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Short-time universal scaling and light-cone dynamics after a quench in an isolated quantum system in d spatial dimensions.

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We investigate the effects of fluctuations on the dynamics of an isolated quantum system represented by a ϕ^4 field theory with $O(N)$ symmetry after a quench in $d > 2$ spatial dimensions. A perturbative renormalization-group approach involving a dimensional expansion in $\epsilon = 4 - d$ is employed in order to study the evolution within a prethermal regime controlled by elastic dephasing. In particular, we focus on a quench from a disordered initial state to the critical point, which introduces an effective temporal boundary in the evolution. At this boundary, the relevant fields acquire an anomalous scaling dimension, while the evolution of both the order parameter and its correlation and response functions display universal aging. Since the relevant excitations propagate ballistically, a light cone in real space emerges. At longer times, the onset of inelastic scattering appears as secularly growing self-generated dissipation in the effective Keldysh field theory, with the strength of the dissipative vertices providing an estimate for the time needed to leave the prethermal regime.

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I. INTRODUCTION

Experimental progress in cold atomic gases^{1–4} and pump-probe spectroscopy of strongly correlated materials^{5,6}, has revived interest in understanding how an isolated quantum many-body system, driven far from equilibrium by an initial perturbation such as an abrupt change of a global parameter of its Hamiltonian (*quench*) H , evolves in time^{7–9}. After almost ten years of intense theoretical and experimental investigation, the post-quench dynamics of the system, initially prepared in the ground state of the pre-quench Hamiltonian H_0 , can be schematically described in the following way. In the generic case of a non-integrable post-quench Hamiltonian, the system relaxes towards a Gibbs thermal state^{10–18} in the sense that the expectation value of a generic local observable can alternatively be determined (in the thermodynamic limit) by an effective thermal ensemble ρ . If, instead, the post-quench Hamiltonian is integrable and therefore it is characterized by an extensive number of mutually commuting (local or quasi-local) integrals of motion which constrain the dynamics, then the eventual asymptotic state is described by the so-called generalized Gibbs ensemble^{19–27} (GGE). The intermediate case in which the post-quench Hamiltonian H is obtained by weakly perturbing (in a non-integrable fashion) an integrable one, displays, after an initial post-quench transient, a quasi-stationary *prethermal* regime^{28–38} determined by the integrable part of H and approximately described by the corresponding GGE. At longer times, the effect of the perturbation becomes predominant, causing the crossover towards the eventual thermal state. The effective strength of the non-integrable perturbation determines the actual duration of this prethermalization. However, a prethermal regime may appear because of a variety of possibly different physical phenomena, resulting in what are collectively re-

ferred to as *non-thermal fixed points*^{39–41}, among which we mention the generation of turbulence^{42–44}, of topological defects^{45–47}, and the presence of conservation laws which slow down the transport of particles and energy⁴⁸.

Remarkably, a novel class of critical-like phenomena may occur before the system thermalizes. These are characterized by a prethermal regime^{49–59} where the non-equilibrium or dynamical “phase transitions” (DPTs) are signalled by a change, upon varying a parameter of the quench, in the qualitative features of the time evolution of physical observables. Qualitatively different temporal behavior play the role of different “phases”. These DPTs are also of fundamental interest because they might be characterized by a scale-invariant behavior with some *universal* features which are independent of the microscopic details of the system, making them resemble critical phenomena encountered in equilibrium and non-equilibrium statistical physics. So far, these transitions have been investigated within a mean-field approximation^{50–52,54,60}, while exact results are available for certain solvable models^{55–57,59,61–63}, all characterized by the absence of thermalization and therefore by a neverending prethermal regime.

In the present work, instead, we explore the dynamical behavior and the DPT beyond the mean-field approximation, i.e., we account for the effects of fluctuations — following the analysis of Ref. 58, — which are relevant in spatial dimensionality d smaller than the upper critical dimensionality d_c of the system. In particular, we detail a perturbative renormalization group (RG) study of the dynamics of the system described in Sec. II and perform a dimensional expansion in the deviation $\epsilon \equiv d_c - d$. While most of the available analytic methods and efficient algorithms for numerical simulations applies to one-dimensional systems⁷, the analysis presented here is not tied to any specific dimensionality d and therefore one can explore its role in determining the features of the ob-

served transition. For the particular model we investigate here, it turns out that the lower critical dimensionality d_l is equal to 2, and therefore our results are expected to be qualitatively correct for $d > d_l = 2$, as discussed further below.

As anticipated in Ref. 58, we identify a novel temporal scaling due to the coarse-grained features of the initial condition affecting the behavior of the system in the early stage of its dynamics, but for times longer than the microscopic scale. Technically this implies that the fields at the time of the quench acquire an anomalous scaling dimension because of the sharp temporal boundary introduced by the quench. Close to the dynamical critical point, this leads to the phenomenon of *quantum aging*, analogous to the one observed in classical systems after a quench to equilibrium^{64,65} and non-equilibrium⁶⁶ critical points.

This aging is characterized by the emergence of the following scaling behavior in the correlation and response functions G_K and G_R , respectively, at two times t and t' after the quench, as a function of the wavevector k ,

$$G_K(k, t, t') = \frac{1}{k^{2-2\theta_N}} \mathcal{G}_K(kt, kt'), \quad (1a)$$

$$G_R(k, t, t') = \frac{1}{k} \left(\frac{t'}{t} \right)^{\theta_N} \mathcal{G}_R(kt, kt'), \quad (1b)$$

with $\mathcal{G}_K(x, y) \sim (xy)^{1-\theta_N}$ for $x, y \ll 1$ and $\mathcal{G}_K(x, y) \sim 1$ for $x, y \gg 1$, while $\mathcal{G}_R(x, y) \sim x$ for $y \ll x \ll 1$. The leading algebraic behaviour is described by the new universal exponent θ_N , called *initial-slip* exponent, which we calculate at the leading order in the dimensional expansion. The scaling forms (1a) and (1b), which emerge here within a perturbative RG analysis, were also confirmed by the exact solution of the model in the limit $N \rightarrow \infty$ of the $O(N)$ model⁵⁹.

The analysis presented here parallel the one for classical dissipative systems⁶⁷, open quantum systems^{68–71}, and imaginary time evolution of quantum systems⁷², but with important differences. First, while our results describe the approach to a prethermal state, those of Refs. 67, 69, and 70 characterize the evolution towards the eventual equilibrium thermal state resulting from coupling the system to an ideal thermal bath. As a consequence of this difference, an anomalous scaling occurs here in the k -dependence of correlation functions, which is discussed in Sec. IIIB. Second, the relevant collective excitations of the system investigated here propagate ballistically and therefore a light cone is present in the spatio-temporal correlation^{36,73,74} and response^{75–77} functions, which are analyzed in Sec. IIIC and which contrast with the diffusive collective excitations of open classical⁶⁷ and quantum^{68–70} systems, as determined by the presence of the thermal bath. Another relevant consequence of this ballistic character is that the effects of the initial condition, i.e., of the “temporal boundary” propagate into the “temporal bulk”, rendering the distinction between “bulk” and “boundary” less marked than in the

case of dissipative classical systems, where the memory of the initial condition is generally lost exponentially fast in time. Third, a major related difference with the classical case is that the quantum dynamics has to preserve the canonical commutation relations, which imposes a constraint on the scaling dimensions of the relevant fields and therefore affects the scaling forms of $G_{R,K}$.

Although most of the qualitative features discussed here are expected to be quite generic to isolated quantum systems, we focus for concreteness on the quench of one described by an N -component order parameter with Hamiltonian $H(r, u)$ characterized by an $O(N)$ symmetry. The system is initially prepared in the symmetric ground state of the same Hamiltonian as the one which rules its subsequent time evolution, but with different values of the parameters r and u , which control the distance from the DPT and the strength of the interaction, respectively. Various properties of this quench protocol have been the object of study^{73,74,78,79}, as summarized in Sec. II. Based on the perturbative renormalization group approach mentioned above and used in Ref. 58, we demonstrate here that a DPT occurs in the prethermal regime after the quench and we characterize its universal features. We argue that this DPT is closely related to the one predicted in the limit $N \rightarrow \infty$ of the present model^{55–57}. In particular, by deriving the equivalent Wilson (Sec. V) and Callan-Symanzik-like (Sec. VI) flow equations for the relevant couplings and correlation functions, respectively, we highlight both similarities and differences between the DPT investigated here and the phase transition occurring within this model in thermodynamic equilibrium.

Remarkably, when the pre-quench value Ω_0^2 of the parameter r is very large on the scale specified further below, Ω_0 plays the role of an effective temperature for the post-quench system and, accordingly, it induces a dimensional crossover on the DPT in complete analogy to what happens in equilibrium quantum phase transitions upon increasing the temperature^{80–82}. In particular, due to this finite effective temperature, the system is taken out of the quantum critical regime⁸² and therefore its upper critical dimensionality d_c is the same as for the classical critical system, i.e., $d_c = 4$.

The rest of the presentation is organized as follows: in Sec. II we introduce the model and discuss the relevant dynamical quantities of the Gaussian theory with Hamiltonian $H(r, u = 0)$ after a quench of the parameter r . In this context, we highlight in Sec. IID the emergence of a light cone in the response and correlation function after a quench at the Gaussian (dynamical) critical point $r = 0$. In Sec. III we account perturbatively for the effects of interactions, identifying the logarithmic corrections associated with the relevant correlation functions. We present the scaling forms although the RG resummation required to justify the scaling forms are presented later, in Sec. VI. We also show in Sec. III that these scaling forms are verified non-perturbatively in the limit $N \rightarrow \infty$ of the present model. In particular, the result-

ing expressions of the response and correlation functions in momentum space at long wavelengths are presented in Sec. III A, of the momentum distribution in Sec. III B, while those of the same functions in real space are discussed in Sec. III C, together with the emerging light cone. Section IV presents results for the time evolution of the average order parameter when a small symmetry-breaking field is added to the pre-quench Hamiltonian, which are obtained using both perturbation theory and the self-consistent Hartree-Fock approximation which is exact for $N \rightarrow \infty$.

The perturbative analysis mentioned above is then complemented by the construction of appropriate RG equations based on both the Wilson approach in Sec. V — which highlights the eventual emergence of dissipative terms — and the Callan-Symanzik-like flow equation for the relevant correlation functions in Sec. VI. The summary and conclusions are then presented in Sec. VII, while intermediate and technical details of the analysis are relegated to the various Appendices.

As a cautionary note, we emphasize here that the present work (together with Refs. 58 and 59) provides convincing evidence of the emergence of scaling behavior in the post-quench dynamics of quantum systems. However, a comprehensive picture of the structure of the ultraviolet singularities of the underlying model on the continuum is far from being achieved, especially beyond the perturbative order considered here. In fact, while the emergence of scaling laws in quantum statistical systems in equilibrium both at zero and finite temperature⁸³, as well as in the static and dynamic behavior of classical statistical systems, both in the bulk and at surfaces^{84–86} is based on a detailed understanding of these properties (and of the renormalizability of the associated continuum theory), the present case is sufficiently complicated that this would require a dedicated study, going well beyond the purposes of this investigation. We emphasize that the present study is already a challenging first step in this direction. Despite the peculiarities of the quantum quench, some similarities (spelled out in Secs. II B, II C and in Ref. 58) between the present case and the well-studied one of quantum systems at finite temperature^{81,82,87}, as well as with the analytic continuation^{73,74,77} of equilibrium classical systems confined within a film geometry,^{83,88,89} suggests that the corresponding theories on the continuum might have a similar structure, though important differences remain due to the boundary conditions imposed by the quench, as discussed in Sec. II E.

II. THE MODEL

A. Quench protocol

We consider a system described by a real vector order parameter $\phi \equiv \phi(\mathbf{x}) = (\phi_1, \dots, \phi_N)$ with N components and the following Hamiltonian in d spatial dimensions,

with $O(N)$ symmetry:

$$H(r, u) = \int_{\mathbf{x}} \left[\frac{1}{2} \mathbf{\Pi}^2 + \frac{1}{2} (\nabla \phi)^2 + \frac{r}{2} \phi^2 + \frac{u}{4!N} (\phi^2)^2 \right], \quad (2)$$

where $\int_{\mathbf{x}} \equiv \int d^d x$ while $\mathbf{\Pi} \equiv \mathbf{\Pi}(\mathbf{x})$ is the momentum conjugate to ϕ , which satisfies the canonical commutation relations $[\phi_i(\mathbf{x}), \Pi_j(\mathbf{x}')] = i\delta^{(d)}(\mathbf{x} - \mathbf{x}')\delta_{ij}$. Note that only scalar products such as $\phi^2 = \phi \cdot \phi = \sum_{i=1}^N \phi_i^2$ enter Eq. (2), in order to guarantee its invariance under $O(N)$ transformations. The coupling $u > 0$ parametrizes the strength of interaction, while r controls the distance from the critical point of DPT, as discussed below. The quench consists of a sudden change of both r and u at time $t = 0$ according to

$$H(\Omega_0^2, 0) \equiv H_0 \rightarrow H(r, u), \quad (3)$$

where we denote the pre-quench value of r as $r(t = 0^-) = \Omega_0^2$ for later convenience. We will assume that this value is positive, so that the system is initially in the disordered phase and any correlation function involving an odd number of fields ϕ vanishes. This protocol is the same as the one studied in Refs. 58 and 59, while in Refs. 51, 55–57 the interaction strength u had the same finite value before and after the quench. For both protocols the system is shown to undergo a DPT at some (protocol-dependent) critical value $r = r_c$. However, one expects the two transitions to be essentially the same. Indeed, as pointed out in Refs. 58 and 59, a non-zero interaction in the pre-quench state renormalizes only the value of Ω_0^2 , leaving unchanged the short-range nature of the correlations in the initial state, which actually determines the features of the DPT, as we argue below: accordingly, the two protocols discussed above are essentially equivalent in this respect. This fact can be regarded as a consequence of the *universality* associated with the DPT: as shown in Sec. VI C, the scaling forms and the value of the critical exponents characterizing the correlation functions are independent of the actual value of Ω_0 .

B. Gaussian model: correlation functions in momentum space

In this section we summarize the analysis of the quantum quench in a bosonic free field theory^{73,74,78} which corresponds to the special case $u = 0$ of the quench we are interested in and which provides the basis for the perturbative calculations discussed in Sec. III. As is clear from the analysis presented below, for the resulting Hamiltonian $H(r, 0)$ to have a spectrum bounded from below it is necessary to assume $r \geq 0$. Introducing the Fourier transform in momentum space of the fields ϕ and $\mathbf{\Pi}$ as $\phi(\mathbf{x}) = \int_{\mathbf{k}} \phi_{\mathbf{k}} e^{i\mathbf{k} \cdot \mathbf{x}}$ and $\mathbf{\Pi}(\mathbf{x}) = \int_{\mathbf{k}} \mathbf{\Pi}_{\mathbf{k}} e^{i\mathbf{k} \cdot \mathbf{x}}$, respectively, where $\int_{\mathbf{k}} \equiv \int d^d k / (2\pi)^d$, Eq. (2) can be written as

$$H(r, 0) = \frac{1}{2} \int_{\mathbf{k}} \left(|\mathbf{\Pi}_{\mathbf{k}}|^2 + \omega_{\mathbf{k}}^2 |\phi_{\mathbf{k}}|^2 \right), \quad (4)$$

where

$$\omega_k \equiv \sqrt{k^2 + r} \quad (5)$$

is the dispersion relation, while $|\mathbf{\Pi}_k|^2 \equiv \mathbf{\Pi}_k \cdot \mathbf{\Pi}_k^\dagger = \mathbf{\Pi}_k \cdot \mathbf{\Pi}_{-\mathbf{k}}$ and analogous for ϕ . The Heisenberg equations of motion for the operators after the quench $\Omega_0^2 \rightarrow r$, derived from $H(r, 0)$, is therefore $\ddot{\phi}_k + \omega_k^2 \phi_k = 0$, with solution

$$\phi_k(t) = \phi_k(0) \cos(\omega_k t) + \mathbf{\Pi}_k(0) \frac{\sin(\omega_k t)}{\omega_k}. \quad (6)$$

In order to calculate the expectation values of the components $\phi_{j,\mathbf{k}}(0)$ and $\Pi_{j,\mathbf{k}}(0)$ (with $j = 1, \dots, N$) on the initial state, it is convenient to introduce the standard bosonic annihilation and creation operators $b_{j,\mathbf{k}}$ and $b_{j,\mathbf{k}}^\dagger$, respectively, defined as

$$\begin{cases} \phi_{j,\mathbf{k}} = \frac{1}{\sqrt{2\omega_k}} (b_{j,\mathbf{k}} + b_{j,-\mathbf{k}}^\dagger), \\ \Pi_{j,\mathbf{k}} = -i\sqrt{\frac{\omega_k}{2}} (b_{j,\mathbf{k}} - b_{j,-\mathbf{k}}^\dagger), \end{cases} \quad (7)$$

with $[b_{j,\mathbf{k}}, b_{j',\mathbf{k}'}^\dagger] = \delta_{jj'} \delta_{\mathbf{k},\mathbf{k}'}$ the standard bosonic commutation relations and $\delta_{\mathbf{k},-\mathbf{k}'} \equiv (2\pi)^d \delta^{(d)}(\mathbf{k} + \mathbf{k}')$. Once expressed in terms of these operators, Eq. (4) becomes

$$H(r, 0) = \sum_{j=1}^N \int_{\mathbf{k}} \omega_k b_{j,\mathbf{k}}^\dagger b_{j,\mathbf{k}}, \quad (8)$$

up to an inconsequential additive constant. Assuming the pre-quench state to be in equilibrium at a temperature $T = \beta^{-1}$, with Hamiltonian $H_0 = H(\Omega_0^2, 0)$, the density matrix of the system is given by $\rho_0 = \mathcal{Z}^{-1} e^{-\beta H_0}$, where $\mathcal{Z} = \text{tr}(e^{-\beta H_0})$. Accordingly, one can evaluate the following statistical averages over ρ_0 by introducing the pre-quench operators $b_{j,\mathbf{k}}^0$ and $b_{j,\mathbf{k}}^{0,\dagger}$ as in Eq. (7):

$$\langle \phi_{i,\mathbf{k}}(0) \phi_{j,\mathbf{k}'}(0) \rangle = \delta_{\mathbf{k},-\mathbf{k}'} \delta_{ij} \frac{1}{2\omega_{0k}} \coth(\beta\omega_{0k}/2), \quad (9a)$$

$$\langle \Pi_{i,\mathbf{k}}(0) \Pi_{j,\mathbf{k}'}(0) \rangle = \delta_{\mathbf{k},-\mathbf{k}'} \delta_{ij} \frac{\omega_{0k}}{2} \coth(\beta\omega_{0k}/2), \quad (9b)$$

$$\langle \{\phi_{i,\mathbf{k}}(0), \Pi_{j,\mathbf{k}'}(0)\} \rangle = 0, \quad (9c)$$

and

$$\omega_{0k} = \sqrt{k^2 + \Omega_0^2} \quad (10)$$

is the pre-quench dispersion relation. Since the initial state does not break the $O(N)$ symmetry one has $\langle \phi_{j,\mathbf{k}}(0) \rangle = \langle \Pi_{j,\mathbf{k}}(0) \rangle = 0$ and therefore Eq. (6) implies that $\langle \phi_{j,\mathbf{k}}(t) \rangle = 0$ at all times t . The correlation functions of the field ϕ during the evolution can be easily determined on the basis of Eqs. (6), (9a), and (9b). Hereafter,

we focus on the retarded and Keldysh Green's functions, defined respectively as

$$iG_{jl,R}(|\mathbf{x} - \mathbf{x}'|, t, t') = \vartheta(t - t') \langle [\phi_j(\mathbf{x}, t), \phi_l(\mathbf{x}', t')] \rangle, \quad (11a)$$

$$iG_{jl,K}(|\mathbf{x} - \mathbf{x}'|, t, t') = \langle \{\phi_j(\mathbf{x}, t), \phi_l(\mathbf{x}', t')\} \rangle, \quad (11b)$$

where $\vartheta(t < 0) = 0$ and $\vartheta(t \geq 0) = 1$. Note that, as a consequence of the invariance of the Hamiltonian under spatial translations and rotations, $G_{R/K}$ depend only on the distance $|\mathbf{x} - \mathbf{x}'|$ between the points \mathbf{x} and \mathbf{x}' at which the fields are evaluated. Accordingly, it is convenient to consider their Fourier transforms which are related to the Fourier components of the field ϕ_k via

$$\delta_{\mathbf{k},-\mathbf{k}'} iG_{jl,R}(k, t, t') = \vartheta(t - t') \langle [\phi_{j,\mathbf{k}}(t), \phi_{l,\mathbf{k}'}(t')] \rangle, \quad (12)$$

$$\delta_{\mathbf{k},-\mathbf{k}'} iG_{jl,K}(k, t, t') = \langle \{\phi_{j,\mathbf{k}}(t), \phi_{l,\mathbf{k}'}(t')\} \rangle. \quad (13)$$

