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Meng Cheng and Cenke Xu

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# A Series of $(2+1)d$ Stable Self-Dual Interacting Conformal Field Theories

Meng Cheng<sup>1</sup> and Cenke Xu<sup>2</sup>

<sup>1</sup>*Department of Physics, Yale University, New Haven, CT 06520-8120, USA*

<sup>2</sup>*Department of Physics, University of California, Santa Barbara, CA 93106, USA*

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Using the duality between seemingly different  $(2+1)d$  conformal field theories (CFT) proposed recently Ref. 1–10, we study a series of  $(2+1)d$  stable self-dual interacting CFTs. These CFTs can be realized (for instance) on the boundary of the  $3d$  bosonic topological insulator protected by  $U(1)$  and time-reversal symmetry ( $\mathcal{T}$ ), and they remain stable as long as these symmetries are preserved. When realized as a boundary system, these CFTs can be driven into anomalous fractional quantum Hall states once  $\mathcal{T}$  is broken. We demonstrate that the newly proposed dualities allow us to study these CFTs quantitatively through a controlled calculation, without relying on a large flavor number of matter fields. We also propose a numerical test for our results, which would provide strong evidence for the originally proposed duality between Dirac fermion and QED<sup>1–5</sup>.

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## I. INTRODUCTION

Analytical studies on interacting  $(2+1)d$  conformal field theories (CFT) usually rely on a large flavor number of matter fields, unless the theory has supersymmetry. For instance, the  $(2+1)d$  quantum electrodynamics (QED) with a large flavor number of massless Dirac fermions is a stable CFT if all gauge-invariant fermion bilinear operators are forbidden by symmetry. The usual wisdom is that, this CFT can be studied reliably through a  $1/N$  expansion ( $N$  is the number of Dirac fermions), as long as  $N$  is larger than some critical number. When  $N$  is small, not only is the  $1/N$  expansion no longer reliable, this CFT could be unstable against spontaneous mass generation<sup>11</sup>.

Recent studies on the bulk duality between gauged topological insulators<sup>2–5</sup> have led to a conjectured duality between a single noninteracting massless  $(2+1)d$  Dirac fermion and a  $(2+1)d$  QED with one flavor ( $N=1$ ) of massless Dirac fermion<sup>49</sup>:

$$\mathcal{L} = \bar{\chi}\gamma_\mu(\partial_\mu - ia_\mu)\chi$$

$$\leftrightarrow \bar{\psi}\gamma_\mu(\partial_\mu - ia_\mu)\psi + \frac{1}{e^2}f_{\mu\nu}^2 + \frac{i}{4\pi}\epsilon_{\mu\nu\rho}a_\mu\partial_\nu A_\rho. \quad (1)$$

The Lagrangian in the second line above was also proposed earlier as a dual description of the half-filled Landau level with a particle-hole symmetry<sup>1</sup>. The symbol “ $\leftrightarrow$ ” stands for “dual to”. In this equation,  $a_\mu$  is a dynamical noncompact  $U(1)$  gauge field, and  $A_\mu$  is an external background  $U(1)$  gauge field. The “flux current” of  $a_\mu$  is dual to the fermion current of  $\chi$ :  $\bar{\chi}\gamma_\mu\chi = \frac{1}{4\pi}\epsilon_{\mu\nu\rho}\partial_\nu a_\rho$ . In the context of topological insulator, the physical meaning of this duality mapping is that, the  $4\pi$  flux of  $a_\mu$ , which is bound with the strength-4 vortex of the “Fu-Kane” superconductor<sup>12</sup> of  $\psi$ , is the fermion  $\chi$ ; and the fermion  $\psi$  can also be viewed as the strength-4 vortex of the Fu-Kane superconductor of  $\chi$ .

The strongest interpretation of this duality is that, the  $(2+1)d$  QED with  $N=1$  and weak coupling constant

$e$  in the ultraviolet will flow to a strongly interacting CFT in the infrared under renormalization group, and this is the same CFT as a free Dirac fermion under duality transformation. Eq. (1) is the fermionic version of the well-known boson-vortex duality<sup>13,14</sup>, which states that the  $O(2)$  Wilson-Fisher fixed point is dual to the Higgs transition of the bosonic QED with  $N=1$  in  $(2+1)d$ . Recently it was shown that Eq. (1) is one example of a bigger “web” of dualities<sup>9,10</sup>.

This new duality sheds light on our understanding of CFTs with a small flavor number of matter fields. Ref. 6 showed that the  $(2+1)d$  QED with  $N=2$  flavors is self-dual. This is a fermionic version of the self-duality of the easy-plane noncompact  $CP^1$  model (a  $N=2$  bosonic QED)<sup>15–17</sup>. The self-duality of the  $N=2$  QED was also verified with different derivations<sup>8,10</sup>. Unlike the previous case with  $N=1$ , there is no equivalent noninteracting description of the  $(2+1)d$  QED with  $N=2$ . Recent numerical studies on the  $N=2$  QED indicates that this theory could indeed be a scale-invariant CFT in the infrared limit<sup>18</sup> (although earlier study suggests a spontaneous mass generation<sup>19</sup>). However, it is difficult to study the  $N=2$  QED quantitatively using analytical methods because it is unclear whether the standard  $1/N$  expansion actually provides useful information for  $N=2$ .

In this paper we study a series of self-dual QEDs with flavor number  $N=2$ . The Lagrangian of these QED reads

$$\mathcal{L} = \bar{\psi}_1\gamma_\mu(\partial_\mu - ika_\mu - i2n_B B_\mu)\psi_1 + \bar{\psi}_2\gamma_\mu(\partial_\mu - ia_\mu)\psi_2$$

$$+ \frac{in_A}{2\pi}\epsilon_{\mu\nu\rho}a_\mu\partial_\nu A_\rho + \dots \quad (2)$$

We take the convention that  $\gamma_0 = \sigma^y$ ,  $\gamma_1 = \sigma^z$ ,  $\gamma_2 = \sigma^x$ . Again,  $a_\mu$  is a dynamical  $U(1)$  gauge field, while  $A_\mu$  and  $B_\mu$  are two external background  $U(1)$  gauge fields. The fermion  $\psi_1$  carries gauge charge  $k$  of  $a_\mu$ , and charge  $2n_B$  of  $B_\mu$ ; the fermion  $\psi_2$  carries charge 1 of  $a_\mu$ . Most importantly, the  $2\pi$ -flux of  $a_\mu$  carries charge  $n_A$  of  $A_\mu$ , hence it is a noncompact  $U(1)$  gauge field when  $n_A \neq 0$ .

