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Triplet FFLO Superconductivity in the doped Kitaev-Heisenberg Honeycomb Model

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We provide analytical and numerical evidence of a spin-triplet FFLO superconductivity in the itinerant Kitaev-Heisenberg model (anti-ferromagnetic Kitaev coupling and ferromagnetic Heisenberg coupling) on the honeycomb lattice around quarter filling. The strong spin-orbit coupling in our model leads to the emergence of 6 inversion symmetry centers for the Fermi surface at non zero momenta in the first Brillouin zone. We show how the Cooper pairs condense into these non-trivial momenta, causing the spatial modulation of the superconducting order parameter. Applying a Ginzburg-Landau expansion analysis, we find that the superconductivity has three separated degenerate ground states with three different spin-triplet pairings. Exact diagonalizations on finite clusters support this picture while ruling out a spin (charge) density wave.

Introduction- Mott insulator and high-T_c superconductor are closely related since the latter can be obtained from doping the half-filled Mott insulator [1–5]. One key element in superconductivity is the emergence of off-diagonal long-range order which results in the Bardeen-Cooper-Schrieffer ground state where Cooper pairs have a zero net momentum. The η pairing, proposed by C. N. Yang [6], binds electrons with momenta $\mathbf{k}$ and $\pi - \mathbf{k}$, and therefore involves a superconductivity with non-zero Cooper pair momentum. This superconductivity is referred to as the Fulde-Ferrell-Larkin-Ovchinnikov (FFLO) superconductivity [7, 8]. The FFLO superconductivity, which supports a spatial modulation for the electron pairing due to the non-trivial Cooper pair momentum, was first proposed in the '60s in a system with significant Zeeman interaction, which shifts the Fermi surfaces for the up and down spins. Experimental realizations of FFLO superconductivity have been proposed, for example, in heavy-fermions [9], ultra-cold atom systems [10–16], BEC analogues [17] and in magnetic analogue materials [18–27]. However, this exotic phase of matter has been observed only in a small number of systems so far[28, 29]. Indeed, the large magnetic field has a strong pair-breaking effect and limits the stability region of the FFLO phase. Models without explicit time reversal symmetry breaking have been considered in the context of superfluid ^{3}He[30] and unconventional superconducting[31] films. Here, we propose a theoretical model where the time reversal symmetry *is not explicitly broken* and *purely two dimensional* (as opposed to Refs. 30 and 31). Thus our approach is suitable for the realization of the FFLO superconductivity in the context of the two-dimensional "iridate" materials.

Lately, the studies of "iridates", a family of materials with significant spin-orbit coupling, have aroused great interests [32–34] partly because of the emergence of topological Mott physics [35] and its connection to the Kitaev anyon model [36, 37]. It has been shown both theoretically and experimentally that the existence of zigzag-magnetic order results from a Kitaev-Heisenberg magnetic coupling in the two-dimensional sodium iridate family [38–42]. An additional symmetric-off diagonal exchange term can also be added in the analysis [43]. Doping these spin-orbit Mott insulators has been addressed theoretically [44–46] and has started to attract some experimental attention [47]. Here, we address superconductivity in the presence of a large Hubbard interaction and adopt a localized magnetism point of view where the Kitaev-Heisenberg spin Hamiltonian originates from super-exchange processes [48]. Such magnetic system with spin-orbit coupling and Kitaev-Heisenberg physics can also be realized in cold atom systems [49–53]. Using both analytical and numerical methods, we provide convincing evidences of a spin-triplet FFLO superconductor thanks to the spin-orbit coupling close to quarter-filling without breaking the time-reversal symmetry. This provides an exotic scenario to reach a spin-triplet FFLO superconductor without breaking time-reversal symmetry with applications in quantum materials and ultra-cold atoms.