In the absence of symmetry breaking, the $O(N)$ symmetry implies that these functions do not vanish only for $j = l$, i.e., $G_{jl,R/K} = \delta_{jl} G_{R/K}$. Accordingly, in what follows, their dependence on the field components is no longer indicated. The Gaussian Green's functions (henceforth denoted by a subscript 0) in momentum space can be immediately determined by using the expression of the time evolution of the field $\phi_{j,\mathbf{k}}$ in Eq. (6) and the averages over the initial condition in Eqs. (9a) and (9b), which yield

$$G_{0R}(k, t, t') = -\vartheta(t - t') \frac{\sin(\omega_k(t - t'))}{\omega_k}, \quad (14a)$$

$$G_{0K}(k, t, t') = -i \frac{\coth(\beta\omega_{0k}/2)}{\omega_k} [K_+ \cos(\omega_k(t - t')) + K_- \cos(\omega_k(t + t'))], \quad (14b)$$

where

$$K_{\pm} = \frac{1}{2} \left(\frac{\omega_k}{\omega_{0k}} \pm \frac{\omega_{0k}}{\omega_k} \right), \quad (15)$$

with ω_k and ω_{0k} given in Eqs. (5) and (10), respectively. While the retarded Green's function G_{0R} , within this Gaussian approximation, does not depend on the initial state and it is therefore time-translation invariant (TTI), the Keldysh Green's function G_{0K} acquires a non-TTI contribution as a consequence of the quantum quench. As we discuss in Secs. III, V, and VIC, this contribution plays an important role in the DPT, as it eventually generates an algebraic behavior in time. Note that, in the absence of a quench, $\omega_k = \omega_{0k}$ and therefore $K_+ = 1$ and $K_- = 0$: correspondingly, the G_{0K} in Eq. (14b) recovers its equilibrium TTI expression. In addition, if the temperature T of the initial state vanishes $T = 0$, i.e., the system is in the ground state of the pre-quench Hamiltonian H_0 , the $G_{R/K}$ in Eqs. (14a) and (14b) at small

wavevectors $k \ll \Omega_0$ and at $r = 0$ read

$$G_{0R}(k, t, t') = -\vartheta(t - t') \frac{\sin(k(t - t'))}{k}, \quad (16a)$$

$$G_{0K}(k, t, t') = -i \frac{\Omega_0}{2k^2} [\cos(k(t - t')) - \cos(k(t + t'))], \quad (16b)$$

which can be cast in the scaling forms in Eqs. (1a) and (1b) with $\theta_N = 0$. For later convenience, we note that these expressions become, at short times $t, t' \ll k^{-1}$,

$$G_{0R}(k, t, t') \simeq -\vartheta(t - t')(t - t'), \quad (17a)$$

$$G_{0K}(k, t, t') \simeq -i\Omega_0 t t'. \quad (17b)$$

The scaling in Eqs. (16a) and (16b) is related to a Gaussian fixed point. In fact, one recognizes from Eqs. (14a) and (14b) that, for momenta $k \ll \Omega_0$, the correlation length is simply given by $\xi = r^{-1/2}$, due to the dependence of this expression on the combination $k^2 + r$. Accordingly, for $r \rightarrow r_c = 0$, this correlation length ξ diverges, thus signalling the onset of a transition and, correspondingly, the emergence of scale invariance into the correlation functions. Moreover, recalling the definition of the critical exponent ν , i.e., $\xi \sim |r - r_c|^{-\nu}$, one immediately realizes that in the present case $\nu = 1/2$, which is also the value expected at the Gaussian fixed point for the corresponding equilibrium model^{82,90,91}.

A relevant quantity we consider below is the number $n_{\mathbf{k}}$ of particles with momentum \mathbf{k} after the quench, defined as $n_{\mathbf{k}} = b_{\mathbf{k}}^\dagger b_{\mathbf{k}}$ in terms of the operators introduced in Eq. (7), where the index of the field component has been omitted for clarity. The operator $n_{\mathbf{k}}$ can be conveniently expressed in terms of the field $\phi_{\mathbf{k}}$ and its conjugate momentum $\Pi_{\mathbf{k}}$ as

$$n_{\mathbf{k}} + \frac{1}{2}\delta_{\mathbf{k},\mathbf{k}} = \frac{1}{4\omega_k} (\{\Pi_{\mathbf{k}}, \Pi_{-\mathbf{k}}\} + \omega_k^2 \{\phi_{\mathbf{k}}, \phi_{-\mathbf{k}}\}). \quad (18)$$

Since each term in this expression is proportional to the (infinite) volume $V = \int d^d x$ of the system, we consider the associated finite *momentum density* $\mathcal{N}_{\mathbf{k}} \equiv \langle n_{\mathbf{k}} \rangle / V$, which can be expressed in terms of the Green's functions in momentum space as

$$\mathcal{N}_{\mathbf{k}} + \frac{1}{2} = \frac{i}{4\omega_k} [G_K^\Pi(k, t, t) + \omega_k^2 G_K(k, t, t)], \quad (19)$$

where we introduced $i\delta_{\mathbf{k},-\mathbf{k}'} G_K^\Pi(t, t') = \langle \{\Pi_{\mathbf{k}}(t), \Pi_{\mathbf{k}'}(t')\} \rangle$ in analogy with G_K in Eq. (11b). Taking into account the Heisenberg equations of motion $\dot{\phi}_{\mathbf{k}} = \Pi_{\mathbf{k}}$, the equal-time G_K^Π can be expressed as

$$G_K^\Pi(k, t, t) = \partial_t \partial_{t'} G_K(k, t, t') \Big|_{t=t'} \equiv \ddot{G}_K(k, t, t), \quad (20)$$

from which it follows that the momentum density $\mathcal{N}_{\mathbf{k}}$ can be expressed in terms of G_K only:

$$\mathcal{N}_{\mathbf{k}} + \frac{1}{2} = \frac{i}{4\omega_k} [\ddot{G}_K(k, t, t) + \omega_k^2 G_K(k, t, t)]. \quad (21)$$

The r.h.s. of this equation can be calculated within the Gaussian approximation (hence the subscript 0) by using Eq. (14b), which yields

$$\mathcal{N}_{0\mathbf{k}} + \frac{1}{2} = \frac{1}{2} K_+ \coth(\beta\omega_{0k}/2). \quad (22)$$

Note that $\mathcal{N}_{0\mathbf{k}}$ does not depend on time, because the post-quench Hamiltonian $H(r, 0)$ can be written as in Eq. (8), i.e., as a linear combination of the momentum densities which are therefore conserved quantities. In addition, the number of excitations after the quench is finite even at $T = 0$ as a consequence of the energy injected into the system upon quenching. In the absence of a quench, instead, $K_+ = 1$ and Eq. (22) renders the Bose equilibrium distribution $\mathcal{N}_{0\mathbf{k}} = 1/[\exp(\beta\omega_{0k}) - 1]$.

C. Deep quenches limit and effective temperature

In the rest of the discussion, we mostly focus on the limit of *deep quench* $\Omega_0^2 \gg r$, i.e., on the case in which there is a significant difference between the pre- and post-quench values of the control parameter r . Since the post-quench value r_c of r at which the DPT takes place scales as⁵⁸ $r_c \sim \epsilon \Lambda^2$ (see also Sec. III) where Λ is some ultraviolet cut-off inherent the microscopic structure of the system, the deep-quench limit can be equivalently expressed as $\Omega_0 \gg \Lambda$. Interpreting Λ as being related to the inverse of the lattice spacing of the underlying microscopic lattice, the condition $\Omega_0 \gg \Lambda$ implies that the correlation length $\simeq \Omega_0^{-1}$ of fluctuations in the pre-quench state is smaller than the lattice spacing, i.e., the system is in a highly disordered state. In turn, as was realized in Refs. 73, 74, 78, and 79, this disordered initial state resembles a high-temperature state and, in fact, the momentum density in Eq. (22) takes the form

$$\mathcal{N}_{0\mathbf{k}} \simeq \frac{T_{\text{eff}}}{\omega_k} \quad (23)$$

of a thermal one in the deep-quench limit, with an effective temperature given by

$$T_{\text{eff}} = \Omega_0/4. \quad (24)$$

This similarity is made even more apparent by considering the fluctuation-dissipation theorem (FDT)⁹² which relates in frequency-momentum space the Keldysh and retarded Green's functions of a system in thermal equilibrium at temperature β^{-1} as

$$G_K(\omega, k) = \coth(\beta\omega/2) [G_R(k, \omega) - G_R(k, -\omega)]. \quad (25)$$

Out of equilibrium, one can *define* an effective temperature $T_{\text{eff}} = \beta_{\text{eff}}^{-1}$ such that $G_{K/R}$ satisfy the FDT^{75,76,93,94}, which generically depends on both frequency ω and momentum k as a consequence of the lack of thermalization. In the present case, considering only

the stationary part of Eq. (14b), the Fourier transform of G_{0K} is related to the one of G_{0R} in Eq. (14a) via

$$G_{0K}(k, \omega) = \frac{\Omega_0}{2\omega} [G_{0R}(k, \omega) - G_{0R}(k, -\omega)], \quad (26)$$

which takes the form of the FDT in Eq. (25) with the same (high) temperature $T = T_{\text{eff}}$ as defined above in Eq. (24) from the behavior of $\mathcal{N}_{0\mathbf{k}}$. As shown in Sec. V, the fact that the system appears to be “thermal” in the deep-quench limit has important consequences on its critical properties. Indeed, it behaves effectively as a d -dimensional classical system rather than a $d + 1$ -dimensional one, the latter expected for a closed quantum system at zero temperature. Accordingly, the effect of a deep quench on a DPT is heuristically the same as that of a non-vanishing temperature on a quantum phase transition, where the temperature is so high that the system falls out of the so-called quantum-critical regime^{80–82}. However, as anticipated in Sec. I and discussed below, this DPT is characterized by novel universal non-equilibrium properties, absent in the transition at equilibrium.

D. Gaussian model: correlation functions in real space and light-cone dynamics for a critical quench

In this section we discuss the properties of the Green’s functions in real space $G_{R/K}(x, t, t')$, with $x = |\mathbf{x}_1 - \mathbf{x}_2|$, highlighting the emergence of a light cone in the dynamics in both correlation⁷³ and response^{76,77} functions, which has been observed experimentally^{95,96} in the correlation function of a one-dimensional quantum gas. The emergence of a light cone is due^{73,76,77} to the fact that the entangled quasi-particle pairs generated by the quench propagate ballistically with a velocity v , causing a qualitative crossover in the behavior of the Green’s functions from short times, at which they behave as in the initial state, to long times at which the effect of the quench dominates; this is accompanied by an enhancement of these functions right on the light cone.

Since $G_{R/K}(x, t, t')$ depend separately on the two times t and t' , there are two kind of light cones emerging in their structure: one for $x = t + t'$ and one for $x = t - t'$. For the Keldysh Green’s function $G_K(x, t, t)$ at equal times, the enhancement at the light cone can be physically interpreted as due to the simultaneous arrival at positions \mathbf{x}_1 and \mathbf{x}_2 of highly entangled excitations generated by the quench⁷³. For the specific Hamiltonian in Eq. (2) with $u = 0$, the present normalization set the velocity of propagation to $v = 1$. While in principle the value of v is affected by the presence of the interaction, this is not the case up to one loop in perturbation theory, as shown in the rest of this section. For the retarded Green’s function $G_R(x, t, t')$ the enhancement at the light cone can also be again understood from the ballistic propagation of excitations with velocity $v = 1$: a perturbation created at $x = 0$ at time t' cannot be felt at position x until

the condition $x = |t - t'|$ is obeyed. After this time, the effect of the initial perturbation decreases upon increasing t for a fixed value of x , and so the response function approaches zero.

Exploiting the spatial isotropy and translational invariance, $G_{R/K}(x, t, t')$ in d spatial dimensions can be calculated from their Fourier transforms $G_{R/K}(k, t, t')$ in Eqs. (14a) and (14b) as

$$G_{R,K}(x, t, t') = \frac{1}{x^{d/2-1} (2\pi)^{d/2}} \int_0^\Lambda dk k^{d/2} \times J_{d/2-1}(kx) G_{R,K}(k, t, t'), \quad (27)$$

where we included a sharp ultra-violet cut-off Λ in the integral over \mathbf{k} and J_α indicates the Bessel function of the first kind. This expression is obtained by exploiting the fact the functions $G_{R,K}$ depend only on the modulus k of the wavevector \mathbf{k} : one can then perform a change of variables using hyperspherical coordinates and then integrate over the angular variables⁹⁷.

In this section we focus on the case of a critical quench, corresponding to a vanishing post-quench value of the parameter $r = 0$. Correspondingly, the correlation length $\xi = r^{-1/2}$ diverges, causing the emergence of universal scaling forms and algebraic decays also in the light-cone structure of the Green’s functions. Let us first consider spatial dimension $d = 4$. The Gaussian retarded Green’s function $G_{0R}^{d=4}(x, t, t')$ follows from Eqs. (27) and (14a):

$$G_{0R}^{d=4}(x, t, t') = -\frac{1}{4\pi^2 x^3} \int_0^{\Lambda x} dy y J_1(y) \sin(y(t - t')/x), \quad (28)$$

where we assume, for simplicity, $t > t'$. Denoting by $\tau = t - t'$ the difference between the two times, for $\Lambda x \gg 1$ and $\Lambda \tau \gg 1$ we find that G_{0R} exhibits a light cone, similarly to what was observed in other models^{75–77}:

$$G_{0R}^{d=4}(x \gg \tau) = \frac{\Lambda^3}{4\pi^{5/2} (\Lambda x)^{5/2}} \sin(\Lambda \tau) \times [\sin(\Lambda x) + \cos(\Lambda x)] \simeq 0, \quad (29a)$$

$$G_{0R}^{d=4}(x = \tau) = -\frac{\Lambda^3}{12\pi^{5/2} (\Lambda \tau)^{3/2}}, \quad (29b)$$

$$G_{0R}^{d=4}(x \ll \tau) = -\frac{\Lambda^2}{4\pi^{5/2} \tau (\Lambda x)^{3/2}} \cos(\Lambda \tau) \simeq 0. \quad (29c)$$

Inside ($x \ll \tau$) and outside ($x \gg \tau$) the light cone, G_{0R} vanishes on average due to the rapidly oscillating terms, while exactly on the light cone $x = \tau = t - t'$ it does not, and actually is characterized by an algebraic temporal decay $\propto \tau^{-3/2}$. Accordingly $G_{0R}(x, \tau)$ is peaked at $x = \tau$ and vanishes away from this point in a manner which depends on the ultraviolet (UV) physics.

In Sec. III C we show that this basic behavior is preserved even in presence of interactions, provided that the dynamics is in the collisionless prethermal regime. However, the algebraic decay at $x = |t - t'|$ will be modified in

two ways: first, it will depend on whether the initial perturbation occurs at a short time or at a long time relative to a microscopic time scale Λ^{-1} , which is not captured by the Gaussian quench discussed here. Second, the algebraic decay will acquire corrections described by the anomalous exponent θ_N for t' at short times.

The Gaussian Keldysh Green's function $G_{0K}^{d=4}(x, t, t)$ at equal times, in real space, and in the deep-quench limit follows from Eqs. (27) and (16b):

$$iG_{0K}^{d=4}(x, t, t) = \frac{\Omega_0}{8\pi^2 x} \int_0^\Lambda dk [1 - \cos(2kt)] J_1(kx). \quad (30)$$

Evaluating the integral, the light cone is observed to emerge for $\Lambda x \gg 1$ and $\Lambda t \gg 1$, upon crossing the line $x = 2t$,

$$iG_{0K}^{d=4}(x \gg 2t) \simeq \mathcal{O}\left(\frac{J_0(\Lambda x)}{x^2}\right) \simeq 0, \quad (31a)$$

$$iG_{0K}^{d=4}(x = 2t) \simeq \frac{\Omega_0}{8\pi^{5/2}} \frac{\Lambda^2}{(\Lambda x)^{3/2}}, \quad (31b)$$

$$iG_{0K}^{d=4}(x \ll 2t) \simeq \frac{\Omega_0}{8\pi^2} \frac{\Lambda^2}{(\Lambda x)^2}. \quad (31c)$$

Outside the light cone, i.e., for $x \gg 2t$, the behavior of G_{0K} is primarily determined by the initial state as the effect of the quench has not yet set in. Since the initial state is gapped, with two-point correlations decaying rapidly upon increasing their distance, G_{0K} vanishes outside the light cone for $\Lambda x \gg 1$. Inside the light cone ($x \ll 2t$), instead, a time-independent value is obtained, which is characterized by an algebraic spatial decay $\propto x^{-2}$. Finally, right on the light cone $x = 2t$, the correlator G_{0K} is enhanced, showing a slower algebraic decay $\propto x^{-3/2}$.