The constants  $k$ ,  $n_A$  and  $n_B$  in Eq. (2) depend on the physical realization of the theory. In section III we will show that the case with an odd integer  $k$  has a natural realization as the boundary of a 3d bosonic SPT state, more precisely it is the boundary of a bosonic topological insulator with U(1) and time-reversal symmetry ( $\mathcal{T}$ ). In this paper we will mostly focus on the case with odd integer  $k$ . The case with even integer  $k$  will also be briefly discussed in section III.

The theories in Eq. (2) parameterized by  $k$  are interacting theories with no free theory dual. However, we will show that the newly proposed dualities mentioned above allow us to study this CFT quantitatively with a  $1/k$  expansion. We will also design a feasible numerical test for our results, which would provide strong evidence for the original conjectured duality in Eq. 1.

## II. SELF-DUAL CONFORMAL FIELD THEORY

### A. The cases with $k > 1$

We now demonstrate that Eq. (2) is self-dual for arbitrary odd integer  $k$ . Following the duality in Eq. (1), Eq. (2) is dual to the following theory:

$$\begin{aligned} \mathcal{L} = & \bar{\chi}_1 \gamma_\mu (\partial_\mu - i b_\mu) \chi_1 + \bar{\chi}_2 \gamma_\mu (\partial_\mu - i c_\mu) \chi_2 \\ & + \frac{i}{4\pi} \epsilon_{\mu\nu\rho} a_\mu \partial_\nu (k b_\rho + c_\rho + 2n_A A_\rho) + \frac{i n_B}{2\pi} \epsilon_{\mu\nu\rho} b_\mu \partial_\nu B_\rho \end{aligned} \quad (3)$$

In the derivation above we have used the duality mapping that the fermion current  $\bar{\psi}_1 \gamma_\mu \psi_1$  is mapped to  $\frac{1}{4\pi} \epsilon_{\mu\nu\rho} \partial_\nu b_\rho$ , and  $\bar{\psi}_2 \gamma_\mu \psi_2 = \frac{1}{4\pi} \epsilon_{\mu\nu\rho} \partial_\nu c_\rho$ . Integrating out  $a_\mu$  will impose the following constraint:

$$c_\mu = -k b_\mu - 2n_A A_\mu. \quad (4)$$

Thus the dual theory of Eq. (2) reads

$$\begin{aligned} \mathcal{L} = & \bar{\chi}_1 \gamma_\mu (\partial_\mu - i b_\mu) \chi_1 + \bar{\chi}_2 \gamma_\mu (\partial_\mu - i k b_\mu - i 2n_A A_\mu) \chi_2 \\ & + \frac{i n_B}{2\pi} \epsilon_{\mu\nu\rho} b_\mu \partial_\nu B_\rho. \end{aligned} \quad (5)$$

In the last equation above we have performed a particle-hole transformation on the dual Dirac fermion  $\chi_2$ . This theory Eq. (5) takes exactly the same form as Eq. (2), except that the gauge charges of the two flavors of fermions are exchanged, and the roles of the two external gauge fields  $A_\mu$  and  $B_\mu$  are also exchanged. (One potential subtlety of the derivation above is the flux quantization of the three dynamical gauge fields  $a_\mu$ ,  $b_\mu$  and  $c_\mu$ , which will be clarified in the next section when we discuss the physical construction of the theory.)

Besides the two U(1) global symmetries, we can also impose discrete symmetries to exclude the Dirac fermion mass terms. For example, we can define the time-reversal transformation for  $\psi_j$  and  $\chi_j$ :

$$\mathcal{T} : \psi_j \rightarrow i\sigma^y \psi_j^\dagger, \quad \chi_j \rightarrow i\sigma^y \chi_j. \quad (6)$$

We can view the duality transformation as a  $Z_2$  transformation, and the CFT under study also has this  $Z_2$  self-dual symmetry. This is analogous to the “mirror symmetry” of the  $(2+1)d$  supersymmetric field theories<sup>20,21</sup>, which also acts on the theories as a duality transformation.

We can rescale  $a_\mu$ , and define  $\tilde{a}_\mu = k a_\mu$ . Then  $\psi_1$  carries charge 1 under  $\tilde{a}_\mu$ , and  $\psi_2$  carries charge  $1/k$ . Thus with large  $k$ ,  $\psi_2$  is effectively decoupled from the gauge field. Another way to understand this statement is that, when  $k \gg 1$ , the dressed propagator of  $a_\mu$  is at order of  $1/k^2$ , thus the effect of  $a_\mu$  on  $\psi_2$  is strongly suppressed with large  $k$ . For the same reason, with large  $k$ , the dual fermion  $\chi_1$  is effectively decoupled from the dual gauge field.

The self-duality of Eq. (2) gives us very helpful quantitative information about the theory, at least for the purpose of a  $1/k$  expansion. For example, to compute physical quantities to the first order of the  $1/k$  expansion, we need to know the gauge field propagator in the large- $k$  limit. In this limit, because  $\psi_2$  decouples from the dynamical gauge field, in order to calculate the fully-dressed  $a_\mu$  propagator one can ignore  $\psi_2$ . Now the theory reduces to a QED with  $N = 1$ , which is dual to a single Dirac fermion  $\chi$ . The duality states that the dual fermion current  $J_\mu^\chi = \bar{\chi} \gamma_\mu \chi = \frac{1}{4\pi} \epsilon_{\mu\nu\rho} \partial_\nu \tilde{a}_\mu = \frac{k}{4\pi} \epsilon_{\mu\nu\rho} \partial_\nu a_\mu$ . In this limit the correlation function of  $J_\mu^\chi$  can be computed exactly because  $\chi$  decouples from any gauge field:

$$\langle J_\mu^\chi(p) J_\nu^\chi(-p) \rangle = \frac{1}{16} |p| \left( \delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} \right). \quad (7)$$

This implies that the full  $a_\mu$  propagator in the large- $k$  limit reads

$$G_{\mu\nu}^a(\vec{p}) = \frac{\pi^2}{k^2 |p|} \left( \delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} \right). \quad (8)$$

We have taken the Landau gauge. The propagator of  $b_\mu$  in the large- $k$  limit takes the same form.