Before showing detailed derivations, we summarize the main points. The Kitaev-Heisenberg coupling entails spin-triplet pairing that engenders spinor-condensates [54–56] in momentum space. One important ingredient

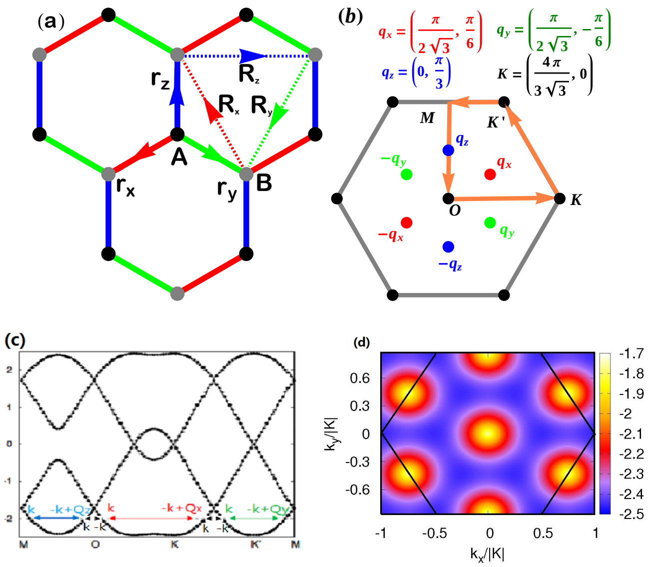

Figure 1. (a). The Kitaev-Heisenberg model on the honeycomb lattice: in Eq. 1 α denotes respectively x on the red links, y on the green links and z on the blue links each of them corresponding to $\mathbf{r}_x = (-\frac{\sqrt{3}}{2}, -\frac{1}{2}); \mathbf{r}_y = (\frac{\sqrt{3}}{2}, -\frac{1}{2}); \mathbf{r}_z = (0, 1)$; the lattice vectors are $\mathbf{R}_x = (-\frac{\sqrt{3}}{2}, \frac{3}{2}), \mathbf{R}_y = (-\frac{\sqrt{3}}{2}, -\frac{3}{2}), \mathbf{R}_z = (\sqrt{3}, 0)$. We have taken the lattice spacing to be 1. (b). The first Brillouin zone, in which, apart from the center of the FBZ, there are six additional centers of inversion symmetry for the Fermi surface of the tight binding part. (c). The band structure of the spin-orbit model ($t = 0, t' = 1$). M, O, K, K' are denoted in (b). When the chemical potential is fixed, electrons on the Fermi surface form triplet Cooper pairs with non-trivial momentum $\mathbf{Q}_x$, $\mathbf{Q}_y$ and $\mathbf{Q}_z$. $\mathbf{Q}_\alpha = 2\mathbf{q}_\alpha$. (d) Energy color plots for the lowest band in units of $t' = 1$.

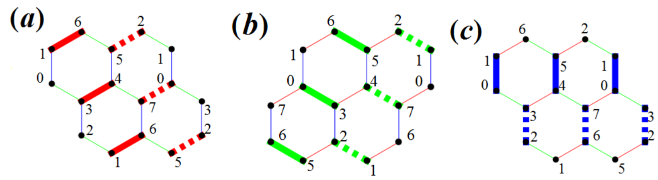

Figure 2. The graphical representation of the 3 times degenerate ground state wave function of the FFLO superconductivity around quarter-filling. The bold line signifies a spin-triplet pairing on the link Δ_{ij}^{α} with the spin-triplet type α (Eq. 4) in correspondence with the type of the link ((a) x red (b) y green and (c) z blue). The dashed line represents the same pairing but with a π phase (opposite sign in the wave function). Here, we only show the nearest-neighbor electron pairing. Long range electron pairing exists and depends on the correlation length of the superconductor [60].

here is the appearance of 6 inversion symmetry centers for the Fermi surface at non zero momenta in the first Brillouin zone. This will allow the Cooper pairs with triplet pairing to condense at non-trivial momenta. In Fig. 1, we show the band structure of the spin-orbit coupling model and the symmetry centers of the Fermi surface. Electron pairs around these symmetry centers with non-trivial momenta $\mathbf{q}_\alpha$ form spin-triplet pairs with Cooper pair momenta $\mathbf{Q}_\alpha = 2\mathbf{q}_\alpha$. We shall study the superconductivity by calculating the Cooper pairs' response in the Ginzburg-Landau theory for both spin-triplet and spin-singlet pairing. We provide compelling evidence of a triplet FFLO superconductor through a Ginzburg-Landau expansion and an exact diagonalization analysis.

