# Magnon-induced long-range correlations and their neutron-scattering signature in quantum magnets

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# Magnon-induced long-range correlations and their neutron-scattering signature in quantum magnets

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We consider the coupling of the magnetic Goldstone modes, or magnons, in both quantum ferromagnets and antiferromagnets to the longitudinal order-parameter fluctuations, and the resulting nonanalytic behavior of the longitudinal susceptibility. In classical magnets it is well known that long-range correlations induced by the magnons lead to a singular wave-number dependence of the form $1/k^{4-d}$ in all dimensions $2 < d < 4$, for both ferromagnets and antiferromagnets. At zero temperature we find a profound difference between the two cases. Consistent with naive power counting, the longitudinal susceptibility in a quantum antiferromagnet scales as $k^{d-3}$ for $1 < d < 3$, whereas in a quantum ferromagnet the analogous result, $k^{d-2}$, is absent due to a zero scaling function. This absence of a nonanalyticity in the longitudinal susceptibility is due to the lack of magnon number fluctuations in the ground state of a quantum ferromagnet; correlation functions that are sensitive to other fluctuations do exhibit the behavior predicted by simple power counting. Also of interest is the dynamical behavior as expressed in the longitudinal part of the dynamical structure factor, which is directly measurable via neutron scattering. For both ferromagnets and antiferromagnets there is a logarithmic singularity at the magnon frequency with a prefactor that vanishes as $T \to 0$. In addition, in the antiferromagnetic case there is a nonzero contribution at $T = 0$ that is missing for ferromagnets. Magnon damping due to quenched disorder restores the expected scaling behavior of the longitudinal susceptibility in the ferromagnetic case; it scales as $k^{d-2}$ if the order parameter is not conserved (magnetic disorder), or as $k^d$ if it is (non-magnetic disorder). Detailed predictions are made for both two- and three-dimensional systems at both $T = 0$ and in the limit of low temperatures, and the physics behind the various nonanalytic behaviors is discussed.

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## I. INTRODUCTION

The collective excitations known as magnons are a characteristic feature of any magnetically ordered state in which a continuous symmetry is spontaneously broken.[1] Common examples are planar, or XY, and Heisenberg magnets, where the spontaneously broken symmetry is $O(2)$ and $O(3)$, respectively. The magnons are the resulting Goldstone modes, which are soft or massless since a uniform rotation of the order parameter does not require any energy. In a ferromagnet, their frequency $\Omega$ scales as the wave number $k$ squared in the long-wavelength limit, $\Omega \sim k^2$; in an antiferromagnet, the frequency is a linear function of the wave number, $\Omega \sim k$. The relevant correlation function is the transverse order-parameter susceptibility, which diverges in the limit of zero frequency and wave number. In a solid, the magnons are gapped at asymptotically small frequencies, and the transverse susceptibility stays finite, due to the underlying lattice that breaks the $O(n)$ symmetry; however, compared to other relevant energy scales this is usually a small effect due to the weakness of the spin-orbit interaction. For our purposes we will ignore the spin-orbit interaction and treat the magnons as gapless.

Magnons can be observed directly via neutron scattering.[1,2] However, via couplings of the transverse order-parameter fluctuations to other modes, they also have profound indirect effects on other observables. An example is the longitudinal spin susceptibility $\chi_{\rm L}$ in a classical Heisenberg ferromagnet or antiferromagnet. It has been known for a long time that the coupling of the longitudinal spin fluctuations to the transverse ones (i.e., the Goldstone modes) leads to a $\chi_{\rm L}$ that diverges for $k \to 0$ everywhere in the ordered phase for all spatial dimensions $2 < d < 4$.[3–5] The leading contribution takes the form of a convolution of two Goldstone modes

$$\chi_{\rm L} \propto \int d\boldsymbol{p}\, \frac{1}{\boldsymbol{p}^2}\, \frac{1}{(\boldsymbol{p}-\boldsymbol{k})^2} \approx \int_{|\boldsymbol{k}|} d\boldsymbol{p}\, \frac{1}{\boldsymbol{p}^4} \propto \frac{1}{|\boldsymbol{k}|^{4-d}} \ . \quad (1.1)$$

It can be represented diagrammatically as shown in Fig. 1. This result, which was originally derived for ferromagnets in perturbation theory, was later shown by renormalization-group (RG) methods to be asymptotically exact.[6] It reflects the scale dimensions that characterize the stable RG fixed point that describes the ordered phase. We stress again that Eq. (1.1) holds for both classical ferromagnets and antiferromagnets. However, in the latter the physical meaning of the longitudinal order-parameter susceptibility $\chi_{\rm L}$ is the correlation

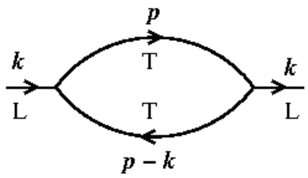


Figure 1: Diagrammatic representation of the coupling between longitudinal and transverse spin flucutuations in the classical case: A longitudinal (L) mode couples to two transverse (T) modes. The resulting contribution to the longitudinal susceptibility $\chi_{\rm L}$ has the form given in Eq. (1.1).

function of the staggered magnetization, rather than the spin susceptibility.

Physically, the nonanalytic dependence of $\chi_{\rm L}$ on the wave number reflects long-range correlations in the system that are due to the massless magnons: In real space, $\chi_{\rm L}$ for large distances $r$ falls off as a power law, $\chi_{\rm L} \propto 1/r^{2d-4}$. This is a particular manifestation of a more general phenomenon: Soft or massless modes lead to long-range correlations that are reflected in nonanalytic wave-number and frequency dependences in the hydrodynamic limit, i.e., the limit of small frequencies and wave numbers, in observables that couple to the soft modes. If the soft modes exists in entire phases, as opposed to, e.g., isolated critical points, then so does the nonanalytic behavior, which usually takes the form of power laws; a phenomenon known as generic scale invariance.[7] In magnets, other soft modes may be present that also couple to any given observable and compete with the magnons in producing long-range correlations, or nonanalytic behavior. For instance, in disordered metals at zero temperature ($T = 0$) there are diffusive excitations known as "diffusons" and "Cooperons" that lead to nonanalyticities in observables known as weak-localization effects.[8,9] A specific example is the nonanalytic frequency dependence of the electrical conductivity. The competing effects of magnons on one hand, and of diffusons and Cooperons on the other, on the conductivity in disordered metallic ferromagnets have been investigated in Ref. 10.

The weak-localization and other zero-temperature effects raise an interesting question: What is the fate of the singular behavior of the classical longitudinal spin susceptibility, Eq. (1.1), in the limit $T \to 0$? Simple considerations show that the singularity must be weaker at $T = 0$. In quantum statistical mechanics the statics and the dynamics are intrinsically coupled. The expression for $\chi_{\rm L}$ at $T = 0$ therefore must include a frequency integration in addition to the wave-number integration, and the integrand must be comprised of the dynamic Goldstone modes. A simple guess, based on power counting only, is that for quantum antiferromagnets at $T = 0$ [11]

$$\chi_{\rm L} \sim \int_{|\boldsymbol{k}|} d\boldsymbol{p} \int_{\Omega} d\omega \; \frac{1}{(\boldsymbol{p}^2 + \omega^2)^2} \sim |\boldsymbol{k}|^{d-3} \sim \Omega^{d-3} \quad (1.2a)$$

for $1 < d < 3$, with a logarithmic singularity in $d = 3$. We will show below that this expectation is indeed borne out by an explicit calculation.

For quantum ferromagnets, the corresponding expression obtained by replacing the denominator in Eq. (1.2a) by $(p^2 + \omega)^2$ is clearly not correct. This can be seen from spin-wave theory, which expresses the spin operators by bosonic operators via a Holstein-Primakoff transformation.[12] In a ferromagnet, the longitudinal spin is given in terms of the magnon-number operator, and $\chi_{\rm L}$ thus is the magnon-number correlation function. At $T = 0$ there are no magnons, and the contribution corresponding to Eq. (1.2a) (which would scale as $k^{d-2}$) therefore has a zero prefactor. An equivalent statement is that in the ground state of a quantum ferromagnet the magnetization has its maximum value, and therefore the ground-state energy has the same value as it does classically and cannot be decreased by quantum fluctuations.[13] In a quantum antiferromagnet, by contrast, this is not true: The classical Neél state is not an eigenstate of the Hamiltonian, and the ground-state energy is lowered below its classical value by quantum fluctuations. The remaining question is how the classical singularity, Eq. (1.1), vanishes as $T \to 0$ in a ferromagnet. As we will see, the leading contribution for $k \to 0$ at a low fixed temperature is given by Eq. (1.1) with a $T$ prefactor,

$$\chi_{\rm L} \propto T \int d\boldsymbol{p} \, \frac{1}{\boldsymbol{p}^2} \, \frac{1}{(\boldsymbol{p} - \boldsymbol{k})^2} \propto \frac{T}{|\boldsymbol{k}|^{4-d}} \; . \quad (1.2b)$$

The above considerations hold for undamped magnons. If the magnons are damped, then in general a nonanalyticity in the hydrodynamic limit is restored, with the exponent depending on the nature of the damping.[14] For instance, magnetic impurities, which lead to a damping coefficient proportional to $p^2$, introduce sufficiently strong fluctuations to invalidate the arguments given below Eq. (1.2a) and lead to a longitudinal susceptibility that does indeed scale as $k^{d-2}$ at $T = 0$. Non-magnetic quenched disorder, which leads to a damping coefficient proportional to $p^4$, leads to a weaker singularily, $\chi_{\rm L} \sim k^d$.

A more general question pertains to the spectrum of the dynamical longitudinal susceptibility or, equivalently, the longitudinal part of the dynamical structure factor, which is directly measurable by neutron scattering, as is the transverse part. For bulk ferromagnets at $T > 0$, the longitudinal structure factor has a logarithmic singularity at the magnon resonance, which gets regularized by a magnetic field.[3] We will show that for an antiferromagnet there is a nonzero contribution even at $T = 0$, which is caused by the same quantum fluctuations that are responsible for Eq. (1.2a) to hold. The singularity at the magnon resonance takes the form of a discontinuous slope in bulk antiferromagnets, and a square-root singularity in two-dimensional systems.

The organization of the paper is as follows. In Sec. II we consider magnets with undamped spin waves by considering nonlinear sigma models (NL$\sigma$Ms) for both quantum ferromagnets and antiferromagnets. This provides a simple and transparent way to understand why the classical nonanalyticity disappears as $T \to 0$ in the ferromagnetic case, while it is just weakened, in agreement with the simple scaling argument given above, in the antiferromagnetic case. In Sec. III we use time-dependent Ginzburg-Landau theory to discuss the effects of damped magnons in ferromagnets. In Sec. IV we conclude with a summary and discussion of our results.

## II. EFFECTS OF UNDAMPED MAGNONS

Nonlinear sigma models (NL$\sigma$Ms) provide a convenient description of the long-wavelength and low-frequency properties of the ordered phase of systems with a spontaneously broken symmetry. They are effective field theories that focus on the Goldstone modes and integrate out all massive fluctuations in the simplest approximation that respects the symmetry. In particular, the classical $O(3)$-symmetric nonlinear sigma model[15] provides a very easy way to demonstrate the divergence of $\chi_{\rm L}$ in a classical Heisenberg ferromagnet, Eq. (1.1). It thus is natural to consider quantum NL$\sigma$Ms to study the corresponding effect in quantum magnets. As we will see, the results are very different for the two types of magnetic order.

