z}_4$ subgroup, lead-

ing to a $\mathbb{Z}_4$ topological order. This does not happen in our case since the vortices $\phi_{1,\pm}^{(A)}$ and $\phi_{1,\pm}^{(B)}$ couple to different $U(1)$ gauge fields.

As we mentioned above, it is also possible to generate a phase with $\mathbb{Z}_2$ topological order by condensing composite vortices in one copy of the $O(4)$ NLSM, and in the other copy simply gapping out one vortex species and condensing the other. We show that such a phase will break either the time-reversal or the inversion symmetry of the BSM model. As an example, consider condensing the composite vortex $\mathcal{O}_A$ in the "A" copy of the NLSM, while in the "B" copy condensing the single vortex $\phi_{2,+}^{(B)}$, and gapping out the vortex $\phi_{2,-}^{(B)}$. The resulting phase has a $\mathbb{Z}_2$ topological order generated by $\phi_{1,+}^{(A)}$ and $\phi_{2,+}^{(A)}$. Note that the "B" copy does not contribute to the topological order since $\phi_{2,+}^{(B)}$ has been condensed (i.e., it is now topologically equivalent to the vacuum quasi-particle) and $\phi_{1,+}^{(B)}$ is confined (it has non-trivial braiding with $\phi_{2,+}^{(B)}$, which is condensed). The electromagnetic response of this phase can be easily calculated using the results contained in this section, and we find that this phase has no time-reversal breaking response, but it does possess the full inversion breaking response of the BSM, as shown in Eq. (5.21). If for the "B" copy we instead chose to condense $\phi_{2,-}^{(B)}$ and gap out $\phi_{2,+}^{(B)}$ (while still condensing $\mathcal{O}_A$ for the "A" copy), we would get a phase with $\mathbb{Z}_2$ topological order which has no inversion breaking response, but the full time-reversal breaking response of the BSM, as in Eq. (5.20).

VI. QUANTIZATION OF POLARIZATION IN GAPPED 2D PHASES AND A CRITERION FOR SEMI-METAL BEHAVIOR

In this section we give a general discussion of the quantization of the charge polarization in gapped phases of 2D quantum many-body systems with translation, inversion, and $U(1)_c$ charge conservation symmetries, with the goal of establishing a criterion for detecting whether a given system is in a semi-metallic phase by measuring its polarization response. The systems in question can be either bosonic or fermionic, and we assume they are made of up some fundamental particles of charge e. For simplicity we focus on systems on a square lattice with lattice spacing a_0, but the result can be easily extended to any Bravais lattice. We consider three broad classes (to be described below) of gapped phases of 2D systems in which one can define a charge polarization, and we show that in these three classes the polarization in (say) the x-direction is quantized in units of

$$P_x^{(min)} = r \frac{e}{2a_0} \, , \qquad (6.1)$$

where $r \in \mathbb{Q}$ is a rational number. This result then implies that *if* a 2D quantum many-body system is found to have a continuously tunable polarization of the form $\alpha \frac{e}{2a_0}$ for a generic real number α, *then* this system cannot be in one of the three classes of gapped phases mentioned above. If these three classes of gapped phases exhaust all possible gapped phases

with translation symmetry which can support a polarization response, and from their definitions below it is clear that they do, then this implies that a polarization of the form $\alpha \frac{e}{2a_0}$ for generic $\alpha \in \mathbb{R}$ is indicative of a gapless semi-metal phase. Therefore our argument in this section provides a direct relation between the gaplessness of a semi-metal and the tunability of its polarization response. As we mentioned in Sec. II, since the polarization response is expected to be reasonably robust, this provides additional evidence for the stability of the semi-metal phase to perturbations which do not destroy its polarization response.

As we show below, the rational number r mentioned above can be related to specific measurable properties of the three types of gapped phases that we consider, so we do not need to worry about the difficulty of "measuring an irrational number", as the number r can be readily obtained for these gapped phases in other ways. Thus, the charge polarization response of a 2D system can be used as a criterion for detecting a gapless semi-metal phase. The reason for focusing on gapped phases *with* translation symmetry is that we know that a semi-metal requires translation symmetry for its stability. Since we are looking for a way to distinguish a semi-metal phase from other phases with a polarization response, we need to compare to other systems with translation symmetry as we know that without translation symmetry the semi-metal phase is not even a possibility. We now give the details of our argument.

The three classes of gapped systems which we consider are (i) systems with a unique ground state and translation symmetry by one site, (ii) systems with a ground state which spontaneously breaks translation symmetry by one site down to translation symmetry by q sites (so q is a positive integer), and (iii) systems with intrinsic topological order as well as translation symmetry by one site. To calculate the charge polarization of these systems we use a many-body formula for the polarization introduced by Resta in Ref. 57, which we now review. We focus on an analysis of the polarization in the x-direction, and so we assume periodic boundary conditions in that direction. This assumption of periodic boundary conditions in at least one direction will also allow us to invoke certain theorems[58,59] which will be crucial for our results in this section.

A. Polarization in 2D, ambiguity with translation invariance, and quantization with inversion symmetry

Consider a quantum many-body system defined on a square lattice with lattice spacing a_0 and lengths L_x and L_y in the x and y directions. Let N_s be the number of sites so that $N_s a_0^2 = L_x L_y$. We label sites on the square lattice by the vector of integers $\mathbf{j} = (j_x, j_y)$, $j_x, j_y \in \mathbb{Z}$. Finally, let $|\Psi_0\rangle$ be the ground state of the system. We assume $|\Psi_0\rangle$ is an eigenstate of the number operator with eigenvalue N_p so that the filling factor in the ground state is $\nu = \frac{N_p}{N_s}$. The total number operator can be expressed as $\hat{N} = \sum_{\mathbf{j}} \hat{n}_{\mathbf{j}}$ where $\hat{n}_{\mathbf{j}}$ is the number operator for site $\mathbf{j}$. Then, assuming that $|\Psi_0\rangle$ is the ground state of a *gapped* system, Resta's formula tell us that

the polarization in the x-direction is given by

$$P_x = \lim_{L_x \to \infty} \frac{e}{2\pi L_y} \text{Im}[\ln\langle\Psi_0|e^{i\frac{2\pi}{L_x}\hat{X}}|\Psi_0\rangle] , \quad (6.2)$$

where the position operator $\hat{X}$ is given by

$$\hat{X} = \sum_{\mathbf{j}} (j_x a_0)\hat{n}_{\mathbf{j}} . \quad (6.3)$$

The polarization in the y-direction has a similar definition.

Let us suppose that the state $|\Psi_0\rangle$ has translation invariance by one site in the x-direction, i.e., $|\Psi_0\rangle$ is an eigenstate of the translation operator $\hat{T}_x$ with some eigenvalue $e^{ik_x^{(0)}}$ (the precise value of $k_x^{(0)}$ will not be important in what follows). Concretely, $\hat{T}_x$ acts as $\hat{T}_x^\dagger \hat{O}_{\mathbf{j}} \hat{T}_x = \hat{O}_{\mathbf{j}+(1,0)}$ on operators $\hat{O}_{\mathbf{j}}$ carrying a position index. If this is the case then one can show that P_x is only well-defined modulo $\frac{e\nu}{a_0}$. To see it we compute the polarization P_x' of $|\Psi_0'\rangle \equiv \hat{T}_x|\Psi_0\rangle$ in two ways. On one hand we can just write $|\Psi_0'\rangle \equiv e^{ik_x^{(0)}}|\Psi_0\rangle$ to find that $P_x' = P_x$. However, we can also use

$$\langle\Psi_0|\hat{T}_x^\dagger e^{i\frac{2\pi}{L_x}\hat{X}}\hat{T}_x|\Psi_0\rangle = \langle\Psi_0|e^{i\frac{2\pi}{L_x}\hat{X}}e^{-i\frac{2\pi a_0}{L_x}\sum_{\mathbf{j}}\hat{n}_{\mathbf{j}}}|\Psi_0\rangle$$
$$= \langle\Psi_0|e^{i\frac{2\pi}{L_x}\hat{X}}|\Psi_0\rangle e^{-i\frac{2\pi a_0 N_p}{L_x}} , (6.4)$$

to show that

$$P_x' = P_x - \frac{e\nu}{a_0} . \quad (6.5)$$

So we conclude that P_x is defined only modulo $\frac{e\nu}{a_0}$ in the presence of translation symmetry by one site.

The last ingredient in the polarization calculation is to enforce inversion symmetry in the system. We consider inversion which acts simply as $\mathbf{j} \to -\mathbf{j}$ for the coordinates on the square lattice. It is clear that under inversion we have $P_x \to -P_x$ and similarly for the polarization in the y-direction. So the polarization in the inversion symmetric system must obey the relation

$$P_x \equiv -P_x \bmod \frac{e\nu}{a_0} . \quad (6.6)$$

The solutions to this relation are

$$P_x \equiv 0 \text{ or } \frac{e\nu}{2a_0} \bmod \frac{e\nu}{a_0} , \quad (6.7)$$

with a similar result for P_y. So in a gapped 2D system with translation and inversion symmetry and filling factor ν, the polarization is quantized in units of

$$P_x^{(min)} = \frac{e\nu}{2a_0} . \quad (6.8)$$

Before moving on let us make a few general comments about this formula for the polarization. First, in a band insulator made out of free fermions the filling ν must be an integer in order for the system to be gapped (i.e., in order to have a completely filled band). This is why the filling ν usually does not appear explicitly in discussions of the polarization in band

insulators. Also, in the discussion above we have assumed that there is only one type of particle. More generally, our system could have several different species of particles, for example spin up and spin down electrons, and in this case one can separately consider the polarization for each species. If we label different particles species by σ then we can compute $P_{x,\sigma}$, the polarization from particles of species σ, by modifying the position operator $\hat{X}$ to

$$\hat{X}_\sigma = \sum_{\mathbf{j}} (j_x a_0)\hat{n}_{\mathbf{j},\sigma} , \quad (6.9)$$

where $\hat{n}_{\mathbf{j},\sigma}$ is the number of particles of species σ on site $\mathbf{j}$. The total polarization is then given by $P_x = \sum_\sigma P_{x,\sigma}$. The importance of computing the polarization in this way is demonstrated by the following example. Suppose we have a band insulator of spinful electrons (so $\sigma = \uparrow, \downarrow$) and we have a completely filled band of up and down electrons. Then we have $\nu_\uparrow = 1$ and $\nu_\downarrow = 1$ and so the total filling is $\nu = 2$. However, in the absence of time-reversal symmetry both bands do not have to have the same polarization. Since each individual band is at filling $\nu_\sigma = 1$ we could have $P_{x,\uparrow} = \frac{e}{2a_0}$ but $P_{x,\downarrow} = 0$, and so $P_x = \frac{e}{2a_0}$. This result could not have been predicted from Eq. (6.8), since that formula does not distinguish between different particle species.