While above we focused on the case $d = 4$, it is interesting to study the behavior of G_{0K} in generic spatial dimensionality d , which we compare further below in Sec. III C with the results of the perturbative dimensional expansion in the presence of interactions. Note that G_{0K} outside the light cone vanishes for the reason indicated above; thus we discuss here its behavior at and inside the light cone. Instead of regularizing the momentum integrals via a sharp cut-off as we did in Eq. (27), we consider below a generic cut-off function $f(k/\Lambda)$ such that $f(x \ll 1) = 1$, with an exponential decay as $x \gg 1$. In view of the asymptotic form of Bessel functions⁹⁸, and noting that they oscillate in phase with $G_{0K}(k, t, t)$ on the light cone, we find (see Eq. (27))

$$\begin{aligned} iG_{0K}(x = 2t) &\simeq \frac{\Omega_0}{x^{d/2-1} (2\pi)^{d/2}} \int_0^\Lambda dk k^{d/2} \frac{f(k/\Lambda)}{k^2 \sqrt{kx}} \\ &\propto \frac{1}{x^{d-2}} \int_0^\Lambda dy y^{(d-5)/2} f\left(\frac{y}{\Lambda x}\right) \\ &\simeq \frac{1}{x^{d-2}} \int_0^{\Lambda x} dy y^{(d-5)/2} \propto \frac{1}{x^{(d-1)/2}}. \end{aligned} \quad (32)$$

Inside the light cone $x \ll 2t$ we find, instead,

$$\begin{aligned} iG_{0K}(x \ll 2t) &\simeq \frac{\Omega_0}{x^{d/2-1} (2\pi)^{d/2}} \\ &\times \int_0^\infty dk k^{d/2} f(k/\Lambda) J_{d/2-1}(kx) \frac{1}{k^2} \\ &\propto \frac{1}{x^{d-2}} \int_0^\infty dy y^{-2+d/2} J_{d/2-1}(y) f\left(\frac{y}{\Lambda x}\right) \propto \frac{1}{x^{d-2}}, \end{aligned} \quad (33)$$

where we replaced $\sin^2(kt)$ appearing in $G_{0K}(k, t, t)$ (see Eq. (16b)) with its temporal mean value $1/2$ and in the last line the cut-off function f has been replaced by 1 because the rapidly oscillating Bessel function suppresses the integral at large arguments. In summary, for a deep quench to the critical point of the Gaussian theory in d spatial dimensions one finds

$$iG_{0K}(x \gg 2t) \simeq 0, \quad (34a)$$

$$iG_{0K}(x = 2t) \propto \frac{1}{x^{(d-1)/2}}, \quad (34b)$$

$$iG_{0K}(x \ll 2t) \propto \frac{1}{x^{d-2}}. \quad (34c)$$

The response function on the other hand can be simply derived by using Eqs. (16a) and (27),

$$G_{0R}(x = t - t') \propto \frac{1}{x^{(d-1)/2}} \quad (35)$$

and vanishes away from $x = t - t'$ as discussed above.

As anticipated above and shown in Sec. III C, these expressions acquire sizable corrections when a finite value of the post-quench interaction u is included, with corrections taking the form of an anomalous scaling which modifies the exponents appearing in Eqs. (34) and (35).

E. Keldysh action for a quench

In the sections which follow we show how the interaction modifies the dynamics after a quench compared to those of the Gaussian model discussed above. To this end, we use perturbation theory and renormalization-group techniques, the formulation of which is particularly simple within the Keldysh functional formalism^{99,100}. For later convenience, in this section we briefly recall how a quench can be describe within it.

1. Construction of the formalism

For simplicity, we assume that the system is prepared in a thermal state described by the density matrix ρ_0 introduced after Eq. (8), where now H_0 is a generic pre-quench Hamiltonian, not necessarily quadratic. After the quench, the dynamics of the system is ruled by the post-quench Hamiltonian H obtained from H_0 by changing

some parameters at time $t = 0$. Accordingly, the expectation value of any operator \mathcal{O} can be expressed in the following functional form¹⁰⁰:

$$\langle \mathcal{O}(t) \rangle = \int \mathcal{D}\phi \mathcal{O}[\phi_f(t)] e^{iS_K}, \quad (36)$$

where $S_K = S_K[\phi_f, \phi_b, \phi_I]$ is the Keldysh action, which is a functional of the fields ϕ_f , ϕ_b and ϕ_I , referred to, respectively, as *forward*, *backward* and *initial* fields; $\mathcal{D}\phi \equiv D[\phi_f, \phi_b, \phi_I]$ is the functional measure. Note that the r.h.s. of Eq. (36) involves N -component vector fields $\phi_{f,b,I}$, with ϕ_f replacing the original operator field ϕ in the formal expression $\mathcal{O}[\phi]$ of \mathcal{O} . The action $S_K = S_s + S_b$ consists of the two parts S_s and S_b corresponding, respectively, to the initial state and the post-quench dynamics, where

$$S_s = i \int_0^\beta d\tau \mathcal{L}_0^E(\phi_I), \quad (37a)$$

$$S_b = \int_0^\infty d\tau [\mathcal{L}(\phi_f) - \mathcal{L}(\phi_b)], \quad (37b)$$

where $\mathcal{L}(\phi)$ is the Lagrangian associated with the Hamiltonian (2), i.e.,

$$\mathcal{L}(\phi) = \int_{\mathbf{x}} \left[\frac{1}{2} \dot{\phi}^2 - \frac{1}{2} (\nabla \phi)^2 - \frac{r}{2} \phi^2 - \frac{u}{4!N} (\phi^2)^2 \right], \quad (38)$$

while $\mathcal{L}_0^E(\phi)$ is the Euclidean Lagrangian associated with the pre-quench Hamiltonian, namely

$$\mathcal{L}_0^E(\phi) = \int_{\mathbf{x}} \left[\frac{1}{2} \dot{\phi}^2 + \frac{1}{2} (\nabla \phi)^2 + \frac{\Omega_0^2}{2} \phi^2 \right]. \quad (39)$$

In order to simplify the notation, hereafter the spatial and temporal dependence of the fields is not explicitly indicated. The fields ϕ_f , ϕ_b and ϕ_I in Eq. (36) are not actually independent but are related by the boundary conditions

$$\begin{cases} \phi_f(t) = \phi_b(t), \\ \phi_b(0) = \phi_I(0), \\ \phi_f(0) = \phi_I(\beta). \end{cases} \quad (40)$$

Note that if the observable \mathcal{O} in Eq. (36) is calculated at $t = 0^-$, then these boundary conditions reduce to the periodic one $\phi_I(0) = \phi_I(\beta)$ usually encountered in a Matsubara path-integral⁹⁹. The functional representation in Eqs. (36), (37a), and (37b) correspond to the Schwinger-Keldysh contour¹⁰⁰: the usual forward-backward branches of the contour of integration are supplemented by an initial branch in imaginary time which appears as a consequence of the initial density matrix ρ_0 represented by S_s in Eq. (37a). The actions S_s and S_b are referred to as *surface* and *bulk* action, respectively, for a reason which will become clear further below. It is convenient to express the forward and backward fields ϕ_f and ϕ_b , respectively, in the so-called retarded-advanced-Keldysh basis¹⁰⁰ via $\phi_f = (\phi_c + \phi_q)/\sqrt{2}$ and

$\phi_b = (\phi_c - \phi_q)/\sqrt{2}$, where the fields ϕ_c and ϕ_q are referred to as *classical* and *quantum* fields, respectively. In these terms, the retarded and Keldysh Green's functions in Eqs. (11a) and (11b) read¹⁰⁰

$$iG_R(|\mathbf{x} - \mathbf{x}'|, t, t') = \langle \phi_c(\mathbf{x}, t) \phi_q(\mathbf{x}', t') \rangle, \quad (41)$$

$$iG_K(|\mathbf{x} - \mathbf{x}'|, t, t') = \langle \phi_c(\mathbf{x}, t) \phi_c(\mathbf{x}', t') \rangle, \quad (42)$$

where, as before, we do not indicate the indices of the field components of $\phi_{c,q}$ because the $O(N)$ symmetry forces them to be equal for the fields inside the expectation values. The expression of $G_{R/K}$ given in Eqs. (14b) and (14a) can alternatively be calculated within the functional formalism introduced above, as we discuss in App. A. For the quench protocol described in Sec. II A, S_s and S_b can be written as

$$\begin{aligned} S_s &= i \int_{\mathbf{x}} \int_0^\beta d\tau \left[\frac{1}{2} (\dot{\phi}_I)^2 + \frac{1}{2} (\nabla \phi_I)^2 + \frac{\Omega_0^2}{2} \phi_I^2 \right], \quad (43) \\ S_b &= \int_{\mathbf{x}} \int_0^{+\infty} dt \left[\dot{\phi}_q \cdot \dot{\phi}_c - (\nabla \phi_q) \cdot (\nabla \phi_c) - r \phi_q \cdot \phi_c \right. \\ &\quad \left. - \frac{2u_c}{4!N} (\phi_q \cdot \phi_c) \phi_c^2 - \frac{2u_q}{4!N} (\phi_c \cdot \phi_q) \phi_q^2 \right]. \end{aligned} \quad (44)$$

Note that the coupling constant u in Eq. (2) is indicated in Eq. (44) as $u_{c,q}$: although, in principle, $u_c = u_q = u$, the couplings of the terms $(\phi_q \cdot \phi_c) \phi_c^2$ and $(\phi_c \cdot \phi_q) \phi_q^2$ deriving from $(\phi^2)^2$ may behave differently under RG and therefore we denote them with different symbols. Given that S_s is Gaussian, the functional integral in Eq. (36) with an observable $\mathcal{O}(t)$ at time $t \neq 0$ can be simplified¹⁰¹ by calculating the integral over ϕ_I . For convenience, we first rewrite Eq. (43) in momentum space, where it reads

$$S_s = \frac{i}{2} \int_{\mathbf{k}} \int_0^\beta d\tau \left(\dot{\phi}_I^2 + \omega_{0k}^2 \phi_I^2 \right), \quad (45)$$

with ω_{0k} given in Eq. (10). In order to integrate out ϕ_I , we solve the saddle-point equation with the boundary conditions $\phi_I(\beta) = \phi_f(0)$ and $\phi_I(0) = \phi_b(0)$, we insert the solution back in S_s and then we perform the integral in τ . The resulting action is

$$S_s = i \int_{\mathbf{k}} \frac{\omega_{0k}}{2} \left[\phi_{0c}^2 \tanh(\beta \omega_{0k}/2) + \phi_{0q}^2 \coth(\beta \omega_{0k}/2) \right], \quad (46)$$

where $\phi_{0c} \equiv \phi_c(t=0)$ and $\phi_{0q} \equiv \phi_q(t=0)$. Accordingly, the initial action S_s can be regarded as a functional which weights the value of $\phi_{c,q}$ at $t = 0$, and therefore it determines the initial conditions for the subsequent evolution. In order to show this, e.g., for $T = 0$, one can consider the saddle-point equations for the action $S_K = S_b + S_s$, with S_b and S_s given in Eqs. (44) and (46), respectively, finding

$$i\omega_{0k} \phi_c(0) = \dot{\phi}_q(0) \quad \text{and} \quad i\omega_{0k} \phi_q(0) = \dot{\phi}_c(0), \quad (47)$$

where the time derivatives on the r.h.s. of these equations come from the integration by parts in the bulk action S_b . Note that in the generalized boundary conditions (47), classical and quantum fields are coupled. In addition, for $\omega_{0k} \rightarrow +\infty$, Dirichlet boundary conditions are effectively recovered. Since the quench induces a breaking of time-translational invariance, the Keldysh action in Eq. (36) can be formally regarded as the action of a semi-infinite system^{84–86,102–104} in $d+1$ dimensions in which the bulk is described by S_b , the d -dimensional surface by S_s , while t measures the distance from that boundary. Note that this is just a suggestive analogy as there is a fundamental difference between these two cases, with far reaching consequences: while in semi-infinite systems the influence of the boundary generically decays exponentially upon increasing the distance from the surface, this does not generically happen in the present non-equilibrium dynamics, due to the oscillatory dependence of the propagators $G_{0R/0K}$ in Eq. (14) on time, which is not qualitatively altered by the one-loop corrections considered further below in Sec. III. However, though somehow artificial, such a separation between bulk and boundary, is anyhow useful.

2. Structure of the resulting action

Before proceeding with the analysis of the perturbation theory and of the RG flow of the coupling associated with Eqs. (43) and (44), we would like to comment briefly below on the structure of the corresponding field theory, comparing it with those of theories emerging in similar contexts.

In the absence of a quench (and therefore with the appropriate S_s , i.e., with the Euclidean Lagrangian density associated with the Hamiltonian in Eq. (2)), time-translational invariance is not broken and it eventually reduces to an equilibrium thermal field theory¹⁰⁰ (characterized by a certain symmetry related to the FDT in Eq. (25), see, e.g., Ref. 105) at the temperature β^{-1} set by the integral along the imaginary time in S_s . The resulting field theory is effectively a $d+1$ -dimensional one ($z=1$ the dynamical exponent), but confined within a film of thickness β and periodic boundary conditions in the direction across the film. Because of these boundary conditions, no breaking of translational invariance occurs and the only complication compared to a bulk theory arises because the zero mode of the transverse fluctuations needs to be accounted for non-perturbatively^{83,88}; as a result, some quantities display a non-analytic dependence^{87,106,107} on the interaction strength $u = u_c = u_q$, while the theory has an upper critical dimensionality $d_c = 4$ for $\beta^{-1} \neq 0$, with a subtle dimensional crossover as $\beta^{-1} \rightarrow 0$.

In the presence of a quench, the situation becomes more subtle due to the breaking of time-translational invariance, and the consequent loss of a number of relationships between the scaling dimensions of the fields

$\phi_{c,q,I}$ which, as discussed further below in Sec. V A, may result in different effective theories depending also on the initial state of the quench. Note that this is never the case both in semi-infinite equilibrium systems^{85,86} or non-equilibrium bulk systems after a temperature quench^{64,67}, as the boundary condition does not affect the effective theory in the bulk. In the limit of the deep quench we focus on here, power counting indicates, for example, that the resulting effective theory has $u_q = 0$, with an upper critical dimensionality $d_c = 4$, somehow in analogy to the equilibrium case at finite temperature mentioned above. Note, however, that in the present case, the formal mapping onto a film of a certain thickness does not apply in the form which is valid for equilibrium systems and thus the problem with the zero mode might not arise, as the perturbative calculations in Sec. III suggest.

However, a similar mapping was introduced in Refs. 73 and 74 for determining the post-quench evolution of observables such as a correlation function and later for a response function in Ref. 77, where a detailed comparison with the Keldysh formalism discussed above was also presented. Within this formalism, the evolution of correlation functions of a certain observable is obtained from the analytic continuation of the corresponding correlation function in the system augmented of one bounded spatial dimension, i.e., in a film, with boundary conditions imposed by the initial state of the quench. The analytic continuation of the coordinate across the film to imaginary values then plays the role of the real time of the evolution. However, this mapping is seemingly limited to the case of dynamics long after the quench and therefore is unable to describe the phenomena described here, though an investigation of the relationship between these two approaches would be certainly fruitful, and is well beyond the scope of the present work.

F. Keldysh Green's function as a propagation of the initial state

In the absence of an initial condition (e.g., assuming the quench to occur at $t = -\infty$), the bulk action S_b in Eq. (44) is characterized by having only the retarded Green's function G_{0R} as the Gaussian propagator. This can be readily seen from the absence of a term ϕ_q^2 in the bulk action (44) when $u_{q,c} = 0$. (In passing we note that this is the reason why an infinitesimally small term $\propto \phi_q^2$ which satisfies the fluctuation-dissipation theorem¹⁰⁰ has to be added to this action in order to recover the equilibrium Keldysh Green's function G_K .) In the presence of the quench at $t = 0$, instead, correlations appear already in the Gaussian theory because of the “forward propagation” of those which are present in the initial state. In fact, the Keldysh Green's function can be generically written as

$$iG_K(1,2) = \langle \phi_c(1)\phi_c(2) e^{-\frac{1}{2} \int_{\mathbf{k}} (\omega_{0k} \phi_{0q}^2 - \dot{\phi}_{0q}^2 / \omega_{0k})} \rangle_b, \quad (48)$$

where $n \equiv (\mathbf{k}_n, t_n)$ and $\langle \dots \rangle_b$ denotes the average taken on the bulk action only, i.e., without the quench. This

focus below on the case in which the post-quench parameter r is tuned at the value $r = r_c$, such that any macroscopic length-scale is removed from the correlation functions, which therefore becomes self-similar. In this sense, r_c corresponds to a critical point where the system becomes scale invariant. As we expect r_c to be perturbatively small in the coupling constant u_c (at least within the dimensional expansion), one can actually set $r = 0$ in the perturbative expression for the one-loop correction. Accordingly, from Eqs. (52) and (53), one finds

$$B_0 = 4\theta_N \Lambda^2, \quad B(t) = \theta_N (2\Lambda)^2 \frac{(2\Lambda t)^2 - 1}{[1 + (2\Lambda t)^2]^2}, \quad (54)$$

where we introduced, for later convenience, the constant

$$\theta_N = \frac{1}{8\pi^2} \frac{N+2}{96N} \Omega_0 u_c. \quad (55)$$

While B_0 in Eq. (54) has no finite limit for $\Lambda \rightarrow \infty$ and therefore its specific value depends on the specific form of the cut-off function f in Eq. (52), $B(t)$ becomes independent of it for $\Lambda t \gg 1$, with

$$B(t \gg \Lambda^{-1}) \simeq \frac{\theta_N}{t^2}. \quad (56)$$

In order to determine the one-loop correction δG_R to G_R , the tadpole $\mathcal{T}(t)$ in Eq. (51) has to be integrated with two Gaussian retarded functions G_{0R} , i.e.,

$$\begin{aligned} \delta G_R(q, t, t') &= \text{Diagram: A horizontal line with a loop (tadpole) in the middle, connected to two points labeled } t \text{ and } t'. \\ &= \int_0^\infty d\tau G_{0R}(q, t, \tau) i\mathcal{T}(\tau) G_{0R}(q, \tau, t'). \end{aligned} \quad (57)$$

Let us first discuss the contribution to δG_R due to the time-independent part B_0 of the tadpole \mathcal{T} , which generates an effective shift of the parameter r . In fact, this can be seen by reverting the argument, i.e., by considering the effect that a perturbatively small shift δr of the parameter r has on $G_R(q, t, t')$ for $t > t'$. Since δr couples to $-i\phi_c \cdot \phi_q$ in the exponential factor appearing in Eq. (36) (see also Eq. (44)) one has, up to first order in δr ,

$$\begin{aligned} G_R(q, t, t')|_{r+\delta r} &= G_R(q, t, t')|_r \\ &+ \delta r \int_0^\infty d\tau G_R(q, t, \tau)|_r G_R(q, \tau, t')|_r, \end{aligned} \quad (58)$$

where G_R has been expressed as the expectation value in Eq. (41). Comparing Eq. (57) with Eq. (58) one can see that the contribution of B_0 is the same as that one of a shift δr of the parameter r , i.e.,

$$r \mapsto r + B_0 + \mathcal{O}(u^2). \quad (59)$$

Accordingly, the resulting critical value r_c of r becomes $r_c = -B_0 + \mathcal{O}(u^2)$.