### A. Quantum ferromagnets

We consider a quantum ferromagnet with a fluctuating magnetization $\boldsymbol{M}(x) = M_0(x)\,\hat{\boldsymbol{m}}(x)$. Here and in what follows $x = (\boldsymbol{x},\tau)$ comprises the real-space position $\boldsymbol{x}$ and the imaginary-time variable $\tau$. $M_0$ is the magnitude of the order parameter, and

$$\hat{\boldsymbol{m}}(x) = (\pi_1(x), \pi_2(x), \sigma(x)) \tag{2.1a}$$

with

$$\hat{\boldsymbol{m}}^2(x) = \pi_1^2(x) + \pi_2^2(x) + \sigma^2(x) \equiv 1 \tag{2.1b}$$

is a unit vector. In a NL$\sigma$M description of a quantum ferromagnet fluctuations of $M_0$ are neglected, $M_0(x) \equiv M_0$, and the partition function can be written[16,17]

$$Z = \int \mathcal{D}[\hat{\boldsymbol{m}}]\ \delta(\hat{\boldsymbol{m}}^2(x) - 1)\ e^{-\int dx\,\mathcal{L}_{\rm FM}[\hat{\boldsymbol{m}}]} \tag{2.2a}$$

Here $\int dx = \int_0^{1/T} d\tau \int_V d\boldsymbol{x}$, with $T$ the temperature and $V$ the system volume, and

$$\begin{aligned}\mathcal{L}_{\rm FM}[\hat{\boldsymbol{m}}] &= -\frac{\rho_s}{2}\,\hat{\boldsymbol{m}}(x)\cdot\boldsymbol{\nabla}^2\hat{\boldsymbol{m}}(x) - M_0\,\mu\boldsymbol{H}\cdot\hat{\boldsymbol{m}}(x)\\ &\quad+\frac{iM_0}{1+\sigma(x)}\big(\pi_1(x)\partial_\tau\pi_2(x) - \pi_2(x)\partial_\tau\pi_1(x)\big)\ .\end{aligned} \tag{2.2b}$$

Here $\rho_s$ is the spin-stiffness coefficient, which is proportional to $M_0^2$, $\boldsymbol{H}$ is an external magnetic field, and $\mu$ is the gyromagnetic ratio. The first two terms on the right-hand side of Eq. (2.2b) are the same as in a classical $O(3)$ NL$\sigma$M.[15] The third term is the Wess-Zumino or Berry-phase term that describes the quantum dynamics.[18] Physically, it describes the Bloch spin precession. The form given in Eq. (2.2b) assumes that the ferromagnet order is along the $z$-direction.

We now expand the action in powers of the fields $\pi_1$ and $\pi_2$. The Gaussian action that governs the transverse fluctuations then reads

$$\mathcal{A}^{(2)}[\pi_1,\pi_2] = \frac{M_0}{2}\sum_k \sum_{i,j=1}^{2} \pi_i(k)\,\Gamma_{ij}(k)\,\pi_j(-k)\ , \tag{2.3a}$$

where $\Gamma_{ij}$ denotes the matrix elements of a $2\times 2$ matrix

$$\Gamma(k) = \begin{pmatrix} D\,\boldsymbol{k}^2 + \mu H & -\Omega_n \\ \Omega_n & D\,\boldsymbol{k}^2 + \mu H\end{pmatrix} \tag{2.3b}$$

where $D = \rho_s/M_0$. Here we have performed a Fourier transform from $x = (\boldsymbol{x},\tau)$ to $k = (\boldsymbol{k}, i\Omega_n)$, with $\boldsymbol{k}$ a wave vector and $\Omega_n = 2\pi T n$ ($n$ integer) a bosonic Matsubara frequency, and we have taken the external field to point in the $z$-direction, $\boldsymbol{H} = (0,0,H)$. The inverse of $\Gamma$ yields the Gaussian transverse susceptibility matrix, i.e., the correlation function

$$M_0^2\langle\pi_i(k)\,\pi_j(-k)\rangle = \chi_{\rm T}^{ij}(k)\ , \tag{2.4a}$$

where

$$\chi_{\rm T}(k) = \frac{M_0}{(D\boldsymbol{k}^2+\mu H)^2 + \Omega_n^2}\begin{pmatrix} D\boldsymbol{k}^2 + \mu H & \Omega_n \\ -\Omega_n & D\boldsymbol{k}^2 + \mu H\end{pmatrix} \tag{2.4b}$$

The non-hermitian nature of the matrix $\Gamma$, with the frequency coupling the magnetization components $M_x$ and $M_y$, reflects the structure of the Bloch spin-precession term in Eq. (2.2b). It shows the quadratic dispersion relation of the ferromagnetic magnons, $i\Omega_n = \pm D\boldsymbol{k}^2$. The spin-wave stiffness coefficient $D$ (not to be confused with a diffusion coefficient) is linear in $M_0$ (since $\rho_s \propto M_0^2$). It is illustrative to diagonalize the Gaussian transverse action. The eigenvalues of $\Gamma(k)$ are $\lambda_\pm(k)$ with

$$\lambda_\pm(k) = \lambda_\mp(-k) = D\boldsymbol{k}^2 + \mu H \mp i\Omega_n\ , \tag{2.5a}$$

and the left and right eigenvectors are

$$\begin{aligned}(u,v)_L &= (1,\mp i)\ ,\\ (u,v)_R &= (1,\pm i)\ .\end{aligned} \tag{2.5b}$$

The Gaussian action can thus be written in terms of fields $\psi_L = (\psi_{L,+},\psi_{L,-})$ and $\psi_R = (\psi_{R,+},\psi_{R,-})$,

$$\mathcal{A}^{(2)}[\psi_L,\psi_R] = \frac{M_0}{2}\sum_k\sum_{\sigma=\pm}\psi_{{\rm L},\sigma}(k)\,\lambda_\sigma(k)\,\psi_{{\rm R},\sigma}(-k) \tag{2.6}$$

In terms of the $\psi_L$ and $\psi_R$ we have

$$\begin{aligned}\pi_1 &= \frac{1}{\sqrt{2}}\left(\psi_{L,+} - i\psi_{L,-}\right) = \frac{1}{\sqrt{2}}\left(\psi_{R,+} + i\psi_{R,-}\right)\ ,\\ \pi_2 &= \frac{1}{\sqrt{2}}\left(-i\psi_{L,+} + \psi_{L,-}\right) = \frac{1}{\sqrt{2}}\left(i\psi_{R,+} + \psi_{R,-}\right)\ .\end{aligned} \tag{2.7}$$

Note that the four fields $\psi_{\mathrm{L},\sigma}$, $\psi_{\mathrm{R},\sigma}$ are not independent; Eqs. (2.7) yield two constraints,

$$\psi_{\mathrm{L},+} = i\psi_{\mathrm{R},-} \quad , \quad \psi_{\mathrm{L},-} = i\psi_{\mathrm{R},+} \ , \tag{2.8}$$

which restore the original number of degrees of freedom. From Eq. (2.6) we obtain the Goldstone mode[19]

$$g_{\pm}(k) = \langle \psi_{L,\pm}(k) \psi_{R,\pm}(-k) \rangle = 1/M_0 \, \lambda_{\pm}(k) \tag{2.9a}$$

which is massless in the absence of the symmetry-breaking field $H$. From Eq. (2.8) we obtain two additional nonzero correlation functions,

$$\begin{aligned} \langle \psi_{\mathrm{L}+}(k) \, \psi_{\mathrm{L},-}(-k) \rangle &= i/M_0 \, \lambda_+(k) \ , \\ \langle \psi_{\mathrm{R}+}(k) \, \psi_{\mathrm{R},-}(-k) \rangle &= -i/M_0 \, \lambda_+(-k) \ , \end{aligned} \tag{2.9b}$$

Now we consider the normalized longitudinal susceptibility $\chi_{\mathrm{L}}(x-y)/M_0^2 = \langle \delta\sigma(x) \delta\sigma(y) \rangle$ with $\delta\sigma(x) = \sigma(x) - \langle \sigma(x) \rangle$. Using the nonlinear constraint, Eq. (2.1b), we expand

$$\langle \delta\sigma(x)\delta\sigma(y) \rangle = \frac{1}{4} \langle (\pi_1^2(x) + \pi_2^2(x))(\pi_1^2(y) + \pi_2^2(y)) \rangle - \frac{1}{4} \langle \pi_1^2(x) + \pi_2^2(x) \rangle^2 + \ldots \tag{2.10a}$$

In terms of $\psi_L$ and $\psi_R$ this can be written

$$\langle \delta\sigma(x)\delta\sigma(y) \rangle = \frac{1}{4} \sum_{\sigma,\sigma'} \Big[ \langle \psi_{L,\sigma}(x) \, \psi_{R,\sigma}(x) \, \psi_{L,\sigma'}(y) \, \psi_{R,\sigma'}(y) \rangle - \langle \psi_{L,\sigma}(x) \, \psi_{R,\sigma}(x) \rangle \langle \psi_{L,\sigma'}(y) \, \psi_{R,\sigma'}(y) \rangle \Big] \ . \tag{2.10b}$$

Using Wick's theorem and Eqs. (2.9) yields the one-loop contribution $\chi_{\mathrm{L}}^{(1)}$ to the longitudinal susceptibility,

$$\begin{aligned} \chi_{\mathrm{L}}^{(1)}(k) &= M_0^2 \, \frac{T}{2V} \sum_p \sum_\sigma g_\sigma(p) \, g_\sigma(p-k) \\ &= \frac{T}{2V} \sum_p \sum_\sigma \frac{1}{\lambda_\sigma(p) \lambda_\sigma(p-k)} \ . \end{aligned} \tag{2.11}$$

This is represented diagrammatically in Fig. 2.

### 1. *Absence of a Goldstone-mode-induced singularity in $\chi_L$ at $T = 0$*

At $T = 0$, where the frequency summation in Eq. (2.11) turns into an integral, it is obvious that this contribution vanishes,[20] in violation of the naive expectation expressed by the ferromagnetic analog of Eq. (1.2a). This null result is readily traced back to the structure of the Bloch spin precession term in the action, which leads to the eigenvalues $\lambda_\sigma(k)$ being odd functions of the frequency. Since the action couples only $\psi_{L,+}$ with $\psi_{R,+}$, and $\psi_{L,-}$ with $\psi_{R,-}$, this results in a final frequency integral where both poles lie on the same side of the real axis. Alternatively, one can easily see this in an operator formalism, see the discussion after Eq. (1.2a) in the Introduction. Adding a frequency dependence to the classical expression therefore has a much stronger effect than increasing the effective dimensionality by two, as the naive power-counting argument suggests, and at $T = 0$ it completely suppresses the effect. It is obvious from this discussion that the absence of a nonanalyticity in the quantum case is a generic property of ferromagnets at $T = 0$ and not an artifact of either the NL$\sigma$M or the one-loop approximation. We also note that the null result is specific to the 2-point correlation of $\sigma(x)$, see the following section.

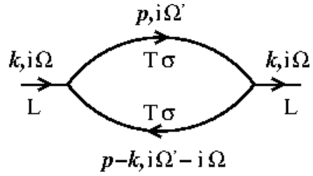


Figure 2: Diagrammatic representation of the coupling between longitudinal and transverse spin flucutations in the quantum case. Note that the two transverse propagators carry the same internal frequency. This leads to the null result discussed in the text.