We now discuss the specific values that ν can take in the three classes of gapped systems discussed above, and in this way constrain the possible values of $P_x^{(min)}$ in such phases.

B. The filling factor ν in the three classes of gapped phases

Now we discuss the possible values of the filling factor ν in the three classes of gapped phases, which will in turn give us the minimum value $\frac{e\nu}{2a_0}$ of the polarization in these systems. To start we go back to a theorem of Oshikawa[58] which was later proven rigorously (under slightly more restrictive assumptions) by Hastings[59]. What Oshikawa/Hastings showed is that if the filling ν of a *gapped* system is a rational number, say $\nu = \frac{p}{q}$ with p and q coprime, then the system will in general have q degenerate ground states (in the thermodynamic limit), each with a different momentum in the (for example) x-direction. For integer ν the ground state is unique. On the other hand, irrational values of ν in the ground state generally imply a gapless system. In Hastings' rigorous proof the condition is actually that $\nu\left(\frac{N_s a_0}{L_x}\right) = \frac{N_p a_0}{L_x} = \frac{p}{q}$, where L_x is the length of the system in the x-direction[59]. In what follows we assume that this result holds for the condition $\nu = \frac{p}{q}$, as is expected on general physical grounds, although the reader should be aware that there is no rigorous proof available in this case (and there are even counterexamples in 2D systems which are long in one direction but short in the other, see e.g., Ref. 60).

Using this theorem we can immediately conclude that in the case of integer filling the minimum value of the polarization in the ground state of a gapped, translation-invariant 2D system

with a unique ground state is

$$P_x^{(\min)} = \frac{e}{2a_0} , \qquad (6.10)$$

which corresponds to the filling $\nu = 1$. This gives the answer for the minimum value of the polarization in a gapped system in class (i) discussed above.

Next we discuss the case of rational filling factor $\nu = \frac{p}{q}$, which will turn out to include gapped systems in classes (ii) and (iii). For rational filling factor $\nu = \frac{p}{q}$ there are two possible physical explanations for the q degenerate ground states[58,59]. The first possibility is that the q degenerate states correspond to a spontaneous breaking of the translation symmetry by one lattice site down to translation symmetry by q lattice sites. In this case the actual ground state in the thermodynamic limit is expected to be a particular linear combination of the q ground states (which each have different momenta in the x-direction) which is an eigenstate of $(\hat{T}_x)^q$ but not of $\hat{T}_x$, thus breaking the symmetry of translation by one site. This corresponds to our class (ii) of gapped phases. If we repeat the analysis from above of the ambiguity of the polarization in the presence of translation symmetry, but replace $\hat{T}_x$ with $(\hat{T}_x)^q$, then we find that the polarization is only well-defined modulo $\frac{qe\nu}{a_0}$. Then in the presence of inversion symmetry the minimum value of the polarization in this case is also

$$P_x^{(\min)} = \frac{e}{2a_0} , \qquad (6.11)$$

corresponding to the choice $\nu = \frac{1}{q}$.

The final possibility is that the system at filling factor $\nu = \frac{p}{q}$ does not break translation symmetry but instead has intrinsic topological order, which can also explain the q-fold ground state degeneracy in the thermodynamic limit. This corresponds to our class (iii) of gapped systems. In this case the filling factor ν can be related to the data describing a 2D symmetry-enriched topological (SET) phase with $U(1)_c$ and translation symmetry[61,62] and so we now give a brief overview of the physical properties of 2D SET phases with $U(1)_c$ and translation symmetry. For more details see Ref. 61.

An SET phase in 2D is a gapped phase possessing intrinsic topological order, but which also has global symmetry of a group G (see Ref. 63 for an in-depth discussion of these phases). The group G can act in various non-trivial ways on the anyons which are present in the topologically ordered system. For example if $G = U(1)_c$ then an anyon can carry a fractional charge under G (i.e., the anyon transforms in a projective representation of G). A more exotic possibility is that the action of G can *exchange*, or permute, two different kinds of anyons. In the case where the symmetry does not permute the anyons it is known that 2D SET phases with symmetry group G are classified by the cohomology group $H^2(G, \mathcal{A})$, where $\mathcal{A}$ is the group of Abelian anyons in the topologically ordered system.

In a 2D SET phase with $U(1)_c$ symmetry, each anyon a can carry a particular fractional charge $e_a = Q_a e$ under the $U(1)$ symmetry, where Q_a is a dimensionless number. The number Q_a can also be expressed in terms of the mutual braiding

statistics $M_{a,v}$ of a with the anyon v, which is the excitation created in the system by threading 2π delta function flux of the $U(1)_c$ gauge field at a point in the system (this excitation was referred to as a *vison* in Ref. 62). Here $M_{a,a'} = e^{i\theta_{a,a'}}$ is the $U(1)$ phase accumulated during a process in which the anyon a makes a complete circuit around the anyon a'. This essentially calculates the Aharonov-Bohm phase of the $U(1)_c$ charge carried by a when dragged around the fundamental flux of $U(1)_c$ carried by v. Hence, we have the relation

$$e^{i2\pi Q_a} = M_{a,v} , \qquad (6.12)$$

or

$$Q_a = \frac{\theta_{a,v}}{2\pi} . \qquad (6.13)$$

A 2D SET phase with translation symmetry is characterized by one additional property. This is the *anyonic flux* b per unit cell, where b is an Abelian anyon in the topologically ordered system under consideration. The physical meaning of the anyonic flux b is that if an anyon a is translated around a unit cell, then the state of the system picks up the phase $M_{a,b}$.

We see that we can characterize a 2D SET phase with $U(1)_c$ and translation symmetry by the data $(\{e_a\}, b)$, which includes the set of charges $\{e_a\}$ of the anyons under the $U(1)_c$ symmetry, and the particular anyon b which provides the anyonic flux per unit cell in the system.

The authors of Refs. 61 and 62 showed that the filling factor ν in a 2D SET phase with translation and $U(1)_c$ symmetry can be expressed in terms of the data of the SET phase as

$$\nu \equiv Q_b \bmod 1 , \qquad (6.14)$$

or, using Eq. (6.13),

$$\nu \equiv \frac{\theta_{b,v}}{2\pi} \bmod 1 . \qquad (6.15)$$

So the filling factor of the 2D SET phase is equal to the $U(1)_c$ charge of the anyon b characterizing the anyonic flux per unit cell in the system, and this is in turn related to the mutual statistical angle $\theta_{b,v}$ between b and the excitation v. The derivation of this equation essentially uses Oshikawa's original flux threading argument and the fact that threading a flux through the hole of the torus is equivalent to wrapping a string operator for v around the cycle of the torus which does not enclose the hole[62]. Note that Q_b must be a rational number since if it were not then the relation between Q_b and ν, combined with the Oshikawa/Hastings argument, would imply that the phase was gapless and not a gapped SET phase. From this relation between Q_b and ν we find that the minimum value of the polarization for systems in our class (iii) is

$$P_x^{(\min)} = \frac{e_b}{2a_0} = \frac{eQ_b}{2a_0} . \qquad (6.16)$$

We have succeeding in showing that for all three classes of gapped phases considered in this section, the polarization is quantized to some rational multiple of $\frac{e}{2a_0}$ and, in particular, is not continuously variable since tuning ν away from

a rational value leads to gapless phase according to the Oshikawa/Hastings argument. Thus, we see that generic non-rational values of the polarization are indicative of a gapless semi-metal phase. Furthermore, we have shown how the polarization in these gapped phases can be simply related to various physical data describing those phases, which means that it should be simple to diagnose whether a given value of the polarization implies a gapped or gapless phase.

VII. COUPLED WIRES CONSTRUCTION OF THE BOSONIC SEMI-METAL

So far we have provided an effective theory for a 2+1-d bosonic semi-metal, and discussed its electromagnetic response properties and stability criteria. In this Section we provide an explicit construction of this phase using a coupled wires approach which is modeled after the coupled wires construction of a single $O(4)$ NLSM with $\theta = \pi$ derived in Ref. 42 (see also Refs. 17 and 43). We are able to find a suitable wire building-block, as well as suitable inter-wire tunneling terms, which together generate our 2D BSM model after taking the continuum limit in the wire stacking direction.

The rationale for a coupled wires construction of the BSM model is provided by the general demonstration in Ref. 7 that free fermion DSMs admit a coupled wires construction in terms of 1+1-d topological insulator wires, each with a charge $\frac{e}{2}$ polarization response. Indeed, one of the most important aspects of the coupled wires construction of the free fermion DSM is the intuitive explanation it provides for the quasi-1D inversion-breaking electromagnetic response of the DSM model, shown in Eq. (2.1).

This Section is organized as follows. We begin by reviewing the coupled wires construction of the free fermion DSM. We then construct a wire building block for the 2D BSM phase using two copies of the Bosonic Integer Quantum Hall (BIQH) edge theory. For our purposes, we require the description of the BIQH edge in terms of an $SU(2)_1$ Wess-Zumino-Witten (WZW) theory, as discussed in Refs. 15 and 17. For completeness, we briefly review this description of the BIQH edge, and carefully discuss how this edge theory couples to an external electromagnetic field. We then review the derivation of Ref. 42 of a single $O(4)$ NLSM with $\theta = \pi$ from coupled wires consisting of a single copy of an $SU(2)_1$ WZW theory.

With all of this information in hand, we go on to present a coupled wires construction of the 2D BSM model, and we include a careful discussion of how to define the action of time-reversal and inversion symmetries in the coupled wires model so that the correct action of these symmetries on the continuum fields is recovered in the continuum limit. Finally, we conclude this section by contrasting the coupled wires constructions of the DSM and BSM phases, and we briefly comment on how the symmetry breaking phases of the BSM model can be accessed within the coupled wires description.