We consider next the term containing the time-dependent part $B(\tau)$ of the tadpole $\mathcal{T}(\tau)$ in Eq. (57). Because of the causality of the retarded Green's functions G_{0R} in the integrand of this equation, the integration domain in τ runs from t' to t , where henceforth we assume $t > t'$. Within this domain, $B(\tau)$ is well approximated by Eq. (56) as soon as $t, t' \gg \Lambda^{-1}$ and, correspondingly, the integral can be easily calculated. Focussing on a quench to the critical point $r = r_c$, the correction δG_R to the retarded Green's function G_R beyond its Gaussian expression can be written as

$$\delta G_R(q, t, t') = -G_{0R}(q, t, t') \theta_N F_R(qt, qt'), \quad (60)$$

with the scaling function

$$\begin{aligned} F_R(x, y) &= \frac{\sin(x+y)}{\sin(x-y)} [\text{Ci}(2x) - \text{Ci}(2y)] \\ &- \frac{\cos(x+y)}{\sin(x-y)} [\text{Si}(2x) - \text{Si}(2y)], \end{aligned} \quad (61)$$

where the sine integral $\text{Si}(x)$ and cosine integral $\text{Ci}(x)$ are defined, respectively, as

$$\text{Si}(x) = \int_0^x dt \frac{\sin t}{t}, \quad \text{Ci}(x) = - \int_x^\infty dt \frac{\cos t}{t}. \quad (62)$$

The scaling function $F_R(x, y)$ is a consequence of the absence of length- and time-scales at the critical point $r = r_c$. At short times $t, t' \ll q^{-1}$, the expansion of the cosine integral $\text{Ci}(x) \simeq \ln x$ renders a logarithmic term and, correspondingly, the leading order of Eq. (60) reads:

$$\delta G_R(q, t > t') = \theta_N (t + t') \ln(t/t'). \quad (63)$$

When these two times are well separated, i.e., $t' \ll t \ll q^{-1}$, the total retarded Green's function $G_R = G_{0R} + \delta G_R$ reads, to leading order,

$$G_R(q, t, t') = -t [1 - \theta_N \ln(t/t') + \mathcal{O}(u^2)], \quad (64)$$

where we used the fact that, at criticality, $G_{0R}(q, t, t') \simeq (t - t')$ for $t' < t \ll q^{-1}$, see Eq. (17a). Equation (64) suggests that the perturbative series can be resummed such that these logarithms result into an algebraic time dependence

$$G_R(q = 0, t \gg t') \simeq -t (t'/t)^{\theta_N}, \quad (65)$$

as is actually proven in Sec. VIC with the aid of the RG approach, which, inter alia, fixes the value of the still arbitrary coupling constant u_c in Eq. (55), providing a value of θ_N only as a function of the dimensionality d . In the analysis above we have assumed that both times t and t' involved in G_R are much longer than the microscopic time scale $\simeq \Lambda^{-1}$. In the opposite case in which the shorter time t' is at the “temporal boundary”, i.e., $t' \ll \Lambda^{-1} \ll t$ or, equivalently, $t' = 0$, it turns out that G_R up to one loop (see App. C)

$$G_R(q, t, 0) \propto -t [1 - \theta_N \ln(\Lambda t)], \quad (66)$$

has a logarithmic dependence on Λt and therefore shows a formal divergence as Λ grows. This divergence can be regularized only by means of a renormalization, i.e., of a redefinition of the fields appearing in the surface action (see Sec. VI) which is responsible for the emergence of the algebraic scaling at small times suggested by Eq. (64).

Consider now the Keldysh Green's function G_K : the one-loop contributions to this function can be expressed in terms of the same tadpole $\mathcal{T}(t)$ as the one in Eq. (50) contributing to the retarded Green's function G_R , the only difference being in the final integrations. In particular, iG_K receives two contributions

$$\begin{aligned} \delta iG_K(q, t, t') &= \\ &= \text{diagram 1} + \text{diagram 2} \\ &= \int_0^\infty d\tau iG_{0R}(q, t, \tau) \mathcal{T}(\tau) iG_{0K}(q, \tau, t') + (t \leftrightarrow t'), \end{aligned} \quad (67)$$

accordingly to the notation explained after Eq. (50). Taking into account the decomposition of \mathcal{T} in Eq. (51) one recognizes, as in the case of G_R , that the term in δiG_K which is proportional to B_0 is similar to the one which would be generated by a shift δr of the parameter r in G_{0K} and therefore it can be absorbed in a redefinition of this parameter as in Eq. (59).

The explicit calculation (reported in App. D) of the most singular correction δiG_K in Eq. (67) renders, to leading order in $q/\Lambda \ll 1$ and for $t, t' \gg \Lambda^{-1}$,

$$\delta iG_K(q, t, t') = iG_{0K}(q, t, t') \theta_N [2 \ln(q/\Lambda) - F_K(qt, qt')], \quad (68)$$

where θ_N is given in Eq. (55), while $F_K(x, y)$ is a scaling function defined as

$$F_K(x, y) = \text{Ci}(2x) + \text{Ci}(2y). \quad (69)$$

Note that the first contribution on the r.h.s. of Eq. (68) contains a term which grows logarithmically as Λ increases. Differently from the case of the retarded Green's function [see Eq. (66)], the divergence for $\Lambda \rightarrow \infty$ occurs even for finite values of t and t' . This is not surprising, because G_K is proportional to a product of two retarded functions $G_R(t, 0)$ with one vanishing time argument [see Eq. (49)], each of them carrying a logarithmic divergence, see Eq. (66). At short times $t, t' \ll q^{-1}$, the scaling function F_K in Eq. (68) reads $F_K(x \ll 1, y \ll 1) \simeq \ln(xy)$, with the logarithm coming from the series expansion of the cosine integral Ci for small arguments [see Eq. (62)]. Accordingly, the leading behaviour of the total Keldysh Green's function $G_K = G_{0K} + \delta G_K$ at short times is given by

$$iG_K(q, t, t') = iG_{0K}(q, t, t') [1 - \theta_N \ln(\Lambda^2 t t')]. \quad (70)$$

As for the retarded Green's function in Eq. (66), this expression suggests that the dependence on time beyond perturbation theory is actually algebraic, of the form

$$iG_K(q = 0, t, t') \sim (t t')^{1-\theta_N}, \quad (71)$$

where $G_{0K}(q, t, t')$ at criticality and for $t, t' \ll q^{-1}$ is given by Eq. (17b). The re-summation of the logarithms emerging in perturbation theory will be justified in Secs. V and VI on the basis of the renormalization-group approach. In the opposite regime of long times $t \gg q^{-1}$ (or, alternatively, large wavevectors $t^{-1} \ll q \ll \Lambda$), the equal-time Keldysh Green's function $G_K(q, t, t)$ (and, consequently, the momentum density in Eq. (21)) acquires an anomalous dependence on the momentum q , as can be seen from the re-summation of the logarithm in Eq. (68):

$$iG_K(q \gg t^{-1}, t, t) \propto q^{-2+2\theta_N}, \quad (72)$$

where we took into account that $F_K(x \gg 1, y \gg 1) \simeq \sin(2x)/(2x) + \sin(2y)/(2y)$. Accordingly, at long times, the stationary part of the Keldysh Green's function acquires the anomalous scaling (72): however, this anomalous scaling is not related to the anomalous dimension of the field $\phi_c(t)$, as it happens in equilibrium systems where the renormalization of $\phi_c(t)$ induces the anomalous dimension η^{111} . Instead, the reason for the scaling observed here turns out to be the renormalization of the initial quantum field ϕ_{0q} : this becomes clear by comparing Eq. (49) with Eq. (66). The initial-slip exponent θ_N thus characterizes not only the short-time critical behaviour of the Green's functions, but also their long-time form, in contrast to what happens in classical diffusive systems⁶⁴. This fact can be regarded as a peculiarity of the prethermal state, which retains memory of the initial state as a consequence of the fact that the dynamics induced by the post-quench Hamiltonian is analytically solvable.

B. Momentum distribution

The one-loop correction to the Gaussian momentum density $\mathcal{N}_{0\mathbf{k}}$ in Eq. (22) can be straightforwardly calculated on the basis of the definition (21), taking into account the perturbative correction δG_K to the Keldysh Green's function G_K in Eq. (68). Accordingly, the total momentum density $\mathcal{N}_{\mathbf{k}} = \mathcal{N}_{0\mathbf{k}} + \delta \mathcal{N}_{\mathbf{k}}$ at small momenta $k \ll \Lambda$ and microscopically long times $t \gg \Lambda^{-1}$ reads

$$\mathcal{N}_{\mathbf{k}}(t) + \frac{1}{2} = \frac{\Omega_0}{4k} \{1 + 2\theta_N [\ln(k/\Lambda) - F_{\mathcal{N}}(kt)]\}, \quad (73)$$

where the scaling function $F_{\mathcal{N}}(x)$ is defined as

$$F_{\mathcal{N}}(x) = \text{Ci}(2x) - \frac{\sin(2x)}{2x}, \quad (74)$$

with $F_{\mathcal{N}}(x \ll 1) \simeq \ln x$, while $F_{\mathcal{N}}(x \gg 1) \simeq -\cos(2x)/(2x)^2$. As discussed above in Sec. III A, the presence of logarithmic corrections suggests that the exact momentum density exhibits an algebraic dependence on momentum and/or time. In fact, after a resummation of the leading logarithms — motivated by the RG

approach presented in Secs. V and VI — the momentum density reads

$$\mathcal{N}_{\mathbf{k}}(t) + \frac{1}{2} \simeq \frac{\Omega_0}{4} \frac{1}{\Lambda} \left(\frac{\Lambda}{k} \right)^{1-2\theta_N} \mathcal{F}(kt), \quad (75)$$

where, consistently with the perturbative expansion, $\mathcal{F}(x) \equiv \exp[-2\theta_N F_N(x)]$ is a scaling function such that (see Eq. (74))

$$\mathcal{F}(x) = \begin{cases} x^{-2\theta_N} & \text{for } x \ll 1, \\ 1 & \text{for } x \gg 1. \end{cases} \quad (76)$$

Accordingly, at a certain time t , the momentum distribution $\mathcal{N}_{\mathbf{k}}(t) + 1/2$ displays different algebraic behaviors as a function of k , i.e., the unperturbed one $\sim k^{-1}$ for $k \ll t^{-1}$ and the anomalous one $\sim k^{-1+2\theta_N}$ for $k \gg t^{-1}$, with the crossover occurring at $k \sim t^{-1}$.

C. Green's functions in real space: light-cone dynamics

Based on the perturbative expressions reported in the previous section, we focus here on the Green's functions $G_{R,K}$ in real space and determine the corrections to their Gaussian expressions $G_{R,K}^0$ reported in Sec. II D. Up to the first order in the perturbative and dimensional expansion they can be written as,

$$G_{R,K} = G_{0R,K} + \delta G_{R,K}^{\delta r} + \delta G_{R,K}^u + \delta G_{R,K}^\epsilon + \mathcal{O}(u_c^2, \epsilon u_c, \epsilon^2) \quad (77)$$

where $\delta G_{R,K}^{\delta r}$ is the correction from the renormalization of the parameter r , i.e., from the constant part B_0 of the tadpole in Eq. (51), $\delta G_{R,K}^u$ contains the universal part coming from the time-dependent part $B(t)$ of the tadpole, while $\delta G_{R,K}^\epsilon$ comes from expanding $G_{0R,K}$ in Eq. (27) up to the first order in the dimensional expansion, with $\epsilon = 4-d$. Some details of the calculation are given in App. E. In what follows we assume that the systems is poised at the critical value $r = r_c$ discussed after Eq. (59), which cancels exactly the term $\delta G_{R,K}^{\delta r}$ in Eq. (77), therefore neglected below.

For the retarded Green's function we find, from Eq. (27), that corrections arise only on the light cone, namely (see App. E)

$$\delta G_R^\epsilon(x = t - t', t, t') = G_{0R}(x = t - t', t, t') \frac{\epsilon}{2} \ln x \quad (78)$$

while, from the Fourier transform of Eq. (60) in hyperspherical coordinates, one finds

$$\begin{aligned} \delta G_R^u(x, t, t') &= \frac{\theta_N}{4\pi^2 x^3} \int_0^{\Lambda x} dy y J_1(y) \\ &\times \left\{ \sin\left(\frac{y(t+t')}{x}\right) \left[\text{Ci}\left(\frac{2ty}{x}\right) - \text{Ci}\left(\frac{2t'y}{x}\right) \right] \right. \\ &\quad \left. - \cos\left(\frac{y(t+t')}{x}\right) \left[\text{Si}\left(\frac{2ty}{x}\right) - \text{Si}\left(\frac{2t'y}{x}\right) \right] \right\}, \end{aligned} \quad (79)$$

where θ_N is given by Eq. (55). For $t' \rightarrow 0$, a logarithmic singularity emerges in the previous expression due to $\text{Ci}(2t'y/x)$ (while $\text{Si}(2t'y/x)$ vanishes), which may be rewritten as:

$$G_R(x, t \gg t') \simeq G_{0R}(x, t \gg t') [1 + \theta_N \ln(\Lambda t')]. \quad (80)$$

Combined with Eq. (78), Eq. (80) and the expression of G_{0R} given in Eq. (29b), this implies

$$G_R(x = t - t', \Lambda t' \gg 1) \propto t^{-3/2+\epsilon/2}, \quad (81a)$$

$$G_R(x = t - t', \Lambda t' \ll 1) \propto t^{-3/2+\epsilon/2} t'^{\theta_N}. \quad (81b)$$

Accordingly, the presence of interaction affects only the behaviour of G_R on the light cone $x = t - t'$, provided the initial perturbation was applied at a very short time t' , modifying the Gaussian scaling (29b) through an anomalous scaling with respect to t' . Note, however, that the scaling law $\propto t'^{\theta_N}$ is visible only for $t' \ll \Lambda^{-1}$, i.e., only before a non-universal microscopic time controlled by lattice effects, and therefore its presence is expected to be masked by non-universal contributions which dominate at such short time scales.

Turning to $G_K(x, t, t')$, for simplicity we focus here only on its behavior at equal times $t' = t$: this allows us to study the structure of G_K as far as the light cone at $x = t + t' = 2t$ is concerned. Outside this light cone, i.e., for $2t \ll x$, iG_K vanishes as it is essentially determined by the pre-quench state which is characterized by very short-range correlation. Accordingly, we consider the expression of iG_K only on and inside the light cone. In particular, on the light cone we find (see App. E for details)

$$\delta G_K^\epsilon(x = 2t, t, t) = G_{0K}(x = 2t, t, t) \frac{\epsilon}{2} \ln x, \quad (82)$$

while inside it (see App. E),

$$\delta G_K^\epsilon(x \ll 2t, t, t) = G_{0K}(x \ll 2t, t, t) \epsilon \ln x. \quad (83)$$

The loop correction δG_K^u to $G_K(x, t, t)$ follows from the Fourier transform in hyperspherical coordinates of Eqs. (68) and (69), and takes the form

$$\begin{aligned} i\delta G_K^u(x, t, t) &= 2\theta_N \frac{\Omega_0}{8\pi^2 x} \int_0^\Lambda dq J_1(qx) \\ &\times \left\{ [1 - \cos(2qt)] [-\text{Ci}(2qt) + \ln(q/\Lambda)] \right. \\ &\quad \left. + \sin(2qt) \text{Si}(2qt) \right\}. \end{aligned} \quad (84)$$

From this expression it is straightforward to see that at long times, inside the light cone (see App. E)

$$i\delta G_K^u(x \ll 2t, t, t) = -2\theta_N G_{0K}(x \ll 2t, t, t). \quad (85)$$

On the light cone $x = 2t$, instead, there are no logarithmic corrections to δG_K^u , as shown in App. E. Combining Eqs. (83) and Eq. (85) with G_{0K} in Eqs. (31c) we obtain the expression of the scaling behavior of G_K inside the

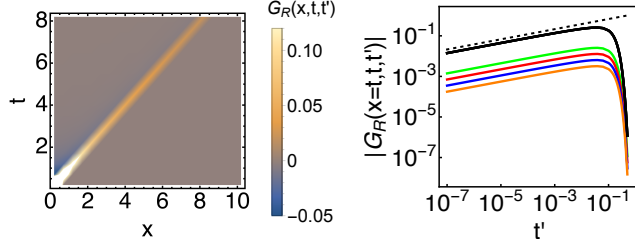


FIG. 1. (Color online). $G_R(x, t, t')$ after a quench to the dynamical critical point for the $O(N)$ model with $N \rightarrow \infty$ and spatial dimension $d = 3$, with an UV cut-off $\Lambda = \pi/2$. Left panel: $G_R(x, t, t')$ as a function of x and t for $t' = 0.1$. The numerical value of G_R is expressed as indicated by the color code in the legend. Right panel: plot of $G_R(x, t, t')$ on a double logarithmic scale as a function of t' for various values of $x = t$. The coloured solid lines correspond to sections with $t = 80$ (lower, orange line), $t = 40$ (lower-middle, blue line), $t = 20$ (upper-middle, red line), and $t = 10$ (upper, green line), respectively. The solid black line corresponds to a rescaling of the solid coloured line with t , i.e., to $tG_R(x, t, t')$, over which the coloured solid lines collapse. For small values of t' , $G_R(x, t, t')$ displays an algebraic growth $\propto t'^{0.25}$, as indicated by the corresponding dashed line, which is in agreement with Eq. (81b).

light cone, while combining Eq. (82) with Eq. (31b), we obtain that one for the scaling on the light cone. The corresponding final expressions are (see also Eq. (31a)):

$$iG_K(x \gg 2t) \simeq 0, \quad (86a)$$

$$iG_K(x = 2t) \simeq t^{-3/2+\epsilon/2}, \quad (86b)$$

$$iG_K(x \ll 2t) \simeq x^{-2+\epsilon-2\theta_N}. \quad (86c)$$

In summary, interactions modify the correlation function $G_K(x, t, t')$ as follows: on the light cone $x = 2t$, G_K still decays $\propto 1/x^{(d-1)/2}$ upon increasing x , as in the Gaussian case (see Eq. (34b)), while inside the light cone, i.e., for $x \ll 2t$, this decay changes qualitatively compared to that case (see Eq. (34c)) and becomes faster $\propto 1/x^{d-2+2\theta_N}$. The latter is consistent with the slower decay⁵⁸ of the momentum distribution $iG_K(q \gg t^{-1})$ upon increasing q , compared to its Gaussian expression, found in the previous section, see Eq. (72). It is also interesting to note that while in quenches of one-dimensional systems ($d = 1$), which belong to the Luttinger liquid universality class, interactions do modify the scaling of G_K on the light cone³⁶, this is not the case for the present system, at least up to the first order in perturbation theory.