### 2. *A singular correlation function at $T = 0$*

It is illustrative to discuss a correlation function other than $\chi_{\mathrm{L}}$. Consider, for instance,

$$\begin{aligned} \Psi(x-y) &= \frac{1}{4} \langle (\pi_1^2(x) - \pi_2^2(x))(\pi_1^2(y) - \pi_2^2(y)) \rangle \\ &= \frac{1}{4} \sum_{\sigma,\sigma'} \sigma\sigma' \langle \psi_{L,\sigma}(x) \, \psi_{L,\sigma}(x) \, \psi_{R,\sigma'}(y) \, \psi_{R,\sigma'}(y) \rangle. \end{aligned} \tag{2.12}$$

Note that this is a physical, if hard to measure, correlation function: It describes the response to a "field" $\Delta$ that renders the exchange coupling $J$ in a Heisenberg model anisotropic in the $x$-$y$-plane: $J_x = J + \Delta$, $J_y = J - \Delta$. Af-

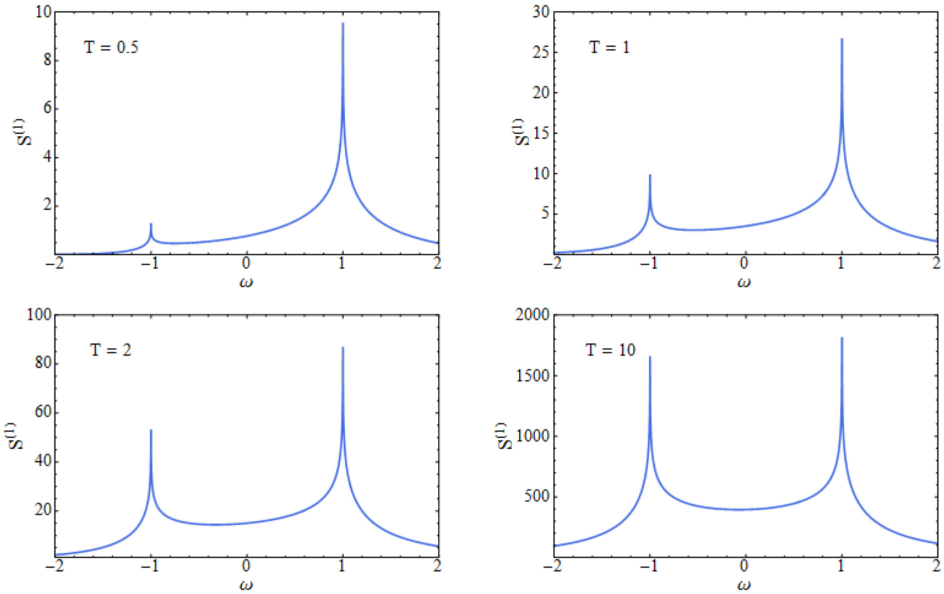


Figure 3: The one-loop contribution to the longitudinal part of the dynamical structure factor for a ferromagnet, Eq. (2.21), normalized by $\sqrt{\omega_{\boldsymbol{k}}}/4\pi D^{3/2}$, for $H=0$ as a function of the frequency $\omega$ for various values of the temperature $T$. $\omega$ and $T$ are measured in units of $\omega_{\boldsymbol{k}}$. On the scale shown, the result for $T/\omega_{\boldsymbol{k}}=10$ is almost indistinguishable from the classical result, Eq. (2.22).

ter a Fourier transform we obtain, instead of Eq. (2.11),

$$\Psi(k) = \frac{T}{2V}\sum_p\sum_\sigma \frac{1}{\lambda_\sigma(p)\lambda_\sigma(k-p)} \; . \tag{2.13}$$

At $T=0$, the frequency integral is now over a function that has poles on either side of the real axis, and the correlation function behaves as simple power counting would suggest, viz.

$$\begin{aligned}\Psi(\boldsymbol{k}, i\Omega_n = 0) &\propto \text{const.} + |\boldsymbol{k}|^{d-2} \; , \\ \Psi(\boldsymbol{k}=0, i\Omega_n) &\propto \text{const.} + |\Omega_n|^{(d-2)/2} \; ,\end{aligned} \tag{2.14}$$

with a logarithmic singularity in $d=2$. This is in complete analogy to Eq. (1.2a). This illustrates that the absence of a singular contribution to $\chi_{\rm L}$, and the related fact that the maximally spin-polarized state is an exact eigenstate of the Heisenberg ferromagnet, is not due to the absence of quantum fluctuations in the ground state, as is sometimes stated in the literature. Rather, it is due to the fact that $\chi_{\rm L}$ can be formulated as a correlation function of the magnon number. Quantum fluctuations do exist in the ground state of a ferromagnet, and they affect correlation functions, such as $\Psi$, that can not be formulated entirely in terms of fluctuations of the magnon number. The same holds for the longitudinal susceptibility in an antiferromagnet, see Sec. II B below. We will come back to this point in Sec. IV B 2.

### 3. *Singularities at $T>0$*

To find the behavior at nonzero temperature we perform the Matsubara frequency sum in Eq. (2.11). This yields

$$\chi_{\rm L}^{(1)}(k,H) = \frac{-1}{V}\sum_{\boldsymbol{p},\sigma}\frac{n(\omega_{\boldsymbol{p}}+\mu H) - n(\omega_{\boldsymbol{p}-\boldsymbol{k}}+\mu H)}{\omega_{\boldsymbol{p}} - \omega_{\boldsymbol{p}-\boldsymbol{k}} + \sigma i\Omega_n}, \tag{2.15}$$

where $n(x) = 1/(e^{x/T}-1)$ is the Bose distribution function (we use units such that $\hbar = k_{\rm B} = 1$), and $\omega_{\boldsymbol{p}} = D\boldsymbol{p}^2$ is the ferromagnetic magnon frequency. We are interested in infrared singularities that arise from the small-momentum behavior of the integrand in Eq. (2.15). Accordingly, to obtain the leading singular behavior as $\boldsymbol{k}\to 0$ for fixed $T$, we can expand the Bose function, $n(x)\approx T/x$.[21] At zero external frequency, $k=(\boldsymbol{k}, i0)$, and zero external field we have

$$\chi_{\rm L}^{(1)}(\boldsymbol{k}, H=0) \approx \left(\frac{M_0}{\rho_s}\right)^2 \frac{2T}{V}\sum_{\boldsymbol{p}}\frac{1}{\boldsymbol{p}^2(\boldsymbol{p}-\boldsymbol{k})^2} \; , \tag{2.16}$$

where we have used $D=\rho_s/M_0$. Note that this leading contribution is necessarily linear in $T$, and that the wavenumber integral is a convolution of two classical Goldstone modes, see Eq. (1.1). In $d=3$ we find ex-

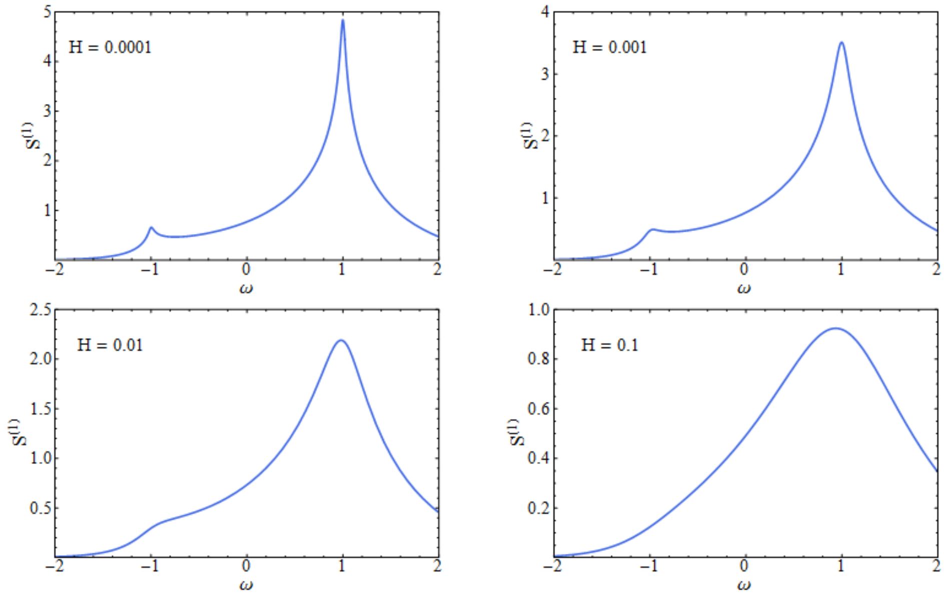


Figure 4: The one-loop contribution to the longitudinal part of the dynamical structure factor for a ferromagnet, Eq. (2.21), normalized as in Fig. 3, for $T/\omega_{\boldsymbol{k}} = 0.5$ as a function of the frequency $\omega$ for various values of the magnetic field $H$. $\omega$ and $H$ are measured in units of $\omega_{\boldsymbol{k}}$ and $\omega_{\boldsymbol{k}}/\mu$, respectively. Notice how even a very weak magnetic field broadens the resonance feature.

plicitly

$$\chi_{\rm L}^{(1)}(\boldsymbol{k}, H=0) = \frac{T}{4D^{3/2}\sqrt{\omega_{\boldsymbol{k}}}}\left[1 + O(\sqrt{\omega_{\boldsymbol{k}}/T})\right] \quad (d=3); \tag{2.17}$$

in generic dimensions $2 < d < 4$ the singularity is proportional to $T/|\boldsymbol{k}|^{4-d}$ with a $d$-dependent prefactor. For $d \leq 2$ the singular integral has a zero prefactor since $M_0 = 0$. This result is valid for $\mu H \ll \omega_{\boldsymbol{k}} \ll T$. The range of validity of Eq. (2.16) thus shrinks with decreasing temperature. In the asymptotic low-temperature limit in a vanishingly small field, i.e., for $\mu H \ll T \ll \omega_{\boldsymbol{k}}$, we find

$$\chi_{\rm L}^{(1)}(\boldsymbol{k}, H=0) = \frac{c_{\rm L}}{\pi^2}\,\frac{T^{3/2}}{D^{3/2}\omega_{\boldsymbol{k}}}\left[1 + O(T/\omega_{\boldsymbol{k}})\right] \quad (d=3), \tag{2.18}$$

where $c_{\rm L} = \sqrt{\pi/2}\,\zeta(3/2) \approx 2.395$, with $\zeta$ the Riemann zeta function. For $T < \omega_{\boldsymbol{k}}$ the $T/\sqrt{\omega_{\boldsymbol{k}}}$ singularity thus crosses over to $T^{3/2}/\omega_{\boldsymbol{k}}$, and for $T \to 0$ the prefactor of the singularity vanishes in agreement with Sec. II A 1.

For $\omega_{\boldsymbol{k}} \ll \mu H \ll T$ an analogous consideration yields

$$\chi_{\rm L}^{(1)}(\boldsymbol{k} \to 0, H) = \frac{T}{4\pi D^{3/2}(\mu H)^{1/2}}\left[1 + O(\sqrt{H/T})\right], \tag{2.19}$$

and for $T \ll \omega_{\boldsymbol{k}}, \mu H$ the leading behavior is

$$\chi_{\rm L}^{(1)}(\boldsymbol{k}, H) = \frac{1}{2\pi^{3/2}}\,\frac{T^{3/2}}{D^{3/2}\omega_{\boldsymbol{k}}}\,e^{-\mu H/T}. \tag{2.20}$$

Both of these results are for $d = 3$. Finally, for $\omega_{\boldsymbol{k}} \ll T \ll \mu H$ the result is proportional to $T^{1/2}e^{-\mu H/T}$ with no singular dependence on $\omega_{\boldsymbol{k}}$.