A. Coupled Wire Construction of a Fermionic Dirac Semi-metal

We begin by reviewing the construction of the free-fermion DSM via a stacking of 1D gapped topological free fermion wires, each with charge $\frac{e}{2}$ polarization. This construction was introduced in Ref. 7, and it provides a clear physical interpretation of the quasi-1D inversion-breaking response of the DSM in terms of the polarization response of the individual wires in the stacking construction. Just as in Sec. III, the degrees of freedom are two-component spinless fermions $\vec{c}_n$ living on a 1D lattice with site index n. The Bloch Hamiltonian for the 1+1-d free fermion topological wire model has the form

$$\mathcal{H}_{1D}(k_x) = \sin(k_x)\sigma^x + (1 - m - \cos(k_x))\sigma^z . \quad (7.1)$$

This model is in a topological phase for $0 < m < 2$, and one can show that charge $\pm \frac{e}{2}$ is trapped at a domain wall between a state with $m \lesssim 0$ and $m \gtrsim 0$[35,64–66]. For our interest, we consider the topological phase of this model to be protected by inversion symmetry[67,68], where the inversion operator $\mathcal{I}$ acts on the lattice fermions as

$$\mathcal{I}\vec{c}_m\mathcal{I}^{-1} = \sigma^z \vec{c}_{-m} . \quad (7.2)$$

To obtain the DSM model we now stack these 1+1-d fermion wires into two dimensions and introduce a hopping term: $t_y(\vec{c}_{\mathbf{n}+\hat{\mathbf{y}}}^\dagger \sigma^z \vec{c}_{\mathbf{n}} + \text{h.c.})$ between fermions on adjacent wires. The Bloch Hamiltonian for the resulting 2D system has exactly the form of Eq. (3.2). Now we note that the 2+1-d model Eq. (3.2) looks like many copies of the 1+1-d model in Eq. (7.1) where the different copies of the 1+1-d wire are labeled by k_y, and with each having a k_y-dependent mass

$$m_{k_y} = m + t_y \cos(k_y) . \quad (7.3)$$

Essentially, the Bloch Hamiltonian for each value of k_y represents a 1+1-d insulator of the type Eq. 7.1, but with a k_y-dependent mass parameter.

Consider the parameter range $m, t_y > 0$, and recall the definition of B_y from Sec. III (it is the positive solution to $m + t_y \cos(B_y) = 0$ with $B_y \in [0, \pi)$). We see that the 1+1-d systems labeled by k_y have $m_{k_y} > 0$ for $k_y \in (-B_y, B_y)$, but $m_{k_y} < 0$ for $k_y \in (B_y, \pi)$ or $k_y \in (-\pi, -B_y)$. So the 1+1-d systems in the range $-B_y < k_y < B_y$ are in the topological phase, while the rest are in the trivial phase.

As we now review, this observation immediately leads to a microscopic description of the quasi-1D response of the DSM. First, note that each *topological* wire contributes a factor $\frac{e}{2} \int dx dt \, F_{tx}$ to the electromagnetic response of the system. Here $F_{tx} = \partial_t A_x - \partial_x A_t = -E_x$ (the electric field in the x-direction), so this response represents a charge polarization of magnitude $\frac{e}{2}$ in the x-direction. The total number of 1+1-d systems in the range $-B_y < k_y < B_y$ is $\frac{2B_y}{\left(\frac{2\pi}{N_y a_0}\right)}$, where N_y is the number of wires that we stack to construct the 2+1-d system, and a_0 is the lattice spacing in the y-direction.

So the total electromagnetic response from all of the topological wires in the range $-B_y < k_y < B_y$ is

$$S_{eff,1D} = \frac{2B_y}{\left(\frac{2\pi}{N_y a_0}\right)} \frac{e}{2} \int dx dt \ F_{tx} \ . \qquad (7.4)$$

Using $N_y a_0 = \int dy$ and the fact that B_y is uniform in this case, we get

$$S_{eff,1D} = \frac{e}{2\pi} \int d^3 x \ B_y F_{tx} \ , \qquad (7.5)$$

which is exactly the response from Eq. (2.1) for the case where only $B_y \neq 0$.

We would like to make one more comment about the free fermion topological wire model of Eq. (7.1). If one linearizes this model near the $m = 0$ critical point, and then takes a continuum limit, the resulting model is a $1 + 1$-d Dirac fermion with a Dirac mass (back-scattering) term acting between the left and right-moving fermions that make up the Dirac fermion. For the BSM construction it will be useful to make the following analogy. We note that the edge theory of the $\nu = 1$ Integer Quantum Hall Effect (IQHE) (for fermions) is a single right-moving fermion. Hence, the fermion topological wire model used to construct the DSM can then be interpreted as being built from the edge theory of a $\nu = 1$ IQH state and a $\nu = -1$ IQH state, with an additional back-scattering mass term introduced to gap out the entire system. Alternatively, we could think of the wire as just a thin strip of $\nu = 1$ IQHE where the opposing edges are close enough to interact with each other. Similarly, in our coupled wires construction of the BSM model, each individual wire will contain the two counter-propagating edge modes of a thin strip of the BIQH system.

B. Edge Theory of the Bosonic Integer Quantum Hall System

In this section we briefly discuss the edge theory of the BIQH system, paying close attention to how the edge theory couples to an external electromagnetic field. This edge theory will help form the basic building block for the 1D bosonic wires we will use to construct our 2D BSM model, just as the edge theory for the fermion IQH system forms the basic building block for the 1+1-d fermionic topological wire considered in the previous subsection. We expect the edge theory for the BIQH system to satisfy (at least) two requirements: (i) the basic fields in the model are bosonic, and (ii) the $U(1)_c$ charge conservation symmetry is realized in an anomalous way so that the variation of the boundary action under a gauge transformation cancels the contribution from the bulk Chern-Simons action for the BIQH system.

The edge theory for the BIQH state can be described using the K-matrix formalism familiar from Abelian quantum Hall systems (in which case it is described by $K = \sigma^x$, c.f. Ref 16), however, we will use the description of the edge theory in terms of an $SU(2)_1$ Wess-Zumino-Witten (WZW) theory, which was proposed in Refs. 15 and 17. Here we review some details of this theory and explicitly show that the anomaly of the edge theory with the correct charge assignment exactly cancels the boundary term we obtain when we perform a gauge transformation on the bulk Chern-Simons effective action for the BIQH system. Indeed, this clearly shows that the BIQH state can be terminated with an $SU(2)_1$ WZW edge theory.

The bulk Chern-Simons effective action for the BIQH system can be written in differential form notation as

$$S_{BIQH} = \frac{1}{2\pi} \int_M A \wedge dA \ , \qquad (7.6)$$

where M is the space-time manifold and $A = A_\mu dx^\mu$. Under a gauge transformation $A \rightarrow A + d\chi$ we have $S_{BIQH} \rightarrow S_{BIQH} + \delta S_{BIQH}$ with

$$\delta S_{BIQH} = \frac{1}{2\pi} \int_{\partial M} A \wedge d\chi \ . \qquad (7.7)$$

Therefore, in order for the system as a whole to be gauge invariant, we should expect that the edge theory has an anomaly when we couple to an electromagnetic field, in order to cancel this term coming from the gauge-variation of the bulk action.

The $SU(2)_1$ WZW theory takes the form (see Ref. 69 for an introduction)

$$S = \frac{1}{8\pi} \int d^2 x \ \text{tr}[(\partial^\mu U^\dagger)(\partial_\mu U)] - S_{WZ}[U] \ , \qquad (7.8)$$

where U is an $SU(2)$ matrix field, and the Wess-Zumino (WZ) term is

$$S_{WZ}[U] = \qquad (7.9)$$
$$\frac{1}{12\pi} \int_0^1 ds \int d^2 x \ \epsilon^{\mu\nu\lambda} \text{tr}[(\tilde{U}^\dagger \partial_\mu \tilde{U})(\tilde{U}^\dagger \partial_\nu \tilde{U})(\tilde{U}^\dagger \partial_\lambda \tilde{U})].$$

As usual, the WZ term involves integration over an auxiliary direction of spacetime. In this expression $\tilde{U}(s,t,x)$ denotes an extension of $U(t,x)$ into the s-direction and $\mu, \nu, \lambda = s, t, x$ in the sum (we take $\epsilon^{stx} = 1$). By convention, one typically chooses boundary conditions $\tilde{U}(0,t,x) = \mathbb{I}$ (i.e., a trivial configuration) and $\tilde{U}(1,t,x) = U(t,x)$, so that the physical spacetime is located at $s = 1$.

The $SU(2)_1$ WZW theory has an $SU(2) \times SU(2)$ symmetry: the action is invariant under the replacement $U \rightarrow g^\dagger U h$, for $g, h \in SU(2)$. The transformation with $g = \mathbb{I}$ is referred to as the *right* $SU(2)$ symmetry, while the transformation with $h = \mathbb{I}$ is referred to as the *left* $SU(2)$ symmetry. The case $g = h$ is called the *diagonal* $SU(2)$ symmetry. Just as in Sec. IV, the matrix U can be written in terms of bosonic fields b_I, $I = 1, 2$, and the physical $U(1)_c$ symmetry $b_I \rightarrow e^{i\chi} b_I$ is realized on U as $U \rightarrow U e^{i\chi \sigma^z}$. Hence, the $U(1)_c$ symmetry is a $U(1)$ subgroup of the right $SU(2)$ symmetry of the $SU(2)_1$ WZW theory.

It is known that one cannot obtain a gauge invariant action by only gauging the right or left $SU(2)$ symmetry of the WZW theory (or a subgroup of one of these symmetry groups)[70]. However, in the case where one chooses to gauge a left or right symmetry of the theory, there is a "best possible" action that one can obtain, in which the gauge transformation

produces a term that only depends on the gauge field itself and the element of the Lie algebra involved in the gauge transformation (instead of a more complicated expression involving the actual field U)[71]. In our case, this "best possible" action takes the form

$$S_{gauged} = \frac{1}{4\pi} \int d^2x \sum_I (D^\mu b_I^*)(D_\mu b_I) - S_{WZ}[U]$$

$$+ \frac{1}{4\pi} \int d^2x \, \epsilon^{\mu\nu} \text{tr}[iA_\mu \sigma^z U^\dagger \partial_\nu U] , \qquad (7.10)$$

where $D_\mu = \partial_\mu - iA_\mu$. In the kinetic term we applied the usual minimal coupling procedure $\partial_\mu \to D_\mu$. The last term, however, is more mysterious. Its purpose is to make the gauge-variation of this action as nice as possible (the WZ term is not gauge invariant). Indeed, under a gauge transformation $U \to U e^{i\chi(x)\sigma^z}$, $A_\mu \to A_\mu + \partial_\mu \chi(x)$, we have

$$\delta S_{gauged} = -\frac{1}{2\pi} \int d^2x \, \epsilon^{\mu\nu} A_\mu \partial_\nu \chi = -\frac{1}{2\pi} \int_{\partial M} A \wedge d\chi . \qquad (7.11)$$

This precisely cancels the gauge variation of the bulk Chern-Simons term, which shows that the $SU(2)_1$ WZW theory with gauged right $U(1)$ symmetry is an appropriate description of the edge of the BIQH system.

In our coupled wires construction of the BSM we will take each wire to consist of two copies of the $SU(2)_1$ WZW theory, with fields U_+ and U_-, but with the two copies having opposite signs on their WZ terms. Based on the form of the gauged action Eq. (7.10) for one WZW theory, it is clear that this doubled system can be gauged in such a way that the total action is completely gauge-invariant. This 1D wire model, which can be interpreted as consisting of two counter-propagating BIQH edge modes, is a completely consistent 1D system and is therefore an appropriate building block for a coupled wires construction of the BSM model.