In order to test the validity of the scaling behavior summarized above (see Eqs. (81) and (86)) beyond perturbation theory, we consider the $O(N)$ model in the exactly solvable limit $N \rightarrow \infty$ which has been considered, e.g., in Ref. 59. In particular, after a deep quench to the critical point, $G_R(x, t, t')$ and $iG_K(x, t, t')$ can be calculated, respectively, via a Fourier transform of $2\text{Im}[f_{\mathbf{k}}(t)f_{\mathbf{k}}^*(t')]$ and $2|f_{\mathbf{k}}(t)|^2$, where the evolution of

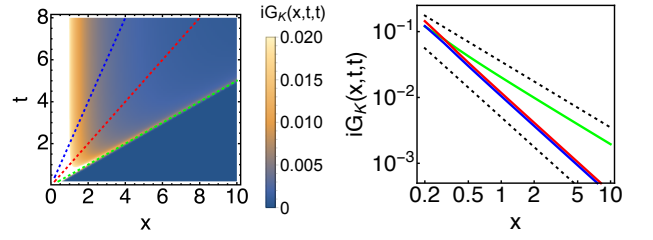


FIG. 2. (Color online). $G_K(x, t, t)$ after a quench to the dynamical critical point for the $O(N)$ model with $N \rightarrow \infty$ and spatial dimension $d = 3$, with an UV cut-off $\Lambda = \pi/2$. Left panel: $G_K(x, t, t)$ as a function of x and t . The numerical value of G_K is expressed as indicated by the color code in the legend. The dashed lines correspond to the sections $x = t/2$ (upper, blue line), $x = t$ (middle, red line) and $x = 2t$ (lower, green line), respectively, which are highlighted in the right panel. Right panel: plot of $G_K(x, t, t)$ on a double logarithmic scale as a function of x along the sections of the (x, t) -plane highlighted in the left panel, i.e., the dashed lines correspond to the sections $x = t/2$ (lower, blue line), $x = t$ (middle, red line) and $x = 2t$ (upper, green line), respectively. Along the first two and the last sections, $G_K(x, t, t)$ displays an algebraic decay $\propto x^{-3/2}$ and $\propto x^{-1}$, respectively, as indicated by the corresponding lower and upper dashed lines. This behaviour is in agreement with Eqs. (86b) and (86c).

the complex coefficients $f_{\mathbf{k}}$ is determined numerically, according to Eqs. (13), (7) and (11) of Ref. 59.

In Fig. 1 we report the resulting $G_R(x, t, t')$ in spatial dimension $d = 3$, as a function of x and t (with fixed $t' \ll t$) on the left panel, and as a function of t' on the right panel. The light cone of $G_R(x, t, t')$ is clearly visible in the left panel: in fact, upon varying x for fixed $t, t' \ll t$, $G_R(x, t, t')$ vanishes for $x \ll t$ and $x \gg t$, while it grows as it approaches the light cone. The colored solid curves on the right panel, instead, show that $G_R(x = t, t, t')$ grows algebraically as $G_R(x = t, t, t') \propto t'^{\theta_N}$ for small values of t' and various values of t , in agreement with Eq. (81b), with the proper value $\epsilon = 1$ and $\theta_N \rightarrow \theta_\infty = 1/4$ (see, c.f., Eq. (107)) in $d = 3$. Moreover, by rescaling the colored solid curves by t , i.e., by plotting $tG_R(x = t, t, t')$ as a function of t' with fixed t , all the curves collapse on the master curve indicated by the solid black line, in agreement with the algebraic dependence on t predicted by Eq. (81b). Analogous agreement with the scaling behaviors in Eq. (81) is found for $d \neq 3$, showing that these relations hold beyond perturbation theory.

In Fig. 2 we report $G_K(x, t, t)$ for the same model in spatial dimensionality $d = 3$, as a function of x and t on the left panel, while on the right panel iG_K is shown as a function of x along the cuts in the (x, t) -plane indicated by the corresponding dashed lines on the left panel. The light cone of $iG_K(x, t, t)$ is clearly visible in the left panel: in fact, $iG_K(x, t, t)$ vanishes for $x > 2t$, while it grows as it approaches either this light cone or the line $x = 0$ (where the white color indicates large values compared to

those displayed in the rest of the plot and indicated by the color code) from within the region with $x < 2t$, and it vanishes upon increasing x and t . The curves on the right panel, instead, show that $G_K(x, t, t) \propto x^{-3/2}$ (lowermost and intermediate solid lines) as x grows within the light cone, i.e., with $x = t/2$ and $x = t$, while $G_K(x, t, t) \propto x^{-1}$ (uppermost line) exactly on the light cone $x = 2t$, in agreement with Eqs. (86b) and (86c), respectively, with the proper value $\epsilon = 1$ and $\theta_N \rightarrow \theta_\infty = 1/4$ (see, c.f., Eq. (107)) in $d = 3$. Analogous agreement with the scaling behaviors in Eq. (86) is found for $d \neq 3$, showing that these relations hold beyond perturbation theory.

IV. MAGNETIZATION DYNAMICS

In the previous sections we focused on the temporal evolution of two-time quantities after a quench, assuming that the $O(N)$ symmetry of the initial state is not broken by the dynamics. However, as it happens in classical systems (see, e.g., Ref. 64), universal features are expected to emerge also in the evolution of one-time quantities such as the order parameter, when the symmetry of the initial state is broken by a suitable small external field $\mathbf{h}(\mathbf{x})$, which is then switched off at $t > 0$. As we show below, the short-time behavior of the mean order parameter $\mathbf{M}(\mathbf{x}, t)$ (henceforth referred to as the magnetization) actually provides one additional direct measure of the short-time exponent θ_N encountered in the previous sections. In Secs. IV A and IV B below, we study the evolution of \mathbf{M} in perturbation theory and by solving the Hartree-Fock equations (which are exact in the limit $N \rightarrow \infty$), respectively.

A. Magnetization dynamics from perturbation theory

The inclusion of a symmetry-breaking field in the pre-quench Hamiltonian H_0 induces a non-vanishing value of the order parameter $\mathbf{M}(\mathbf{x}, t) \equiv \langle \phi(\mathbf{x}, t) \rangle$, which evolves after the quench. If the post-quench Hamiltonian H is tuned to its critical point, one expects this evolution to be characterized by universal exponents. More precisely, in this section we investigate the effect of modifying $H_0 = H(\Omega_0^2, 0)$ in Eq. (2) as:

$$H_0 = \int_{\mathbf{x}} \left[\frac{1}{2} \mathbf{\Pi}^2 + \frac{1}{2} (\nabla \phi)^2 + \frac{\Omega_0^2}{2} \phi^2 - \mathbf{h} \cdot \phi \right], \quad (87)$$

where $\mathbf{h} \equiv \mathbf{h}(\mathbf{x})$ is the external field which breaks explicitly the $O(N)$ symmetry of H_0 and therefore of the initial state. Within the Keldysh formalism, this corresponds to a change in the initial action (46), which now reads, in

momentum space:

$$S_s = i \int_{\mathbf{k}} \frac{\omega_{0k}}{2} \left[\left(\phi_{0c, \mathbf{k}} - \sqrt{2} \frac{\mathbf{h}_{\mathbf{k}}}{\omega_{0k}^2} \right)^2 \tanh(\beta \omega_{0k}/2) + \phi_{0q, \mathbf{k}}^2 \coth(\beta \omega_{0k}/2) - 2\beta \frac{|\mathbf{h}_{\mathbf{k}}|^2}{\omega_{0k}^3} \right], \quad (88)$$

where $\mathbf{h}_{\mathbf{k}}$ indicates the Fourier transform of $\mathbf{h}(\mathbf{x})$ and ω_{0k} is given in Eq. (10). Recalling that the integration by parts of $\dot{\phi}_q \cdot \dot{\phi}_c$ in the bulk action (44) generates a term proportional to $\dot{\phi}_{0q} \cdot \phi_{0c}$ in the initial action S_s , one can easily see from the change of variables $\phi_{0c, \mathbf{k}} \rightarrow \phi_{0c, \mathbf{k}} + \sqrt{2} \mathbf{h}_{\mathbf{k}} / \omega_{0k}^2$ that taking the functional average $\langle \dots \rangle_{\mathbf{h}}$ with $\mathbf{h} \neq \mathbf{0}$ is equivalent to calculating it with $\mathbf{h} = \mathbf{0}$, but with a modified weight, i.e.,

$$\langle \dots \rangle_{\mathbf{h}} = \langle \dots e^{-i\sqrt{2} \int_{\mathbf{k}} \mathbf{h}_{\mathbf{k}} \cdot \dot{\phi}_{0q, \mathbf{k}} / \omega_{0k}^2} \rangle_{\mathbf{h}=\mathbf{0}}. \quad (89)$$

This means that any average in the presence of the field \mathbf{h} can be calculated as in its absence by considering the insertion of the operator shown in the r.h.s. of Eq. (89). Let us first consider the Gaussian approximation, i.e., set $u = 0$. Using either Eq. (89) or by repeating the calculations of Sec. II B with the pre-quench Hamiltonian (87), one finds that

$$\langle \phi_{\mathbf{k}}(t) \rangle = \frac{\mathbf{h}_{\mathbf{k}}}{\omega_{0k}^2} \cos(\omega_k t). \quad (90)$$

Moreover, while G_R is not affected by $\mathbf{h} \neq \mathbf{0}$, the Keldysh Green's function is modified because of the presence of a non-vanishing $\langle \phi_{\mathbf{k}}(t) \rangle$ as:

$$\langle \{ \phi_{i, \mathbf{k}}(t), \phi_{j, \mathbf{k}'}(t') \} \rangle = \delta_{i,j} \delta_{\mathbf{k}, -\mathbf{k}'} i G_K(k, t, t') + 2 \langle \phi_{i, \mathbf{k}}(t) \rangle \langle \phi_{j, \mathbf{k}'}(t') \rangle, \quad (91)$$

where $G_K(k, t, t')$ is the same as in Eq. (14b), i.e., the Keldysh Green's function corresponding to $\langle \phi_{\mathbf{k}}(t) \rangle = \mathbf{0}$. In the following, we assume the initial external field \mathbf{h} to be spatially homogeneous and aligned with the first component of the field, i.e.,

$$h_i(\mathbf{x}) = \delta_{i,1} M_0 \Omega_0^2. \quad (92)$$

Note that the residual symmetry of the initial state under $O(N-1)$ transformations involving the components $i = 2, \dots, N$ of the field which are transverse to \mathbf{h} (collectively indicated by \perp) implies that the only non-vanishing post-quench component of the magnetization is the one along $i = 1$ (referred to as longitudinal and indicated by \parallel), i.e.,

$$M_i(\mathbf{x}, t) = \delta_{i,1} M(t). \quad (93)$$

Accordingly, from Eq. (90), one finds

$$M(t) = M_0 \cos(\sqrt{r} t), \quad (94)$$

which shows that the order parameter M oscillates indefinitely with angular frequency \sqrt{r} and that at the Gaussian critical point $r = 0$ it does not evolve.

Beyond the Gaussian approximation $u = 0$ considered above, one has to take into account the presence of interaction in the post-quench Hamiltonian, which affects the temporal evolution of M in Eq. (93). Focussing, for concreteness, on the critical point $r = r_c$ (with $r_c \neq 0$ due to the presence of interaction, see Sec. III A, in particular Eq. (59)) and on a deep quench, the same perturbative expansion illustrated in Sec. III renders, at one loop, a correction $M(t) = M_0 + \delta M(t)$ with

$$\delta M(t) = \int_0^t dt' G_{0R}(k=0, t, t') \times \left[B(t') M_0 + \frac{2u_c}{4!N} M_0^3 \right], \quad (95)$$

where only the time-dependent part $B(t')$ of the tadpole $i\mathcal{T}(t')$ in Eq. (51) contributes to Eq. (95) because the time-independent contribution B_0 is cancelled by assuming $r = r_c$, as explained for the correlation functions in Sec. III A. Taking into account the explicit expression of $B(t)$ in Eq. (54) and of G_{0R} in Eq. (16a) it is easy to find that δM exhibits a logarithmic dependence on t for $\Lambda t \gg 1$, and a quadratic growth in time:

$$\delta M(t) \simeq M_0 \theta_N \ln(\Lambda t) - M_0 \frac{u_c}{4!N} (M_0 t)^2, \quad (96)$$

where θ_N is given in Eq. (55). This expression indicates that the dynamics of the magnetization $M(t)$ is characterized by two regimes: for $t \ll t_i \simeq 1/(M_0 \sqrt{u_c})$ the term proportional to t^2 is negligible, and therefore δM is dominated by the logarithmic term. This suggests that, after a proper resummation of the logarithms (which is justified further below in Sec. VI), the magnetization grows algebraically as a function of time as

$$M(t) \propto M_0 (\Lambda t)^{\theta_N}. \quad (97)$$

On the other hand, when $t \gtrsim t_i$ the second term in the r.h.s. of Eq. (96) becomes dominant, signalling a crossover to a different dynamical regime. Notice that, for these two regimes to be distinguishable, one must require that $\Lambda t_i \gg 1$, i.e., $\Lambda \gg M_0 \sqrt{u_c}$. In Sec. VID, we assess the existence of these two dynamical regimes by showing that a general scaling form can be derived for the magnetization M after a quench to the critical point, which involves the scaling variable associated with M_0 as in Eq. (172).

B. Magnetization dynamics from a self-consistent Hartree-Fock approximation

In the previous subsection, the post-quench dynamics of the model in Eq. (88) has been studied via a perturbative dimensional expansion around the upper critical

dimensionality $d = d_c$. Within this approach, the temporal evolution of $\mathbf{M}(\mathbf{x}, t)$ turns out to be compatible with the algebraic law in Eq. (97), which is actually derived in Sec. VID on the basis of suitable RG equations. However, in order to check the emergence of such an algebraic behavior beyond perturbation theory, we consider here the case $N \rightarrow \infty$ of the present model which, as discussed in detail in Ref. 59 for $\mathbf{h} = \mathbf{0}$, is exactly solvable beyond perturbation theory, i.e., for a generic value of the spatial dimensionality d . In fact, in this limit, the Hartree-Fock or self-consistent approximation where the $(\phi^2)^2$ interaction is approximated by a quadratic “mean-field” term $\propto \phi^2 \langle \phi^2 \rangle$ becomes exact. The latter corresponds to an effective, time-dependent square “mass” $r(t)$ which is calculated self-consistently, see, e.g., Refs. 55–57, and 59. As in Sec. IV A, the initial magnetic field \mathbf{h} is taken along the direction of the first component of the field, see Eq. (92), and therefore the post-quench order parameter is given by Eq. (93) with $M(0) = \sqrt{N} M_0$, where the multiplicative factor \sqrt{N} is needed in order to have a well-defined limit $N \rightarrow \infty$, as discussed below.

The one-loop equations governing the time evolution of the system are⁵⁵:

$$\left[\partial_t^2 + r(t) - \frac{u}{3} M^2(t) \right] M(t) = 0, \quad (98a)$$

$$\left[\partial_t^2 + q^2 + r(t) - \frac{u}{6N} i G_K^\parallel(x=0, t, t) \right] G_K^\parallel(q, t, t') = 0, \quad (98b)$$

$$\left[\partial_t^2 + q^2 + r(t) - \frac{u}{3} M^2(t) \right] G_K^\perp(q, t, t') = 0, \quad (98c)$$

$$r(t) = r + \frac{u}{2} \left[M^2(t) + \frac{1}{2N} i G_K^\parallel(x=0, t, t) + \frac{N-1}{6N} i G_K^\perp(x=0, t, t) \right], \quad (98d)$$

where G_K^\parallel and G_K^\perp are the Keldysh Green’s functions of the longitudinal and of a generic transverse component of the field, respectively, while $r(t)$ is the time-dependent effective parameter self-consistently determined from Eq. (98d)

To solve this system of equations we need to specify initial conditions for the magnetization and the Green’s functions. Since, from the Heisenberg equations of motion $\dot{\phi} = \Pi$, with ϕ and Π the components of the fields and its conjugate momentum along \mathbf{h} , and since in the initial state $\langle \Pi(t=0) \rangle = 0$, one concludes that $\dot{M}(t)|_{t=0} = 0$. The initial conditions for the $G_K^{\parallel, \perp}(q, t, t')$, $\partial_t G_K^{\parallel, \perp}(q, t, t')$ and $\partial_t \partial_{t'} G_K^{\parallel, \perp}(q, t, t')$ are⁵⁵:

$$G_K^{\parallel, \perp}(q, t=0, t'=0) = \frac{1}{\sqrt{q^2 + r_{\parallel, \perp}(0)}}, \quad (99a)$$

$$\partial_t G_K^{\parallel, \perp}(q, t=0, t'=0) = 0, \quad (99b)$$

$$\partial_t \partial_{t'} G_K^{\parallel, \perp}(q, t=0, t'=0) = \sqrt{q^2 + r_{\parallel, \perp}(0)}, \quad (99c)$$

where we introduced

$$r_{\parallel}(0) = r(0), \quad (100a)$$

$$r_{\perp}(0) = r(0) - \frac{u}{3}M_0^2. \quad (100b)$$

The set of Eqs. (98) is not analytically solvable and in general one has to resort to approximations or to a numerical evaluation. However, in the limit $N \rightarrow \infty$, one can see from Eqs. (98) that the quadratic term $\propto M^2(t)$ can be neglected for times $t \ll t_i$, with t_i defined by the condition

$$r + \frac{u}{12}iG_K^{\perp}(x=0, t_i, t_i) \simeq \frac{u}{3}M^2(t_i); \quad (101)$$

with these assumptions, Eqs. (98) simplify to:

$$[\partial_t^2 + r(t)] M(t) = 0, \quad (102a)$$

$$[\partial_t^2 + q^2 + r(t)] G_K^{\parallel, \perp}(q, t, t') = 0, \quad (102b)$$

$$r(t) = r + \frac{u}{12}iG_K^{\perp}(x=0, t, t). \quad (102c)$$

The critical value r_c of the parameter r can be obtained via the ansatz suggested in Ref. 57, namely:

$$r_c = -\frac{u}{4!} \int_q e^{-q/\Lambda} \frac{2q^2 + \Omega_0}{q^2 \sqrt{q^2 + \Omega_0}}, \quad (103)$$

where the integral is regularized through the smooth cut-off function $\exp(-q/\Lambda)$. This regularization is used to evaluate all the integrals over momentum in the self-consistent Eqs. (102): the choice of a smooth cutoff is crucial in order to avoid spurious non-universal oscillations which would mask the real universal behaviour of the dynamical quantities⁵⁹. For $r = r_c$, the effective parameter $r(t)$ at long times $\Lambda t \gg 1$ has (up to rapidly oscillatory terms) the following universal behavior⁵⁹

$$r(t) = \frac{a}{t^2}, \quad (104)$$

in which the value of a can be obtained analytically and for $2 < d < 4$ is given by⁵⁹

$$a = \frac{d}{4} \left(1 - \frac{d}{4} \right). \quad (105)$$

By inserting this expression into Eq. (102a), we then obtain

$$M(t \gg \Lambda^{-1}) \propto (\Lambda t)^{\theta_{\infty}}, \quad (106)$$

where⁵⁹

$$\theta_{\infty} = 1 - \frac{d}{4}, \quad (107)$$

which agrees with Eq. (55) in the limit $N \rightarrow \infty$ and up to the first order in $\epsilon = 4 - d$ after the fixed-point value of the coupling constant u_c is introduced in that expression, according to what we discuss further below in

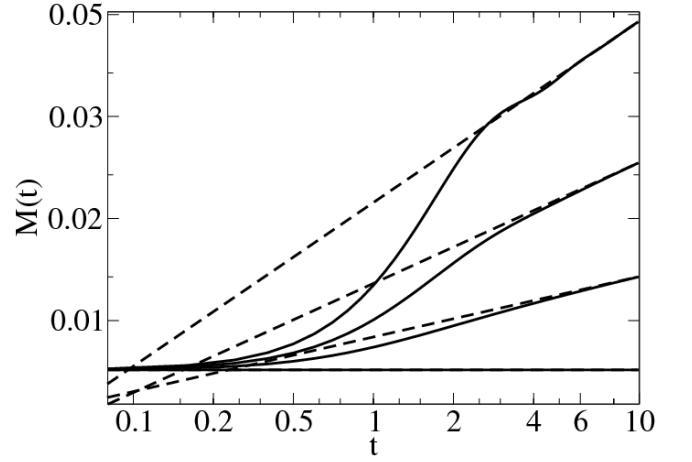


FIG. 3. Magnetization $M(t)$ as a function of time t (on a double logarithmic scale) for various values of the spatial dimensionality $d = 2.5, 3, 3.5$, and 4 , from top to bottom. The solid lines indicate the numerical solution of the self-consistent equations (102) with $r = r_c$ and $M_0 = 0.005$. The dashed lines, instead, indicate the corresponding algebraic increase $\propto t^{\theta_{\infty}}$ (see Eq. (106)) with θ_{∞} reported in Eq. (107). These numerical solutions correspond to the case $\Omega_0 = 25$, $\Lambda = 1$ and to an exponential cutoff function e^{-q} introduced for the regularization of the integrals as in Eq. (103) (see also the discussion in Ref. 59).