### 4. *The dynamical structure factor*

Also of interest is the longitudinal part of the dynamical structure factor $S_{\rm L}(\boldsymbol{k},\omega) = (2/(1-e^{-\omega/T}))\chi''_{\rm L}(\boldsymbol{k},\omega)$, with $\chi''_{\rm L}$ the spectrum of the susceptibility $\chi_{\rm L}$. From Eq. (2.15) we find for the one-loop contribution[3]

$$S_{\rm L}^{(1)}(\boldsymbol{k},\omega) = \frac{1}{1-e^{-\omega/T}}\,\frac{T}{4\pi D^{3/2}\sqrt{\omega_{\boldsymbol{k}}}} \times \ln\left(\frac{1-e^{-(\omega+\omega_{\boldsymbol{k}})^2/4T\omega_{\boldsymbol{k}}-\mu H/T}}{1-e^{-(\omega-\omega_{\boldsymbol{k}})^2/4T\omega_{\boldsymbol{k}}-\mu H/T}}\right). \tag{2.21}$$

The leading behavior for small $\boldsymbol{k}$, $\omega$, and $H$ for fixed $T$ is

$$S_{\rm L}^{(1)}(\boldsymbol{k},\omega) \approx \frac{T^2}{4\pi D^{3/2}\omega\sqrt{\omega_{\boldsymbol{k}}}} \times \ln\left(\frac{(\omega+\omega_{\boldsymbol{k}})^2/4T\omega_{\boldsymbol{k}}+\mu H/T}{(\omega-\omega_{\boldsymbol{k}})^2/4T\omega_{\boldsymbol{k}}+\mu H/T}\right). \tag{2.22}$$

As in the case of Eq. (2.16), this is also what one obtains in the classical limit, $\hbar \to 0$ (see also Ref. 21, and note that $\mu/\hbar$ is independent of $\hbar$).

The structure factor is shown in Fig. 3 for several values of $T/\omega_{\boldsymbol{k}}$. Notable features are as follows: (1) There is a logarithmic singularity at $\omega = \pm\omega_{\boldsymbol{k}}$. This leads to a broad feature, even for undamped magnons, whose width is independent of the normalized temperature. (2) There is a marked decrease in the overall value of $S_{\rm L}$ with decreasing temperature, and (3) $S_{\rm L}$ becomes strongly asymmetric at low temperature due to the detailed-balance factor. A nonzero magnetic field removes the logarithmic singularity, and even a rather small magnetic field substantially broadens the resonance feature, see Fig. 4. We will further discuss the dynamical structure factor in Sec. IV.

We also note that the minus first frequency moment of $\chi''_{\rm L}$ yields the static susceptibility: $\chi_{\rm L}(\boldsymbol{k}) = \int_{-\infty}^{\infty} d\omega\, \chi''_{\rm L}(\boldsymbol{k},\omega)/\pi\omega$. Performing the frequency integral we recover the results given in Eqs. (2.17) - (2.20).

## B. Quantum antiferromagnets

We now consider quantum antiferromagnets, whose spin dynamics are very different from their ferromagnetic counterparts. The NL$\sigma$M for an antiferromagnet can be written[16,17]

$$Z = \int \mathcal{D}[\hat{\boldsymbol{n}}]\, \delta(\hat{\boldsymbol{n}}^2(x) - 1)\; e^{-\int dx\, \mathcal{L}_{\rm AFM}[\hat{\boldsymbol{n}}]} \tag{2.23a}$$

with an action density

$$\mathcal{L}_{\rm AFM}[\hat{\boldsymbol{n}}] \;=\; \frac{\rho_s}{2}\Big[-\hat{\boldsymbol{n}}(x)\cdot\boldsymbol{\nabla}^2\hat{\boldsymbol{n}}(x) + \frac{1}{c^2}\big(\partial_\tau\hat{\boldsymbol{n}}(x) - i\mu\boldsymbol{H}\times\hat{\boldsymbol{n}}(x)\big)^2\Big]\ . \tag{2.23b}$$

Here $\hat{\boldsymbol{n}}(x)$ is the normalized staggered magnetization. It obeys

$$\hat{\boldsymbol{n}}^2(x) \equiv 1 \tag{2.24a}$$

and we parameterize it as

$$\hat{\boldsymbol{n}}(x) = (\pi_1(x), \pi_2(x), \sigma(x)) \tag{2.24b}$$

in analogy to the ferromagnetic case. The physical staggered magnetization is $\boldsymbol{N}(x) = N_0\,\hat{\boldsymbol{n}}(x)$ with an amplitude $N_0$. $\rho_s$ is the spin stiffness, $c$ is the spin-wave velocity and $\boldsymbol{H}$ is a homogeneous external magnetic field. Notice that in the absence of an external field the dynamics are given by a $(\partial_\tau\hat{\boldsymbol{n}})^2$ term, in contrast to the linear dependence on $\partial_\tau$ in the ferromagnetic case, Eq. (2.2b).[22] Putting the external field equal to zero, and proceeding as in the ferromagnetic case, we obtain a transverse Gaussian fluctuation action that is diagonal in the $\pi_1$-$\pi_2$ basis:

$$\mathcal{A}^{(2)}[\pi_1,\pi_2] = \frac{\rho_s}{2c^2}\sum_k\sum_{i=1}^{2}\pi_i(k)\,\mu(k)\,\pi_i(-k)\ , \tag{2.25a}$$

with an eigenvalue

$$\mu(k) = \omega_{\boldsymbol{k}}^2 - (i\Omega_n)^2\ . \tag{2.25b}$$

Here $\omega_{\boldsymbol{k}} = c|\boldsymbol{k}|$ is the antiferromagnetic magnon frequency. The one-loop contribution to the longitudinal susceptibility $\chi_{\rm L}(x-y) = N_0^2\langle\delta\sigma(x)\,\delta\sigma(y)\rangle$ now has the form

$$\chi_{\rm L}^{(1)}(k) = \left(\frac{N_0\,c^2}{\rho_s}\right)^2 \frac{T}{V}\sum_p \frac{1}{\mu(p)\mu(p-k)}\ . \tag{2.26}$$

Notice that this is the longitudinal order-parameter susceptibility, which describes the response to a staggered magnetic field, rather than to a homogeneous one.

### 1. The Goldstone-mode-induced singularity at $T = 0$

The one-loop contribution to the longitudinal susceptibility given by Eq. (2.26) is still represented by the diagram shown in Fig. 2, but now the frequency integration at $T = 0$ involves poles on either side of the real axis. The frequency integral thus does not vanish, and we obtain

$$\chi_{\rm L}^{(1)}(\boldsymbol{k}, i\Omega_n = 0) \;=\; \left(\frac{N_0\,c^2}{\rho_s}\right)^2 \frac{1}{2V}\sum_{\boldsymbol{p}}\frac{1}{\omega_{\boldsymbol{p}+\boldsymbol{k}/2}\,\omega_{\boldsymbol{p}-\boldsymbol{k}/2}} \times\frac{1}{\omega_{\boldsymbol{p}+\boldsymbol{k}/2}+\omega_{\boldsymbol{p}-\boldsymbol{k}/2}}\ , \tag{2.27a}$$

$$\chi_{\rm L}^{(1)}(\boldsymbol{k}=0, i\Omega_n) \;=\; \left(\frac{N_0\,c^2}{\rho_s}\right)^2 \frac{1}{V}\sum_{\boldsymbol{p}}\frac{1}{\omega_{\boldsymbol{p}}}\,\frac{1}{4\omega_{\boldsymbol{p}}^2+\Omega_n^2}\ . \tag{2.27b}$$

This yields the result expected from naive power counting, Eq. (1.2a):

$$\begin{aligned}\chi_{\rm L}^{(1)}(\boldsymbol{k}, i\Omega_n = 0) \;&\propto\; |\boldsymbol{k}|^{d-3}\ ,\\ \chi_{\rm L}^{(1)}(\boldsymbol{k}=0, i\Omega_n) \;&\propto\; |\Omega_n|^{d-3}\end{aligned} \tag{2.28}$$

for $1 < d < 3$. In time space the latter result corresponds to a $1/t^{d-2}$ long-time tail, see Appendix B 1. The above derivation makes it clear that the striking difference between the behavior of this correlation function for ferromagnets and antiferromagnets, respectively, is a direct consequence of the different spin dynamics in the two cases.

In $d = 3$ the divergence is logarithmic. Calculating the prefactor we obtain, keeping only the leading terms,

$$\chi_{\rm L}^{(1)}(\boldsymbol{k}, i0) \;=\; \frac{N_0^2 c}{8\pi^2\rho_s^2}\,\log(\omega_0/\omega_{\boldsymbol{k}})\ , \tag{2.29a}$$

$$\chi_{\rm L}^{(1)}(\boldsymbol{k}=0, i\Omega_n) \;=\; \frac{N_0^2 c}{8\pi^2\rho_s^2}\,\log(\omega_0/|\Omega_n|)\ , \tag{2.29b}$$

$$\chi_{\rm L}^{(1)}(\boldsymbol{k}=0, i\Omega_n \;\to\; \Omega + i0) = \frac{N_0^2 c}{8\pi^2\rho_s^2}\Big[\log(\omega_0/|\Omega|) + i\,\frac{\pi}{2}\,\mathrm{sgn}\,\Omega\Big]\ , \tag{2.29c}$$

where $\omega_0$ is an ultraviolet cutoff wave frequency. In $d=2$ the explicit result is

$$\chi_{\rm L}^{(1)}(\boldsymbol{k},i0) = \frac{N_0^2 c^2}{8\rho_s^2}\,\frac{1}{\omega_{\boldsymbol{k}}}\ . \tag{2.30a}$$

$$\chi_{\rm L}^{(1)}(\boldsymbol{k}=0,i\Omega_n) = \frac{N_0^2 c^2}{8\rho_s^2}\,\frac{1}{\vert\Omega_n\vert}\ , \tag{2.30b}$$

$$\chi_{\rm L}^{(1)}(\boldsymbol{k}=0,i\Omega_n\to\omega+i0) = \frac{N_0^2 c^2}{8\rho_s^2}\left[\frac{i}{\omega}+\pi\delta(\omega)\right]\ . \tag{2.30c}$$

Note that in time space Eq. (2.30b) implies a correlation function that does not decay for long times, but rather is constant, see Appendix B 3. We will get back to this in Sec. IV.

#### 2. Singularities at $T>0$

We now demonstrate that at a nonzero temperature we obtain the same result as in the ferromagnetic case. Performing the frequency summation in Eq. (2.26) we obtain

$$\chi_{\rm L}^{(1)}(k) = \left(\frac{N_0\,c^2}{\rho_s}\right)^2 \frac{-1}{2V}\sum_{\boldsymbol{p},\sigma}\frac{1}{\omega_{\boldsymbol{p}}}\,\frac{n(\omega_{\boldsymbol{p}})-n(-\omega_{\boldsymbol{p}})}{(\omega_{\boldsymbol{p}}+\sigma i\Omega_n)^2-\omega_{\boldsymbol{p}+\boldsymbol{k}}^2}\ . \tag{2.31}$$

The leading infrared behavior again comes from the small-momentum behavior of the integrand, so we approximate $n(x)\approx T/x$. The resulting expression at zero external frequency can be rewritten to yield

$$\chi_{\rm L}^{(1)}(\boldsymbol{k},i0) \approx \left(\frac{N_0}{\rho_s}\right)^2 \frac{T}{V}\sum_{\boldsymbol{p}}\frac{1}{\boldsymbol{p}^2(\boldsymbol{p}-\boldsymbol{k})^2}\ . \tag{2.32}$$

As in the ferromagnetic case, Eq. (2.16), this indeed reproduces Eq. (1.1). In $d=3$ we have explicitly

$$\chi_{\rm L}^{(1)}(\boldsymbol{k},i0) = \frac{N_0^2 c}{8\rho_s^2}\,\frac{T}{\omega_{\boldsymbol{k}}}\,\left[1+O\big((\omega_{\boldsymbol{k}}/T)\log(\omega_0/\omega_{\boldsymbol{k}})\big)\right] \quad (d=3)\ , \tag{2.33}$$

which is valid for $\omega_{\boldsymbol{k}}\ll T\ll\omega_0$.