There is a slightly different way of deriving the propagator of  $a_\mu$ : by varying with  $A_\mu$  on both Eq. (2) and Eq. (5), we can conclude that  $2\bar{\chi}_2 \gamma_\mu \chi_2 = \frac{1}{2\pi} \epsilon_{\mu\nu\rho} \partial_\nu a_\rho$ . Then varying  $b_\mu$  leads to the constraint  $k\bar{\chi}_2 \gamma_\mu \chi_2 = -\bar{\chi}_1 \gamma_\mu \chi_1$ . Since  $\chi_1$  decouples from the gauge field in the large- $k$  limit, the  $a_\mu$  propagator can be computed through the correlation of the fermion current  $\bar{\chi}_1 \gamma_\mu \chi_1$ , and the result will be the same as Eq. (8).

According to Ref. 22,23, a  $(2+1)d$  CFT with a U(1) global symmetry should have a universal conductivity. Due to the last term of Eq. (2), a  $2\pi$ -flux of  $a_\mu$  carries global U(1) charge  $n_A$  of the external gauge field  $A_\mu$ . Hence the global U(1) charge current of  $A_\mu$  is  $J_\mu^A = \frac{n_A}{2\pi} \epsilon_{\mu\nu\rho} \partial_\nu a_\rho$ . We can use the propagator of  $a_\mu$  to compute the universal conductivity of the global U(1) charge transport:

$$\langle J_\mu^A(p) J_\nu^A(-p) \rangle = \tilde{\sigma}_A |p| \left( \delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} \right), \quad (9)$$

where  $\tilde{\sigma}_A$  is the universal conductivity. Comparing Eq. (9) and Eq. (8) leads to the conclusion that with large  $k$  the leading order universal conductivity is

$$\tilde{\sigma}_A = \frac{n_A^2}{4k^2}. \quad (10)$$

For the same reason, the universal conductivity of the current of  $B_\mu$  is  $\tilde{\sigma}_B = \frac{n_B^2}{4k^2}$ .

The self-duality also identifies local operators on the two sides of the duality. For example we can identify  $\bar{\psi}_1\psi_1 \leftrightarrow -\bar{\chi}_1\chi_1$ ,  $\bar{\psi}_2\psi_2 \leftrightarrow -\bar{\chi}_2\chi_2$ . All these operators are odd under time-reversal. This identification of operators will be further clarified in section III. Using the fully-dressed gauge field propagator, we can compute the scaling dimension of these fermion mass operators at the leading order of  $1/k$  expansion:

$$\Delta[\bar{\psi}_2\psi_2] = \Delta[\bar{\chi}_1\chi_1] = 2 - \frac{4}{3k^2}. \quad (11)$$

In the large- $k$  limit,  $\psi_2$  and  $\chi_1$  effectively decouple from any gauge field, hence the scaling dimension of their mass operators are exactly 2. The  $1/k$  correction comes from the fermion wave function renormalization and vertex correction due to the coupling to the gauge field, like the calculation in (for example) Ref. 24. Then due to the self-duality of the theory,  $\bar{\psi}_1\psi_1$  ( $\bar{\chi}_2\chi_2$ ) should have the same scaling dimension as  $\bar{\chi}_1\chi_1$  ( $\bar{\psi}_2\psi_2$ ). When the time-reversal symmetry is preserved, these mass operators are all forbidden in the Lagrangian. But four-fermion interaction terms are still allowed. However, because the fully dressed gauge field propagators of  $a_\mu$  and  $b_\mu$  both acquire a  $1/k^2$  suppression, when  $k$  is large enough, these four-fermion operators are always irrelevant. Thus at least for large enough  $k$ , Eq. (2) describes a stable  $(2+1)d$  CFT.

In Eq. (2), a  $2\pi$ -flux of  $a_\mu$  carries charge  $n_A$  of the external gauge field  $A_\mu$ . In the dual theory, the smallest local gauge invariant operator that carries charge under  $A_\mu$  is (schematically)  $(\chi_1)^k \chi_2^\dagger$ . This operator should also have space-time derivatives due to the fermi statistics of  $\chi_1$ . This operator carries charge  $2n_A$ , and hence can be viewed as the double-monopole operator of  $a_\mu$ . It also has a power-law correlation at this CFT, but its scaling dimension is proportional to  $k$  with large  $k$ .

### B. The case with $k = 1$

As was first discussed in Ref. 25, the case  $k = 1$  of Eq. (2) has an enlarged  $O(4)$  global symmetry, if we ignore the external gauge fields. Ref. 25 demonstrated this  $O(4)$  global symmetry by deriving the low energy effective field theory of the  $N = 2$  QED with  $k = 1$ , and the gauge invariant degrees of freedom of this model turns out to be a four component bosonic vector. This  $O(4)$  symmetry becomes very natural knowing the self-duality of the theory. First of all, when  $k = 1$ , each side of the duality has a manifest  $SU(2)$  global flavor symmetry<sup>6,8,10</sup>, thus the symmetry of the system is at least

$SO(4) \sim SU(2) \times SU(2)$ . The fermion  $\psi_j$  and dual fermion  $\chi_j$  carry representation  $(1/2, 0)$  and  $(0, 1/2)$  under the  $SO(4)$ , while the four component vector introduced in Ref. 25 carries a vector representation of  $SO(4)$ .

The  $Z_2$  duality transformation is a symmetry that exchanges the two  $SU(2)$  groups. When the two  $SU(2)$  subgroups are exchanged, an  $O(4)$  vector  $(n_1, n_2, n_3, n_4)$  becomes  $(-n_1, -n_2, -n_3, n_4)$ , hence the  $Z_2$  duality transformation is an improper rotation of the four component vector. Hence the full symmetry of the CFT (assuming  $k = 1$  is indeed a stable CFT, as suggested by Ref. 18) is  $O(4) = SO(4) \times Z_2$ , and time-reversal  $\mathcal{T}$ .

A mass term  $\sum_{j=1}^2 m \bar{\psi}_j \psi_j$  not only breaks  $\mathcal{T}$ , but also breaks the  $Z_2$  duality symmetry, *i.e.* it breaks the  $O(4)$  symmetry down to  $SO(4)$ . This is because if we couple the system to two external  $SU(2)$  gauge fields, the  $SO(4)$  invariant mass term will generate different Chern-Simons terms for the two external  $SU(2)$  gauge fields, so it breaks the equivalence between the two  $SU(2)$  symmetries. In the nonlinear sigma model derived in Eq. 25, this fermion mass term corresponds to tuning the  $\Theta$ -term away from  $\Theta = \pi$ .