Sec. VB (see, in particular, Eq. (127)). Accordingly, for times smaller than t_i estimated below (but much longer than the microscopic scale Λ^{-1}), the order parameter increase is controlled by the universal exponent θ_{∞} . The time dependence of $M(t)$ obtained from the numerical solution of Eqs. (102a), (102b), and (102c) with $r = r_c$ and various values of the spatial dimensionality d are shown in Fig. 3, where they are also compared with the analytical prediction in Eq. (106).

In order to estimate the timescale t_i beyond which Eqs. (102) are no longer valid, we analyse Eq. (101). First, note that, as $d \rightarrow 4$, $M(t) \simeq \mathcal{A}(\Lambda t)^{\theta_{\infty}}$ becomes almost independent of t and equal to M_0 (see Fig. 3) and therefore $\mathcal{A} \simeq M_0$. Hence, for d sufficiently close to 4 we may safely approximate $M(t) \simeq M_0(\Lambda t)^{\theta_{\infty}}$. Using this approximation in Eq. (101) and solving for t_i with the aid of Eq. (104), we obtain, to leading order in θ_{∞} , the following expression for t_i :

$$\Lambda t_i \simeq \frac{\sqrt{3}\Lambda}{M_0} \sqrt{\frac{a}{u}} \propto \frac{1}{M_0}. \quad (108)$$

This estimate of t_i is compatible for $d \rightarrow 4$ with the scaling derived from the RG analysis in Sec. VID, see Eq. (172).

V. RENORMALIZATION GROUP: WILSON APPROACH

In the previous sections we discussed the emergence of scaling behaviors on the basis of the results of perturbative calculations at the lowest order in the interaction strength, with the sole exception of the exact results presented in Sec. IV B. Here and in Sec. VI, instead, we demonstrate that such scaling behaviors follow from the existence of a fixed point in the renormalization-group (RG) flow which we determine within the Wilson and Callan-Symanzik approach, respectively. In particular, this allows us to determine the fixed-point value of u_c on which θ_N in Eq. (55) depends and which is not fixed by the perturbative calculations above.

A. Canonical dimensions

In order to determine the canonical dimensions of the relevant coupling constants and parameters in the Keldysh actions (43) and (44) which describe the non-equilibrium dynamics of the model, we consider the scaling of the corresponding Gaussian theory in the case of a deep quench $\Omega_0 \gg \Lambda$, with a vanishing temperature $T = 0$ of the initial state. Accordingly, the only operators appearing in the initial action S_s in Eq. (46) are:

$$\omega_{0k} \phi_{c,q}^2 = \left[\Omega_0 + \frac{k^2}{2\Omega_0} + O(k^4/\Omega_0^3) \right] \phi_{c,q}^2. \quad (109)$$

Since we consider $\Omega_0 \gg \Lambda$ and the leading term in the expansion of ω_{0k} (see Eq. (10)) is the most relevant in the RG sense, we can approximate $\omega_{0k} \simeq \Omega_0$. However, as the classical and quantum fields $\phi_{c,q}$ can in principle have different flows under RG, we allow the coefficients of the corresponding terms ϕ_c^2 and ϕ_q^2 in the initial action to be different, denoting them by Ω_{0c} and Ω_{0q} , respectively.

In order to study the canonical scaling of the model, we introduce an arbitrary momentum scale μ , such that space and time coordinates scale as $\mathbf{x} \sim \mu^{-1}$ and $t \sim \mu^{-z}$, where z is the dynamical critical exponent with $z = 1$ within the Gaussian approximation. While in thermal equilibrium, the canonical dimensions of the fields and of the couplings follow immediately from requiring the action to be dimensionless, this requirement is no longer sufficient in the present case, given the large number of fields involved in the total Keldysh action $S_s + S_b$. Accordingly, in order to fix these dimensions we impose the following two conditions: (i) the canonical dimensions of the fields at the surface $t = 0$ are the same as in the bulk $t > 0$ and (ii) Ω_{0q} is dimensionless. Condition (i) — which do not necessarily hold beyond the Gaussian approximation — basically amounts at requiring the continuity of the fields during the quench and, in fact, it is the functional translation of the operatorial conditions $\phi(0^-) = \phi(0^+)$ and $\Pi(0^-) = \Pi(0^+)$ ^{59,73,74}. Condition (ii), instead, is based on the fact that Ω_{0q} plays the role of

an effective temperature — see the discussion in Sec. II C — and temperatures are known to be scale-invariant in equilibrium quantum systems outside the quantum critical regime⁸⁰. Using these conditions, from Eqs. (44) and (43) one finds the scaling of the fields

$$\phi_c(x) \sim \mu^{(d-2)/2}, \quad \phi_q(x) \sim \mu^{d/2}, \quad (110)$$

and of the couplings

$$\Omega_{0c} \sim \mu^2, \quad r \sim \mu^2, \quad u_c \sim \mu^{4-d}, \quad u_q \sim \mu^{2-d}, \quad (111)$$

from which it follows that the upper critical dimensionality is $d = d_c = 4$. Notice that the quantum vertex u_q is irrelevant for $d > 2$. This conclusion is consistent with the dimensional crossover presented and discussed in Sec. II C: the upper critical dimensionality, which is 3 for a quantum system in equilibrium at $T = 0$, is increased to 4 by the effect of a deep quench. Note, in addition, that the positive canonical dimension of the classical initial “mass” Ω_{0c} implies that it grows indefinitely under RG and therefore its stable fixed-point value is $\Omega_{0c} = +\infty$.

B. One-loop corrections

In order to account for the effects of the interaction, in this subsection we derive the RG equations in perturbation theory up to one loop, based on a momentum-shell integration¹¹². It turns out that this is sufficient to highlight the emergence of an initial-slip exponent, while additional effects, such as the emergence of dissipative terms and thermalization, can be captured only by including contributions at two or more loops, which account for inelastic processes^{94,108,109,113}.

We assume that the actions in Eqs. (43) and (44) are defined in momentum space with momenta of modulus $k < \Lambda$, where Λ is a cutoff set by the inverse of some microscopic length scale, e.g., a possible lattice spacing. In order to implement the RG transformation in momentum space^{90,91} we decompose each field $\phi(k) \in \{\phi_c, \phi_q\}$ as a sum of a “slow” and a “fast” component $\phi_<(k)$ and $\phi_>(k)$, respectively, i.e., as $\phi = \phi_< + \phi_>$, where $\phi_<(k)$ involves only modes with $0 \leq k < \Lambda - d\Lambda$, while $\phi_>(k)$ involves the remaining ones within the momentum shell $\Lambda - d\Lambda \leq k \leq \Lambda$ of small thickness $d\Lambda = \delta\ell\Lambda$, with $0 < \delta\ell \ll 1$. Then, the fast fields are integrated out from the action: this generates new terms which renormalize the bare couplings of the action of the remaining slow fields. Here we do this integration perturbatively in the coupling constant and therefore the exponential factor in Eq. (36) is first split into the Gaussian and interaction parts $S_G + S_{\text{int}}$, then expanded up to second order in the interaction terms, with the remaining expectation values of the fast fields calculated with respect to their Gaussian action S_G . Finally, the result is re-exponentiated using a

cumulant expansion:

$$\begin{aligned} & \int \mathcal{D}[\phi_>, \phi_<] e^{i(S_G + S_{\text{int}})} \\ & \simeq \int \mathcal{D}[\phi_>, \phi_<] e^{iS_G} \left[1 + i S_{\text{int}} - \frac{1}{2} S_{\text{int}}^2 \right] \\ & \simeq \int \mathcal{D}\phi_< \exp \left\{ i S_G^< + i \langle S_{\text{int}} \rangle_> - \frac{1}{2} \langle S_{\text{int}}^2 \rangle_>^c \right\}, \end{aligned} \quad (112)$$

where $S_G^< = S_G[\phi_<]$ is the Gaussian action calculated on the slow fields, $\langle \dots \rangle_>$ denotes the functional integration with respect to $S_G[\phi_>]$, while the superscript c indicate the connected average. For the case we are interested in,

$$\begin{aligned} S_{\text{int}} = & -\frac{2u_c}{4!N} \int_{\mathbf{x}, t > 0} (\phi_q \cdot \phi_c) |\phi_c|^2 \\ & -\frac{2u_q}{4!N} \int_{\mathbf{x}, t > 0} (\phi_q \cdot \phi_c) |\phi_q|^2. \end{aligned} \quad (113)$$

After the integration, coordinates and fields are rescaled in order to restore the initial value of the cutoff:

$$x \rightarrow bx, \quad t \rightarrow b^z t, \quad \phi_i \rightarrow b^{\zeta_i} \phi_i, \quad \phi_{0,i} \rightarrow b^{\zeta_{0,i}} \phi_{0,i}, \quad (114)$$

where $i = c, q$ and $b = \Lambda/(\Lambda - d\Lambda) \simeq 1 + \delta\ell$. The field scaling dimensions ζ_i and $\zeta_{0,i}$ — the Gaussian values of which can be read directly from Eq. (110) — generically acquire an anomalous contribution due to the interactions. The resulting action for the slow fields (which, after rescaling are renamed as the original fields ϕ) then acquires contributions from both integration and rescaling and the expression of the new effective parameters (denoted below by $'$) in the action in terms of the original ones constitute the so-called RG recursion relations.

In passing, we mention that, strictly speaking, the functional integral on the l.h.s. of Eq. (112) is actually equal to one and therefore the renormalization procedure outlined above — which in equilibrium is typically implemented on the partition function of the statistical system under study — would look unnecessary. However, this is no longer the case if one either adds external sources for the field (as done, e.g., in App. A) or, equivalently, implements integration and rescaling directly on correlation functions.

Assuming, for the sake of clarity, a vanishing temperature $T = 0$ in the initial state, we find the following RG recursion relations (see App. F for details) at the lowest order in perturbation theory

$$r' = b^2 (r + u_c I_1), \quad (115a)$$

$$u'_c = b^{2-2\zeta_c} u_c (1 - u_c I_2), \quad (115b)$$

$$u'_q = b^{2-2\zeta_q} u_q (1 - u_q I_2), \quad (115c)$$

where the integrals $I_{1,2}$ can be decomposed in time-independent and time-dependent parts as $I_{1,2}(t) = I_{1,2}^b +$

$I_{1,2}^s(t)$, with the time-independent terms

$$I_1^b = \delta\ell c_{1,N} \frac{\Lambda^d}{\omega_\Lambda} K_+, \quad (116a)$$

$$I_2^b = \delta\ell c_{2,N} \frac{\Lambda^d}{\omega_\Lambda^4} \omega_{0\Lambda}, \quad (116b)$$

while the time-dependent ones are

$$I_1^s(t) = \delta\ell c_{1,N} \frac{\Lambda^d}{\omega_\Lambda} K_- \cos(2\omega_\Lambda t), \quad (117a)$$

$$\begin{aligned} I_2^s(t) = & \delta\ell c_{2,N} \frac{\Lambda^d}{\omega_\Lambda^3} \left[K_- - 2\omega_\Lambda t \sin(2\omega_\Lambda t) \right. \\ & \left. - \frac{\omega_{0\Lambda}}{\omega_\Lambda} \cos(2\omega_\Lambda t) \right], \end{aligned} \quad (117b)$$

where K_\pm , $\omega_\Lambda = \omega_{k=\Lambda}$ and $\omega_{0\Lambda} = \omega_{0,k=\Lambda}$, with ω_k and ω_{0k} given in Eqs. (5) and (10), respectively. Notice that, in terms of Ω_{0q} and Ω_{0c} , the functions K_\pm now reads (see also App. A)

$$K_\pm(k) = \frac{1}{2} \left(\sqrt{\frac{k^2 + r}{k^2 + \Omega_{0c}^2}} \pm \sqrt{\frac{k^2 + \Omega_{0q}^2}{k^2 + r}} \right). \quad (118)$$

In the expressions above we introduced the coefficients

$$c_{1,N} = \frac{N+2}{12N} a_d \quad \text{and} \quad c_{2,N} = \frac{N+8}{24N} a_d, \quad (119)$$

where $a_d = \Omega_d/(2\pi)^d$, with $\Omega_d = 2\pi^{d/2}/\Gamma(d/2)$ is the d -dimensional solid angle, $\Gamma(x)$ being the gamma function. Note that $I_{1,2}$ are time-dependent quantities as a consequence of the non-TTI term of G_{0K} in Eq. (14b).

In order to cast the RG equations in a convenient differential form, we take into account that $b \simeq 1 + \delta\ell$, with $\delta\ell \ll 1$ and we keep from Eq. (115) only the first order in $\delta\ell$. Neglecting for the time being the time-dependent terms I_1^s and I_2^s (see further below for a discussion), we find the differential equations:

$$\frac{dr}{d\ell} = 2r + u_c \frac{c_{1,N}}{2} \frac{\Lambda^d}{\omega_\Lambda} \left[\sqrt{\frac{\Lambda^2 + r}{\Lambda^2 + \Omega_{0c}^2}} + \sqrt{\frac{\Lambda^2 + \Omega_{0q}^2}{\Lambda^2 + r}} \right], \quad (120a)$$

$$\frac{du_c}{d\ell} = u_c \left[(d_c - d) - c_{2,N} u_c \frac{\Lambda^d}{\omega_\Lambda^4} \sqrt{\Lambda^2 + \Omega_{0q}^2} \right], \quad (120b)$$

$$\frac{du_q}{d\ell} = u_q \left[(d_q - d) - c_{2,N} u_c \frac{\Lambda^d}{\omega_\Lambda^4} \sqrt{\Lambda^2 + \Omega_{0q}^2} \right]. \quad (120c)$$

For later convenience, we defined here the classical and quantum “upper critical dimensions” d_c and d_q , respectively, as $d_{c,q} = 2 + d - 2\zeta_{c,q}$.

While Eqs. (120) are valid in the generic case, henceforth we focus on a deep quench. As noted in Sec. V A, Ω_{0c} has a stable fixed point at $\Omega_{0c}^* = +\infty$ for a deep quench and, correspondingly, the first term in brackets in Eq. (120a) vanishes. Moreover, from the Gaussian

scaling we find $d_c = 4$, $d_q = 2$, and therefore the quantum vertex is irrelevant for $d > 2$ and can be neglected. Finally, using the fact that $\Omega_{0q} \gg \Lambda$, Eqs. (120a) and (120b) become

$$\frac{dr}{d\ell} = 2r + u_c \Omega_{0q} \frac{c_{1,N}}{2} \frac{\Lambda^{4-\epsilon}}{\Lambda^2 + r}, \quad (121a)$$

$$\frac{du_c}{d\ell} = u_c [\epsilon - u_c \Omega_{0q} c_{2,N} \Lambda^{-\epsilon}], \quad (121b)$$

where $\epsilon = 4 - d$, while $c_{1,N}$ and $c_{2,N}$ can be calculated here for $d = 4$. These flow equations admit a nontrivial fixed point $Q_{dy}(\Omega_{0q})$ in the (r, u_c) -plane, which, at the lowest order in the dimensional expansion, is given by

$$\begin{aligned} Q_{dy}(\Omega_0) &= \left(-\epsilon \Lambda^2 \frac{c_{1,N}}{4c_{2,N}}, \frac{\epsilon}{\Omega_{0q} c_{2,N}} \right) \\ &= \left(-\frac{\epsilon \Lambda^2 (N+2)}{2(N+8)}, \epsilon \frac{192\pi^2 N}{\Omega_{0q} (N+8)} \right), \end{aligned} \quad (122)$$

which corresponds to the dynamical transition discussed in Sec. I. This point turns out to be stable for $\epsilon > 0$, while it becomes unstable for $\epsilon < 0$, i.e., $d > d_c$ such that, as it happens in equilibrium, the resulting theory is effectively Gaussian. Right before Eqs. (120) we mentioned that these equations have been determined by neglecting the time-dependent parts $I_{1,2}^s$ of the integrals $I_{1,2}$ in Eqs. (115). These parts are responsible for rapid oscillations on the temporal scale $\sim \omega_\Lambda^{-1} \simeq \Lambda^{-1}$ set by the microscopic structure of the model, which however do not contribute to time averages of the slow fields on much longer time scales. We emphasize that these terms are not universal as they strongly depend on the regularization. For instance, if a soft cut-off was used instead of a sharp one, these fast-oscillating terms would be replaced by functions which decay smoothly to zero on a temporal scale $\simeq \Lambda^{-1}$ (cf. Ref 59), and therefore they would clearly not contribute to Eqs. (120).