Upon taking the $T\to 0$ limit in Eq. (2.31), when $n(\omega_{\boldsymbol{p}})-n(-\omega_{\boldsymbol{p}})\to 1$, we correctly recover the integrals given in Eqs. (2.27). In particular, Eq. (2.33) crosses over to Eq. (2.29a), which is valid for $T\ll\omega_{\boldsymbol{k}}$.

#### 3. The dynamical structure factor

Calculating the spectrum of the susceptibility from Eq. (2.31) we obtain the one-loop contribution to the longitudinal part of the dynamical structure factor. In $d=3$ we find

$$S_{\rm L}^{(1)}(\boldsymbol{k},\omega) = \frac{N_0^2 c}{4\pi\rho_s^2}\,\frac{T/\omega_{\boldsymbol{k}}}{1-e^{-\omega/T}}\,\ln\!\left(\frac{\sinh(\vert\omega_{\boldsymbol{k}}+\omega\vert/4T)}{\sinh(\vert\omega_{\boldsymbol{k}}-\omega\vert/4T)}\right)\ . \tag{2.34}$$

It is illustrative to rewrite this as

$$S_{\rm L}^{(1)}(\boldsymbol{k},\omega) = \frac{N_0^2 c}{16\pi\rho_s^2}\,\frac{1}{1-e^{-\omega/T}}\left[\left\vert 1+\frac{\omega}{\omega_{\boldsymbol{k}}}\right\vert - \left\vert 1-\frac{\omega}{\omega_{\boldsymbol{k}}}\right\vert + \frac{4T}{\omega_{\boldsymbol{k}}}\ln\left(\frac{1-e^{-\vert\omega_{\boldsymbol{k}}+\omega\vert/2T}}{1-e^{-\vert\omega_{\boldsymbol{k}}-\omega\vert/2T}}\right)\right]\ . \tag{2.35}$$

This separates $S_{\rm L}^{(1)}$ into a contribution that survives the $T\to 0$ limit, and another one that is qualitatively very similar to the corresponding result in the ferromagnetic case, see Eq. (2.21). The former represents the quantum fluctuations that are responsible for the singular behavior of $\delta\chi_{\rm L}^{(1)}(\boldsymbol{k})$ at $T=0$, and the latter has again has a logarithmic singularity at the magnon resonance frequency $\omega=\omega_{\boldsymbol{k}}$. Note that the zero-temperature contribution does not fall off as $\omega\to\infty$, but is constant. This statement is equivalent to the logarithmic divergence in the static susceptibility: Calculating the minus first frequency moment of the spectrum $\chi''_{\rm L}(\boldsymbol{k},\omega) = (1-e^{-\omega/T})S_{\rm L}(\boldsymbol{k},\omega)/2$ in the limit $T\to 0$ we recover Eq. (2.29a). The difference between the antiferromagnetic and ferromagnetic cases becomes pronounced for temperatures $T\ll\omega_{\boldsymbol{k}}$; Fig. 5 shows the respective results for $T/\omega_{\boldsymbol{k}}=0.05$.

In the classical limit we have

$$S_{\rm L}^{(1)}(\boldsymbol{k},\omega) = \frac{N_0^2 c}{4\pi\rho_s^2}\,\frac{T^2}{\omega\,\omega_{\boldsymbol{k}}}\,\ln\!\left(\frac{\omega_{\boldsymbol{k}}+\omega}{\omega_{\boldsymbol{k}}-\omega}\right)^2\ , \tag{2.36}$$

which is analogous to Eq. (2.22).

In $d=2$ at $T=0$ the result is

$$S_{\rm L}^{(1)}(\boldsymbol{k},\omega) = \frac{N_0^2 c^2}{4\rho_s^2}\,\Theta(\omega^2-\omega_{\boldsymbol{k}}^2)\,\frac{\Theta(\omega)}{\sqrt{\omega^2-\omega_{\boldsymbol{k}}^2}}\ , \tag{2.37}$$

and calculating the minus first frequency moment recovers Eq. (2.30a). For $T>0$ there is no long-range order in $d=2$.

#### 4. Quantum antiferromagnets in an external magnetic field

So far we have considered the case of a vanishing external magnetic field. We now briefly discuss the effects of keeping the field $\boldsymbol{H}$ in the action density, Eq. (2.23b). The $(\boldsymbol{H}\times\hat{\boldsymbol{n}})^2$ term in the action implies that in the ground state the order-parameter vector $\boldsymbol{n}$ is perpendicular to $\boldsymbol{H}$. Let $\boldsymbol{H}$ point in the $x$-direction, $\boldsymbol{H}=(H,0,0)$, and we parameterize $\boldsymbol{n}$ as before in Eq. (2.24b). The we find a Gaussian action

$$\mathcal{A}^{(2)}[\pi_1,\pi_2] = \frac{\rho_s}{2c^2}\sum_k\sum_{i=1}^{2}\pi_i(k)\,\mu_i(k)\,\pi_i(-k)\ , \tag{2.38a}$$

where

$$\mu_1(k) = \mu(k)+(\mu H)^2 \quad , \quad \mu_2(k)=\mu(k)\ , \tag{2.38b}$$

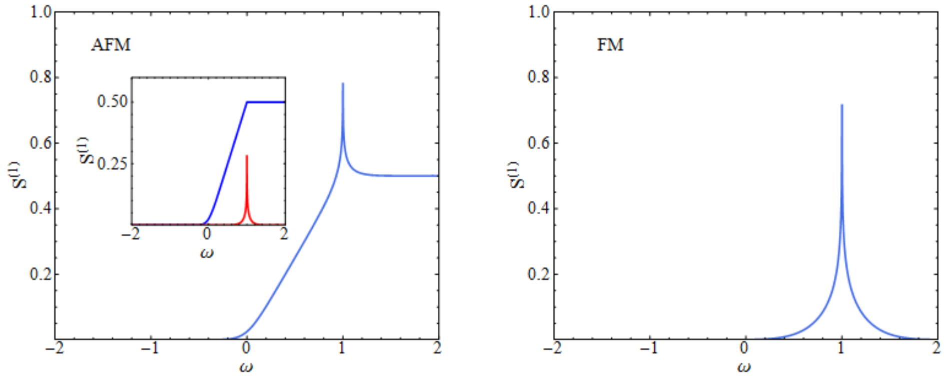


Figure 5: The one-loop contribution to the longitudinal part of the dynamical structure factor for an antiferromagnet (left panel) and a ferromagnet (right panel), normalized by $N_0^2 c/4\pi\rho_s^2$ and as in Figs. 4 and 3, respectively, for $T/\omega_{\boldsymbol{k}} = 0.05$ as functions of the frequency $\omega$ measured in units of $\omega_{\boldsymbol{k}}$. The inset in the left panel separately shows the $T=0$ contribution to the antiferromagnetic structure factor (blue curve) and the contribution that vanishes as $T \to 0$ (red curve). The structure factor shown in the main panel is the sum of these two, see Eq. (2.36).

with $\mu(k)$ from Eq. (2.27b). Of the two Goldstone modes, one is thus unchanged, whereas the other one acquires a mass. Equation (2.26) thus gets generalized to

$$\chi_{\rm L}^{(1)}(k) = \left(\frac{N_0\, c^2}{\rho_s}\right)^2 \frac{T}{2V} \sum_p \sum_i \frac{1}{\mu_i(p)\mu_i(p-k)}\ , \quad (2.39)$$

and there is a singularity for $\boldsymbol{k} \to 0$ even for $H \neq 0$. At $T=0$ in $d=3$ we find, to leading logarithmic accuracy,

$$\chi_{\rm L}^{(1)}(\boldsymbol{k}, i0) = \frac{N_0^2 c}{16\pi^2\rho_s^2}\left[\log\left(\frac{\omega_0}{\omega_{\boldsymbol{k}}}\right) + \log\left(\frac{\omega_0}{\sqrt{\omega_{\boldsymbol{k}}^2 + (2\mu H)^2}}\right)\right]. \quad (2.40)$$

The corresponding result in $d=2$ is

$$\chi_{\rm L}^{(1)}(\boldsymbol{k}, i0) = \frac{N_0^2 c^2}{16\rho_s^2}\frac{1}{\omega_{\boldsymbol{k}}}\left[1 + \frac{2}{\pi^2}\, g(\omega_{\boldsymbol{k}}/\sqrt{\omega_{\boldsymbol{k}}^2 + (2\mu H)^2})\right]. \quad (2.41a)$$

where

$$g(x) = \int_{-1}^{1} d\eta\, \frac{\ln(1+x\eta)}{\eta\sqrt{1-\eta^2}} = \begin{cases} \pi^2/2 & \text{for } x = 1 \\ \pi x & \text{for } x \to 0 \end{cases}. \quad (2.41b)$$

For $\mu H \ll \omega_{\boldsymbol{k}}$ we recover Eq.(2.30a); for $\mu H \gg \omega_{\boldsymbol{k}}$ we have $\chi_{\rm L}^{(1)}(\boldsymbol{k}, i0) \propto 1/H$. Corresponding results are obtained for $\chi_{\rm L}^{(1)}$ as a function of the frequency.

## III. EFFECTS OF DAMPED FERROMAGNETIC MAGNONS

So far we have ignored the effects of damping on the magnons. In this section we will consider the effects of magnon damping on the longitudinal susceptibility and the longitudinal dynamical structure factor in ferromagnets. We restrict ourselves to the ferromagnetic case, where magnon damping has a qualitative effect.

### A. Time-dependent Ginzburg-Landau theory

We need to determine the effects of damping on the ferromagnetic Goldstone mode, Eq. (2.9a). To this end we use the standard time-dependent Ginzburg-Landau theory for a ferromagnet[23–25]

$$\partial_t \boldsymbol{M}(\boldsymbol{x}, t) = \boldsymbol{M}(\boldsymbol{x}, t) \times \left.\frac{\delta S}{\delta \boldsymbol{M}(\boldsymbol{x})}\right|_{\boldsymbol{M}(\boldsymbol{x},t)} - \int d\boldsymbol{y}\, \Gamma(\boldsymbol{x}-\boldsymbol{y})\left.\frac{\delta S}{\delta \boldsymbol{M}(\boldsymbol{y})}\right|_{\boldsymbol{M}(\boldsymbol{y},t)}. \quad (3.1a)$$

Here $\Gamma(\boldsymbol{x})$ is the damping operator, which we will specify below, and $S$ is a suitable action for the static magnetization $\boldsymbol{M}(\boldsymbol{x})$. Very general considerations yield, to linear order in $\boldsymbol{M}$,

$$\delta S/\delta \boldsymbol{M}(\boldsymbol{x}) = -(\rho_s/M_0^2)\boldsymbol{\nabla}^2\boldsymbol{M}(\boldsymbol{x}) - \mu \boldsymbol{H}\ . \quad (3.1b)$$

Here we use the same notation as in Sec. II for the prefactor of the gradient-squared term.