C. Review: Coupled wires model for one $O(4)$ NLSM with $\theta = \pi$

Before presenting the coupled wires construction of the BSM model, we first review the coupled wires construction of a single $O(4)$ NLSM with $\theta = \pi$, which was first derived in Ref. 42 (see also Refs. 17 and 43). In this construction each 1D wire consists of just one copy of the $SU(2)_1$ WZW theory. We note briefly that in accordance with the discussion in the previous subsection, if each wire contains only one copy of the $SU(2)_1$ WZW theory, then the left or right $SU(2)$ symmetry of each wire cannot be consistently gauged. This was not a problem in the physical context of Refs. 42 and 43, where the $SU(2)_1$ WZW theory was considered in connection with 1D spin chains. In that case the $SU(2)$ subgroup of the theory which one might consider gauging is actually the diagonal subgroup ($U \to h^\dagger U h$), and this subgroup can be consistently gauged[70].

We label the individual wires in the wire model by the discrete coordinate $j = 0, \ldots, N-1$. The lattice spacing in the stacking direction is a_0, and the continuum coordinate for the stacking direction will be $y = ja_0$. The unperturbed action for the collection of wires is

$$S_0 = \sum_j \left\{ \frac{1}{8\pi} \int d^2x \, \text{tr}[(\partial^\mu U_j^\dagger)(\partial_\mu U_j)] + (-1)^j S_{WZ}[U_j] \right\} . \qquad (7.12)$$

We see that the sign of the WZ term alternates between adjacent wires. The coupling between the wires takes the form

$$S_\perp = \frac{t_\perp}{2} \sum_j \int d^2x \sum_{I=1}^2 (b_{I,j}^* b_{I,j+1} + \text{c.c.}) , \qquad (7.13)$$

where $t_\perp > 0$, and b_1 and b_2 are the matrix elements of U. This term is proportional to $\text{tr}[U_j^\dagger U_{j+1} + \text{h.c.}]$. We now Fourier transform in the stacking direction

$$b_{I,j} = \frac{1}{\sqrt{N}} \sum_k b_{I,k} e^{ikja_0} , \qquad (7.14)$$

to get

$$S_\perp = t_\perp \sum_k \cos(ka_0) \int d^2x \sum_I b_{I,k}^* b_{I,k} . \qquad (7.15)$$

The key point now is that we should expand this term near its lowest energy point. This should be contrasted with the free fermion case, where the correct procedure was to expand the dispersion near the band touchings at zero energy (which is where the low energy excitations are located when the lattice is at half-filling). The potential energy associated with $S_\perp$ is

$$H_\perp = -t_\perp \sum_k \cos(ka_0) \int dx \sum_I b_{I,k}^* b_{I,k} , \qquad (7.16)$$

which has its minimum value at $k = 0$ for $t_\perp > 0$. Expanding around $k = 0$ gives

$$S_\perp \approx \text{const.} - \frac{t_\perp}{2} \sum_k (ka_0)^2 \int d^2x \sum_I b_{I,k}^* b_{I,k} . \qquad (7.17)$$

Since this interaction tends to align the fields U_j and U_{j+1} (if we think of them as four component unit vector fields), it makes sense to introduce the slowly varying continuum fields $b_I(t, x, y)$, which are obtained from $b_{I,j}(t, x)$ by keeping only the modes near $k = 0$. We have[72]

$$b_{I,j}(t,x) \approx b_I(t,x,y) = \frac{1}{\sqrt{N}} \int \frac{dk}{\left(\frac{2\pi}{Na_0}\right)} b_{I,k}(t,x) e^{iky} , \qquad (7.18)$$

where $y = ja_0$, and we have expressed the continuum field $b_I(t, x, y)$ as an integral over a continuous set of wavenumbers k. The continuum fields $b_I(t, x, y)$ then become the components of the continuum matrix field $U(t, x, y)$. Back in real space, $S_\perp$ becomes the y derivative term $(\partial_y U^\dagger)(\partial_y U)$ in the continuum limit.

Finally, the theta term comes from a careful evaluation of the alternating sum of Wess-Zumino terms. We have

$$\sum_j (-1)^j S_{WZ}[U_j] \approx \frac{1}{2} \int dy \, \partial_y S_{WZ}[U] , \qquad (7.19)$$

where U in $S_{WZ}[U]$ is the continuum field $U(t,x,y)$. It remains to evaluate $\partial_y S_{WZ}[U]$. One method for evaluating this quantity is to simply use the definition of the derivative,

$$\partial_y S_{WZ}[U] = \lim_{\epsilon \to 0} \frac{S_{WZ}[U(t,x,y+\epsilon)] - S_{WZ}[U(t,x,y)]}{\epsilon}.$$
(7.20)

We then expand $U(t,x,y+\epsilon) \approx U(t,x,y) + \epsilon \partial_y U(t,x,y)$ and use the formula for the variation of the Wess-Zumino term with δU set equal to $\epsilon \partial_y U$. The variation of the WZ term is

$$\delta S_{WZ}[U] = \frac{1}{4\pi} \int d^2 x \, \epsilon^{\bar\mu\bar\nu} \text{tr}[(U^\dagger \partial_{\bar\mu} U)(U^\dagger \partial_{\bar\nu} U)(U^\dagger \delta U)],$$
(7.21)

where $\bar\mu, \bar\nu = t, x$ only. Setting $\delta U = \epsilon \partial_y U$, we obtain for the y derivative,

$$\partial_y S_{WZ}[U] = \frac{1}{4\pi} \int d^2 x \, \epsilon^{\bar\mu\bar\nu} \text{tr}[(U^\dagger \partial_{\bar\mu} U)(U^\dagger \partial_{\bar\nu} U)(U^\dagger \partial_y U)].$$
(7.22)

A small amount of algebra then gives the final result

$$\sum_j (-1)^j S_{WZ}[U_j] = \pi S_\theta[U],$$
(7.23)

where $S_\theta[U]$ is the theta term for the $O(4)$ NLSM from Eq. (4.4). Note that the theta angle θ works out to be exactly π.

D. Coupled wires construction of the 2D BSM model

In this section we give a coupled wires construction of the 2D BSM model. Specifically, the construction presented here

yields our 2D BSM model with only the y-component of the field B_μ non-zero. As we have discussed, our coupled wires construction uses two $SU(2)_1$ WZW theories in each unit cell j in the stacking direction. We label the fields for the two copies of the WZW model in each unit cell as $U_{\pm,j}$. Below, we will see how the "A" and "B" fields for the 2D BSM model emerge from these initial $\pm$ fields (they are not the same). In order to accommodate the inversion transformation in the stacking direction, we take the wires to be numbered as $j = -\frac{N}{2}, \ldots, \frac{N}{2} - 1$ (so there are still N unit cells). We take N even and assume periodic boundary conditions in the stacking direction so that $j = \frac{N}{2}$ is identified with $j = -\frac{N}{2}$. The unperturbed action for the decoupled collection of wires is

$$S_0 = \sum_j \left\{ \sum_{s=\pm} \frac{1}{8\pi} \int d^2 x \, \text{tr}[(\partial^\mu U_{s,j}^\dagger)(\partial_\mu U_{s,j})] \right.$$
$$\left. + S_{WZ}[U_{+,j}] - S_{WZ}[U_{-,j}] \right\},$$
(7.24)

which consists of two $SU(2)_1$ WZW theories in each unit cell j, but with the $\pm$ copies having opposite signs on their respective WZ terms. We add two kinds of inter-wire coupling terms, which take the form

$$S_{\perp,1} = \frac{t_1}{2} \sum_j \int d^2 x \sum_I \left\{ b_{I,+,j}^* b_{I,-,j+1} + b_{I,-,j+1}^* b_{I,+,j} + b_{I,-,j}^* b_{I,+,j+1} + b_{I,+,j+1}^* b_{I,-,j} \right\}$$
(7.25)

and

$$S_{\perp,2} = -i\frac{t_2}{2} \sum_j (-1)^j \int d^2 x \sum_I \left\{ b_{I,+,j}^* b_{I,-,j+1} - b_{I,-,j+1}^* b_{I,+,j} - (b_{I,-,j}^* b_{I,+,j+1} - b_{I,+,j+1}^* b_{I,-,j}) \right\}.$$
(7.26)

The hopping term $S_{\perp,1}$ is proportional to $\text{tr}[(U_{+,j}^\dagger U_{-,j+1} + \text{h.c.}) + (U_{-,j}^\dagger U_{+,j+1} + \text{h.c.})]$, while the term $S_{\perp,2}$ is proportional to $\text{tr}\left[\left((iU_{+,j}^\dagger U_{-,j+1} + \text{h.c.}) - (iU_{-,j}^\dagger U_{+,j+1} + \text{h.c.}) \right) \sigma^z \right]$. When $t_1 > 0$ the term $S_{\perp,1}$ will tend to align $U_{+,j}$ with $U_{-,j+1}$ and $U_{-,j}$ with $U_{+,j+1}$. We therefore define the new fields $b_{I,A,j}$ and $b_{I,B,j}$ by

$$b_{I,A,j} = \begin{cases} b_{I,+,j}, & j = \text{even} \\ b_{I,-,j}, & j = \text{odd} \end{cases}$$
(7.27)

and

$$b_{I,B,j} = \begin{cases} b_{I,-,j}, & j = \text{even} \\ b_{I,+,j}, & j = \text{odd} \end{cases}.$$
(7.28)

It is these fields which have a nice continuum limit for the chosen hopping terms. In terms of these fields the hopping terms take the simpler form

$$S_{\perp,1} = \frac{t_1}{2} \sum_j \int d^2x \sum_I \left\{ b^*_{I,A,j} b_{I,A,j+1} + b^*_{I,A,j+1} b_{I,A,j} + b^*_{I,B,j} b_{I,B,j+1} + b^*_{I,B,j+1} b_{I,B,j} \right\} , \tag{7.29}$$

and

$$S_{\perp,2} = -i\frac{t_2}{2} \sum_j \int d^2x \sum_I \left\{ b^*_{I,A,j} b_{I,A,j+1} - b^*_{I,A,j+1} b_{I,A,j} - (b^*_{I,B,j} b_{I,B,j+1} - b^*_{I,B,j+1} b_{I,B,j}) \right\} . \tag{7.30}$$

Now we Fourier transform the "A" and "B" fields as in Eq. (7.14), and also make a specific choice of hopping parameters, $t_1 = t\cos(B_y a_0)$ and $t_2 = t\sin(B_y a_0)$. In terms of the Fourier-transformed fields the inter-wire coupling now takes the form (with $S_\perp = S_{\perp,1} + S_{\perp,2}$)

$$S_\perp = t \sum_k \left\{ \cos[(k - B_y)a_0] \sum_I b^*_{I,A,k} b_{I,A,k} \right.$$
$$\left. + \cos[(k + B_y)a_0] \sum_I b^*_{I,B,k} b_{I,B,k} \right\} . \tag{7.31}$$

It is clear that the additional imaginary hopping terms with amplitude t_2 have cause the minima of the cosine potentials to shift from $k = 0$ to $k = \pm B_y$.