Equations (120a) and (120b) are remarkably similar to those for the same model *in equilibrium* at a finite temperature $T = \beta^{-1}$ (see, e.g., Ref. 80) which, in turn, admit a non-trivial fixed point $Q_{eq}(T)^{81,82}$. In fact, both of them can be cast in the form

$$\frac{dr}{d\ell} = 2r + \frac{c_{1,N}}{2} u F(r), \quad (123a)$$

$$\frac{du}{d\ell} = u [(d_c - d) + c_{2,N} u F'(0)], \quad (123b)$$

where u indicates, in equilibrium, the coupling constant of the model in Eq. (2), while out of equilibrium, it stands for u_c . For a quench, $F(r)$ reads

$$F^{\text{neq}}(r) = \frac{\Lambda^d}{\sqrt{\Lambda^2 + r}} \left(\sqrt{\frac{\Lambda^2 + r}{\Lambda^2 + \Omega_{0c}^2}} + \sqrt{\frac{\Lambda^2 + \Omega_{0q}^2}{\Lambda^2 + r}} \right), \quad (124)$$

while in equilibrium,

$$F^{\text{eq}}(r) = \frac{\Lambda^d}{\sqrt{\Lambda^2 + r}} 2 \coth \left(\beta \sqrt{\Lambda^2 + r} / 2 \right). \quad (125)$$

The relationship between the thermal and the dynamical fixed points Q_{eq} and Q_{dy} has been extensively discussed in Ref. 58. Note that the upper critical dimension is $d_c = 3$ both in the case of a vanishing initial mass (i.e., $\Omega_{0q}^2 = \Omega_{0c}^2 = r$) and of the equilibrium theory at temperature $T = 0$; instead, the upper critical dimension is $d_c = 4$ for both the case of a deep quench and of equilibrium at a finite temperature. These similarities provide further support to the observations made in Sec. II C that the stationary state after the quench resembles an equilibrium state, with the initial value Ω_{0q} of the parameter r in the former case playing the role of an effective temperature in the latter. Accordingly, the deep quench is responsible for a change of the upper critical dimensionality of the dynamical transition, in the very same way as a finite temperature modifies the upper critical dimensionality for an equilibrium quantum phase transition.

Although the analysis here reported only applies in dimensions above the lower critical one d_l , this “dimensional crossover” induced by the quench appears to be a generic feature of non-equilibrium isolated quantum systems. In fact, the same effect was shown to appear for a quench in the integrable one-dimensional Ising chain in transverse field^{114–116}, where the eventual (non-thermal) stationary state does not display any critical behaviour. Correspondingly, the equilibrium Ising chain in a transverse field⁸² displays a quantum phase transition at $T = 0$ (as it can be mapped on a classical Ising model in two spatial dimensions), while no phase transition occurs for $T > 0$ (as the corresponding classical Ising model would be in one spatial dimension). In addition, a recent experiment¹¹⁷ on a one-dimensional Bose gas with two components suggested the absence of scaling behaviour in the long-time dynamics of the system. In this case, the dynamics was shown to be captured by the effective Hamiltonian (2) with $N = 1$, which, at equilibrium, is characterized by a quantum phase transition at $T = 0$ which however disappears for $T > 0$. Accordingly, we do not expect any DPT in this case.

A remarkable consequence of the structure of RG equations (123) is that the value of the critical exponent ν — which can be obtained as the eigenvalue of the linearized RG flow around the fixed point^{90,91} $u = u^* = -\epsilon/[c_{2,N} F'(0)] + \mathcal{O}(\epsilon^2)$ — is independent of the specific form of $F(r)$ and it is given by $\nu^{-1} = 2 - c_{1,N}\epsilon/(2c_{2,N}) + \mathcal{O}(\epsilon^2)$, i.e., by

$$\nu = \frac{1}{2} + \frac{N+2}{N+8} \frac{\epsilon}{4} + \mathcal{O}(\epsilon^2), \quad (126)$$

with $\epsilon = d_c - d$. Accordingly, both the dynamical and the equilibrium transitions are characterized by the same exponent ν , a fact which was already noticed in Ref. 57 for the $O(N)$ model in the limit $N \rightarrow \infty$. However, one may wonder if also the other independent critical exponent of the equilibrium theory, i.e., the anomalous dimension η , is the same as the one of the dynamical phase transition. The answer to this question is far from being trivial, since the computation of η requires a two-loop

calculation which would also generate dissipative terms, responsible also for the destabilization of the prethermal state.

The analysis presented above reveals the existence of the fixed point Q_{dy} (see Eq. (122)) of the RG flow, which is approached upon considering increasingly long-distance properties whenever the value of the parameter r in Eq. (2) equals a certain critical value r_c , which depends, inter alia, on u_c and which corresponds to the stable manifold of the fixed point Q_{dy} . Accordingly, the coupling constant u_c which appears in the perturbative calculations discussed in Sec. III with an unknown value can be substituted with its fixed-point value u_c^* corresponding to Q_{dy} . In particular, inserting Eq. (122) into Eq. (55), and noting that $\Omega_{0q} = \Omega_0$, the initial-slip exponent eventually reads:

$$\theta_N = \frac{\epsilon N + 2}{4N + 8} + \mathcal{O}(\epsilon^2). \quad (127)$$

The value of θ_N in the limit $N \rightarrow \infty$ agrees with the exact results presented in Ref. 59 and reported here in Eq. (107).

C. Dissipative and secular terms

The integration procedure outlined in Sec. VB generates a number of terms in addition to those which are present in the action before integration. Physically, as the system evolves in time, inelastic collisions induce a thermalization process, and the system starts acting as a bath for itself. It is then natural to wonder whether dissipative terms might emerge¹⁰⁸. In particular, they would be encoded of the form $ig_1\phi_q^2$, $ig_2^A(\phi_q \cdot \phi_c)^2$ and $ig_2^B\phi_q^2\phi_c^2$ in the bulk action, which indeed correspond to having, respectively, additive (g_1) and multiplicative ($g_2^{A,B}$) Gaussian noise acting on the system¹⁰⁰. Based on the canonical power counting associated with the deep quench (see Sec. VB, with $d_c = 4$), these couplings turn out to scale as

$$g_1 \sim \mu, \quad g_2^A \sim g_2^B \sim \mu^{3-d}, \quad (128)$$

accordingly g_1 is expected to be relevant in all spatial dimensions while $g_2^{A,B}$ are relevant only for $d < 3$. If these terms are generated under RG, their effect is to change significantly the properties of the resulting action and to induce a crossover towards a different fixed point, with a different canonical scaling. However, in order for g_1 to be generated one needs to consider at least a two-loop contribution and therefore one is allowed to neglect it in our one-loop analysis. On the other hand, corrections to $g_2^{A,B}$ are generated at one loop (see App. F) as

$$\delta S_{(\text{diss})}^{(2)} = i \int_{\mathbf{x},t} \left[\delta g_2^A(t) (\phi_c^< \cdot \phi_q^<) ^2 + \delta g_2^B(t) (\phi_c^<)^2 (\phi_q^<)^2 \right], \quad (129)$$

with

$$\delta g_2^A(t) = \delta \ell \frac{a_d}{144} \frac{N+6}{N^2} \frac{\Lambda^d}{\omega_\Lambda^2} [u_c^2 (K_+^2 + K_-^2) - u_c u_q] t, \quad (130a)$$

$$\delta g_2^B(t) = \delta \ell \frac{a_d}{72} \frac{1}{N^2} \frac{\Lambda^d}{\omega_\Lambda^2} [u_c^2 (K_+^2 + K_-^2) - u_c u_q] t, \quad (130b)$$

with K_\pm given in Eq. (118) and a_d after Eq. (119). In the absence of a quench of the parameter r , i.e., with $r = \Omega_0^2$ one has $K_+ = 1$, $K_- = 0$ and $u_c = u_q$; accordingly, the dissipative vertices $\delta g_2^{A,B}(t)$ vanish at this order in perturbation theory, because the quench of the coupling constant u (occurring from a vanishing to a non-vanishing value) does not affect the Gaussian propagators and has consequences only at higher-order terms in perturbation theory. The dissipation is also absent in the limit $N \rightarrow \infty$, as it can be seen from the scaling $\propto N^{-1}$ of the terms in Eq. (130): this implies that the prethermal state is actually the true steady-state of the model after the quench, in agreement with the fact that the Hamiltonian H becomes solvable in this limit⁵⁹. In all the other cases, the dissipative correction increases upon increasing the time t and accordingly, though formally irrelevant for $d > 3$, $\delta g_2^{A,B}(t)$ grows indefinitely, spoiling the perturbative expansion. This kind of divergence is due to the presence of the so-called secular terms in the simple perturbative expansion which has been done here. Although several techniques exist to avoid these terms^{15,118–120}, we emphasize that they affect significantly only the long-time dynamics, while being essentially inconsequential as far as the short-time behaviour we are considering here is concerned. An estimate of the time at which the contributions of Eq. (130) become important can be obtained by requiring them to be of the same order as the other couplings present in the action. Considering, for simplicity, the values of r , u_c and u_q at their fixed points in the deep-quench limit (122), one can readily derive the RG equations for $g_2^{A,B}$ from Eq. (130). Then, by identifying the dimensionless RG flow parameter ℓ with the time t elapsed since the quench, i.e., by setting $\ell = \ln(\Lambda t)$, one finds

$$\Lambda^{d-3} \frac{dg_2^A}{d(\Lambda t)} = \epsilon^2 \frac{N+6}{(N+8)^2} \frac{2}{a_d}, \quad (131)$$

$$\Lambda^{d-3} \frac{dg_2^B}{d(\Lambda t)} = \epsilon^2 \frac{1}{(N+8)^2} \frac{1}{a_d}. \quad (132)$$

Accordingly, for large N , the dissipative couplings $g_2^{A,B}(t)$ become of order one at time scales $\Lambda t_A^* \sim N/\epsilon^2$ and $\Lambda t_B^* \sim N^2/\epsilon^2$, respectively. Note, however, that corrections to $g_2^{A,B}$ are generated also for $d > 4$ (i.e., $\epsilon < 0$), when the coupling u_c is irrelevant: in fact, u_c still generates anyhow terms which are responsible for thermalization, providing thus an instance of a sort of *dangerously irrelevant* operator¹⁰⁸. In this case, the time scales t_A^* and t_B^* are modified to $\Lambda t_A^* \sim N/(u_c^0)^2$ and

$\Lambda t_B^* \sim N^2/(u_c^0)^2$, where u_c^0 indicates the microscopic value of the coupling constant u_c . Accordingly, for large values of N or small values of $\max\{\epsilon, u_c^0\}$, the dissipative vertices can be disregarded for quite long times after the quench.

VI. RENORMALIZATION GROUP: CALLAN-SYMANZIK APPROACH

In spite of its transparent physical interpretation, the Wilson RG discussed in the previous section is not very practical for carrying out actual calculations. In fact, as we showed in Section V, it still leaves a certain degree of arbitrariness in fixing the dimensions of all the fields and therefore one has to supplement the analysis with additional physical arguments, as we did in order to fix the upper critical dimensionality to $d_c = 4$. More severely, Wilson RG is limited in providing predictions of some critical exponents beyond one loop¹²¹ and therefore different techniques have been developed for extending the RG analysis. In view of these difficulties, in this section we discuss the issue of the emergence of the short-time universal scaling within an alternative renormalization scheme, inspired by the well-established field-theoretical approach (see, e.g., Ref. 111). The idea is to regularize the correlation functions of the relevant fields by redefining the couplings and the fields so as to remove the terms in perturbation theory which would be divergent upon increasing the ultraviolet cutoff Λ , suitably introduced into the original (bare) action. As a result, the effective action as well as all the correlation functions generated by it, once expressed in terms of these renormalized couplings and fields, is free of divergences. This renormalized action depends on the arbitrary scale μ which defines the momentum scale of the effective theory. Using the fact that the original correlation functions — determined by the bare action — are actually independent of μ , one can derive a Callan-Symanzik flow equation for the correlation functions which involves also the RG equations for the relevant couplings of the theory. Note that, because of the breaking of TTI induced by the quench, it is no longer viable to define a Fourier transform of the correlation functions and consequently the Callan-Symanzik equations discussed below will be derived for the correlation functions rather than the vertex functions which are usually introduced when TTI is not broken.

A. Renormalization of the initial fields

As anticipated above, the divergences upon increasing the cutoff Λ which emerge in perturbation theory, can be cancelled by a suitable renormalization of coupling constants and fields: in practice, this is achieved by calculating the relevant correlation functions at particular values of the times and momenta — the so-called normalization point (NP) — of the involved fields and by

absorbing the resulting divergences in a redefinition of coupling and fields. In particular, we define the NP as times $t = \mu^{-1}$ and vanishing momenta $q = 0$ for all fields involved in the correlation function. For simplicity, we focus below on the critical point $r = r_c$, which, as shown in Sec. III, is shifted away from its Gaussian value $r_c = 0$ because of the interaction.

We consider first the retarded Green's function $G_R(k, t, t') = \langle \phi_c(t) \phi_q(t') \rangle$, where, for the sake of simplicity, the indices of the field components have been suppressed and the fields inside the correlators are consequently assumed to refer to the same component. For simplicity, the dependence on k is understood on the l.h.s. of the previous equality. As shown in App. C, G_R is finite in the formal limit $\Lambda \rightarrow \infty$ as long as both times $t > t'$ do not vanish, while it grows $\propto \ln(\Lambda t)$ when the earlier time t' vanishes, i.e., when the involved quantum field is the one at the boundary ϕ_{0q} . In fact, at the normalization point,

$$\langle \phi_c(t) \phi_{0q} \rangle|_{\text{NP}} = [1 - \theta_N \ln(\Lambda/\mu)] \times \text{finite terms}, \quad (133)$$

where by “finite terms” we mean an expression which is finite as $\Lambda \rightarrow \infty$. In order to reabsorb the logarithmic divergence in Eq. (133), we redefine the initial quantum field as $\phi_{q0} = Z_0^{1/2} \phi_{0q}^R$, where ϕ_{q0}^R is the corresponding renormalized field and the renormalization constant Z_0 is determined such that the renormalized retarded Green's function $G_R^R(t, 0) \equiv \langle \phi_c(t) \phi_{0q}^R \rangle = Z_0^{-1/2} G_R(t, 0)$, remains finite as $\Lambda \rightarrow \infty$. This requires

$$Z_0^{1/2} = 1 - \theta_N \ln(\Lambda/\mu), \quad (134)$$

up to the lowest order in perturbation theory considered in this work. Since Eq. (49) holds up to the same order, the renormalized Keldysh Green's function G_K^R can be simply defined as:

$$G_K^R(k, t, t') = \Omega_{0q} G_R^R(k, t, 0) G_R^R(k, t', 0) = Z_0^{-1} G_K(k, t, t'). \quad (135)$$

As shown in Sec. III A, the perturbative correction to $G_K(k, t, t')$ generates a term $\propto \ln(\Lambda/\mu)$ for generic values of the times t and t' . After the introduction of the renormalization constant as in Eq. (135), this logarithmic dependence is actually removed from G_K^R .

B. Renormalization of the coupling constant

In addition to the renormalizations introduced above, which render finite $G_{K,R}^R$, one has also to consider that the coupling constant u_c is renormalized by the interactions and that this renormalization is essential for determining the RG flow of the whole system. In order to determine it, we study the perturbative corrections to the four-point function $\langle \phi_{cj}(1) \phi_{ql}(2) \phi_{qm}(3) \phi_{qn}(4) \rangle$ to second order in u_c , where, with a more convenient notation, $\phi_{cj,qj}(n)$ denotes the j -component of the N -component

field $\phi_{c,q}$ calculated at the space-time point $n \equiv (\mathbf{x}_n, t_n)$. Taking into account the possible diagrams, one finds:

$$\begin{aligned}
& \langle \phi_{cj}(1) \phi_{ql}(2) \phi_{qm}(3) \phi_{qn}(4) \rangle \\
&= \text{Diagram 1} + \text{Diagram 2} \\
&= \langle \phi_{cj}(1) \phi_{ql}(2) \phi_{qm}(3) \phi_{qn}(4) \left[1 - i \frac{2u_c}{4!N} \int_{1'} (\phi_q \cdot \phi_c) \phi_c^2 \right. \\
&\quad \left. - \frac{1}{2} \left(\frac{2u_c}{4!N} \right)^2 \int_{1'} (\phi_q \cdot \phi_c) \phi_c^2 \int_{2'} (\phi_q \cdot \phi_c) \phi_c^2 \right] \rangle_0 \\
&\equiv I_0 + I_1 + I_2,
\end{aligned} \tag{136}$$

where the integrations run on the points indicated as $1'$ and $2'$ at which the arguments of the corresponding integrands are calculated, while I_m indicates the contribution of order u_c^m to that expansion. Note that I_0 vanishes because, as a consequence of Wick's theorem, it is proportional to a two-point correlation of quantum fields $\langle \phi_q \phi_q \rangle_0 = 0$. The first order term I_1 , instead, can be written as

$$\begin{aligned}
I_1 &= -i \frac{2u_c}{4!N} \int_{1'} \langle \phi_{cj}(1) \phi_{ql}(2) \phi_{qm}(3) \phi_{qn}(4) (\phi_q \cdot \phi_c) \phi_c^2(1') \rangle_0 \\
&= -i \frac{u_c}{6N} F_{jlmn} \\
&\quad \times \int_{1'} G_{0R}(1, 1') G_{0R}(1', 2) G_{0R}(1', 3) G_{0R}(1', 4),
\end{aligned} \tag{137}$$

where $F_{jlmn} = \delta_{jl} \delta_{mn} + \delta_{jm} \delta_{ln} + \delta_{jn} \delta_{lm}$. The calculation of I_2 is slightly more involved and it is reported in App. G (see Eq. (F9)). Among the many terms generated by the repeated use of Wick's theorem, we need to retain only those which renormalize the local vertex. After discarding the terms which do not fulfil this requirement, one finds (see Eq. (G2))

$$\begin{aligned}
I_2 &= \frac{N+8}{36N} F_{jlmn} u_c^2 \\
&\quad \times \int_{1'} G_{0R}(1, 1') G_{0R}(1', 2) G_{0R}(1', 3) G_{0R}(1', 4) \\
&\quad \times \int_{2'} G_{0K}(1', 2') G_{0R}(1', 2').
\end{aligned} \tag{138}$$

Accordingly, adding I_1 to I_2 , the four-point correlation function reads:

$$\begin{aligned}
& \langle \phi_{cj}(1) \phi_{ql}(2) \phi_{qm}(3) \phi_{qn}(4) \rangle = -i \frac{u_c}{6N} F_{jlmn} \\
&\quad \times \int_{1'} G_{0R}(1, 1') G_{0R}(1', 2) G_{0R}(1', 3) G_{0R}(1', 4) (1 + J),
\end{aligned} \tag{139}$$

where the integral J , for $d = 4$ and $\Lambda t \gg 1$ (see App. G), is given by (see Eqs. (G3) and (G4))

$$\begin{aligned}
J &= \frac{N+8}{6N} u_c \int_{\mathbf{x}} \int_0^t dt' i G_{0K}(\mathbf{x}, t, t') G_{0R}(\mathbf{x}, t - t') \\
&= -\frac{N+8}{24N} a_4 u_c \Omega_{0q} \ln(\Lambda t).
\end{aligned} \tag{140}$$

Accordingly, by evaluating the four-point function in Eq. (136) at the normalization point, one finds

$$\begin{aligned}
\langle \phi_c \phi_q \phi_q \phi_q \rangle \Big|_{\text{NP}} &= u_c \left[1 - u_c \Omega_{0q} \frac{N+8}{24N} a_4 \ln(\Lambda/\mu) \right] \\
&\quad \times \text{finite terms.}
\end{aligned} \tag{141}$$

In order to render this correlation function finite as $\Lambda \rightarrow \infty$, one introduces the dimensionless renormalized coupling constant g_R as

$$\frac{a_4}{16} \Omega_{0q} u_c = \mu^{4-d} Z_g g_R, \tag{142}$$

where the factor $a_4 \Omega_{0q}/16$ (with $a_4 \equiv a_{d=4}$, as defined below Eq. (119)) is introduced for later convenience. Note that up to this order in perturbation theory, it is not necessary to introduce a renormalization of the bulk classical and quantum fields. Accordingly, one concludes that Z_g has to be chosen in such a way as to cancel the logarithmic dependence in Eq. (141) and therefore

$$Z_g = 1 + u_c \Omega_{0q} \frac{N+8}{24N} a_4 \ln(\Lambda/\mu). \tag{143}$$

C. Renormalization-group (Callan-Symanzik) equations

Within the renormalization scheme discussed above, one can infer the renormalization-group equations by exploiting the arbitrariness of the scale μ which was introduced in order to renormalize the critical model. In fact, the original (bare) action is actually independent of the (infrared) scale μ and therefore the logarithmic derivative with respect to μ of the corresponding bare coupling constants and correlation functions, once expressed in terms of the renormalized ones, has to vanish when taken with fixed bare coupling constants and cutoff Λ . By taking the logarithmic derivative $\mu \partial/\partial \mu$ of Eq. (142) one finds

$$\mu \frac{\partial g_R}{\partial \mu} \equiv \beta(g_R) = -\epsilon g_R + \frac{2}{3} \frac{N+8}{N} g_R^2, \tag{144}$$

where we used the fact that, from Eq. (142), $a_4 \Omega_{0q} u_c \mu^{d-4}/16 = g_R + O(g_R^2)$, and we introduced $\epsilon \equiv 4 - d$. Henceforth, the derivative with respect to μ is always understood as taken with fixed bare parameters. This β -function determines the flow of the coupling constant as the critical theory is approached for $\mu \rightarrow 0$ and

it is characterized by a fixed-point value g_R^* such that $\beta(g_R^*) = 0$, with

$$g_R^* = \frac{3}{2} \frac{N}{N+8} \epsilon + O(\epsilon^2), \quad (145)$$

which is infrared (IR) stable for $d < 4$, while the Gaussian fixed point $g_R^* = 0$ becomes stable for $d > 4$.