Model Hamiltonian- For the doped Kitaev-Heisenberg model, we consider the following Hamiltonian on the honeycomb lattice:

$$
\begin{aligned}
H =& H_0 + H_J \\
H_0 =& -\sum_{\langle i,j\rangle} P_i [t c_{i\sigma}^\dagger d_{j\sigma} + t' c_{i\sigma}^\dagger d_{j\sigma'} \tau_{\sigma\sigma'}^{\alpha} + h.c.] P_j \\
H_J =& J_1 \sum_{\langle i,j\rangle} \mathbf{S}_i \cdot \mathbf{S}_j + J_2 \sum_{\langle i,j\rangle} [S_i^\alpha S_j^\alpha - S_i^\beta S_j^\beta - S_i^\gamma S_j^\gamma],
\end{aligned}
\tag{1}
$$

here i and j refer to the site index, $c_{i\sigma}$ and $d_{j\sigma}$ to electron operators on the lattices A and B in Fig. 1a. σ and σ' are the spins of the electrons and τ the Pauli matrix with $\alpha = x, y, z$ respectively for red, green and blue links ($\mathbf{r}_i - \mathbf{r}_j = \mathbf{r}_\alpha$) and β, γ take other components than α (See Fig. 1a). We note the Gutzwiller projectors as $P_i = (1 - \sum_\sigma c_{i\sigma}^\dagger c_{i\sigma})$ or $P_j = (1 - \sum_\sigma d_{j\sigma}^\dagger d_{j\sigma})$ according to the sub-lattice [57–59]. The filling factor n and the doping level δ are connected by the relation: $n = \frac{1}{2} - \delta$. In contrast to previous analyses [44, 45], we include a spin-orbit term of the (doped) model [48], such that the anti-ferromagnetic Kitaev and ferromagnetic Heisenberg couplings at half-filling are microscopically obtained from second-order super-exchange processes: $J_1 = \frac{4t^2}{U}, J_2 = \frac{4t'^2}{U}$ with U the Hubbard interaction. Due to the sign conventions in Eq. 1, positive J2 values favor ferromagnetic correlations. The singlet component would rather involve small-Q wavevectors. Setting $J = J_1 - J_2$ and $K = J_2$, we recover the model used in Ref. [38] describing the half-filled system. One shall assume that t' is real to avoid an induced Dzyaloshinskii-Moriya interaction. However, an imaginary t' does not change the physics in the limit of $t = 0$. With a purely imaginary t', the time-reversal symmetry (TRS) is restored and we will show the presence of FFLO superconductivity with TRS in this limit.

Band structure around quarter-filling- Around quarter-filling, which is sufficiently away from half-filling, one can assume that the effect of the Gutzwiller weights on the values of t' is weak and neglect the renormalization of t'.

We can then diagonalize H_0:

$$
\begin{aligned}
H_0 =& \sum_k \Psi_k^\dagger \mathcal{H}_0(k)\Psi_k, \quad \Psi_k^\dagger = (c_{k\uparrow}^\dagger, c_{k\downarrow}^\dagger, d_{k\uparrow}^\dagger, d_{k\downarrow}^\dagger) \\
\mathcal{H}_0(\mathbf{k}) =& \begin{pmatrix} 0 & M^\dagger(\mathbf{k}) \\ M(\mathbf{k}) & 0 \end{pmatrix} \\
M(\mathbf{k}) =& tg(\mathbf{k})\tau^0 + \sum_{\alpha=x,y,z} t' g_\alpha(\mathbf{k})\tau^\alpha \\
h_\alpha(\mathbf{k}) =& 2t'^2 \sin \mathbf{k}\cdot\mathbf{R}_\alpha \\
&+ 2tt'[1+\cos\mathbf{k}\cdot(\mathbf{r}_\alpha - \mathbf{r}_\beta) + \cos\mathbf{k}\cdot(\mathbf{r}_\alpha-\mathbf{r}_\gamma)]
\end{aligned} \tag{2}
$$