The existence of an  $O(4)$  breaking but  $SO(4)$  invariant mass term implies that, the QED with  $N = 2$  is a CFT without any  $O(4)$  invariant relevant perturbation, but once one breaks the  $O(4)$  down to  $SO(4)$ , there will be a relevant perturbation (if we do not assume an extra  $\mathcal{T}$ ). This is a signature of this CFT that one can look for with various numerical methods, and it is fundamentally different from the ordinary  $O(4)$  Wilson-Fisher fixed point<sup>50</sup>. At the ordinary  $3d$   $O(4)$  Wilson-Fisher fixed point, there is one relevant  $O(4)$  invariant perturbation, but weakly breaking the  $O(4)$  down to  $SO(4)$  does not generate any new relevant perturbation.

The effect of the mass term  $\sum_{j=1}^2 m \bar{\psi}_j \psi_j$  can be inferred from Ref. 6. Ref. 6 constructed Eq. (2) with  $k = 1$  on the boundary of a  $3d$  system, and showed that the mass term  $\sum_{j=1}^2 m \bar{\psi}_j \psi_j$  generates level  $+1$  and  $-1$  Chern-Simons terms for the two  $U(1)$  external gauge fields, where the two  $U(1)$  symmetries are subgroups of the  $SU(2)$  global symmetries. Formally the level  $\pm 1$   $U(1)$  Chern-Simons terms correspond to level  $\pm 1/2$  Chern-Simons terms of the  $SU(2)$  gauge fields, where the half-integer level is the sign of anomaly of the boundary of a  $3d$  SPT state, and the anomaly can be cancelled by the bulk  $\Theta$ -term. If this theory is realized in a pure  $2d$  system, then the external  $SU(2)$  gauge fields must receive at least another level  $\pm 1/2$  Chern-Simons terms to cancel the anomaly. This is the physical meaning of the “counterterms” introduced in Ref. 10. For example, if we consider a thin film of the  $3d$  system constructed in Ref. 6, then this theory on one boundary can receive another level  $\pm 1/2$   $SU(2)$  Chern-Simons terms (or level  $\pm 1$   $U(1)$  Chern-Simons terms) from the opposite boundary.

### C. Entanglement Entropy

The entanglement entropy of a  $(2+1)d$  CFT across a circle of radius  $R$  takes the general form

$$S = \alpha \frac{R}{\epsilon} - F, \quad (12)$$

where  $\epsilon$  is the short-distance cutoff. The radius-independent universal term  $F$  must decrease under the renormalization group flow<sup>26</sup> (the F-theorem). Ref. 27 computed  $F$  for a series of  $(2+1)d$  CFT with a large flavor number of matter fields, as  $F$  is related to the correctly regularized free energy of the CFT on a three dimensional sphere<sup>28</sup>.

The F-theorem provides us a lower bound for the universal entanglement entropy  $F$  for the CFT in Eq. (2). As will be discussed in section III, a Dirac fermion mass term, which is a relevant perturbation according to Eq. (11), will drive the CFT into a gapped topological order with  $(k^2+1)/2$  different Abelian anyons (our physical construction in section III guarantees that  $k$  be an odd integer, thus  $(k^2+1)/2$  is also an odd integer). Such Abelian topological order will have a topological entanglement entropy  $F_{\text{topo}} = \frac{1}{2} \log(\frac{k^2+1}{2})$ <sup>29,30</sup>. This implies that  $F$  of the CFT in Eq. (2) must satisfy

$$F \geq \frac{1}{2} \log \left( \frac{k^2+1}{2} \right). \quad (13)$$

It is also possible to find an upper bound for  $F$ , which requires identifying a UV fixed point that can flow to Eq. (2) through a relevant perturbation. Such candidate UV fixed point could be a supersymmetric version of Eq. (2) (which was the strategy used in Ref. 31). This supersymmetric CFT presumably flows to Eq. (2) through a supersymmetry breaking perturbation.

When  $k = 1$ , a different lower bound of  $F$  can be found. In section III we will see the mass term  $m\bar{\psi}_1\psi_1$  of fermion  $\psi_1$  will drive the system into another CFT that is conjectured to be dual to the  $3d$  XY Wilson-Fisher fixed point<sup>32,33</sup>. If this conjecture is correct, then it implies that  $F_{k=1}$  must be no smaller than that of the  $3d$  XY Wilson-Fisher fixed point.

A direct calculation of  $F$  seems to be more involved than the case studied in Ref. 27. The reason is that, after formally integrating out the fermions, the gauge field  $a_\mu$  will acquire an effective Lagrangian, which can be expanded as a polynomial of  $a_\mu$ , whose schematic form is  $L_{\text{eff}} \sim \sum_n c_n (a_\mu)^n$ , and  $c_n \sim k^n$  with large  $k$ . In Ref. 27, one can simply keep the quadratic term of the polynomial as the leading order approximation, because higher order terms are suppressed in the large- $N$  limit, since the gauge field fluctuation is at the order of  $O(1/N)$ . But in our case, all the higher terms in the polynomial will be at the same order in the large- $k$  limit, thus one cannot compute  $F$  by truncating the polynomial.

### D. Numerical Test

In the next section we will see that the field theory in Eq. 2 can be realized at the boundary of a  $3d$  bosonic SPT state with  $U(1)$  and time-reversal symmetry. But if we give up some of the symmetries, we can realize Eq. 2 in a  $2d$  system, which provides a feasible numerical test for the analytical calculations in this section. The original proposed duality Eq. 1 has not been confirmed quantitatively yet. Our proposed numerical effort can provide strong evidence for the duality in Eq. 1, because all of our calculations were based on the assumption that the two equations in Eq. 1 were dual to each other in the infrared limit.