We now use Eqs. (3.1) to calculate the linear response of the transverse magnetization components to the external field $\boldsymbol{H}$, i.e., the transverse magnetic susceptibility $\chi_T$. The result is Eq. (2.4b) with the substitution $\Omega_n \to \Omega_n + \Gamma_{\boldsymbol{k}}\, \boldsymbol{k}^2 \,\text{sgn}\,(\Omega_n)$, where $\Gamma_{\boldsymbol{k}}$ is the Fourier transform of $\Gamma(\boldsymbol{x})$. The one-loop contribution to the longitudinal susceptibility is still given by Eq. (2.11), but with

$\lambda_\pm$ replaced by

$$\lambda_\pm(\boldsymbol{k}, i\Omega_n) = D\boldsymbol{k}^2 + \mu H \mp i\Omega_n \mp i\Gamma_{\boldsymbol{k}}\, \boldsymbol{k}^2\, \mathrm{sgn}\,(\Omega_n)\ . \quad (3.2)$$

The sgn $(\Omega_n)$ in the damping term follows from causality requirements. In the absence of damping, $\Gamma_{\boldsymbol{k}} \equiv 0$, we recover the expressions given in Sec. II A.

We expand the damping coefficient in the long-wavelength limit as

$$\Gamma_{\boldsymbol{k}\to 0} = \gamma_0 + \gamma_2\, \boldsymbol{k}^2 \quad (3.3)$$

and distinguish between two physically distinct cases:[25] (1) A non-conserved order parameter, in which case $\gamma_0 > 0$, and (2) a conserved order parameter, in which case $\gamma_0 = 0$. The former case is realized, e.g., by magnetic impurities;[26,27] the latter, by, e.g., damping by electron-magnon and/or magnon-magnon interactions at $T > 0$ [28] or by nonmagnetic quenched disorder at any temperature, including $T = 0$.[27,29,30]

### B. Non-Conserved order parameter

We now perform the integral in Eq. (2.11) with $\lambda_\pm$ given by Eq. (3.2). For a non-conserved order parameter, $\Gamma_{\boldsymbol{p}} = \gamma_0$, and again keeping only the leading terms, we find for $d = 3$

$$\chi_{\mathrm{L}}^{(1)}(\boldsymbol{k}\to 0, i0) \;=\; \text{const.} - \frac{1}{32\pi}\,\frac{\gamma_0/\sqrt{D}}{\gamma_0^2 + D^2}\,\sqrt{\omega_{\boldsymbol{k}}}\ , \quad (3.4a)$$

$$\chi_{\mathrm{L}}^{(1)}(\boldsymbol{k} = 0, i\Omega_n) \;=\; -\frac{\gamma_0}{\pi^3 D^{5/2}}\, f(\gamma_0/D)\, |\Omega_n|^{1/2}\ , \quad (3.4b)$$

$$\chi_{\mathrm{L}}^{(1)}(\boldsymbol{k} = 0, i\Omega_n \to \omega + i0) \;=\; \text{const.} - \frac{\gamma_0}{\sqrt{2}\pi^3 D^{5/2}}\, f(\gamma_0/D)\, [1 - i\,\mathrm{sgn}\,(\omega)]\, |\omega|^{1/2}\ . \quad (3.4c)$$

The function $f$ can be expressed in terms of elementary functions; however, both the derivation and the result are lengthy, see Appendix A. Here we give only the power-series expansion for small damping, which reads

$$f(x \to 0) = \frac{\sqrt{2}\pi}{5} + \frac{3\pi}{7\sqrt{2}}\, x + O(x^2)\ . \quad (3.4d)$$

In $d = 2$ there is a logarithmic singularity,

$$\chi_{\mathrm{L}}^{(1)}(\boldsymbol{k}\to 0, i0) \;=\; \frac{1}{2\pi^2}\,\frac{\gamma_0}{\gamma_0^2 + D^2}\, \ln(\omega_0/\omega_{\boldsymbol{k}})\ , \quad (3.5a)$$

$$\chi_{\mathrm{L}}^{(1)}(\boldsymbol{k} = 0, i\Omega_n) \;=\; \frac{1}{2\pi^2}\,\frac{\gamma_0}{D^2 + \gamma_0^2}\, \ln(\omega_0/|\Omega_n|)\ , \quad (3.5b)$$

$$\chi_{\mathrm{L}}^{(1)}(\boldsymbol{k} = 0, i\Omega_n \to \omega + i0) \;=\; \frac{1}{2\pi^2}\,\frac{\gamma_0}{D^2 + \gamma_0^2}\left[\ln(\omega_0/|\omega|) + i\,\frac{\pi}{2}\,\mathrm{sgn}\,\omega\right]\ . \quad (3.5c)$$

In generic dimensions the nonanalytic contribution is proportional to $|\boldsymbol{k}|^{d-2}$ and $\Omega_n^{(d-2)/2}$, respectively. In time space the latter corresponds to a $1/t^{d/2}$ long-time tail, see Appendix B 1.

### C. Conserved order parameter

For a conserved order parameter, $\Gamma_{\boldsymbol{p}} = \gamma_2 \boldsymbol{p}^2$, the calculations are analogous but more involved and we give the results only to linear order in $\gamma_2$. For $d = 3$ we find

$$\chi_{\mathrm{L}}^{(1)}(\boldsymbol{k}\to 0, i0) \;=\; \text{const.} + O(\boldsymbol{k}^2) + \frac{\gamma_2\left[1 + O(\gamma_2^2)\right]}{64\pi D^{7/2}}\,\omega_{\boldsymbol{k}}^{3/2}, \quad (3.6a)$$

$$\chi_{\mathrm{L}}^{(1)}(\boldsymbol{k} = 0, i\Omega_n) = \text{const.} - \frac{\sqrt{2}\,\gamma_2\left[1 + O(\gamma_2^2)\right]}{7\pi^2 D^{7/2}}\,|\Omega_n|^{3/2}\ , \quad (3.6b)$$

$$\chi_{\mathrm{L}}^{(1)}(\boldsymbol{k} = 0, i\Omega_n \;\to\; \omega + i0) = \text{const.} + \frac{\gamma_2\left[1 + O(\gamma_2^2)\right]}{7\pi^2 D^{7/2}}\,[1 + i\,\mathrm{sgn}\,(\omega)]\,|\omega|^{3/2}\ . \quad (3.6c)$$

In $d = 2$ the leading singularity is

$$\chi_{\mathrm{L}}^{(1)}(\boldsymbol{k}\to 0, i0) = \text{const.} - \frac{\gamma_2\left[1 + O(\gamma_2^2)\right]}{48\pi^2 D^3}\,\omega_{\boldsymbol{k}}\ln(\omega_0/\omega_{\boldsymbol{k}})\ , \quad (3.7a)$$

$$\chi_{\mathrm{L}}^{(1)}(\boldsymbol{k} = 0, i\Omega_n) = \text{const.} - \frac{\gamma_2\left[1 + O(\gamma_2^2)\right]}{6\pi D^3}\,|\Omega_n|\ . \quad (3.7b)$$

$$\chi_{\mathrm{L}}^{(1)}(\boldsymbol{k} = 0, i\Omega_n \to \omega + i0) = \text{const.} + \frac{i\gamma_2\left[1 + O(\gamma_2^2)\right]}{12\pi D^3}\,\omega\ . \quad (3.7c)$$

In generic dimensions the nonanalytic contribution is proportional to $|\boldsymbol{k}|^d$ and $\Omega_n^{d/2}$, respectively. In time space this corresponds to a $1/t^{(d+2)/2}$ long-time tail, see Appendix B 1.

## IV. DISCUSSION

In this final section we give a summary of our results and conclude with a discussion of various physical points that underly them.

### A. Summary

In summary, we have investigated the coupling of magnons in quantum ferromagnets and antiferromagnets to other correlation functions, in particular the longitudinal susceptibility and the longitudinal part of the dynamical structure factor. In the case of ferromagnets with undamped magnons the longitudinal susceptibility vanishes at $T = 0$. In $d = 3$, and in the absence of an external magnetic field, an interpolating expression that correctly describes the leading behavior for both $T > \omega_{\boldsymbol{k}}$ and $T < \omega_{\boldsymbol{k}}$ is

$$\chi_{\mathrm{L}}^{(1)}(\boldsymbol{k}, H = 0) = \frac{T}{4D^{3/2}\sqrt{\omega_{\boldsymbol{k}}}}\,\frac{1}{1 + (\pi^2/c_{\mathrm{L}})\sqrt{\omega_{\boldsymbol{k}}/T}}\ , \quad (4.1)$$

where $\omega_{\boldsymbol{k}} = D\boldsymbol{k}^2$ is the ferromagnetic magnon frequency and $c_{\rm L}$ is the constant given after Eq. (2.18). For $T > \omega_{\boldsymbol{k}}$ one has the classical $1/|\boldsymbol{k}|$ singularity, Eq. (2.17), whereas for $T < \omega_{\boldsymbol{k}}$ $\chi_{\rm L}$ vanishes as $T^{3/2}$, Eq. (2.18). For a quantum antiferromagnet, the corresponding interpolating expression is

$$\chi_{\rm L}^{(1)}(\boldsymbol{k}, i0) = \frac{N_0^2 c}{8\rho_s^2}\,\frac{T}{\omega_{\boldsymbol{k}}}\,\left[1 + (\omega_{\boldsymbol{k}}/\pi^2 T)\log(\omega_0/\omega_{\boldsymbol{k}})\right] \ , \tag{4.2}$$

see Eqs. (2.29a) and (2.33). Here $\omega_{\boldsymbol{k}} = c|\boldsymbol{k}|$ is the antiferromagnetic magnon frequency. This reflects the expected scaling behavior, viz., $1/|\boldsymbol{k}|$ for high temperature, and $\ln|\boldsymbol{k}|$ for low temperature. Similarly, the longitudinal dynamical structure factor for a ferromagnet vanishes at $T = 0$, see Eq. (2.22) and Fig. 3, whereas in the antiferromagnetic case there is a nonvanishing contribution even at $T = 0$, see Eq. (2.35) and Fig. 5. Quenched disorder introduces additional fluctuations, leads to magnon damping, and qualitatively changes the ferromagnetic results. Magnetic impurities, which lead to a non-conserved magnetization, results in the longitudinal susceptibility scaling as $|\boldsymbol{k}|^{d-2}$, where the zero exponent in $d = 2$ signifies a logarithmic divergency, see Sec. III B. Non-magnetic disorder leads to a weaker scaling behavior, $|\boldsymbol{k}|^d$, see Sec. III C. For $T > 0$ the longitudinal dynamical structure factor has a logarithmic singularity at the magnon frequency in both ferromagnets and antiferromagnets.

## B. Discussion

We conclude with a discussion of various physical points raised by our results.

### 1. *Predictions for experiments*

*a) Longitudinal susceptibility and dynamical structure factor:* The classical singularity of $\chi_{\rm L}$ in the ferromagnetic case as a function of an external magnetic field, Eq. (2.19), has been observed by Kötzler et al.[31] The theoretical prediction is that in the limit of low temperatures, $T \ll \mu H$, $\chi_{\rm L}$ becomes exponentially small, see Eq. (2.20) and the paragraph following it.