in which $\alpha \neq \beta,\gamma$ and $g(\mathbf{k}) = \sum_\alpha e^{i\mathbf{k}\cdot\mathbf{r}_\alpha}$, and $g_\alpha(\mathbf{k}) = e^{i\mathbf{k}\cdot\mathbf{r}_\alpha}$ ($\alpha = x,y,z$). We see that in the spin-orbit coupling limit ($t=0$) the Fermi surface has six additional inversion symmetry centers, apart from the inversion symmetry center O with trivial momentum $\mathbf{Q}_0 = 0$, in the first Brillouin zone (FBZ) $\mathbf{k} \leftrightarrow 2\mathbf{q}_\alpha - \mathbf{k}$ ($\alpha = x,y,z$) as indicated in Fig. 1b. This derives from the Sine function remaining invariant under the change of $\mathbf{k}\cdot\mathbf{R}_\alpha \leftrightarrow \pi - \mathbf{k}\cdot\mathbf{R}_\alpha$. In Fig. 1c, we show the band structure at the spin-orbit coupling limit $t=0, t'\neq 0$: the four bands have a conic structure for the Fermi surface at half and quarter filling.

Superconducting Instability- The doped itinerant Kitaev-Heisenberg model in the spin-orbit limit ($t=0$) has 7 symmetry centers around quarter-filling with momenta: $\pm\mathbf{q}_\alpha$ ($\alpha = x,y,z$) and $\mathbf{q}_0 = 0$. There are 4 kinds of Cooper pairs around these symmetry centers [61, 62]:

$$
\hat{\Delta}^\dagger_{\alpha\mathbf{Q}_\alpha}(\mathbf{k}) = i\tau^y_{\sigma\sigma"}\tau^\alpha_{\sigma"\sigma'} c^\dagger_{\mathbf{k}\sigma} d^\dagger_{-\mathbf{k}+\mathbf{Q}_\alpha\sigma'} \quad (\alpha = 0,x,y,z) \tag{3}
$$

In the direct space, the three types of spin-triplet pairing and the spin-singlet pairing in competition are:

$$
\begin{aligned}
\hat{\Delta}^x_{ij} &= c_{i\uparrow}d_{j\uparrow} - c_{i\downarrow}d_{j\downarrow}; \hat{\Delta}^y_{ij} = i(c_{i\uparrow}d_{j\uparrow} + c_{i\downarrow}d_{j\downarrow}); \\
\hat{\Delta}^z_{ij} &= c_{i\uparrow}d_{j\downarrow} + c_{i\downarrow}d_{j\uparrow}; \hat{\Delta}^0_{ij} = c_{i\uparrow}d_{j\downarrow} - c_{i\downarrow}d_{j\uparrow}.
\end{aligned} \tag{4}
$$

The Kitaev-Heisenberg coupling involves the density channel $\hat{\chi}_\alpha = c^\dagger_{i\sigma} d_{j\sigma'}\tau^\alpha_{\sigma\sigma'} + h.c.$ besides the superconductivity pairing. We have checked that around quarter-filling the density channel renormalizes the spin-orbit coupling term t' and such renormalization is negligible [62]. Then we can decompose the Kitaev-Heisenberg coupling at the mean-field level as:

$$
\begin{aligned}
&J_2 \sum_{\langle i,j\rangle} [S_i^\alpha S_j^\alpha - S_i^\beta S_j^\beta - S_i^\gamma S_j^\gamma] \\
=& \frac{3J_2 N_s}{4}\sum_{\alpha,\mathbf{Q}} |\Delta_{\alpha\mathbf{Q}}|^2 - J_2 \sum_{\alpha,\mathbf{k},\mathbf{Q}} [g_\alpha(\mathbf{k})\Delta_{\alpha\mathbf{Q}}\hat{\Delta}^\dagger_{\alpha\mathbf{Q}}(\mathbf{k}) \\
&- g_\alpha(\mathbf{k})\Delta_{0\mathbf{Q}}\hat{\Delta}^\dagger_{0\mathbf{Q}}(\mathbf{k}) + h.c.],
\end{aligned} \tag{5}
$$