To get Eq. 2, we can define four flavors of fermions on a lattice model:  $f_1$ ,  $f_2$ ,  $f_\uparrow$  and  $f_\downarrow$ . These fermions couple to a lattice  $U(1)$  gauge field with gauge charges  $k, 1, 1, 1$  respectively. Then we design the following band structure for all the fermions:  $f_\uparrow$  forms a Chern insulator with Chern number  $C_\uparrow = +1$ ;  $f_\downarrow$  forms a Chern insulator with Chern number  $C_\downarrow = -1$ , *i.e.*  $f_{\uparrow,\downarrow}$  forms a quantum spin Hall insulator; then we tune  $f_1$  to the critical point between two insulators with Chern number  $C_1 = 0$  and  $C_1 = +1$ , and tune  $f_2$  to the critical point between two states with Chern number  $C_2 = -(k^2-1)/2$  and  $C_2 = -(k^2+1)/2$ . All the Chern-Simons terms of the dynamical gauge field cancel out. And if we assume there is a  $U(1)$  global symmetry associated with the conservation of  $S^z = \sum_j (f_{j,\uparrow}^\dagger f_{j,\uparrow} - f_{j,\downarrow}^\dagger f_{j,\downarrow})$ , then by design the  $2\pi$  flux of the dynamical  $U(1)$  gauge field must carry a conserved quantum number  $S^z = 2$ , due to the quantum spin Hall response.

We could just define a lattice model with a dynamical  $U(1)$  gauge field coupled to the fermion band structure described above. Another standard method to engineer dynamical  $U(1)$  gauge field in condensed matter theory is the Gutzwiller projected wave function<sup>34,35</sup>. For example, Ref. 36 used the Gutzwiller projected wave function of fermionic spinons to engineer  $(2+1)d$  QED in a quantum spin-1/2 system defined on a  $2d$  Kagome lattice. In Ref. 36 (and related works), a spin liquid wave function with  $U(1)$  gauge field fluctuation is obtained by performing Gutzwiller projection on a mean field state of spinon  $f_{j,\uparrow}$  and  $f_{j,\downarrow}$ :  $|\Psi\rangle = P(f_{j,\uparrow}^\dagger f_{j,\uparrow} + f_{j,\downarrow}^\dagger f_{j,\downarrow} = 1)|\text{MF}\rangle$ .

To be concrete, we can define our fermionic partons mentioned above on  $(k^2+1)/2$  layers of honeycomb lattice. Parton  $f_2$  moves on all layers, and on each layer  $f_2$  has a band structure of the Haldane's model of quantum anomalous Hall effect<sup>37</sup>, while all the other partons only move on the first layer. Then the Gutzwiller projected wave function we consider is given by

$$|\Psi\rangle = P(kn_{j,1} = n_{j,2} + n_{j,\uparrow} + n_{j,\downarrow})|\text{MF}\rangle, \quad (14)$$

where  $|\text{MF}\rangle$  is the multi-critical state discussed above. Here  $j$  labels the cluster of  $(k^2+1)/2$  sites of the honeycomb layers.

To compare with our analytical calculation, one can

tune the mean field state  $|\text{MF}\rangle$  away from the multicritical point, into the insulating states. The tuning parameter is just the fermion mass operators in the field theory. Assuming the fermion mass is  $m$ , then the correlation length  $\xi$  of the system should scale with  $m$  as

$$\xi \sim m^{-1/(1+\frac{4}{3k^2})}, \quad (15)$$

based on our  $1/k$  expansion of the scaling dimension of the fermion mass operators. This result can be directly compared with numerical calculations proposed in the previous paragraphs.

We would like to point out that directly simulating the  $N = 1$  QED in Eq. 1 would be difficult, because on a  $2d$  lattice model there will always be another massive Dirac fermion coupled to the dynamical  $U(1)$  gauge field, which contributes another level- $1/2$  Chern-Simons term, and changes the dynamics of the gauge field completely. One can also try to design a numerical test for the more complete version of the dual  $N = 1$  QED discussed in Ref. 9, but more gauge fields need to be involved which increase the technical complexity. Thus our lattice construction above provides the most feasible numerical test for the original duality Eq. 1.

### III. PHYSICAL REALIZATION

#### A. Bulk Construction

The theory Eq. (2) could have various physical realizations. Here we will construct a  $(3+1)d$  gapped bulk state whose boundary is described by Eq. (2) with odd integer  $k$ . This construction of the bulk state follows similar steps as Ref. 2,6.

(1) We start with a  $U(1)$  spin liquid in the  $3d$  bulk with a gapless photon  $a_\mu$  and fermionic parton  $\psi_\alpha$ . Under time-reversal symmetry,  $\psi_\alpha$  transforms as  $\mathcal{T} : \psi \rightarrow i\sigma^y \psi^\dagger$ , namely the symmetry group of  $\psi$  is  $U(1)_g \times \mathcal{T}$ , where  $U(1)_g$  is the gauge symmetry. The electric and magnetic field of  $a_\mu$  are odd and even under time-reversal respectively, which is opposite to the transformation of the external  $U(1)$  gauge field  $A_\mu$ .

(2) In order to construct Eq. (2) as the boundary theory, we assume that the fermion  $\psi$  and the  $k$ -body bound state of  $\psi$  with odd integer  $k$  (we denote this bound state as  $\psi^k$ ) both form a topological insulator in the AIII class, at the mean field level, *i.e.* when the gauge field fluctuations are ignored. Thus at the mean field level, the most natural boundary state of the system is two flavors of fermions with gauge charge 1 and  $k$  respectively. A topological  $\Theta$ -term for  $a_\mu$  is generated if we integrate out the gapped partons, with the theta angle  $\Theta = (k^2 + 1)\pi$ . Due to the Witten's effect of the topological insulator<sup>38</sup>, a  $2\pi$ -monopole of  $a_\mu$  will carry a polarization gauge charge  $\frac{\Theta}{2\pi} = (k^2 + 1)/2$ , which is an odd integer, as long as  $k$  is odd. In general, a dyonic excitation in this spin liquid can be labelled as  $(q, m)$  where

$q$  is the total gauge charge and  $m$  the monopole number. The Witten's effect then implies  $q = n + \left(\frac{k^2+1}{2}\right)m$ , where  $n \in \mathbb{Z}$  is the number of partons  $\psi$  attached to the dyon. Thus a  $2\pi$ -monopole can be neutralized by binding with  $(k^2 + 1)/2$  holes of  $\psi$ . We label this neutralized monopole as the  $(0, 1)$  monopole.

(3) In the bulk we condense the bound state of a  $(0, 1)$  monopole and a physical boson that carries no gauge charge but one global  $U(1)$  charge<sup>51</sup>, *i.e.* it carries charge  $+1$  under the external gauge field  $A_\mu$ . This entire bound state is a gauge neutral, time-reversal invariant, and charge-1 boson. Following the notation in Ref. 2, we can label all excitations in terms of their quantum numbers  $(q, m, Q, M)$ , where  $q$  is the gauge charge under  $a_\mu$ ,  $m$  is the monopole number of  $a_\mu$ ,  $Q$  is the global  $U(1)$  symmetry charge, and  $M$  is the monopole number of the external  $U(1)$  gauge field  $A_\mu$ . Under this notation, the condensed bound state has quantum number  $(0, 1, 1, 0)$ .

