Proximity effect and Majorana bound states in clean semiconductor nanowires coupled to disordered superconductors

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Proximity effect and Majorana bound states in clean semiconductor nanowires coupled to disordered superconductors

William S. Cole,* Jay D. Sau, and S. Das Sarma
Condensed Matter Theory Center, Joint Quantum Institute, and Station Q Maryland, Department of Physics, University of Maryland, College Park, MD 20742, USA
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We model a semiconductor wire with strong spin-orbit coupling which is proximity-coupled to a superconductor with chemical potential disorder. When tunneling at the semiconductor-superconductor interface is very weak, disorder in the superconductor does not affect the induced superconductivity nor, therefore, the effective topological superconductivity that emerges above a critical magnetic field. Here we demonstrate nonperturbatively how this result breaks down with stronger proximity coupling by obtaining the low-energy (i.e., subgap) excitation spectrum through direct numerical diagonalization of an appropriate BdG hamiltonian. We find that the combination of strong proximity coupling and superconductor disorder suppresses the (non-topological) induced gap at zero magnetic field by disordering the induced pair potential. In the topological superconducting phase at large magnetic field, strong proximity coupling also reduces the localization length of Majorana bound states, such that the induced disorder eliminates the topological gap while *bulk* zero modes proliferate, even for short wires.

One-dimensional topological superconductors (TS) supporting isolated, edge-localized Majorana zero modes (MZMs) can in principle be synthesized in strongly spin-orbit coupled semiconductor (SM) wires via the proximity effect from a conventional superconductor (SC) [1, 2], and several groups have recently reported measurements consistent with this 1D TS state in proximitized InAs and InSb nanowires [3–8]. However, lingering disagreements between theory and experiments have motivated the development of an increasingly sophisticated theoretical picture.

A simplified model for the induced superconductivity consists of adding a uniform s-wave pair potential to a non-superconducting hamiltonian, assuming any further effects of coupling to the SC may be ignored. This approximation turns out to be qualitatively, and even quantitatively, accurate for "weak" proximity coupling [9–11]. Explicitly, if the proximity coupling is characterized by an energy scale γ and the parent superconductor gap is Δ_{sc}, this model is viable when $\gamma/\Delta_{sc} \ll 1$. Unfortunately for applications, the induced gap in this regime is also approximately γ, while a large induced gap (relative to the energy resolution, temperature, etc., of the specific experiment) is a necessary requirement for robust signatures of in-gap modes. An additional requirement is that the induced gap be *hard*, that is, free of sub-induced-gap states that contribute to low-bias conductance. For 1D TS, this requires wires that are relatively free of intrinsic disorder, as well as a uniform SC-SM coupling along the length of the wire [12].

However, in an effort to fabricate TS devices with large and hard induced gaps, epitaxial SM-SC hybrids have recently been produced and characterized in this strong proximity coupling regime [13] (a similar platform relies on metal-on-superconductor hybrids which are *intrinsically* in this regime [14, 15].) It is, therefore, necessary to understand the signatures of deviations from this simplified model in the current context of intense experimental activity [16, 17] for the laboratory realization of synthetic TS.

One such implication, which motivates this Letter, is the possibility of an *induced disorder* arising from strong coupling to a disordered parent SC. To lowest order in γ/Δ_{sc}, the induced superconductivity is known to be "protected" against any nonmagnetic disorder in the parent SC [18, 19]. At intermediate or strong proximity coupling, though, this perturbative result must break down. Here we demonstrate this explicitly, calculating the combined nonperturbative effects of strong proximity coupling and parent superconductor disorder.

We begin with the standard minimal model of a single-channel, disorder-free nanowire with spin-orbit coupling and Zeeman splitting with a large g factor [1, 2]. This wire is coupled through a uniform interface hopping to a conventional s-wave superconducting bulk (represented by a ribbon, with width much greater than the SC coherence length) with on-site chemical potential disorder. Our goal is to treat the composite wire and SC system in a nonperturbative, approximation-free manner to the extent that the wire is noninteracting and the parent superconductor can be treated within the BdG mean-field approximation. Because the model is fundamentally quadratic, it seems that it should be amenable to an exact numerical solution. However, there are length scale and energy scale considerations that make a full diagonalization unreasonable *in practice*, and this has generally motivated the use of simpler effective models. For SM-SC hybrids, the Fermi wavelength of electrons in the SM wire is typically of the order of tens of nm, while the Fermi wavelength of electrons in the SC is a fraction of a nm. A typical SC coherence length could be $\sim \mu$m for Aluminum or tens of nm for Niobium. Additionally, the wire is expected to have a long mean-free path, while the superconductor mean-free path is generally much shorter than the coherence length.