A remarkable feature in the longitudinal dynamical structure factor is the logarithmic singularity at the magnon resonance frequency, see Eqs. (2.21) and (2.34), and Fig. 3. In a clean system at low temperature the magnon damping is very weak, and the magnon peaks in the transverse dynamical susceptibility are very narrow. The longitudinal susceptibility or structure factor, by contrast, shows an intrinsically broad feature at the magnon frequency. Even a rather small magnetic field substantially broadens and suppresses this feature, see Fig. 4.

*b) Other correlation functions:* We stress again that the behavior of the longitudinal susceptibility is not generic, but rather restricted to a class of correlation functions that can be expressed entirely in terms of magnon number fluctuations. Other correlation functions do show the expected $\omega^{(d-2)/2}$ frequency scaling, see the example in Sec. II A 3.

An example of a correlation function that belongs to the same class as the longitudinal susceptibility is the electrical conductivity in a metallic quantum ferromagnet; they both share the same scaling behavior. This implies that undamped magnons do *not* lead to an $\omega^{(d-2)/2}$ frequency dependence of the conductivity at $T = 0$, or a $\ln\omega$ singularity in $d = 2$. The latter conclusion was reached correctly in Ref. 10, but a sign error incorrectly led to the prediction of an $\omega^{(d-2)/2}$ nonanalyticity in $d > 2$. A corrected analysis of the conductivity in itinerant ferromagnets will be given elsewhere.[32]

### 2. *Comments on the results for ferromagnets*

*a) Fluctuations and entanglement entropy:* Let us come back to the issue of fluctuations in the ground state of a ferromagnet, see the remarks at the end of Sec. II A 3. A global measurement of fluctuations in a system is given by the entanglement entropy, defined as the von Neumann entropy of a subsystem of linear size $L$. At zero temperature the entropy vanishes in the thermodyamic limit, and for $L \to \infty$ it grows more slowly than the volume $L^d$. In systems that do not contain a Fermi surface the leading contribution is in general given by an "area-law" term that grows as $L^{d-1}$;[33] this term is due to short-range entanglement and has a non-universal prefactor. The leading universal contribution, which is a measure of long-range fluctuations, in systems with Goldstone modes grows as $\ln L$. This is true for both quantum ferromagnets[34,35] and antiferromagets[36–38] for $d = 2, 3$, although the area-law term is missing in the former.[35] This is another indication that fluctuations exist in the ferromagnetic ground state, although they may or may not be probed by a specific correlation function.

In metallic magnets, and more generally in systems with a Fermi surface, there is an area-law term with a multiplicative logarithm that is due to long-range fluctuations in the fermionic degrees of freedom. This is one of many indications of fundamental differences between metallic and insulating magnets. We briefly discuss some of these next.

*b) Spin models vs. itinerant magnets:* There are important differences between the fluctuations in quantum ferromagnets vs. antiferromagnets, the qualitatively similar universal parts of the entanglement entropies discussed above notwithstanding. For instance, a spin model for a Heisenberg ferromagnet, with Hamiltonian $H = J\sum_{<ij>} \boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_j$ with $J < 0$, has no quantum phase transition as a function of $J$, since the ground state is fully spin-polarized for any $J < 0$. In this sense the quantum fluctuations in a ferromagnet, while present, are weaker than those in a quantum antiferromagnet. This argument must survive nonmagnetic quenched disorder, which makes $J$ a random function of spatial position, as long as the distribution of $J$ is restricted to negative values, since the spins will still be locally maximally polarized. Since the physical reason for the absence of a nonanalyticity in $\chi_{\rm L}$ is the same as that for the absence of a quantum phase transition, it follows that nonmagnetic disorder with a such restricted distribution cannot lead to magnon damping; the damping coefficient $\gamma_2$ in Sec. III C must vanish at $T = 0$. These considerations raise interesting questions about the strength of quantum fluctuations, as well as ways to measure them, see, e.g., Refs. 36,39.

These aspects change qualitatively in a metallic ferromagnet, and in particular in an itinerant one: The coupling of the magnetic degrees of freedom to the fermionic ones leads to a large increase in fluctuations. As a result, the entanglement entropy has an area-law term multiplied by a logarithm, as is typical for systems with a Fermi surface, and as a function of the exchange coupling there is a quantum phase transition, as first described by Stoner.[40] It is also likely that the presence of nonmagnetic disorder leads to magnon damping irrespective of the shape of the disorder distribution. For a recent review of metallic ferromagnets, see Ref. 41. An explicit discussion of $\chi_{\rm L}$ and related correlation functions in a model of itinerant ferromagnets will be given elsewhere.[32]

*c) Effects of quenched disorder:* We now discuss the fact that quenched disorder, and the resulting damping of the magnons, leads to a nonanalyticity in $\chi_{\rm L}$, and demonstrate that the result is consistent with scaling and renormalization-group considerations and is indeed asymptotically exact as far as the exponent of the nonanalyticity is concerned.

First of all, we recall that the absence of a nonanalyticity for systems with undamped magnons is due to the absence of fluctuations that couple to the longitudinal magnetization fluctuations. Disorder introduces additional fluctuations, which makes it plausible that it will lead to a nonanalyticity. Furthermore, magnetic disorder, which couples directly to the order parameter, will have a stronger effect than nonmagnetic disorder, and thus result in a stronger singularity. The results in Secs. III B and III C thus are physically plausible.

In order to deduce the explicit results from general arguments, we consider the Gaussian action written in the form of Eq. (2.3) or (2.6), and add damping according to the prescription given above Eq. (3.2). In a schematic notation that shows only what is necessary for power counting the Gaussian action then takes the form

$$\mathcal{A}^{(2)} = \int dx\, \pi(x) \left[D\partial_{\boldsymbol{x}}^2 + \partial_\tau + H + \gamma_n\, \partial_{\boldsymbol{x}}^{n+2}\right] \pi(x)\ . \tag{4.3}$$

Here $n = 0$ and $n = 2$ correspond to the cases of a non-conserved and conserved order parameter, respectively. Additional terms in the action fall into two classes: (1) Gaussian with additional gradients, with the leading terms of the form

$$\delta\mathcal{A}^{(2)} = \int dx\, \partial_{\boldsymbol{x}}^4\, \pi^2(x)\ , \tag{4.4a}$$

or equivalent in terms of scale dimensions. (2) Of higher order in $\pi$, with the leading terms of the form

$$\delta\mathcal{A}^{(4)} = \int dx\, \partial_{\boldsymbol{x}}^2\, \pi^4(x) \tag{4.4b}$$

or equivalent.

We now sketch a renormalization-group analysis of this action. In doing so, we follow a scheme pioneered by Ma,[24] see also Refs. 42 and 6 for applications of this scheme in different contexts. We assign scale dimensions $[L] = -1$ and $[\tau] = -2$ to lengths and imaginary times, respectively. Then there is a stable Gaussian fixed point where $\pi$ has a scale dimension $[\pi(x)] = d/2$. In Fourier space this corresponds to $[\pi(k)] = -1$. We thus have $\langle\pi(k)\pi(-k)\rangle \sim 1/\boldsymbol{k}^2 \sim 1/\Omega_n$. This scaling behavior describes the magnons, see Eqs. (2.4), and the Gaussian fixed point describes the ordered phase where the symmetry is broken. The field $H$ is relevant with respect to this fixed point with a scale dimension $[H] = 2$. For a non-conserved order parameter the damping coefficient $\gamma_0$ is dimensionless, $[\gamma_0] = 0$, and the damping term is part of the fixed-point Hamiltonian. The free-energy density $f$, the magnetization $m$, and the scaling part $\delta\chi_{\rm L}$ of the longitudinal susceptibility $\chi_{\rm L} = \partial m/\partial H$ then have scale dimensions $[f] = d-2$, $[m] = d$, and $[\delta\chi_{\rm L}] = d-2$ respectively. For the latter this implies a homogeneity law

$$\delta\chi_{\rm L}(\boldsymbol{k}, i\Omega_n) = b^{2-d}\, F_\chi(\boldsymbol{k}b, i\Omega_n b^2, \gamma_0)\ , \tag{4.5}$$

with $b$ an arbitrary length rescaling factor and $F_\chi$ a scaling function. The latter has the property $F_\chi(x, y, \gamma_0 \to 0) = 0(\gamma_0)$, as we have discussed in the main part of this paper. We thus obtain the scaling behavior

$$\delta\chi_{\rm L}(\boldsymbol{k}, i\Omega_n) \sim \gamma_0 |\boldsymbol{k}|^{d-2} \sim \gamma_0 |\Omega_n|^{(d-2)/2}\ , \tag{4.6}$$

in agreement with Sec. III B. The leading correction terms to the fixed-point action are irrelevant by power counting, with scale dimensions $-2$ for the operator in Eq. (4.4a) and $-d$ for the one in Eq. (4.4b), respectively. These arguments show that the one-loop results obtained in Sec. III B are exact as far as the exponents are concerned; higher terms in the loop expansion will change the prefactor of the nonanalyticity, but not the power.

In the case of a conserved order parameter the damping term is not part of the fixed-point action; it is an irrelevant operator with a scale dimension $[\gamma_2] = -2$ which is the same as the least irrelevant operators represented by, e.g., Eq. (4.4a). The homogeneity equation for $\delta\chi_{\rm L}$ now reads

$$\delta\chi_{\rm L}(\boldsymbol{k}, i\Omega_n) = b^{2-d}\, F_\chi(\boldsymbol{k}b, i\Omega_n b^2, \gamma_2 b^{-2}) \ , \qquad (4.7a)$$

where we do not show the other irrelevant operators. Even though $\gamma_2$ is irrelevant, the scaling function still vanishes for $\gamma_2 = 0$, and we obtain, to linear order in $\gamma_2$,

$$\delta\chi_{\rm L}(\boldsymbol{k}, i\Omega_n) = b^{-d}\, \gamma_2\, \tilde{F}_\chi(\boldsymbol{k}b, i\Omega_n b^2) \ , \qquad (4.7b)$$

with $\tilde{F}$ another scaling function. This yields

$$\delta\chi_{\rm L}(\boldsymbol{k}, i\Omega_n) \sim \gamma_2 |\boldsymbol{k}|^d \sim \gamma_2 |\Omega_n|^{d/2} \ , \qquad (4.8)$$

in agreement with Sec. III C. Again, this is the exact leading scaling behavior.

*d) Magnon damping:* An interesting aspect of ferromagnetic magnons is that these excitations cannot be overdamped, irrespective of the magnitude of the damping coefficient. Consider Eq. (2.4b) with $\Omega_n \to \Omega_n + \Gamma_{\boldsymbol{k}}\, \boldsymbol{k}^2\, {\rm sgn}\,(\Omega_n)$. The poles of $\chi_{\rm T}(\boldsymbol{k}, z)$ with $z$ the complex frequency, always have a real part given by $\pm\omega_{\boldsymbol{k}}$, independent of $\Gamma_{\boldsymbol{k}}$. This is in contrast to a damped harmonic oscillator, where the resonance frequency has no real part if the damping coefficient is larger than a threshold value, and also to sound waves in fluids,[2] antiferromagnetic magnons, see Eqs. (2.25) with a damping coefficient added, and helimagnons in helical magnets,[43] all of which have the same structure as a damped harmonic oscillator.