in which $\Delta_{\alpha\mathbf{Q}} = \frac{1}{N_s}\sum_{\langle i,j\rangle} e^{i\mathbf{Q}\cdot\mathbf{r}_j}\left\langle \hat{\Delta}^\alpha_{ij}\right\rangle$ is the Fourier transform of the order parameter $\left\langle \hat{\Delta}^\alpha_{ij}\right\rangle$ in Eq. 4 with spatial phase modulation $e^{i\mathbf{Q}\cdot\mathbf{r}_j}$. N_s denotes here the number of unit cells.

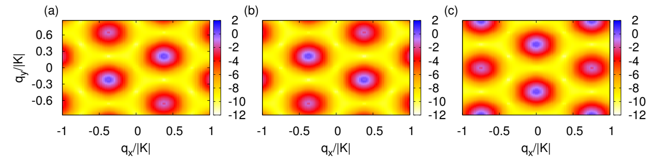

Figure 3. The inverse of the vertex function $\Gamma^{-1}_\alpha(\mathbf{Q},T)$ at quarter-filling $\alpha = x,y,z$ as a function of $\mathbf{q} = \frac{\mathbf{Q}}{2} \in$ FBZ (first Brillouin zone) at quarter-filling for the spin-triplet pairing $\Delta_{x\mathbf{Q}}$ (a), $\Delta_{y\mathbf{Q}}$ (b) and $\Delta_{z\mathbf{Q}}$ (c) at temperature $k_BT = 0.01t'$ and $t'=1$.

We constitute the Nambu spinor for the four Cooper pairs $\Phi_{\mathbf{kQ}} = (\Psi_\mathbf{k}, \Psi^\dagger_{\mathbf{Q}-\mathbf{k}})$ ($\Psi_\mathbf{k}$ is defined in Eq. 2) and write down their Gor'kov-Green function $G^{-1}_\alpha(\omega,\mathbf{k},\mathbf{Q})$ ($\alpha = 0,x,y,z$, $\mathbf{Q}/2 \in$ FBZ). We then pursue the Landau expansion [63]. In the spin-orbit coupling limit ($t=0$, $t'\neq 0$), we have the second order Landau expansion (here we fix $U=6$ following Ref. [44]):

$$
F_{BCS} \approx -\sum_{\alpha,\beta=0,x,y,z}\sum_{\mathbf{Q}} N_s \Gamma^{-1}_{\alpha\beta}(\mathbf{Q},T)\Delta_{\alpha\mathbf{Q}}\Delta^*_{\beta\mathbf{Q}} \tag{6}
$$

in which F_{BCS} is the free energy and to the lowest (second) order is proportional to the inverse of the Cooper pair vertex function $\Gamma^{-1}_{\alpha\beta}(\mathbf{Q},T)$ [63]. When $\alpha\neq\beta$, we have checked that $\Gamma^{-1}_{\alpha\beta}(\mathbf{Q},T)$ is negligible because of frustration in the momentum space; therefore we focus our attention on the diagonal part of the inverse of the Cooper pair vertex function that we denote as $\Gamma^{-1}_\alpha(\mathbf{Q},T) \equiv \Gamma^{-1}_{\alpha\alpha}(\mathbf{Q},T)$. When $\Gamma^{-1}_\alpha(\mathbf{Q},T) > 0$, the triplet superconductor pairing $\Delta_{\alpha\mathbf{Q}}$ is stable [64]. In Fig. 3, we show $\Gamma^{-1}_\alpha(\mathbf{Q},T)$ as a function of $\mathbf{q} = \mathbf{Q}/2 \in$ FBZ at temperature $k_BT = 0.01t'$, in which we remark the condensation of spin-triplet Cooper pairs $\Delta_{\alpha\mathbf{Q}}$ into the peaks at wave vector $\mathbf{q}_\alpha = \frac{\mathbf{Q}_\alpha}{2}$. We have three spin-triplet condensates at different momenta as shown in Fig. 2 a, b and c.