In the following, we do not attempt a detailed quantitative simulation of any existing experiment, instead taking an approach that maintains this realistic hierarchy of length scales without making any additional approximations that may obscure the effects of induced disorder. We overcome the practical limitation of diagonalizing the correspondingly large hamiltonian matrix by abandoning full diagonalization outright. We are interested in a small subset of states with energy eigenvalues below (or on the order of) the parent SC gap, which is itself a small energy scale compared to band structure scales (i.e., chemical potential in the SC and hopping matrix elements). With this understanding, we can leverage the sparse matrix structure of the model and use the Lanczos method, with the shift-and-invert scheme appropriate to interior eigenvalue problems, to resolve only the part of the spectrum corresponding to the induced superconductivity. Additional details on the model and method of solution are relegated to the supplementary material.

Our goal is instead to elucidate the impact of nonmagnetic disorder in the parent SC nonperturbatively, for any strength of SM-SC coupling. We find that SC disorder can introduce "pair-breaking" behavior in the induced superconductivity of a perfectly clean SM wire with a perfectly smooth SM-SC interface, providing an alternative origin for observed states below the induced gap [12]. In the TS phase, the "topological gap" which protects MZMs becomes extremely fragile against SC disorder with increasing SM-SC coupling, corresponding to a proliferation of near-zero energy but bulk-localized states [20]. Observation of non-Abelian statistics would be essentially impossible in this regime, even though other signatures of MZMs (such as zero bias conductance peaks) may still be present. We conclude by highlighting some strategies to mitigate this induced disorder effect.

Spectral gap at zero field.—First we calculate the excitation gap E_G, the smallest eigenvalue of the full (wire + SC bulk) hamiltonian, as a function of the interface coupling γ and a dimensionless measure of the disorder strength in the parent superconductor, $d = \sqrt{\xi/(k_F \lambda^2)}$, with ξ, k_F, λ the clean BCS coherence length, Fermi wavevector, and mean free path respectively (more details available in the supplementary information). Our exact numerical results for E_G, shown in Fig. 1(a) for zero field, agree qualitatively with a recent self-consistent Born approximation (SCBA) calculation [21]. We also recover the perturbative result [18] that SC disorder does not affect E_G for $\gamma/\Delta_{sc} \ll 1$, however, at stronger coupling disorder does substantially suppress the induced gap (though it has no effect on the gap of the parent SC, which remains Δ_{sc} regardless of the disorder by virtue of Anderson's theorem [22]). Fluctuations around an averaged E_G arise from sampling over independent disorder realizations. Vanishing sample-to-sample fluctuations for $\gamma/\Delta_{sc} \ll 1$ further confirm the disorder independence of E_G in that regime.