Finally, we take the continuum limit in the stacking direction. For the "A" fields we expand the cosine around $k = B_y$, and for the "B" fields around $k = -B_y$, which is where the potential energy (which is proportional to $-\cos[(k \pm B_y)a_0]$) has its minimum. The lattice fields now take the approximate form

$$b_{I,A,j}(t,x) \approx e^{iB_y y} b_{I,A}(t,x,y) \tag{7.32a}$$
$$b_{I,B,j}(t,x) \approx e^{-iB_y y} b_{I,B}(t,x,y) , \tag{7.32b}$$

where the slowly varying continuum fields are now given by

$$b_{I,A}(t,x,y) = \frac{1}{\sqrt{N}} \int \frac{dk}{\left(\frac{2\pi}{Na_0}\right)} b_{I,A,k+B_y}(t,x) e^{iky} \tag{7.33}$$

$$b_{I,B}(t,x,y) = \frac{1}{\sqrt{N}} \int \frac{dk}{\left(\frac{2\pi}{Na_0}\right)} b_{I,B,k-B_y}(t,x) e^{iky} \tag{7.34}$$

where the integration over wavenumbers k is now centered at the modes with wavenumber $\pm B_y$ instead of at $k = 0$. The term $S_\perp$ will give the terms $|(\partial_y - iB_y)b_{I,A}|^2$ and $|(\partial_y + iB_y)b_{I,B}|^2$ in the continuum limit, so this construction gives the correct minimal coupling of the bosonic fields to the "gauge field" B_y.

Now we look at how the alternating sums of WZ terms transform into the theta terms for the "A" and "B" copies of the $O(4)$ NLSM. We first define the matrix lattice fields $U_{A,j}$ and $U_{B,j}$, whose matrix elements are the lattice fields $b_{I,A,j}$ and $b_{I,B,j}$. In the continuum limit these are expressed in terms of the continuum matrix fields $U_A(t,x,y)$ and $U_B(t,x,y)$ (whose matrix elements are the continuum fields $b_{I,A}(t,x,y)$

and $b_{I,B}(t,x,y)$), as

$$U_{A,j}(t,x) \approx U_A(t,x,y) e^{i(B_y y)\sigma^z} \tag{7.35}$$
$$U_{B,j}(t,x) \approx U_B(t,x,y) e^{-i(B_y y)\sigma^z} . \tag{7.36}$$

The matrix phase factors $e^{\pm i(B_y y)\sigma^z}$ attach the appropriate phase to the lattice bosons, as shown in Eq. (7.32). Because of the form of the WZ term, the matrix phase factors $e^{\pm i(B_y y)\sigma^z}$ completely cancel each other, and the evaluation of the theta terms from alternating sums of WZ terms proceeds exactly as in the case of one copy of the $O(4)$ NLSM. In addition, we have

$$\sum_j (S_{WZ}[U_{+,j}] - S_{WZ}[U_{-,j}]) =$$
$$\sum_j (-1)^j S_{WZ}[U_{A,j}] - \sum_j (-1)^j S_{WZ}[U_{B,j}] , \tag{7.37}$$

so the theta angles for the "A" and "B" copies of the $O(4)$ NLSM will have opposite sign.

E. Symmetry transformations

We now define transformations for the lattice bosonic fields $b_{I,\pm,j}$ under inversion $Z_2^\mathcal{I}$ and time-reversal $Z_2^\mathcal{T}$ in such a way that in the continuum limit we get the transformations shown in Eq. (5.4) and Eq. (5.5) for the fields $b_{I,A}$ and $b_{I,B}$ of the 2D BSM model. For time-reversal, we take

$$\mathcal{T} b_{I,\pm,j} \mathcal{T}^{-1} = b_{I,\mp,j} . \tag{7.38}$$

It is easy to see that the term $S_{\perp,1}$ has this symmetry. The term $S_{\perp,2}$ picks up a minus sign under the swap $+ \to -$, but the factor of i in that term is also negated since $\mathcal{T}$ is anti-unitary. These two signs cancel each other, and so the term $S_{\perp,2}$ is also symmetric under this time-reversal symmetry. We also see that $\mathcal{T} b_{I,A,j} \mathcal{T}^{-1} = b_{I,B,j}$, which then translates over to the correct continuum transformation $\mathcal{T} b_{I,A} \mathcal{T}^{-1} = b_{I,B}$, as can be seen from Eq. (7.32).

Next we consider the action of inversion symmetry. We take $\mathcal{I}$ to act on the lattice fields as

$$\mathcal{I} b_{I,\pm,j}(x) \mathcal{I}^{-1} = b_{I,\mp,-j}(-x) , \tag{7.39}$$

which is just an inversion about the origin $x = 0, j = 0$. Again, it is easy to see that $S_{\perp,1}$ has this inversion symmetry. Although it is not obvious, one can explicitly check

that $S_{\perp,2}$ also has this symmetry. For example the terms $b^*_{I,+,1}b_{I,-,2}$ and $b^*_{I,-,-1}b_{I,+,-2}$, which are partners under inversion, appear in $S_{\perp,2}$ with the same sign. We also see that $\mathcal{I}b_{I,A,j}(x)\mathcal{I}^{-1} = b_{I,B,-j}(-x)$ since $j \equiv -j$ mod 2. In the continuum limit this inversion symmetry then translates into $\mathcal{I}b_{I,A}(\mathbf{x})\mathcal{I}^{-1} = b_{I,B}(-\mathbf{x})$, as can be seen from Eq. (7.32), and this is exactly the inversion transformation for the continuum fields in the 2D BSM model.

Finally, we discuss the emergence of the $U(1)_t$ translation symmetry for the continuum fields. We saw that after expanding the cosines near $k = \pm B_y$ and taking the continuum limit in the y direction, the term $S_\perp$ gave the kinetic terms $|(\partial_y - iB_y)b_{I,A}|^2$ and $|(\partial_y + iB_y)b_{I,B}|^2$ for the continuum fields $b_{I,A}$ and $b_{I,B}$. We can see from the form of these terms that the continuum action is invariant under the transformation $b_{I,A} \to e^{i\xi}b_{I,A}$, $b_{I,B} \to e^{-i\xi}b_{I,B}$ and $B_y \to B_y + \partial_y\xi$. This transformation is exactly the $U(1)_t$ gauge transformation shown in Eq. (5.3) and discussed in the paragraphs following that equation (in the special case where only the y-component of B_μ is non-zero).

F. Discussion

In this section we have shown how to construct our 2D BSM model from a quasi-1D coupled wires model. Let us now contrast the coupled wires model for the BSM phase with the coupled wires model for the DSM phase (derived in Ref. 7).

In the DSM case, we considered fermions on the square lattice at half-filling. The Bloch Hamiltonian for the model in question featured two bands with energies $E_\pm(\mathbf{k})$ shown in Eq. (3.5). At half-filling, the low-energy excitations of that model were at the locations in the BZ where the two bands touched, i.e., at the locations where $E_+(\mathbf{k}) = E_-(\mathbf{k}) = 0$. For this reason we expanded the Bloch Hamiltonian where $E_\pm(\mathbf{k}) = 0$ to obtain the low energy description of the system. If the band was just a cosine, e.g., $\cos(k_y)$, then we would expand at $k_y = \pm\frac{\pi}{2}$ (so *two* locations), which are the locations of the two Dirac points. From this discussion it is clear why the form $m \pm t_y\cos(k_y)$ for the dispersion was appropriate for the construction of the DSM model: the addition of the intra-wire mass m shifts the cosine *vertically*, which changes the positions of the zeros of energy, and hence shifts the locations of the Dirac nodes in the BZ.

Now we compare to the BSM case. For bosons there is no notion of filling a band or of expanding a dispersion near band touchings. Instead, the appropriate method for finding the low energy description of the system was to expand the potential about its minimum. For a potential which is just a cosine, e.g., $-\cos(k_y)$, we expand around $k_y = 0$ (so a *single* location). From this discussion it is apparent that in order to move the low-energy physics of the bosonic system away from $k_y = 0$, we need to shift the minimum of the cosine potential, i.e., we need a *horizontal* shift of the cosine, as in $-\cos(k_y - B_y)$. In our coupled wires construction of the BSM model this horizontal shift was accomplished using an imaginary inter-wire hopping term, not an intra-wire mass term as in the DSM

case.

It seems that the essential difference between the coupled wires constructions of the DSM and BSM models comes from the simple fact that fermions fill a band structure, while bosons do not. Therefore a different mechanism is needed in the two cases to shift the low-energy physics to the points $(0, \pm B_y)$ in momentum space.

Finally, we note that our coupled wires model for the BSM can be driven into time-reversal or inversion breaking phases by adding dimerization to the inter-wire tunneling terms. As we discussed in Sec. V (see the discussion in the paragraph above Eq. (5.26)), the time-reversal and inversion breaking perturbations to the BSM model correspond to correlated shifts of the theta angles θ_A and θ_B away from their original values $\theta_A = -\theta_B = \pi$. In Ref. 43, Tanaka and Hu have shown that incorporating dimerization into the inter-wire interactions in the coupled wires construction (of Ref. 42) of the $O(4)$ NLSM at $\theta = \pi$ leads to an $O(4)$ NLSM with θ shifted away from π. It is therefore possible to investigate the time-reversal and inversion breaking phases of the BSM model within its quasi-1D description in terms of coupled wires, just by adding suitable dimerization to the inter-wire tunneling terms. However, we do not carry out this analysis here as we have already investigated these phases within the continuum description in Sec. V and we do not expect the results to be modified in an essential way.

VIII. CONCLUSION

We have constructed an effective theory and a coupled-wire model for a bosonic analog of a topological DSM, in which the Dirac cones of the DSM are replaced with copies of the $O(4)$ NLSM with topological theta term and theta angle $\theta = \pm\pi$. We computed the time-reversal and inversion symmetry breaking electromagnetic responses of this BSM model, and showed that they are twice the value of the responses obtained in the fermionic DSM case. We also examined the stability of our BSM model to many kinds of perturbations, and found that the same composite $Z_2^{\mathcal{T}\mathcal{I}}$ symmetry which protects the local stability of the DSM also plays an important role in the local stability of the BSM. Finally, we provided a quasi-1D construction of the BSM model using an array of coupled 1D wires in which each individual wire is made up of two copies of the $SU(2)_1$ WZW conformal field theory.