### 3. *Comments on correlation functions that do not decay*

We finally discuss the physical meaning of the constant long-time behavior implied by Eq. (2.30b), see Eq. (B17). Let $T_{\rm max}$ be the maximum time scale, which can be, e.g., the total duration of the experiment, or $L$ divided by the relevant characteristic velocity. $\chi_{\rm L}$ then depends on two times, $t_1$ and $t_2$. As long as $t_1$, $t_2$, and $|t_1 - t_2|$ all are small compared to $T_{\rm max}$, $\chi_{\rm L}$ will not decay if $|t_1 - t_2|$ increases. In position space, by contrast, $\chi_{\rm L}$ does decay, but only as a power: The $1/|\boldsymbol{k}|$ divergence in the 2-$d$ quantum antiferromagnet, which is the same as the one in a 3-$d$ classical magnet, Eq. (1.1), implies that in real space the correlation function decays as $1/r$. For a general discussion of power-law decays of correlation functions, see, e.g., Ref. 7.

These results are examples of an effect that can be even stronger: In classical non-equilibrium fluids, and in Fermi liquids even in equilibrium, there are correlation functions that *increase* with increasing length or time scales in a well-defined sense, see Refs. 44–46.

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## Appendix A: Frequency dependence of $\chi_{\rm L}$ due to damped ferromagnetic magnons

Here we sketch the derivation of Eq. (3.4b) and give the full expression for the function $f$. Performing the frequency sum in Eq. (2.15) at $T = 0$, with $\lambda_\pm$ given by Eq. (3.2), we find

$$\begin{aligned}
\chi_{\rm L}(\boldsymbol{k}=0, i\Omega_n) &= \frac{2}{\pi V}\sum_{\boldsymbol{p}} \Gamma_{\boldsymbol{p}}\, \boldsymbol{p}^2 \int_{\omega_{\boldsymbol{p}}^2}^\infty dx\, \frac{1}{x + (\Gamma_{\boldsymbol{p}}\,\boldsymbol{p}^2)^2}\, \frac{1}{x + (\Gamma_{\boldsymbol{p}}\,\boldsymbol{p}^2 + \Omega_n)^2} \\
&= \frac{2}{\pi V}\sum_{\boldsymbol{p}} \Gamma_{\boldsymbol{p}}\, \boldsymbol{p}^2\, \frac{1}{\Omega_n^2 + 2\Omega_n \Gamma_{\boldsymbol{p}}\,\boldsymbol{p}^2} \ln\left(1 + \frac{\Omega_n^2 + 2\Omega_n\Gamma_{\boldsymbol{p}}\,\boldsymbol{p}^2}{\omega_{\boldsymbol{p}}^2 + (\Gamma_{\boldsymbol{p}}\,\boldsymbol{p}^2)^2}\right) \\
&= \frac{2}{\pi}\int_0^1 d\alpha\, \frac{1}{V}\sum_{\boldsymbol{p}} \frac{\Gamma_{\boldsymbol{p}}\,\boldsymbol{p}^2}{\alpha\Omega_n^2 + 2\alpha\Omega_n\Gamma_{\boldsymbol{p}}\,\boldsymbol{p}^2 + \omega_{\boldsymbol{p}}^2 + (\Gamma_{\boldsymbol{p}}\,\boldsymbol{p}^2)^2} \ ,
\end{aligned} \qquad (A1)$$

where in the last line we have expressed the logarithm in terms of an auxiliary integral. This procedure is also useful for deriving the prefactors of the nonanalytic wave-number dependence at $\Omega_n = 0$ that are given in Eqs. (3.4a) and (3.6a).

We now consider the case of a non-conserved order parameter, $\Gamma_{\boldsymbol{p}} = \gamma_0$. Splitting off the constant contribution at $\Omega_n = 0$ in $d = 3$, and scaling out the frequency, we obtain Eq. (3.4b) with the function $f$ given by

$$f(x) = \frac{1}{(1+x^2)^2}\int_0^1 d\alpha\, \alpha \int_0^\infty dy\, \frac{1 + 2xy^2}{y^4 + 2y^2\alpha x/(1+x^2) + \alpha/(1+x^2)} \ . \qquad (A2)$$

The integration over $y$ can now be easily performed, and the final integral over $\alpha$ can be expressed in terms of algebraic and inverse hyperbolic functions. We find

$$f(x) = \frac{\pi}{6\sqrt{2}}\,\frac{1}{x^{5/2}(1+x^2)^{3/2}}\left\{\left[3+7x^2-2x\sqrt{1+x^2}\right]\sqrt{x^2+x\sqrt{1+x^2}}-3(1+x^2)^{3/2}\sinh^{-1}(\sqrt{x}/(1+x^2)^{1/4})\right\}. \quad \text{(A3)}$$

An expansion for $x \to 0$ yields Eq. (3.4d). In $d = 2$ the logarithmic singularity is the leading term, and from Eq. (A1) one readily obtains Eq. (3.5b).

For a conserved order parameter, $\Gamma_{\boldsymbol{p}} = \gamma_2\,\boldsymbol{p}^2$, the integrals are more involved, but to linear order in $\gamma_2$ one easily obtains Eqs. (3.6b) and (3.7b) from Eq. (A1).

## Appendix B: Causal functions, and long-time tails

Here we list, without proofs, some properties of the class of causal functions that the longitudinal susceptibility belongs to. For general properties of causal functions see, e.g., Ref. 2. For derivations of the long-time tails see, e.g., Ref. 47.

### 1. Non-integer powers

Consider a causal function $\chi$ of complex frequency $z$ that behaves, for $z \to 0$, as

$$\chi(z) = \frac{1}{\cos(\alpha\pi/2)}\,\left[z^\alpha + (-z)^\alpha\right]\ , \quad \text{(B1)}$$

with $\alpha$ real and not integer. Here and in what follows we consider even functions of $z$, since the magnetic susceptibility has that property. We also give the asymptotic small-frequency, or long-time, behavior only; for $z \to \infty$ $\chi$, or any causal function, must vanish. On the imaginary axis $\chi$ then takes the values

$$\chi(i\Omega_n) = |\Omega_n|^\alpha\ , \quad \text{(B2)}$$

and the spectrum $\chi''$ and the reactive part $\chi'$, respectively, of $\chi$ read

$$\chi''(\omega) = -\sin(\pi\alpha/2)\,|\omega|^\alpha\,\mathrm{sgn}\,\omega\ , \quad \text{(B3a)}$$

$$\chi'(\omega) = \cos(\pi\alpha/2)\,|\omega|^\alpha\ . \quad \text{(B3b)}$$

The real-time behavior of $\chi$ is given by the Fourier transform of $\chi''(\omega)$,

$$\chi(t) = \int_{-\infty}^{\infty}\frac{d\omega}{\pi}\,e^{-i\omega t}\,\chi''(\omega)\ . \quad \text{(B4)}$$

In the long-time limit the Hardy-Littlewood tauberian theorem yields a long-time tail:

$$\chi(t\to\infty) = i\,\frac{\Gamma(\alpha+1)}{\pi}\,\sin(\alpha\pi)\,\frac{1}{|t|^{\alpha+1}} \quad \text{(B5)}$$

The ferromagnet with damped magnons in $d = 3$ is an example of this behavior, with $\alpha = 1/2$ and $\alpha = 3/2$ for a non-conserved and a conserved order parameter, respectively, see Secs. III B and III C. It is also realized by both ferromagnets and antiferromagnets in generic dimensions.

### 2. Even powers

Now consider

$$\chi(z) = \frac{(-)^m}{2}\,z^{2m}\left[\ln z + \ln(-z)\right]\ , \quad \text{(B6)}$$

with $m$ integer. On the imaginary axis this yields

$$\chi(i\Omega_n) = |\Omega_n|^{2m}\,\ln|\Omega_n|\ . \quad \text{(B7)}$$

The spectrum and the reactive part are

$$\chi''(\omega) = \frac{(-)^{m+1}\pi}{2}\,\omega^{2m}\,\mathrm{sgn}\,\omega\ , \quad \text{(B8a)}$$

$$\chi'(\omega) = (-)^m\,\omega^{2m}\ln|\omega|\ , \quad \text{(B8b)}$$

and the long-time behavior is

$$\chi(t\to\infty) = i\,\frac{(2m)!}{|t|^{2m+1}}\ . \quad \text{(B9)}$$

Examples of this behavior are the antiferromagnet in $d = 3$, Sec. II B 1, and the ferromagnet in $d = 2$ with a non-conserved order parameter, Sec. III B.

### 3. Odd powers

Finally, consider

$$\chi(z) = \frac{(-)^{m+1}}{\pi}\,z^{2m+1}\left[\ln z - \ln(-z)\right]\ , \quad \text{(B10)}$$

with $m$ integer, which leads to

$$\chi(i\Omega_n) = |\Omega_n|^{2m+1}\ , \quad \text{(B11)}$$

and

$$\chi''(\omega) = (-)^{m+1}\,\omega^{2m+1}\ , \quad \text{(B12)}$$

We now need to distinguish between $m \geq 0$ and $m < 0$. For $m \geq 0$ the spectrum is analytic, the reactive part vanishes,

$$\chi'(\omega) = 0\ , \quad \text{(B13)}$$

and there is no long-time tail in the real-time domain. However, there is a long-time tail in the limit of large imaginary time $\tau \to \infty$. $\chi(\tau)$ is given by

$$\chi(\tau) = T \sum_{i\Omega_n} e^{-i\Omega_n \tau}\, \chi(i\Omega_n) \ . \tag{B14}$$

At $T = 0$ the sum turns into an integral and we find

$$\chi(\tau \to \infty) = \frac{1}{\pi}\, (-)^{m+1} (2m+1)!\, \frac{1}{\tau^{2(m+1)}} \ . \tag{B15}$$

An example for this behavior is the ferromagnet with damped magnons in $d = 2$ with a conserved order parameter, see Sec. III C.

For $m < 0$ the spectrum is singular at $\omega = 0$ and there is a long-time tail even in the real-time domain. We consider only $m = -1$, in which case

$$\chi'(\omega) = \delta(\omega) \ , \tag{B16}$$

and the long-real-time behavior is a constant,

$$\chi(t) = -i \ . \tag{B17}$$

An example is the antiferromagnet in $d = 2$, see Sec. II B 1.


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[13] This is true for a Heisenberg model of spins on a lattice. In itinerant ferromagnets there are many more fluctuating degrees of freedom and the situation is different. We will come back to this point in Secs. III and IV.

[14] There are some subtleties in this context that have to do with differences between insulating and itinerant magnets, see the discussion in Sec. IV.

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[18] The spin precession term can be written in various ways. It often is expressed in terms of an integral over an auxiliary variable in addition to the imaginary time, see, e.g., Refs. 16,17. For the form quoted in Eq. (2.2b), which is local in space and imaginary time, see Refs. 48 or 49. The factor $1/(1+\sigma)$ is often approximated by 1/2, but this is only valid deep inside the ordered phase. Note the prefactor $M_0$, which reflects the fact that the spin precession term is cubic in the order-parameter field.

[19] Equation (2.8) represents only one Goldstone mode, since the two eigenvectors are not independent, in contrast to the antiferromagnetic case, where there are two Goldstone modes. This is obvious in spin-wave theory (see, e.g., Ref. 12), but less so in a field-theoretic approach. For discussions of this point, see Refs. 50–52.

[20] The contribution is zero if one integrates over all frequencies, which requires an ultraviolet cutoff on the momentum integral. With an ultraviolet frequency cutoff it is an analytic function of the external frequency or wave number.

[21] Restoring $\hbar$, we see that this is also the form of the Bose distribution function in the classical limit, $\hbar \to 0$. Subtracting this classical contribution leads to Eq. (2.15) with $n(x)$ replaced by $N(x) = n(x) - T/x$. The momentum integral is then infrared convergent in all dimensions. This is another way of saying that there is no singularity in a quantum ferromagnet.

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