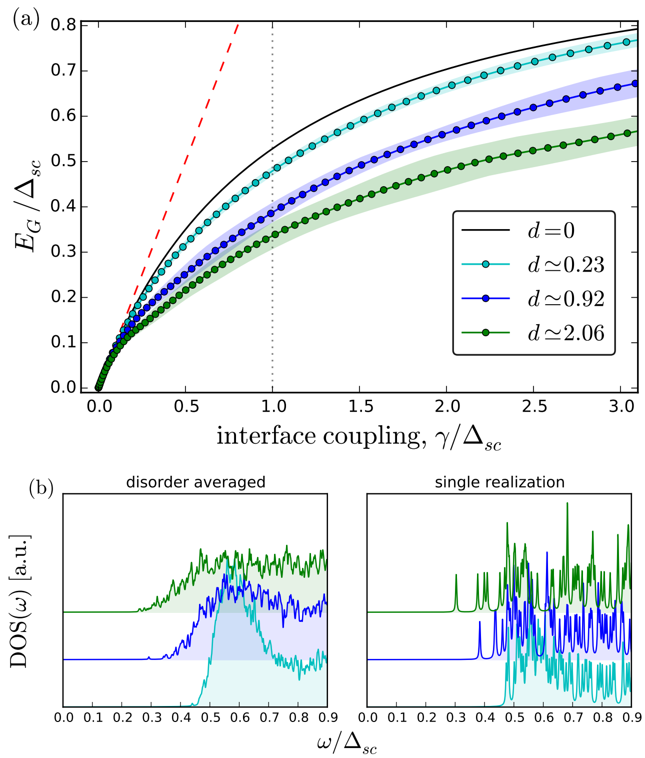

FIG. 1. Proximity-induced gap without magnetic field. (a), Induced gap (in units of parent superconductor gap) versus interface coupling for several values of disorder. We see that (i) disorder suppresses the induced gap substantially at intermediate coupling, $\gamma/\Delta_{sc} \simeq 1$, while (ii) for sufficiently small γ, all of the curves recover the expected $E_G = \gamma$ limit (red dashed line) as $\gamma \to 0$. In this limit the wire is "protected" from bulk SC disorder. (b), Disorder averaged and single realization subgap densities of states at intermediate coupling ($\gamma/\Delta_{sc} \simeq 1$). Each disorder averaged curve is obtained from the same set of 30 independent samples. Disorder in the parent SC broadens the subgap "coherence peak" and produces bound states below the clean-limit induced gap (thus suppressing E_G).

In Fig. 1(b), we fix $\gamma = \Delta_{sc}$ and show characteristic densities of states (averaged over 30 disorder realizations) for the same values of d. In the disorder-free case, the subgap spectrum exhibits a "coherence peak" of states accumulating at E_G. However, even for weak disorder, this peak is broadened with the lowest-lying states shifting to energies below the clean E_G. Alternatively, we also show the DOS evaluated for a *single* disorder realization. Here, we see that the high-energy states (i.e., near the parent gap) form a broad continuum, while the lowest energy states (which determine E_G) are energetically isolated and correspondingly spatially localized. Because of this localization, these states will not contribute to the wire edge LDOS, probed via the conductance of a normal metal-superconductor junction, unless by accident they are within a localization length of the wire edge.

At zero field, this suppression of the induced gap and "pair-breaking" appearance of the DOS is an initially surprising result in light of Anderson's theorem, which suggests that conventional s-wave superconductors are immune to disorder that preserves time-reversal symmetry (TRS) [22]. However, this apparent pair-breaking effect is limited to the induced superconductivity in the wire.

It is not universally appreciated that Anderson's theorem relies both on TRS *and* an approximately spatially uniform pair potential $\Delta(\boldsymbol{r}) \simeq \Delta$ [23]. An inhomogeneous pair potential also can produce bound states, even in the presence of TRS. In conventional superconductors with a small gap and a correspondingly large correlation length, a self-consistent pair potential disorder is energetically costly, so this physics is not relevant even for very dirty samples of, say, Aluminum. There is no such penalty for a strongly inhomogeneous *induced* pair potential in a hybrid device with no intrinsic attractive interaction in the semiconductor, as is clear from our numerical calculation despite a spatially uniform wire-SC coupling and uniform pair potential in the SC. This inhomogeneous proximity-induced pair potential leads to a "violation" of Anderson's theorem in the clean wire, even without any spin-orbit coupling or spin splitting (i.e. even in the non-topological s-wave SC regime). This is qualitatively similar to an effective model for pair-potential disorder studied previously in this context [12], although the mechanism here is quite different. Here, the pairing inhomogeneity appears even though the SC-wire interface is perfectly smooth. This establishes the possibility that the experimentally observed soft superconducting gap arises (at least partially) from parent SC disorder.

Spectral gap in the topological regime.—We consider now the effect of SC disorder on the subgap spectrum with a very large, fixed spin splitting in the wire, $V_{nw}^Z = 4\Delta_{sc}$. Coarsely, topological superconductivity is achieved (in the clean limit) when $V_{nw}^Z > \gamma$ [24]. The gap here cannot be a monotonically increasing function of γ, since increasing γ in turn reduces the effective spin splitting, so crossing $\gamma > V_{nw}^Z$ drives a gap-closing transition back to the trivial state. In our numerical results, this actually occurs at a slightly larger value of γ, attributable to including the small, typically neglected, Zeeman energy in the SC.