Along the way we have been able to clarify many aspects of the $O(4)$ NLSM with $\theta = \pi$ which have been discussed in the literature. In particular we provided a detailed analysis of the stability of the BTI surface theory to symmetry-allowed perturbations, which were only briefly discussed in Ref. 17. We were also able to prove the results on the charges and statistics of vortices in the $O(4)$ NLSM with theta term which were argued for in Refs. 17 and 42. We also conjectured a relationship between the descriptions of the BTI surface discussed in this paper, in particular the dual vortex description of Ref. 17 and the description in terms of Abanov-Wiegmann fermions, and the recently proposed dual description in terms of $N = 2$ QED$_3$[40]. As we discussed in Sec. IV, one interesting direc-

tion for future work would be to give a direct derivation of the $N = 2$ QED$_3$ description of the BTI surface, starting from the description in terms of the $O(4)$ NLSM with $\theta = \pi$.

Another interesting direction for future work would be to explore bosonic analogues of Weyl semi-metals in three spatial dimensions. In particular, it would be interesting to understand the requirements for the local stability of a bosonic analogue of a Weyl semi-metal, since in the fermion case the local stability of the Weyl nodes does not depend on any discrete symmetry[73]. This is quite different from the DSM case in 2D, in which the composite symmetry $Z_2^{\mathcal{TI}}$ was necessary to ensure the local stability of the Dirac nodes. One possibility for a bosonic analogue of a Weyl semi-metal would be to try replacing each Weyl node with a copy of the $O(5)$ NLSM with theta term and theta angle $\theta = \pm\pi$.

Finally, there is still more to be learned about the $O(4)$ NLSM at $\theta = \pi$. The disordered (symmetry-preserving) phase of this model was first argued to be gapless in Ref. 42. Qualitative arguments about the RG flows of this model also indicate the existence of a fixed point (representing the putative gapless phase) at $\theta = \pi$ at a large but finite value of the coupling constant g[46]. Very recently, numerical simulations on a (fermionic) honeycomb lattice model whose low energy sector is described by the $O(4)$ NLSM with $\theta = \pi$ have shown that this model is indeed gapless[74,75]. It would be very interesting to understand how the vortex braiding processes we described in Appendix C, which at $\theta = \pi$ lead to destructive interference between the different field configurations summed over in the path integral of the $O(4)$ NLSM, lead to this gapless behavior. In addition, it would be interesting to calculate the scaling dimension of the $O(4)$ field $\mathbf{N}$ at the disordered fixed point.

ACKNOWLEDGMENTS

Appendix A: Canonical quantization of the $O(4)$ non-linear sigma model

In this appendix we briefly discuss the canonical quantization of the $O(4)$ NLSM. We use these commutation relations in Sec. IV to understand the effects of symmetry-allowed perturbations on the surface theory of the BTI. Since the $O(4)$ NLSM is a constrained system, it is necessary to use the Dirac bracket formalism to obtain the canonical commutators of this system[76,77]. Let ψ_i, $i = 1, \ldots, M$, be the second class constraints of the system in question. Then the Dirac bracket is

given by

$$\{f(\mathbf{x}), g(\mathbf{y})\}_D = \{f(\mathbf{x}), g(\mathbf{y})\} \tag{A1}$$

$$- \sum_{i,j=1}^{M} \int d^2\mathbf{z}\, d^2\mathbf{z}'\, \{f(\mathbf{x}), \psi_i(\mathbf{z})\} C_{ij}^{-1}(\mathbf{z}, \mathbf{z}')\{\psi_j(\mathbf{z}'), g(\mathbf{y})\}\,,$$

where the $C_{ij}(\mathbf{z}, \mathbf{z}')$, which are the matrix elements of a matrix with discrete indices i, j and continuous spatial indices $\mathbf{z}$ and $\mathbf{z}'$, are given by

$$C_{ij}(\mathbf{z}, \mathbf{z}') = \{\psi_i(\mathbf{z}), \psi_j(\mathbf{z}')\}\,, \tag{A2}$$

and where $\{\,,\,\}$ is the ordinary Poisson bracket.

In the case of the $O(4)$ NLSM, one possible choice of coordinates and momenta is just the fields N^a and their canonically conjugate momenta $\Pi^a = \frac{\partial \mathcal{L}}{\partial (\partial_t N^a)}$. In terms of these variables the Poisson bracket reads

$$\{f(\mathbf{x}), g(\mathbf{y})\} = \tag{A3}$$

$$\sum_{a=1}^{4} \int d^2\mathbf{z} \left(\frac{\delta f(\mathbf{x})}{\delta N^a(\mathbf{z})} \frac{\delta g(\mathbf{y})}{\delta \Pi^a(\mathbf{z})} - \frac{\delta f(\mathbf{x})}{\delta \Pi^a(\mathbf{z})} \frac{\delta g(\mathbf{y})}{\delta N^a(\mathbf{z})} \right)\,,$$

where $\frac{\delta}{\delta N^a(\mathbf{z})}$ is a functional derivative. This system has two second class constraints, which take the form

$$\psi_1 = \sum_a N^a N^a - 1 \tag{A4}$$

$$\psi_2 = \sum_a N^a \Pi^a\,. \tag{A5}$$

Using this data one finds, for example, that the Dirac bracket for N^a and Π^b is

$$\{N^a(\mathbf{x}), \Pi^b(\mathbf{y})\}_D = \left(\delta^{ab} - N^a(\mathbf{x})N^b(\mathbf{y})\right)\delta^{(2)}(\mathbf{x} - \mathbf{y})\,. \tag{A6}$$

The rest of the Dirac brackets for this system are shown explicitly in Ref. 78. The commutator for the quantum field theory is then obtained by replacing $\{N^a(\mathbf{x}), \Pi^b(\mathbf{y})\}_D$ with $-i[N^a(\mathbf{x}), \Pi^b(\mathbf{y})]$ in the previous expression.

In this paper we discuss the $O(4)$ NLSM using the variables b_1 and b_2 defined in Eq. (4.5). In the canonical formalism we now have the coordinates b_I and b_I^* and momenta $\pi_I = \frac{\partial \mathcal{L}}{\partial (\partial_t b_I)}$ and $\pi_I^* = \frac{\partial \mathcal{L}}{\partial (\partial_t b_I^*)}$ for $I = 1, 2$. In these variables the second class constraints are

$$\psi_1 = \sum_I b_I^* b_I - 1 \tag{A7}$$

$$\psi_2 = \sum_I (b_I \pi_I + b_I^* \pi_I^*)\,. \tag{A8}$$

The Dirac bracket for b_I and π_J takes the form

$$\{b_I(\mathbf{x}), \pi_J(\mathbf{y})\}_D = \left(\delta_{IJ} - \frac{1}{2}b_I(\mathbf{x})b_J^*(\mathbf{y})\right)\delta^{(2)}(\mathbf{x} - \mathbf{y})\,. \tag{A9}$$

For the quantum theory this yields the commutation relation

$$[b_I(\mathbf{x}), \pi_J(\mathbf{y})] = i\left(\delta_{IJ} - \frac{1}{2}b_I(\mathbf{x})b_J^\dagger(\mathbf{y})\right)\delta^{(2)}(\mathbf{x} - \mathbf{y})\,, \tag{A10}$$

where the function $b_I^*(\mathbf{x})$ has been replaced with the operator $b_I^\dagger(\mathbf{x})$ on the Hilbert space. One can also show that the operators $b_I(\mathbf{x})$ and $b_J^\dagger(\mathbf{x})$ all commute with each other. These are the only commutation relations we require for this paper, but the others can also be derived using the Dirac bracket formalism.

Appendix B: Vortices in the $O(4)$ NLSM and their quantum numbers

In this appendix we study vortex solutions of the equations of motion for the $O(4)$ NLSM, and we also perform a collective coordinate quantization of the global excitations on the background of a single vortex. This allows us to show very directly that vortices in the phase of b_1 carry charge $\frac{\theta}{2\pi}$ of b_2 and vice-versa, as was argued in Ref. 17. A more precise statement is that in the presence of a vortex in b_1, the charge spectrum of b_2 is shifted by $\frac{\theta}{2\pi}$. Our analysis (in particular, the collective coordinate quantization) closely parallels the analysis in Ref. 78 of solitons in the $O(3)$ NLSM with Hopf term. In Ref. 78, the authors showed that a soliton of topological charge Q carries angular momentum $\frac{\theta}{2\pi}Q^2$, where θ is the coefficient of the Hopf term (the result for $Q = 1$ was originally worked out in Ref. 50).

1. Finite energy vortex solutions

We start by discussing a class of finite energy vortex solutions to the NLSM equations of motion. To the best of our knowledge, these solutions have not appeared in the literature. They are, however, closely related to solitons in the $O(3)$ NLSM, due to the fact that they involve only three components of the $O(4)$ field. Exact soliton solutions for the $O(3)$ NLSM were obtained long ago by Belavin and Polyakov[79]. Our vortex solutions, however, involve different boundary conditions than those considered in the soliton case. Indeed, in the study of solitons in an $O(3)$ NLSM, with field $\mathbf{m}$, one imposes the boundary condition that $\mathbf{m}$ tends to a fixed configuration $\mathbf{m}_0$ at spatial infinity. This boundary condition has the effect of compactifying 2D space to the sphere S^2. For the vortex configurations considered here, we will instead regard 2D space as a large disk of radius R, and only take R to infinity at the end of the calculation.

If we vary the $O(4)$ NLSM action in Eq. (4.1) with respect to U and use $\delta U^\dagger = -U^\dagger \delta U U^\dagger$ (since U is an $SU(2)$ matrix) we find the equation of motion

$$\Box U - U(\Box U^\dagger)U = 0 , \qquad (B1)$$

where $\Box = \partial_t^2 - \nabla^2$. The theta term does not contribute to the equation of motion since its variation is a total derivative. We work in polar coordinates (r, ϕ) for the plane, but with an upper cutoff R for the radial direction, i.e., $r \in [0, R]$, and take $R \to \infty$ at the end of the calculation. Let $z = (b_1, b_2)^T$, where b_1 and b_2 are the elements of U as shown in Eq. (4.5).

We make the time-independent vortex ansatz,

$$z = \begin{pmatrix} \cos(f(r))e^{i\alpha\phi} \\ \sin(f(r)) \end{pmatrix} \qquad (B2)$$

where $\alpha \in \mathbb{Z}$ (so that the solution is single-valued) and we take the boundary conditions $f(0) = \frac{\pi}{2}$ and $f(R) = 0$, so that the amplitude of b_1 vanishes in the vortex core. One can actually take α to be any real number in what follows. Solutions with general values of α might be relevant for the study of braiding statistics of excitations in gauged NLSM's as considered in Ref. 49. Plugging this ansatz into the equations of motion yields a differential equation for $f(r)$

$$f''(r) + \frac{1}{r}f'(r) + \alpha^2 \frac{\sin(f(r))\cos(f(r))}{r^2} = 0 , \qquad (B3)$$

whose exact solution for the given boundary conditions is

$$f(r) = \mathrm{am}[\log\left(\frac{R}{r}\right)^{|\alpha|}, 1] = -\frac{\pi}{2} + 2\tan^{-1}\left[\left(\frac{R}{r}\right)^{|\alpha|}\right] . \quad (B4)$$

In this expression, $\mathrm{am}[u, k]$ is the Jacobi Amplitude function. When $k = 1$, this function reduces to a much more manageable form.