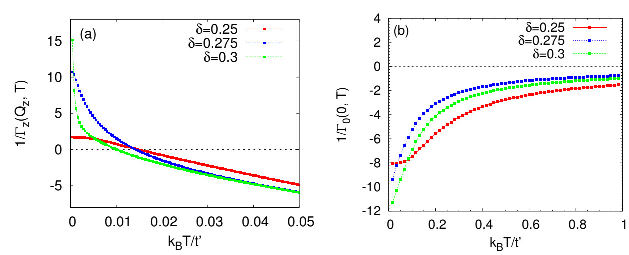

Figure 4. In the limit $t = J_1 = 0$: (a) The peak of Cooper pair vertex function $\Gamma^{-1}_\alpha(\mathbf{Q}_\alpha,T)$ as a function of temperature at different doping level ($\delta = 0.25$ is the quarter-filling). (b) The vertex function of singlet Cooper pair $\Gamma^{-1}_0(0,T)$ as a function of temperature at different doping level.

We also study the peak of the static Cooper pair response $\Gamma^{-1}_\alpha(\mathbf{Q}_\alpha,T)$ as a function of temperature at different doping levels δ: the peak remains finite at quarter-filling, while it has logarithmic divergence at zero temperature when the doping diverts from quarter-filling (Fig. 4a). Here, $\Gamma_\alpha^{-1}(\mathbf{Q}_\alpha, T)$ is proportional to the density of states at the Fermi level, which vanishes linearly as $\delta \to 1/4$, which means that at quarter-filling superconductivity disappears and we have a free electron system, assuming J_2 is not too large compared to t'. Indeed, we have checked that at quarter-filling, the critical value of J_2 to reach a superconducting instability is $J_{2C} \simeq 0.6t'$ as shown in Fig. 5b. At low temperature, the peak of the condensate profile $\Gamma_x^{-1}(\mathbf{Q}_x, T) = \Gamma_y^{-1}(\mathbf{Q}_y, T) = \Gamma_z^{-1}(\mathbf{Q}_z, T)$ stays positive while the peak of the spin-singlet condensate profile $\Gamma_0^{-1}(0, T)$ remains negative at all temperature (Fig. 4b). This indicates that in the spin-orbit coupling limit, the doped itinerant Kitaev-Heisenberg model hosts only the three spin-triplet ground states. Since the phase related to $\mathbf{Q}_\alpha$ is π, the analysis for $-\mathbf{Q}_\alpha$ remains the same.

The three spin-triplet condensates may interact with each other and we have calculated the box diagram to study this effect by extending the Landau expansion to the fourth order. We note

$$
\begin{aligned}
b_{xq}^\dagger &= \frac{1}{N_s}\sum_{\mathbf{k}}(c_{\mathbf{k}\uparrow}^\dagger d_{-\mathbf{k}+\mathbf{q}\uparrow}^\dagger - c_{\mathbf{k}\downarrow}^\dagger d_{-\mathbf{k}+\mathbf{q}\downarrow}^\dagger) \\
b_{yq}^\dagger &= -i\frac{1}{N_s}\sum_{\mathbf{k}}(c_{\mathbf{k}\uparrow}^\dagger d_{-\mathbf{k}+\mathbf{q}\uparrow}^\dagger + c_{\mathbf{k}\downarrow}^\dagger d_{-\mathbf{k}+\mathbf{q}\downarrow}^\dagger) \qquad (7) \\
b_{zq}^\dagger &= -\frac{1}{N_s}\sum_{\mathbf{k}}(c_{\mathbf{k}\uparrow}^\dagger d_{-\mathbf{k}+\mathbf{q}\downarrow}^\dagger + c_{\mathbf{k}\downarrow}^\dagger d_{-\mathbf{k}+\mathbf{q}\uparrow}^\dagger)
\end{aligned}
$$

the creation operators for the three Cooper pairs. Since the three Cooper pairs condense at different momenta $\mathbf{Q}_\alpha$, the box diagram is actually the only one respecting momentum conservation. To the fourth order, we obtain the free energy of the three condensates:

$$
\begin{aligned}
F_{BCS} =& N_s \sum_{\alpha=x,y,z} \{-\Gamma_\alpha^{-1}(\mathbf{Q}_\alpha, T)|\Delta_{\alpha\mathbf{Q}_\alpha}|^2 + C_1|\Delta_{\alpha\mathbf{Q}_\alpha}|^4\} \\
&+ N_s C_2 \sum_{\alpha\neq\beta} |\Delta_{\alpha\mathbf{Q}_\alpha}|^2 |\Delta_{\beta\mathbf{Q}_\beta}|^2, \qquad (8)
\end{aligned}
$$

in which C_1 and C_2 are positive numbers obtained from the calculation of the box diagram in the left panel of Fig. 5a. We have checked that $C_2 > C_1 > 0$ and thus we deduce that mixing of the three superconducting condensates is not energetically favorable, and there is phase separation among the three types of fermionic pairs. Consequentially, the ground state wave function at zero temperature is three times degenerate (See Fig. 2): the modulated Δ_{ij}^α (Eq. 4) are represented by bold and dashed lines ((a) red for X, (b) green for Y and (c) blue for Z).

When t and J_1 are small compared to t' and J_2, the three FFLO states are still stable when the temperature is low enough ($\Gamma_x^{-1}(\mathbf{Q}_x, T) = \Gamma_y^{-1}(\mathbf{Q}_y, T) = \Gamma_z^{-1}(\mathbf{Q}_z, T) > 0$) . The FFLO phase remains stable as long as the energy related to the critical temperature is bigger than the gap of the free electron system around quarter-filling opened by the t term i.e. $k_B T_c(\delta) > t$ [65]. In Fig. 5b, we have plotted the critical value of J_{2C} for the superconductivity instability as a function of temperature T.

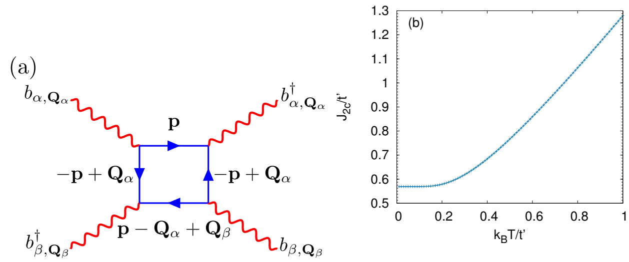

Figure 5. *Left panel*: The box diagram of the $4th$ order Landau expansion describing the interaction between the triplet pairing. *Right panel*: J_{2C}/t' as a function of temperature at quarter filling $\delta = 0.25$. The critical value of J_{2C} below which superconductivity instability is induced in the limit $t = J_1 = 0$.

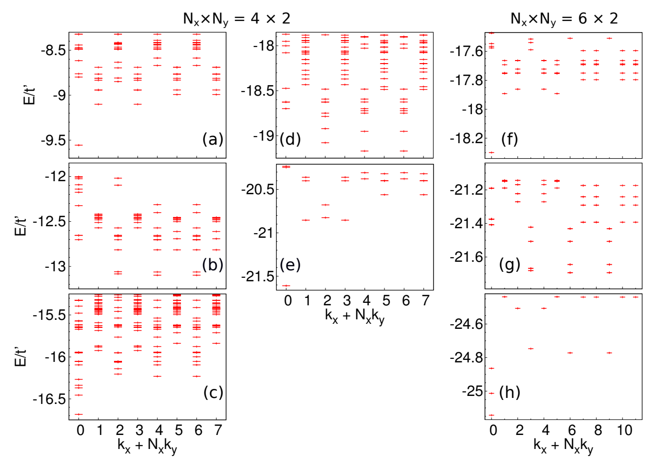

Figure 6. Energy spectra as a function of the linearized momentum $k_x + N_x k_y$ of the Hamiltonian in Eq. 1 with periodic boundary conditions and $t = J_1 = 0$, $t' = 1, J_2 = 0.667$. The left column and middle column show a system of $N_x \times N_y = 4 \times 2$ plaquettes with particle numbers (a) $N = 4$ (b) $N = 6$ (c) $N = 8$ or quarter-filling (d) $N = 10$ and (e) $N = 12$. The right column only provides the (f) $N = 8$, (g) $N = 10$, (h) $N = 12$ spectra on a 6×2 system (the largest Hilbert space dimension involved for (h) is $\simeq 1.7.10^8$). Note that for this system, only the few first energy levels are shown.