(4) The condensate of the aforementioned bound state  $(0, 1, 1, 0)$  will confine all the excitations that have non-trivial statistics with it, including the  $\psi$  fermions. But this condensate does not break any global symmetry. The global  $U(1)$  symmetry is still preserved because the condensed bound state is also coupled to the dynamical gauge field  $a_\mu$ . The condensate does not have any gapless Goldstone mode, which would be a signature of spontaneous continuous symmetry breaking. If we move a  $2\pi$  Dirac monopole of  $A_\mu$  into the bulk, to avoid confinement caused by the condensate of the  $(0, 1, 1, 0)$  bound state, this  $A_\mu$  monopole will automatically pair with a fermion  $\psi$  to form a bound state with quantum number  $(1, 0, 0, 1)$ , so it has trivial mutual statistics with the condensed  $(0, 1, 1, 0)$  bound state. This deconfined Dirac monopole  $(1, 0, 0, 1)$  is neutral under the global  $U(1)$  symmetry, but it is a fermion. This neutral fermionic Dirac monopole of the external gauge field  $A_\mu$  is the characteristic statistical Witten's effect of the bosonic SPT state with  $U(1)$  and time-reversal symmetry discussed in Ref. 39.

Following Ref. 2,6, we can derive the surface theory of the system. We will skip most of the details since they are straightforward generalizations of those in Ref. 2,6, and just mention that since we condense the bound state  $(0, 1, 1, 0)$  in the bulk, the surface must have a  $\frac{i}{2\pi}\epsilon_{\mu\nu\lambda}a_\mu\partial_\nu A_\lambda$  Chern-Simons term, *i.e.*  $n_A = 1$ .

Our bulk construction indicates that  $\psi_1$  can be viewed as a  $k$ -body bound state of  $\psi_2$ . Thus a highly irrelevant interaction term  $\sim \psi_1^\dagger \psi_2^k$  is allowed in Eq. (2) (this term must also have space-time derivatives due to fermion statistics, which renders this term even more irrelevant). Thus in the IR limit the CFT described in Eq. (2) has an emergent global  $U(1)$  symmetry in addition to the conservation of  $a_\mu$  flux:  $\psi_1$  and  $\psi_2$  are conserved separately. The emergent  $U(1)$  symmetry in this construction can also be made an explicit  $U(1)$  symmetry on the lattice, if we assume  $\psi_1$  and  $\psi_2$  are two separately conserved fermions. Also, a  $2\pi$  monopole of  $a_\mu$  carries half-integer polarized fermion number of  $\psi_1$  and  $\psi_2$  respectively, thus

the flux number of  $b_\mu$  and  $c_\mu$  in Eq. (3) (which are the dual gauge fields of fermion currents of  $\psi_1$  and  $\psi_2$  respectively) must sum to be a multiple of  $4\pi$ , while each can be a multiple of  $2\pi$ . These conditions guarantee that Eq. (3) is gauge invariant, and  $a_\mu$ ,  $b_\mu$  in Eq. (2) and Eq. (5) both allow  $2\pi$  fluxes.

The construction for Eq. (2) with even integer  $k$  is more involved. We need to assume the existence of another gauge neutral fermion  $\Psi$ , in addition to the gauged fermion  $\psi$  in the  $3d$  bulk. We assume  $\psi$ , and the bound state of  $\psi^k$  and  $\Psi$  both form an AIII topological insulator. The  $2\pi$ -monopole of  $a_\mu$  carries half-integer polarization gauge charge. Then in order to confine the gauge field  $a_\mu$  in the bulk, we need to condense the bound state constructed with a gauge-neutralized  $4\pi$ -monopole of  $a_\mu$  and a physical electron. Now the boundary of the system is described by Eq. (2) with even integer  $k$  and  $n_A = 1/2$ . Thus just like the case discussed in Ref. 2, this theory only allows for fluxes of  $a_\mu$  that are multiples of  $4\pi$ .

### B. Anomalous Fractional Quantum Hall states

The mass terms for the fermions are allowed once the time-reversal symmetry is broken, which drives Eq. (2) and its dual Eq. (5) into topological orders with a nonzero Hall conductivity. There appears to be four different gauge invariant mass operators:  $m_1\bar{\psi}_1\psi_1$ ,  $m_2\bar{\psi}_2\psi_2$ ,  $m'_1\bar{\chi}_1\chi_1$ ,  $m'_2\bar{\chi}_2\chi_2$ , but they are not independent from each other. After the masses are turned on, in general the system is driven into a  $\mathcal{T}$ -symmetry-breaking fractional quantum Hall (FQH) phase.

We first compute the Hall conductance of this gapped surface state. When  $m_1 > 0$ ,  $m_2 > 0$ , formally the response of the surface state to the external  $A_\mu$  field is described by the following field theory:

$$\mathcal{L} = \frac{k^2 + 1}{2} \frac{i}{4\pi} \epsilon_{\mu\nu\rho} a_\mu \partial_\nu a_\rho + \frac{i}{2\pi} \epsilon_{\mu\nu\rho} a_\mu \partial_\nu A_\rho. \quad (16)$$

In the dual side, when  $m'_1 < 0$ ,  $m'_2 < 0$ , the system is described by

$$\begin{aligned} \mathcal{L} = & -\frac{k^2 + 1}{2} \frac{i}{4\pi} \epsilon_{\mu\nu\rho} b_\mu \partial_\nu b_\rho \\ & - \frac{ki}{2\pi} \epsilon_{\mu\nu\rho} b_\mu \partial_\nu A_\rho - \frac{2i}{4\pi} \epsilon_{\mu\nu\rho} A_\mu \partial_\nu A_\rho. \end{aligned} \quad (17)$$

Both Eq. (16) and Eq. (17) give the same Hall conductivity

$$\sigma_H = \frac{2}{k^2 + 1}. \quad (18)$$

When  $m_1 > 0$ ,  $m_2 < 0$  (or equivalently  $m'_1 < 0$  and  $m'_2 > 0$ ), the response is also described by a CS theory similar to Eq. (16) and Eq. (17):

$$\mathcal{L} = \frac{k^2 - 1}{2} \frac{i}{4\pi} \epsilon_{\mu\nu\rho} a_\mu \partial_\nu a_\rho + \frac{i}{2\pi} \epsilon_{\mu\nu\rho} a_\mu \partial_\nu A_\rho. \quad (19)$$

The Hall conductivity of this state is

$$\sigma_H = \frac{2}{k^2 - 1}. \quad (20)$$

We notice that when  $k = 1$ , the CS term of  $a_\mu$  in Eq. (19) vanishes, and this phase with  $m_1 > 0$ ,  $m_2 < 0$  become a superfluid phase with spontaneous U(1) symmetry breaking, or equivalently a gapless photon phase in the dual picture. In this case, if we fix  $m_1$  positive, and change the sign of  $m_2$ , this boundary system goes through a transition from a quantum Hall state with  $\sigma_H = 1$  to a superfluid phase. This transition was conjectured to be dual to a  $3d$  XY transition<sup>32,33</sup>, which as we discussed in the previous section, has given us a lower bound of  $F$  for the CFT with  $k = 1$ .