Next we show that this solution has finite energy. We will see that the energy of the solution is actually independent of the long-distance cutoff R. The topological term does not contribute to the energy, so we just have ($i = x, y$ and we sum over i)

$$E_\alpha = \int d^2\mathbf{x} \frac{1}{g}(\partial_i z^\dagger)(\partial_i z)$$
$$= \int d^2\mathbf{x} \frac{1}{g}\left\{(\partial_i f(r))^2 + \frac{\cos^2(f(r))}{r^2}\right\} . \quad (B5)$$

Next we go to polar coordinates and use the fact that $(\partial_i f(r))^2 = \frac{4\alpha^2}{r^2}\frac{R^{2|\alpha|}r^{2|\alpha|}}{(r^{2|\alpha|}+R^{2|\alpha|})^2}$ and $\frac{\cos^2(f(r))}{r^2} = \frac{4}{r^2}\frac{R^{2|\alpha|}r^{2|\alpha|}}{(r^{2|\alpha|}+R^{2|\alpha|})^2}$ for the solution for $f(r)$ in Eq. (B4) to find

$$E_\alpha = \frac{1}{g}\int_0^{2\pi} d\phi \int_0^R rdr \frac{4(\alpha^2+1)}{r^2}\frac{R^{2|\alpha|}r^{2|\alpha|}}{(r^{2|\alpha|}+R^{2|\alpha|})^2}$$
$$= \frac{2\pi}{g}\left(|\alpha| + \frac{1}{|\alpha|}\right) . \qquad (B6)$$

So we find that the vortex solution has finite energy, and that the energy is independent of the upper cutoff R. The energy increases essentially linearly with the "vortex strength" α. For the case of $\alpha = 1$, we just get $E_1 = \frac{4\pi}{g}$.

It is interesting to note that this theory admits finite energy vortex solutions without requiring coupling to a dynamical gauge field, as is necessary in the case of an ordinary complex scalar field in 2D (see, for example, the discussion of the Abelian Higgs model in Ref. 80). These vortex solutions are, however, somewhat pathological, in the sense that the size of the vortex core grows without bound as the upper cutoff R is pushed to infinity. Vortex-anti-vortex pairs, however, do not have this problem. This is because the energy density of such a pair falls of faster than $\frac{1}{r^2}$ at long distances, so these objects are well-defined when the system size is infinite.

2. $O(2)$ NLSM for phase excitations of b_2 on a vortex background

We now study the global excitations of the phase of the boson b_2 on the background of a vortex in b_1. Note that the classical energy E_α of the vortex ansatz in Eq. (B2) is invariant under the replacement $\sin(f(r)) \to \sin(f(r))e^{i\bar\theta_2}$ where $\bar\theta_2$ is any *constant* phase. To study the global excitations about the vortex solution, we promote $\bar\theta_2$ to a time-dependent phase $\theta_2(t)$,

$$z = \begin{pmatrix} \cos(f(r))e^{i\phi} \\ \sin(f(r))e^{i\theta_2(t)} \end{pmatrix} \tag{B7}$$

where $f(r)$ is the vortex solution from Eq. (B4) with $\alpha = 1$. We then evaluate the action on this configuration and quantize the motion of $\theta_2(t)$. This type of analysis is referred to as *collective coordinate* quantization (see Refs. 78 and 81) and is useful for understanding how quantum fluctuations can lift the classical degeneracy of global fluctuations about the vortex solution.

On this field configuration the theta term in the action reduces as

$$S_\theta = \frac{1}{24\pi^2} \int d^3x \, \epsilon^{\mu\nu\lambda} \mathrm{tr}[(U^\dagger \partial_\mu U)(U^\dagger \partial_\nu U)(U^\dagger \partial_\lambda U)]$$
$$\to \frac{1}{2\pi} \int dt \, \partial_t \theta_2(t) \,, \tag{B8}$$

which is precisely the theta term for an $O(2)$ NLSM in $0+1$-d[82]. The kinetic term in the action reduces to

$$S_{kin} = \int d^3x \, \frac{1}{g}(\partial^\mu z^\dagger)(\partial_\mu z)$$
$$\to \int dt \left\{ \frac{2\pi R^2 J}{g}(\partial_t\theta_2)^2 - E_1 \right\} , \tag{B9}$$

where $E_1 = \frac{4\pi}{g}$ is the energy of the vortex solution and J is the convergent integral

$$J = \int_0^\infty dw \, e^{-2w}\mathrm{sn}^2[w,1] = \frac{3}{2} - \ln(4) \,. \tag{B10}$$

In this expression $\mathrm{sn}[w,1] = \sin(\mathrm{am}[w,1])$ is one of the Jacobi Elliptic functions. An important point here is that it does not make physical sense to evaluate the action on a vortex solution with infinite energy, therefore it is crucial for our analysis that the vortex solutions do have finite energy.

The full action for the phase excitation $\theta_2(t)$ is (neglecting the constant E_1)

$$S_{\theta_2} = \int dt \left\{ \frac{2\pi R^2 J}{g}(\partial_t\theta_2)^2 - \frac{\theta}{2\pi}\partial_t\theta_2 \right\} . \tag{B11}$$

This is exactly the action for an $O(2)$ NLSM with theta term in $0+1$ dimensions. We can now canonically quantize the action for θ_2. We define the canonical momentum

$$p_2 = \frac{\partial \mathcal{L}_{core}}{\partial(\partial_t\theta_2)} = \frac{4\pi R^2 J}{g}(\partial_t\theta_2) - \frac{\theta}{2\pi} \,, \tag{B12}$$

from which we derive the Hamiltonian

$$H_{core} = \frac{1}{2m}\left(p_2 + \frac{\theta}{2\pi}\right)^2 , \tag{B13}$$

where $m = \frac{4\pi R^2 J}{g}$ is the "mass" of the degree of freedom inside the vortex. In canonical quantization we set $p_2 = -i\frac{\partial}{\partial\theta_2}$, and so we find that the eigenfunctions of the vortex Hamiltonian are

$$\psi_n(\theta_2) = \frac{1}{\sqrt{2\pi}}e^{in\theta_2} \,, n \in \mathbb{Z} \tag{B14}$$

with energies

$$E_n = \frac{1}{2m}\left(n + \frac{\theta}{2\pi}\right)^2 . \tag{B15}$$

We see that there is generally a unique ground state except for when $\theta = \pi$, in which case the $n = 0$ and $n = -1$ states are degenerate. The energies of these states do, however, all collapse to zero in the thermodynamic limit $R \to \infty$.

3. Spectrum of charges

Finally, we can look at the charge spectrum of $\theta_2(t)$ fluctuations on the background of a vortex in b_1. We start by considering the conserved charge for boson species 2,

$$Q_2 = \int d^2\mathbf{x} \, \frac{i}{g}(\partial_t b_2^* b_2 - b_2^* \partial_t b_2) \,, \tag{B16}$$

where the integration is taken over all of space. This is the conserved charge for the Noether current of the $O(4)$ NLSM associated with the invariance of the action under the symmetry $b_2 \to e^{i\chi}b_2$. After canonical quantization, Q_2 will become the number operator for the b_2 bosons. Evaluating this expression on our vortex solution gives

$$Q_2 = \frac{4\pi R^2 J}{g}(\partial_t\theta_2) \,, \tag{B17}$$

and replacing $\partial_t\theta_2$ with the canonical momentum p_2 gives

$$Q_2 = p_2 + \frac{\theta}{2\pi} \,. \tag{B18}$$

This shows that the charge spectrum of b_2 is shifted by $\frac{\theta}{2\pi}$ in the presence of a vortex in b_1, which means that a half-charge of b_2 may be associated to vortices in b_1 at $\theta = \pi$. An analogous result holds for vortices in b_2. It follows that a vortex in b_1 will carry half of *any* $U(1)$ charge carried by b_2, for example the $U(1)_c$ and $U(1)_t$ charges considered in this paper. Thus, we have been able to prove the result of Ref. 17, which is that vortices on the surface of the BTI carry charge $\pm\frac{1}{2}$, directly from the description of the surface in terms of the $O(4)$ NLSM with $\theta = \pi$.

Appendix C: Theta term and the Minkowski space path integral for the $O(4)$ NLSM

In this Appendix we discuss the role of the theta term in the Minkowksi spacetime (i.e., *real* time) path integral of the $O(4)$ NLSM. Recall from Sec. IV that in Euclidean spacetime (compactified to the sphere S^3 via appropriate boundary conditions), the theta term was quantized due to the non-trivial homotopy group $\pi_3(S^3) = \mathbb{Z}$. In that case the theta term contributed a phase $e^{i\theta n_I}$ to the Euclidean path integral, where $n_I \in \mathbb{Z}$ was the instanton number of the field configuration (see Eq. (4.9)). It then followed that the time-reversal symmetric values of θ are $\theta = n\pi$, $n \in \mathbb{Z}$, at which the phase $e^{i\theta n_I}$ is real. In Minkowski spacetime these arguments no longer hold, and it is illuminating to develop a separate understanding of the role of the theta term in the real time path integral.

In this Appendix we show that in the real time path integral the theta term gives a weight $e^{i\theta}$ to spacetime configurations of the $O(4)$ field in which a vortex in the field b_2 makes a complete circuit around around a vortex in b_1. This result was anticipated by the Euclidean spacetime arguments of Senthil and Fisher (Ref. 42), but in this Appendix we derive this result using only the properties of the theta term in Minkowski spacetime. In addition, following an argument used by Wilczek and Zee in Ref. 50 in their analysis of solitons in the $O(3)$ NLSM with Hopf term, our result implies that a bound state of a vortex in b_1 and a vortex in b_2 carries intrinsic angular momentum $J = \frac{\theta}{2\pi}$. When $\theta = \pi$, we have $J = \frac{1}{2}$, which means that the bound state is a fermion. This result was also argued for in Ref. 17.