In Fig. 2(a), we show the disorder-averaged excitation gap E_G. The dashed lines are to highlight the *minimum* value of E_G over several disorder realizations. (We have used periodic boundary conditions to avoid the contribution from edge MZMs.) We emphasize that these are both length-dependent quantities. In the limit of an infinite length of wire both of these measures must coincide and tend to zero. A finite wire length and fixed number of disorder samples is, however, reflective of the distribution of gap measurements from a finite set of devices in an experiment. That the average E_G is substantially larger than the minimum E_G suggests that the finite system disorder-averaged DOS is characterized by a long tail of low-energy but unlikely states, verified in Fig. 2(b).

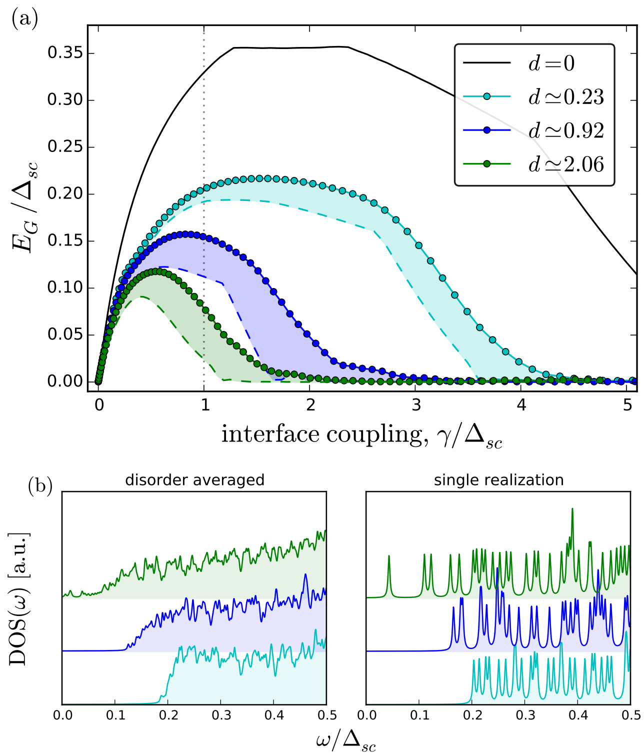

FIG. 2. Proximity-induced topological gap. (a), Induced *topological* gap versus interface coupling energy for several values of disorder, with a bare wire Zeeman energy $V_{nw}^Z = 4\Delta_{sc}$, such that the wire is topological over the entire coupling window in the clean limit. Compared to Fig. 1, the gap suppression is much more substantial. The topological gap however also exhibits "protection" from bulk SC disorder at small γ. The data points correspond to disorder averaging, while the dashed lines designate the *minimum* E_G over all sampled disorder realizations. (b), Disorder averaged subgap densities of states for several representative values of disorder at intermediate coupling ($\gamma/\Delta_{sc} \simeq 1$), as in Fig. 1. For sufficiently strong disorder, the DOS acquires a long tail such that the disorder averaged DOS closes, even though most individual realizations have a noticeable gap.

The general fragility of the underlying 1D effective p-wave superconductivity against disorder is well understood [20, 25–29], however, the specific dependence on the proximity coupling is interesting and has not been studied before. For any value of disorder, at sufficiently strong coupling the average E_G in Fig. 2(a) vanishes. That is, $E_G \sim 0$ regardless of the particular disorder configuration. However, this closing is not sharp. For each

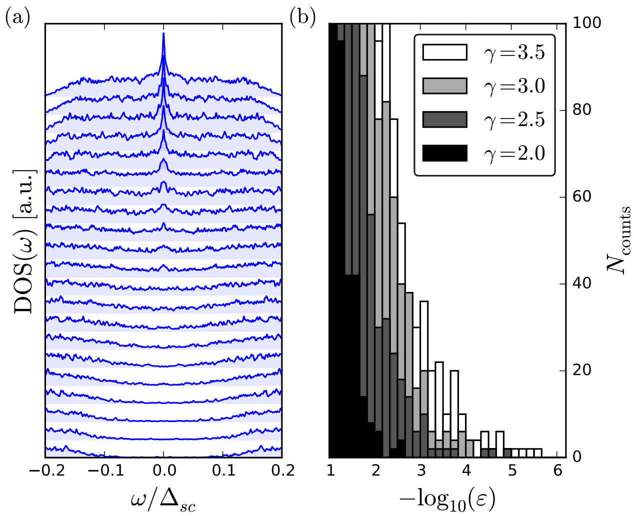

FIG. 3. Strong SM-SC coupling effect on the topological gap. (a), Disorder averaged low-energy DOS for $d \simeq 0.92$ with increasing $\gamma = 1.5 - 3.5\Delta_{sc}$ from bottom to top, demonstrating the closure of the gap and eventual emergence of a singularity at zero-energy. (b), Distribution of the negative logarithm of low-energy eigenvalues for several values of coupling. Exponentially small eigenvalues are exponentially rare, however the *relative* probability of obtaining an exponentially small eigenvalue increases substantially with γ.