Exact Diagonalization of the Kitaev-Heisenberg model- We have done an exact diagonalization of the Kitaev-Heisenberg model of Eq. 1 in the spin-orbit coupling limit $t = 0, t' = 1$. The exact diagonalization treats the Gutzwiller projectors exactly in Eq. 1. We fix the parametrization $J_1 = \frac{4t^2}{U}$, $J_2 = \frac{4t'^2}{U}$ (here we choose

$U = 6$ as suggested by Ref. [44]). The system has $N_s = N_x \times N_y$ plaquettes with periodic boundary conditions in both directions, and is filled with N electrons on the $2N_x \times N_y$ sites. N_x and N_y are both even numbers in order to avoid frustration of the FFLO condensates. Due to computational constraints, we reduce our study to three system sizes: $N_y = 2$, $N_x = 2,\ 4,\ 6$. For an odd number of Cooper pairs (doped system), the lowest energy eigenstates appear in momentum sectors $\mathbf{k}_x = (N_x/2, N_y/2)$ $\mathbf{k}_y = (N_x/2, 0)$ and $\mathbf{k}_z = (0, N_y/2)$ (in the bases of $\mathbf{k}_1 = \frac{1}{N_x}(0, \frac{4\pi}{3})$, $\mathbf{k}_2 = \frac{1}{N_y}(\frac{2\pi}{\sqrt{3}}, -\frac{2\pi}{3})$.) as shown in Figs. 6b, d, and g. The degeneracy for the three spin-triplet states is partially lifted when $N_x \neq N_y$ which breaks the symmetry of a $2\pi/3$ rotation followed by a permutation of spin components. For an even number of Cooper pairs, the ground state appears in momentum sector $\mathbf{k}_0 = (0, 0)$ as shown in Figs. 6a, c, e, f and h. In agreement with the theory, $\mathbf{k}_\alpha$ coincides with the three discrete version of the FFLO Cooper pair momenta $\mathbf{q}_\alpha$ ($\alpha = x, y, z$) for an odd number of Cooper pairs while for even number of Cooper pairs $\mathbf{k}_0 \equiv 2\mathbf{k}_\alpha = 2\mathbf{q}_\alpha$ mod (N_x, N_y). This alternation of ground state momentum sector as a function of particle numbers distinguishes the FFLO superconductivity here from other modulated orders like spin or charge density waves [66]. The quasidegeneracy in Fig.6b is yet to be understood and might just be a finite size effect.

Conclusion- We have provided both analytical and numerical evidence of a pure spin-triplet FFLO superconductor in the doped itinerant Kitaev-Heisenberg model in the spin-orbit coupling limit ($t, J_1 \to 0$). When t' is purely imaginary, the time-reversal symmetry (TRS) is restored, which might overcome the difficulties of the experimental realization of the FFLO phase. The key ingredient of the FFLO superconductivity here is the symmetry centers of the Fermi surface at non-trivial momenta instead of a Zeeman field. The ground state is three times degenerate with respectively the three spin-triplet pairing Δ_{ij}^α in the p-wave state with non-trivial Cooper pair momentum $\mathbf{Q}_\alpha = 2\mathbf{q}_\alpha$ and spatial modulation of π phase in the direction of lattice vector $\mathbf{R}_\alpha$ for the order parameter. These results may have relevance for doped iridate honeycomb materials or in ultra-cold atom systems. This FFLO state could be detected by possible Josephson effect measurements, by coupling such an FFLO material with a usual superconductor as proposed in several works such as [67]. This FFLO state could also reveal interesting (short-range) magnetic fluctuations in connection with the zig-zag phase at half-filling, which is beyond the scope of the present work.

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