These fractional quantum Hall states are anomalous, in the sense that they cannot be realized in a pure  $2d$  system with bosons. To see why this is the case, we need to learn more about the topological order of these anomalous FQH states. The most important fact one needs to keep in mind is that the fundamental matter field that couples to  $a_\mu$  is a fermion. The Chern-Simons term in Eq. (16) and (17) attaches a  $\frac{4\pi}{k^2 \pm 1} a_\mu$ -flux to the fermion, so the topological twist (or the self statistics) of this excitation is  $\tau = e^{i\theta} = -e^{\frac{2\pi i}{k^2 \pm 1}}$ . From the standard flux attachment picture, we also know that the excitation carries  $\frac{2}{k^2 \pm 1}$  charge of the global U(1) symmetry. Other anyonic excitations of this FQH state can be constructed by fusing multiples of this fundamental anyons together, and there are in total  $\frac{k^2 \pm 1}{2}$  of them that form a  $\mathbb{Z}_{(k^2 \pm 1)/2}$  fusion group. We will label the anyon types by an integer  $[n]$  (defined mod  $\frac{k^2 \pm 1}{2}$ ), with the topological twist and the fractional U(1) charge given by

$$\tau_{[n]} = e^{i\theta_{[n]}} = (-1)^n e^{\frac{2\pi i n^2}{k^2 \pm 1}}, \quad q_{[n]} = \frac{2n}{k^2 \pm 1}. \quad (21)$$

The statistics of these Abelian anyons are consistent with the  $2d$  topological order described by the  $SU(\frac{k^2 \pm 1}{2})_{-1}$  Chern-Simons theory.

Let us start from the state with masses  $m_1 > 0$ ,  $m_2 > 0$ . To understand the anomaly on the surface, we imagine adiabatically inserting a  $\Phi = (k^2 + 1)\pi = \frac{k^2 + 1}{2} \cdot 2\pi$  flux of the external U(1) gauge field  $A_\mu$  into the surface. Since  $\Phi$  is an integer multiple of  $2\pi$ , the adiabatic flux insertion should create a local excitation of the system. Based on the Hall conductance  $\sigma_H = \frac{2}{k^2 + 1}$ , such a flux binds  $Q = 1$  charge of  $A_\mu$ , and therefore has a self statistics  $e^{i\Phi Q/2} = (-1)^{\frac{k^2 + 1}{2}}$ . For odd  $k$ ,  $\frac{k^2 + 1}{2}$  is also an odd integer. So inserting  $\Phi$  flux creates a fermionic excitation. On the other hand, the full braiding of this excitation with an anyonic excitation with U(1) charge  $q_{[n]} = \frac{2n}{k^2 + 1}$  in the system is just given by the Aharonov-Bohm phase  $e^{i\Phi q_{[n]}} = 1$ . So what we have obtained is a fermionic excitation that has trivial braiding statistics with all other anyons, which is impossible in a  $2d$  system of bosons. But on the surface of a  $3d$  system, this inconsistency is

precisely circumvented by the statistical Witten effect: insertion of a  $2\pi$  flux of  $A_\mu$  can be thought of as passing a charge-neutral  $2\pi$ -monopole of  $A_\mu$  through the surface, which leaves behind an extra fermion on the surface<sup>39</sup>.

This argument for anomaly does not lead to any direct inconsistency for the  $m_1 > 0, m_2 < 0$  case since  $\frac{k^2-1}{2}$  is an even integer, so a more elaborate argument is demanded. As we have explained, adiabatically inserting a  $2\pi$  flux of  $A_\mu$  creates an excitation of the system. We denote the anyon type of this excitation by  $[v]$ . One can generalize the previous argument to show that in a pure two-dimensional bosonic system, the following two relations must hold<sup>40,41</sup>:

$$M_{v,j} = e^{2\pi i q_{[j]}}, \quad (22)$$

$$\tau_v = e^{i\theta_v} = e^{\pi i \sigma_H}. \quad (23)$$

where  $j$  denotes the anyon type  $[j]$ .  $M_{v,j}$  is the mutual braiding statistics between  $v$  and  $[j]$ , and  $\theta_v$  is the self-statistics angle of  $v$ . These relations allow one to unambiguously determine  $v$  in a  $2d$  bosonic system.

Now coming back to the surface FQH state. If we assume that this surface FQH state can be realized in a pure  $2d$  bosonic system, then based on the data given in Eq. (21) and the general relation Eq. (22), we can determine  $v = [1]$ . Then Eq. (23) would imply that  $\sigma_H = \frac{2}{k^2 \pm 1} + 1 + 2\mathbb{Z}$ , which differ from Eq. (18) and Eq. (20) by exactly 1 (mod  $2\mathbb{Z}$ ). This difference/anomaly can be amended by the  $\Theta$ -term response in the bulk with  $\Theta = 2\pi$ <sup>42</sup>. Or equivalently, based on the Hall conductivity Eq. (18) and Eq. (20), the  $v$  anyon we would derive in a pure  $2d$  bosonic system differs from the anyon type  $[1]$  in our surface FQH state by one local fermion that comes from the statistical Witten's effect in the  $3d$  bulk.