d, the average E_G shows a "foot" well after the minimum E_G has gone essentially to zero. This is the rare-region-induced Griffiths effect - in this crossover window, any *individual* disorder realization will most likely have a comparably large E_G, but in some rare disorder configurations (or, equivalently, for a sufficiently long wire) the smallest E_G is exponentially small.

To explore this further, in Fig. 3(a), we track the disorder averaged density of states for $d \simeq 0.92$ along this foot, from $\gamma/\Delta_{sc} = 1.5 - 3.5$. We see that the gap in the DOS closes and is replaced by a Griffiths singularity at zero energy, as every disorder realization provides near-zero energy states.

This Griffiths effect provides useful insight into Fig. 2(a). The probability of the disorder effectively producing a segment of wire (topological embedded in nontopological or vice versa) with two MZMs separated by a length L is $P(L) \propto e^{-cL}$ for some model-dependent constant c. If such a segment exists, it contributes an excitation of energy $E \propto e^{-L/\xi}$, with ξ the localization length of the MZMs. It has only recently been realized that the localization length of MZMs is strongly renormalized by the proximity coupling [30, 31]. Since $\xi \propto \gamma^{-1}$, the required L to produce a state of some fixed (low) energy decreases as γ increases. The relative probability of generating this segment likewise increases, to the point where one (or more) of these states will reliably occur in the finite system for *any* particular disorder potential. That is, as γ is increased, the destruction of the topological gap and onset of a zero-energy DOS singularity, *even for a short wire*, is driven by the exponentially increasing probability of making a domain that contributes a near-zero eigenvalue to the spectrum. In Fig. 3(b), we show the distribution of low-energy eigenvalues in a histogram over several disorder realizations, demonstrating this exponential dependence and its scaling with γ.

Of course, this has unfortunate consequences for the application of wire edge MZMs in quantum information. Although a zero bias conductance peak may still arise from MZMs localized near the wire edges, the system is by no means protected since there are many other MZMs localized at random spatial locations along the wire (in fact, the conductance signature may be similar to a random Majorana chain with non-generic zero-bias peaks depending on various microscopic details [32]). These bulk low-energy modes can then participate in braiding in an uncontrollable way. One upside, however, is that this situation cannot persist down to arbitrarily small γ (again, in a finite wire), since we eventually return to the protected regime, where the TS gap is small ($\propto \gamma$) but unaffected by SC disorder.

Discussion.—Strong proximity coupling is experimentally desirable for maximizing the induced pair correlations and spectral gap in the nanowire, however, this also necessarily eliminates the protection, predicted for weak coupling, of the induced superconductivity in the nanowire against SC disorder. Realistically, the parent SC materials in present experiments are quite disordered and the resulting induced disorder in the nanowire can easily be sufficient to close the spectral gap and, in particular, produce an accumulation of low-energy MZMs in the wire bulk, consistent with previously studied effective models of disordered semiconductor nanowires. One new element of this work is a technique addressing this issue directly, beyond effective models or Born approximation, which also characterizes the entire wire spectrum, rather than just the edge LDOS. We observe that the disorder-averaged induced DOS gap as a function of tunneling strength, $E_G(\gamma, d)$, has a particular shape in both topological and nontopological regimes, and it would be interesting in future work to identify if there is some universal scaling that relates the different curves.

The expected benefits of strong proximity coupling vanish in the presence of SC disorder, even if the wire itself has zero disorder. Aside from eliminating disorder in the parent SC, alternative solutions for experiments are to aim for the protected limit of $\gamma \ll \Delta_{sc}$, possibly through the introduction of a tunnel barrier between the SM and SC, or to make use of larger gap (but still BCS-like) parent superconductors, both to increase the overall energy scale and to decrease disorder effects through the shorter coherence length. While our results are obtained for a quasi-one-dimensional model, the general principle extends just as well to synthetic TS in two dimensional

SM-SC hybrids [33–35] and to Fu-Kane TI-SC hybrids [36].

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* wcole1@umd.edu

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