To start, we express the components b_1 and b_2 of the NLSM field U in Hopf coordinates as in Sec. IV. In these coordinates the bosonic fields are expressed as $b_1 = \sin(\eta)e^{i\vartheta_1}$ and $b_2 = \cos(\eta)e^{i\vartheta_2}$ with $\eta \in [0, \frac{\pi}{2}]$ and $\vartheta_1, \vartheta_2 \in [0, 2\pi)$. The theta term can be written in the form (compare to Eq. (4.44))

$$S_\theta[U] = \frac{1}{4\pi^2} \int d^3x \, \epsilon^{\mu\nu\lambda} \partial_\mu \left(\sin^2(\eta)\right) \partial_\nu \vartheta_1 \partial_\lambda \vartheta_2 \, . \quad \text{(C1)}$$

Now we integrate by parts, for the moment ignoring boundary terms. Later we will comment on the boundary conditions necessary to justify ignoring these boundary terms. We get

$$S_\theta[U] = \frac{1}{2\pi} \int d^3x \, \sin^2(\eta) \left(\partial_\mu \vartheta_1 K_2^\mu - \partial_\mu \vartheta_2 K_1^\mu\right) \, , \quad \text{(C2)}$$

where we have introduced the vortex currents $K_I^\mu = \frac{1}{2\pi}\epsilon^{\mu\nu\lambda}\partial_\nu\partial_\lambda\vartheta_I$ for vortices in the phase of the field b_I. If ϑ_I has vortices of vorticity $q_{I,j}$ ($q_{I,j} \in \mathbb{Z}$) at locations $\mathbf{r}_{I,j}(t)$ (i.e., $\mathbf{r}_{I,j}(t)$ is the location of the vortex core), then the components of the vortex current K_I^μ take the form

$$K_I^t = \sum_j q_{I,j}\delta^{(2)}(\mathbf{x} - \mathbf{r}_{I,j}(t)) \quad \text{(C3)}$$

$$\mathbf{K}_I = \sum_j q_{I,j}\mathbf{v}_{I,j}(t)\delta^{(2)}(\mathbf{x} - \mathbf{r}_{I,j}(t)) \, , \quad \text{(C4)}$$

where $\mathbf{K}_I = (K_I^x, K_I^y)$ and $\mathbf{v}_{I,j}(t) = \frac{d\mathbf{r}_{I,j}(t)}{dt}$. Since $\sin(\eta) = 0$ at the core of vortices in b_1, and $\sin(\eta) = 1$ at

the core of vortices in b_2, the theta term reduces further to

$$S_\theta[U] = \frac{1}{2\pi} \int d^3x \, \partial_\mu \vartheta_1 K_2^\mu \, . \quad \text{(C5)}$$

We now show that the theta term gives a phase of $e^{i\theta}$ in the real time path integral whenever a vortex (of strength $q = 1$) in the phase of b_2 makes a complete circuit around a vortex (also of strength $q = 1$) in the phase of b_1. We take the vortex in ϑ_1 to be located at $\mathbf{r}_1(t)$, and the vortex in ϑ_2 to be located at $\mathbf{r}_2(t)$, and we restrict the time integration in the action to be on the interval $[0, T]$, where T is the time it takes for the vortices to complete their circuit. From the form of the components of the vortex current shown above, we find the result

$$\int d^3x \, \partial_\mu \vartheta_1 K_2^\mu = \int_0^T dt \frac{d}{dt}\vartheta_1(t, \mathbf{r}_2(t)) \, , \quad \text{(C6)}$$

where the integrand is the *total* time derivative of $\vartheta_1(t, \mathbf{r}_2(t))$,

$$\frac{d}{dt}\vartheta_1(t, \mathbf{r}_2(t)) = \partial_t\vartheta_1(t, \mathbf{r}_2(t)) + \mathbf{v}_2(t) \cdot \nabla\vartheta_1(t, \mathbf{x})\Big|_{\mathbf{x}=\mathbf{r}_2(t)} \, . \quad \text{(C7)}$$

Here the function $\vartheta_1(t, \mathbf{r}_2(t))$ is the phase of b_1 evaluated at the core of the vortex in b_2.

Now we integrate the total time derivative of $\vartheta_1(t, \mathbf{r}_2(t))$ from $t = 0$ to $t = T$ to obtain

$$S_\theta[U] = \frac{1}{2\pi}\left(\vartheta_1(T, \mathbf{r}_2(T)) - \vartheta_1(0, \mathbf{r}_2(0))\right) \, . \quad \text{(C8)}$$

Finally, since the core of the vortex in b_2 makes one full circuit around the core of the vortex in b_1 as t varies from 0 to T, we have $\vartheta_1(T, \mathbf{r}_2(T)) - \vartheta_1(0, \mathbf{r}_2(0)) = 2\pi$. We then get $S_\theta[U] = 1$, which means that in the real time path integral we get a phase $e^{i\theta S_\theta[U]} = e^{i\theta}$ for every field configuration in which a vortex in b_2 makes a complete circuit around a vortex in b_1. More generally, if a vortex of strength q_2 in b_2 makes a complete circuit around a vortex of strength q_1 in b_1, we get a phase of $e^{iq_1 q_2 \theta}$.

The result obtained above can also be used to investigate the intrinsic angular momentum and statistics of the bound state of vortices in b_1 and b_2. In Ref. 50, the authors calculated the intrinsic angular momentum J of a soliton in the $O(3)$ NLSM with Hopf term by calculating the action corresponding to an adiabatic rotation of the soliton by 2π. If the time it takes to complete the rotation is T, then the action should evaluate to $S = 2\pi J + O(\frac{1}{T})$. The topological term in the action is responsible for the value of J, and the terms of order $\frac{1}{T}$ are produced by the other terms in the action. From our result above, we immediately see that $2\pi J = \theta$ for the vortex bound state, so $J = \frac{\theta}{2\pi}$. At $\theta = \pi$ we get $J = \frac{1}{2}$, which means that the vortex bound state is a fermion.

Finally, a few words are in order about the conditions necessary to justify ignoring the boundary terms produced when we integrated the theta term by parts. First, the boundary terms in the time direction can be neglected if the field configurations at the initial and final time are chosen to be the same. This is the usual choice in field theory, where the path integral represents a matrix element of the form $\langle\psi|e^{-iHT}|\psi\rangle$, in which

the time evolution operator e^{-iHT} is sandwiched between the same initial and final state $|\psi\rangle$ (usually the vacuum, or ground state).

Now we discuss the spatial boundary terms. One way to ensure that the spatial boundary terms vanish is to require the phases ϑ_1 and ϑ_2 to tend to constants at spatial infinity. This means that these two phases cannot wind at spatial infinity, which means that if vortices are present in ϑ_1 or ϑ_2, there must also be an equal number of anti-vortices present to completely cancel the winding of the phase at spatial infinity. In other words, the sum over configurations of the $O(4)$ field U in the path integral should be restricted to include only those configurations which contain an equal number of vortices and anti-vortices in the phase of each boson b_I. This requirement makes physical sense since, as we saw in Appendix B, isolated vortices have some undesirable properties (their core size grew without bound as the system size was taken to infinity). As we discussed in Appendix B, vortex-anti-vortex pairs do not have this problem.

Appendix D: Abanov-Wiegmann fermions and the relation to the $O(4)$ NLSM with theta term

We mentioned in Sec. IV that the Abanov-Wiegmann method seems to be more closely connected to an $O(4)$ NLSM in the ordered (small g) phase, whereas we are interested in studying the disordered (large g) phase of the model. Nevertheless, our response calculation using Abanov-Wiegmann fermions completely agrees with the response calculation of Ref. 17 using the dual vortex theory (which we reviewed in Sec. IV). In this Appendix we use the Abanov-Wiegmann formula to argue that the topological part of the electromagnetic response of the $O(4)$ NLSM with $\theta = \pi$ must be exactly equal to the topological part of the response of the theory of four massless fermions ψ_a, where ψ_a are the four Abanov-Wiegmann fermions which can be coupled to the $O(4)$ field to produce an $O(4)$ NLSM at $\theta = \pi$.

As discussed above, the Abanov-Wiegmann method cannot produce an $O(4)$ NLSM in the disordered, or large g phase, because the expansion in powers of M^{-1} would not be reliable at such low orders if M was taken to be small. Let us instead consider a completely different scenario, in which we start out with a system containing bosonic *and* fermionic degrees of freedom. The ingredients in this theory are (i) an $O(4)$ NLSM in the disordered phase with a theta angle $\theta = -\pi$, and (ii) the four massless fermions ψ_a introduced in the discussion of the Abanov-Wiegmann method in Sec. IV. The action for these two decoupled theories takes the form $S = S_b + S_f$ with

$$S_b = \int d^3x \left[\frac{1}{g}(\partial^\mu N^a)(\partial_\mu N^a) \right] + \pi S_\theta[\mathbf{N}] , \qquad (D1)$$

and

$$S_f = \int d^3x \, i\bar{\Psi}\slashed{\partial}\Psi , \qquad (D2)$$

where $\Psi = (\psi_1, \psi_2, \psi_3, \psi_4)^T$. We now turn on a strong interaction between these two theories of the form

$$S_{int} = -M \sum_{a=1}^{4} \int d^3x \, \bar{\Psi}N^a\Gamma^a\Psi , \qquad (D3)$$

with $M > 0$ and large (so the coupling is strong).

If we integrate out the fermions in this theory (using the Abanov-Wiegmann formula), then the theta term for the $O(4)$ NLSM will be canceled (recall that the original theta angle was $-\pi$), and we are left with the action

$$S = \int d^3x \, \frac{1}{\tilde{g}}(\partial^\mu N^a)(\partial_\mu N^a) , \qquad (D4)$$

where $\tilde{g}$ is very small, since $\tilde{g}^{-1} = g^{-1} + \frac{M}{\text{const.}}$. The result is an $O(4)$ NLSM with no theta term which is in its ordered phase. We see that strong coupling to the four massless fermions ψ_a has completely destroyed the topological properties of the original $O(4)$ NLSM with $\theta = -\pi$.

Our interpretation of this is as follows. The theory of four massless fermions in Eq. (D2) (in which the fermions carry the charges q_a calculated in Sec. IV) should be regarded, in some sense, as the *inverse* of the $O(4)$ NLSM with $\theta = -\pi$, since strong coupling between the two theories completely destroys the topological properties of the latter theory. In particular, the topological part of the electromagnetic responses of these two theories should have opposite signs. Now the $O(4)$ NLSM with $\theta = \pi$ is also, in this same sense, the inverse of the $O(4)$ NLSM with $\theta = -\pi$. To see this, suppose we had two $O(4)$ NLSM's with theta term, with fields $\mathbf{N}$ and $\mathbf{M}$, with the first copy having $\theta = \pi$ and the second copy having $\theta = -\pi$. Then a strong dot product coupling of the form $\mathbf{N} \cdot \mathbf{M}$ between these two theories will have the effect of setting $\mathbf{N} = \pm\mathbf{M}$, which will in turn cause the theta terms for the two theories to cancel. We therefore conclude that the topological part of the electromagnetic response of the fermion theory in Eq. (D2) should be exactly *equal* to the topological part of the response of the $O(4)$ NLSM with $\theta = \pi$. This explains why we were able to calculate the electromagnetic response of the $O(4)$ NLSM with $\theta = \pi$ by instead coupling the fermion theory in Eq. (D2) to the external field A_μ.

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