We can in fact construct explicitly  $2D$  fractional quantum Hall states with  $SU(\frac{k^2 \pm 1}{2})_{-1}$  topological order, and  $\sigma_H = \frac{2}{k^2 \pm 1} + 1 + 2\mathbb{Z}$ . In the following denote  $N = \frac{k^2 \pm 1}{2}$ . We can write down an Abelian Chern-Simons theory:

$$\mathcal{L}_{CS} = \frac{\varepsilon^{\mu\nu\lambda}}{4\pi} a_{I\mu} K_{IJ} \partial_\nu a_{J\lambda} + \frac{\varepsilon^{\mu\nu\lambda}}{2\pi} t_I A_\mu \partial_\nu a_{I\lambda}. \quad (24)$$

Here  $a_I, I = 1, 2, \dots, N-1$  are dynamical  $U(1)$  gauge fields,  $A$  stands for the external  $U(1)$  gauge field.  $K$  is the following  $(N-1) \times (N-1)$  integer matrix:

$$K = \begin{pmatrix} -2 & 1 & 0 & \cdots & 0 \\ 1 & -2 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & -2 \end{pmatrix}, \quad (25)$$

which is just minus the Cartan matrix for  $SU(N)$ . We also define the following charge vector:

$$\mathbf{t} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \quad (26)$$

We can compute the Hall conductance:

$$\sigma_H = \mathbf{t}^\top K^{-1} \mathbf{t} = \frac{1}{N} - 1. \quad (27)$$

One can further show that the charge fractionalization of anyons agrees with the surface theory. Thus the anomalous FQH surface state we obtained can be viewed as stacking the non-anomalous  $2D$  FQH state described by Eq. (24) on top of the standard time-reversal-symmetry breaking surface with  $\sigma_H = 1$ <sup>39,42</sup> of the bosonic SPT state.

Besides the statistical Witten's effect, some of the boundary FQH states are anomalous in a different way. The FQH state with  $m_1, m_2 > 0$  is nonchiral because the Hall conductance of  $A_\mu$  and  $B_\mu$  in Eq. 2 are opposite. If we create a domain wall between two regions with  $m_1, m_2 > 0$  and  $m_1, m_2 < 0$ , at the domain wall the chiral central charge is zero. However, if we want to realize a topological order in  $2d$  with the same fusion rule and statistics as the anyons of the boundary FQH state with  $m_1, m_2 > 0$ , this  $2d$  topological order is described by a  $SU(\frac{k^2+1}{2})_{-1}$  Cherns-Simons field theory, whose bound-

ary has chiral central charge  $c = -\frac{k^2-1}{2} \bmod 8$ , and one can readily check that  $c = 0 \pmod{8}$  for  $k = \pm 1 \pmod{8}$ , and  $c = 4 \pmod{8}$  for  $k = \pm 3 \pmod{8}$ . One can drive a purely  $2d$  boundary phase transition to increase the chiral central charge  $c$  by a multiple of 8 without changing the topological order, due to the existence of a bosonic state in  $2d$  without any topological order (the so called  $E_8$  state<sup>43,44</sup>), but to change  $c$  by 4 one needs to drive a  $3d$  bulk transition, or attaching the bulk to another bosonic SPT state with time-reversal symmetry whose boundary is "half" of the  $E_8$  state<sup>42</sup>.

Similarly, when  $m_1 > 0$  and  $m_2 < 0$ , one finds that the surface state actually has chiral central charge  $c = 1$  (assuming  $k \neq 1$ ), but the  $2d$  realization of these anyon types is the  $SU(\frac{k^2-1}{2})_{-1}$  Cherns-Simons field theory, which has  $c = 1 - \frac{k^2-1}{2}$ . The difference between them is again  $\frac{k^2-1}{2} \bmod 8$ . So the states with  $k = \pm 3 \pmod{8}$  have another anomaly independent from the statistical Witten's effect, which can be amended by a bulk gravitational  $\Theta$ -term. A similar anomaly related to the chiral central charge mismatch was discussed previously in Ref. 45.

The fact that the anomalous FQH state with  $m_1 > 0$  and  $m_2 < 0$  has chiral central charge  $c = 1$ , can be understood as the following. We first create a thin film of the system, so the entire system is a true  $2d$  state without anomaly. On the top surface, we create a domain wall with  $m_1 > 0, m_2 < 0$  on the left, and  $m_1 < 0, m_2 > 0$  on the right; On the bottom surface, we create a domain wall with  $m_1, m_2 < 0$  on the left, and  $m_1, m_2 > 0$  on the right. Then for the entire  $2d$  thin film, the left side is described by the CS field theory  $SU(\frac{k^2-1}{2})_{-1} \times SU(\frac{k^2+1}{2})_{+1}$ , and the right side is described by the CS field theory  $SU(\frac{k^2-1}{2})_{+1} \times SU(\frac{k^2+1}{2})_{-1}$ . The chiral central

charge on the domain wall is 2, which means that the left side contributes a chiral central charge  $c = 1$ . This chiral central charge can only come from the anomalous FQH state on the top surface, because from the previous discussion we know that the anomalous FQH state on the bottom surface is nonchiral.

#### IV. SUMMARY

In this paper we studied a series of self-dual  $(2+1)d$  CFTs parameterized by an odd integer  $k$ . These CFTs are stable for large enough  $k$ , *i.e.* there is no relevant perturbation as long as certain symmetries (such as  $U(1)$ , time-reversal, etc.) are imposed. Unlike the usual cases in  $(2+1)d$ , the stability of these CFTs under study do not rely on supersymmetry or a large number of matter fields. Our  $1/k$  expansion calculation can be directly compared

with numerical simulation proposed in section IID, which would provide a strong evidence for the original duality Eq. 2.

Some questions remain open, and deserve further study. For example, a direct calculation of the entanglement entropy  $F$  is still demanded. It would also be interesting to search for other anomalous topological orders adjacent to the CFT under study. Sometimes the duality transformation of a theory can be enlarged to  $SL(2, Z)$ , as is the case for the  $(3+1)d$  supersymmetric Yang-Mills theory<sup>46,47</sup>, and also some  $(2+1)d$  models motivated from condensed matter systems<sup>48</sup>. Searching for a generalized version of our theory with an enlarged duality group would be another interesting direction.

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<sup>49</sup> For simplicity and convenience, we will use the formalism developed in Ref. 1–5, instead of the version in Ref 9 which does not require a special quantization of the flux of  $a_\mu$ .

<sup>50</sup> The existence of a relevant  $SO(4)$  invariant deformation of

this CFT was first pointed out to us by T. Senthil, during a private discussion. Here we identify this perturbation as the mass of the Dirac fermions.

<sup>51</sup> We notice that the physical boson must be a Kramers doublet under  $\mathcal{T}$