In order to highlight the consequences of the existence of this fixed point on the correlation functions, we consider one which involves N_c and N_q classical and quantum fields in the bulk (i.e., with non-vanishing times), respectively, and N_0 initial quantum fields, which we schematically indicate as

$$G_{\{N\}} \equiv \langle [\phi_c]^{N_c} [\phi_q]^{N_q} [\phi_{0q}]^{N_0} \rangle = Z_c^{N_c/2} Z_q^{N_q/2} Z_0^{N_0/2} G_{\{N\}}^R. \quad (146)$$

Here $\{N\} = (N_c, N_q, N_0)$, $G_{\{N\}}^R$ indicates the same quantity $G_{\{N\}}$ expressed in terms of the renormalized fields, while $Z_{c,q}$ are the renormalization constants of the bulk fields with $Z_{c,q} = 1$ up to the order in perturbation theory which we are presently interested in. Note that any correlation function which involves a classical field taken at the boundary $t = 0$ vanishes in a deep quench because of the effective initial conditions on the classical field, see Eq. (47). By requiring that the logarithmic derivative of the l.h.s. of Eq. (146) with respect to the renormalization scale μ vanishes when taken at fixed bare parameters, and by expressing the r.h.s. in terms of renormalized quantities, one finds the Callan-Symanzik equation⁸³

$$\left\{ \mu \partial_\mu + \frac{N_c}{2} \gamma_c + \frac{N_q}{2} \gamma_q + \frac{N_0}{2} \gamma_0 + \beta \partial_g \right\} G_{\{N\}}^R = 0, \quad (147)$$

where we introduced the functions

$$\gamma_{c,q,0} \equiv \mu \partial_\mu \ln Z_{c,q,0}|_{\text{bare}}. \quad (148)$$

At the lowest order in perturbation theory one has $\gamma_{c,q} = 0$, while

$$\gamma_0 = \frac{N+2}{3N} g_R + O(g_R^2), \quad (149)$$

which follows from Eq. (134). Equation (147) can be solved in full generality by employing the method of characteristics and, in conjunction with dimensional analysis, renders the scaling behaviour of the renormalized correlation function $G_{\{N\}}^R$. Note that, in principle, $G_{\{N\}}^R$ still depends on the cut-off Λ . However, dimensional analysis done by using μ as the reference scale implies that $G_{\{N\}}^R$ actually depends on Λ/μ , which diverges as one explores the long-time and large-scale properties of the theory by letting $\mu \rightarrow 0$. However, the renormalizations introduced above in Secs. VI A and VI B were in fact determined such that this limit (formally $\Lambda \rightarrow \infty$) renders finite quantities. Accordingly, the leading scaling behaviour of the renormalized quantities can actually be obtained by removing their dependence on Λ , i.e., by assuming $\Lambda \gg \mu$

For simplicity, we focus here on the scaling behaviour emerging at the fixed point g_R^* (see Eq. (145)) of the coupling constant. Accordingly, Eq. (147) simplifies as

$$\left\{ \mu \partial_\mu + \frac{N_0}{2} \gamma_0^* \right\} G_{\{N\}}^R(\{k, t\}; \mu) = 0, \quad (150)$$

where

$$\gamma_0^* = \frac{\epsilon}{2} \frac{N+2}{N+8} + O(\epsilon^2) \quad (151)$$

indicates the value of γ_0 calculated at the fixed point. In order to exploit the consequences of dimensional analysis, we define the dimensionless renormalized Green's functions $\hat{G}_{\{N\}}^R$ in the momentum-time representation

$$G_{\{N\}}^R(\{k, t\}; \mu) = \mu^{d_{\{N\}}^k} \hat{G}_{\{N\}}^R(\{k/\mu, \mu t\}), \quad (152)$$

where $\{k, t\}$ indicates the set of times and momenta at which the various fields are calculated, while $d_{\{N\}}^k$ takes into account the canonical scaling of the fields in momentum space and of the delta functions which ensure momentum conservation, i.e.,

$$d_{\{N\}}^k = d + N_c \zeta_c^k + N_q \zeta_q^k + N_0 \zeta_{q0}^k + \dots \quad (153)$$

Note that the r.h.s. of Eq. (152) is calculated with the dimensionless parameters corresponding to those appearing on the l.h.s. The factors ζ_c^k, ζ_q^k and ζ_{q0}^k denote the canonical scaling of the classical, quantum in the bulk and quantum at the surface fields in momentum space and, from Sec. V A, they read

$$\zeta_c^k = -\frac{d+2}{2}, \quad \zeta_q^k = \zeta_{q0}^k = -\frac{d}{2}. \quad (154)$$

The dots in Eq. (153) account for the possibility of including derivatives of the fields into the correlation functions (146): they scale differently from the fields and therefore they would add new contributions to $d_{\{N\}}^k$. Accordingly, by inserting Eq. (152) in Eq. (150), one finds

$$\left\{ \mu \partial_\mu + d_{\{N\}}^k + \frac{N_0}{2} \gamma_0^* \right\} \hat{G}_{\{N\}}^R(\{k/\mu, \mu t\}) = 0. \quad (155)$$

Using the method of characteristics it is possible to transform this partial differential equation into an ordinary one in terms of an arbitrary dimensionless parameter l :

$$\left\{ l \frac{d}{dl} + d_{\{N\}}^k + \frac{N_0}{2} \gamma_0^* \right\} \hat{G}_{\{N\}}^R(\{k/(\mu l), \mu l t\}) = 0, \quad (156)$$

the solution of which is simply given by

$$l^{N_0 \gamma_0^*/2 + d_{\{N\}}^k} \hat{G}_{\{N\}}^R\left(\left\{\frac{k}{\mu l}, \mu l t\right\}\right) = \hat{G}_{\{N\}}^R\left(\left\{\frac{k}{\mu}, \mu t\right\}\right), \quad (157)$$

which expresses nothing but the fact that the correlation function is a homogeneous (i.e., scale-invariant) function

of its argument, as expected at the critical point. Specializing this general expression to the case of the retarded function G_R^R with two fields in the bulk (i.e., $G_{\{N\}}^R$ with $N_0 = 0$, $N_c = N_q = 1$) one finds, at sufficiently long times:

$$G_R^R(k, t, t') = l^{-1} G_R^R(k/l, lt, lt'). \quad (158)$$

When, instead, one of the fields is at the boundary ($N_q = 0$ and $N_c = N_0 = 1$) one finds

$$G_R^R(k, t, 0) = l^{-1+\gamma_0^*/2} G_R^R(k/l, lt, 0). \quad (159)$$

The scaling of G_K can be thus inferred on the basis of Eq. (135) with the aid of Eq. (159), which yield

$$G_K^R(k, t, t') = l^{-2+\gamma_0^*} G_K^R(k/l, lt, lt'). \quad (160)$$

The scaling forms in Eqs. (158), (159), and (160) are consistent with those inferred from the perturbation theory in Sec. III A, with the identification

$$\theta_N = \frac{\gamma_0^*}{2} = \frac{\epsilon}{4} \frac{N+2}{N+8} + O(\epsilon^2). \quad (161)$$

For example, by choosing $l \sim (\Lambda t)^{-1} \ll 1$ in Eq. (159), one finds $G_R^R(k, t, 0) = t^{1-\gamma_0^*/2} G_R^R(kt, 1, 0)$, which is nothing but the resummed version of the perturbative result reported in Eq. (66).

The Callan-Symanzik equation (147) alone does not provide information on the two-time scaling suggested in Eq. (64), which actually emerges as a consequence of the fact that the smaller time t' approaches the “surface” $t' = 0$ at which the anomalous dimension of the field ϕ_{0q} differs from the one of the quantum field ϕ_q in the bulk. In order to work out the consequences of this fact, we consider here the equivalent of the standard short-distance expansion of quantum field theory^{83,85} adapted to $\phi_q(t')$ at short times⁶⁷, which we assume to be valid for $t' \rightarrow 0$:

$$\phi_q(t') = \sum_{i \geq 0} \sigma_{i,q}(t') O_i, \quad (162)$$

where O_i indicates operators located at the surface, ordered such that their scaling dimension increases upon increasing i , while $\sigma_{i,q}(t')$ are functions of t' . In the limit $t' \rightarrow 0$, this sum is dominated by the most relevant term O_0 , which is ϕ_{0q} , and therefore we can simply write Eq. (162) as $\phi_q(t') \simeq \sigma_q(t') \phi_{0q}$, where we dropped the terms of the expansion which are less singular as $t' \rightarrow 0$. The scaling form of $\sigma_q(t')$ can be derived by inserting this expansion in Eq. (157) and by reading out the scaling behavior, which turns out to be

$$\sigma_q(t') = l^{-\gamma_0^*/2} \sigma_q(lt'), \quad (163)$$

at the critical point. Finally, by using Eqs. (159) and (163), one finds:

$$\begin{aligned} G_R(k, t, t' \rightarrow 0) &\simeq \sigma_q(t') G_R(k, t, 0) \\ &= t \left(\frac{t'}{t} \right)^{\gamma_0^*/2} G_R(kt, 1, 0), \end{aligned} \quad (164)$$

which is the resummed version of Eq. (64), taking into account the relationship between γ_0^* and θ_N in Eq. (161). Equations (159), (160) and (164) demonstrate that the resummation of the leading logarithms which emerge in perturbation theory and which we did somewhat arbitrarily in Sec. III, is fully justified by the existence of an IR-stable fixed point for the flow of the coupling constant.

D. Scaling of magnetization

The approach developed above can be conveniently used in order to determine the scaling form of the magnetization $M(t) \equiv M_1(t) = \langle \phi_1(\mathbf{x}, t) \rangle$ in the presence of a symmetry-breaking term in the pre-quench Hamiltonian H_0 , which was discussed in Sec. IV either within perturbation theory or in the exactly solvable limit $N \rightarrow \infty$. In fact, Eq. (89) (which is valid beyond perturbation theory) can be expanded in powers of M_0 , assuming a homogeneous external field \mathbf{h} with the components given in Eq. (92); focussing on the components of the fields along direction 1 of the initial field \mathbf{h} , this yields

$$\begin{aligned} M(t) &= \sum_{n=1}^{+\infty} \frac{(-i\sqrt{2}M_0)^n}{n!} \int d^d x_1 \cdots d^d x_n \\ &\quad \times \langle \phi_c(\mathbf{x}, t) \dot{\phi}_{0q}(\mathbf{x}_1) \cdots \dot{\phi}_{0q}(\mathbf{x}_n) \rangle_{h=0}, \end{aligned} \quad (165)$$

(the term with $n = 0$ in the previous expansion vanishes due to the choice of the initial condition) from which it follows that the scaling of $M(t)$ is related to that of correlation functions containing the initial field $\dot{\phi}_{0q} \equiv \dot{\phi}_q(t = 0)$. Note that due to spatial translational invariance, the r.h.s. of Eq. (165) is actually independent of \mathbf{x} . In fact, the logarithmic dependence on Λ of the one-loop correction to the magnetization pointed out in Sec. IV (see, in particular, Eq. (96)) can be suitably cancelled (according to the approach discussed in this section) with a renormalization of $\dot{\phi}_{0q} = Z_0^{1/2} \dot{\phi}_{0q}^R$, where a simple calculation gives

$$Z_0 = 1 + 2\theta_N \ln(\Lambda/\mu). \quad (166)$$

Note that Z_0 differs from Z_0 in Eq. (134), which means that the anomalous dimension (defined as in Eq. (148))

$$\gamma_0 \equiv \mu \partial_\mu \ln Z_0|_{\text{bare}} = -\frac{N+2}{3N} g_R + O(g_R^2) \quad (167)$$

of $\dot{\phi}_{0q}$ is different from that one of ϕ_{0q} , i.e., from γ_0 in Eq. (149). Accordingly, when γ_0 is evaluated at the fixed point g_R^* reported in Eq. (145), it renders:

$$\gamma_0^* = -\frac{\epsilon}{2} \frac{N+2}{N+8} + O(\epsilon^2) = -2\theta_N. \quad (168)$$

Following the same line of argument as in Sec. VI C, one can easily derive a Callan-Symanzik equations for the correlations functions appearing on the r.h.s. of Eq. (165),

which are of the form of Eq. (146) but with the fields ϕ_{0q} replaced by $\dot{\phi}_{0q}$ and, correspondingly, γ_0 by $\gamma_{\dot{0}}$ in the equations which follow. At the critical point, one eventually finds the scaling equation

$$l^{d_{\{N\}}^{\mathbf{x}} - n\theta_N} \langle \phi_c(lt, l\mathbf{x}) \dot{\phi}_{0q}(l\mathbf{x}_1) \cdots \dot{\phi}_{0q}(l\mathbf{x}_n) \rangle = \langle \phi_c(t, \mathbf{x}) \dot{\phi}_{0q}(\mathbf{x}_1) \cdots \dot{\phi}_{0q}(\mathbf{x}_n) \rangle, \quad (169)$$

where $d_{\{N\}}^{\mathbf{x}}$ accounts for the canonical scaling dimension of the fields in real space and reads:

$$d_{\{N\}}^{\mathbf{x}} = N_c \zeta_c^{\mathbf{x}} + N_0 \zeta_0^{\mathbf{x}} = [d - 2 + n(d + 2)]/2, \quad (170)$$

for $N_c = 1$ and $N_0 = n$, while $\zeta_c^{\mathbf{x}} = (d - 2)/2$ and $\zeta_0^{\mathbf{x}} = (d + 2)/2$ are, respectively, the canonical scaling dimensions of the fields ϕ_c and $\dot{\phi}_q$ in real space, which can easily be determined from Sec. V A by noticing that $\dot{\phi}_{0q}$ has the same dimension as $\mu\phi_{0q}$, where μ is an arbitrary momentum scale, see Eq. (110). Substituting Eq. (169) into Eq. (165) one obtains, after a change of variables in the spatial integrals:

$$M(t) = l^{d/2-1} \sum_{n=1}^{+\infty} \frac{(-i\sqrt{2})^n}{n!} \left(l^{-d/2+1-\theta_N} M_0 \right)^n \times \int d^d x_1 \cdots d^d x_n \langle \phi_c(lt, l\mathbf{x}) \dot{\phi}_{0q}(l\mathbf{x}_1) \cdots \dot{\phi}_{0q}(l\mathbf{x}_n) \rangle_{h=0} \quad (171)$$

from which, by choosing $l = (\Lambda t)^{-1}$ at long times $t \gg \Lambda^{-1}$, we find a scaling form for the magnetization:

$$M(t) = M_0 t^{\theta_N} \mathcal{M} \left(t^{d/2-1+\theta_N} M_0 \right). \quad (172)$$

This result entails the existence of a time scale $t_i \sim M_0^{1/(d/2-1+\theta_N)}$ at which the increase of the magnetization reported in Eq. (97) (supported by both the perturbative and non-perturbative analysis presented in Sec. IV) crosses over to a different behaviour. For $d \rightarrow 4$, this result is in agreement with the t_i estimated in Sec. IV for the critical quench of the $O(N)$ model for $N \rightarrow \infty$ (see Eq. (108)).

VII. CONCLUSIONS

In this work we investigated the effects of fluctuations on the dynamical transition beyond mean-field theory which is observed after a quench of the parameters of an isolated quantum many-body system with $O(N)$ symmetry. In particular, we accounted for fluctuations at the lowest order in the post-quench coupling constant, extending the analysis of Ref. 58 to the spatio-temporal structure of response and correlation functions and to the dynamics of the order parameter, while providing the details of the results anticipated therein. We found that, before thermalization eventually occurs as a consequence of the interaction which is switched on upon quenching,

the system approaches a pre-thermal state within which it undergoes a dynamical transition. We have characterized this transition within a Wilson's renormalization-group approach, showing that it is associated with a stable fixed point of the RG flow of the relevant control parameters and that it displays remarkable analogies with the corresponding equilibrium phase transition at finite temperature. Nevertheless, the breaking of the invariance of the dynamics under time translations caused by the quench induces a novel algebraic and universal behavior at short times with new features compared to the classical case. This universal short-time behavior is characterized by a new critical exponent θ_N calculated here to the leading order in a dimensional expansion [see Sec. V, Eq. (127)]. This result is found in agreement with the expression of the one for a critical quench of the $O(N)$ model in the limit $N \rightarrow \infty$ [see Sec. IV, Eq. (107)], in which the model is exactly solvable. The correlation functions after the quench are characterized by scaling forms which depend on the exponent θ_N and which we determined by using Callan-Symanzik equations [see Sec. VI C, Eq. (156)]. Moreover, when an external field breaks the $O(N)$ symmetry of the initial state (restored after the quench), the dynamics of the order parameter exhibits an algebraic growth, controlled by the initial-slip exponent θ_N , up to a certain macroscopic time t_i [see Sec. VI D, Eq. (172)].

This slow dynamics, which is referred to as aging, is very similar to the one occurring in both classical dissipative systems and open quantum system, though it belongs to a different universality class and is characterized by different scaling forms (see Appendix of Ref. 59). Among these differences, the most marked ones concern the spatio-temporal dependence of the correlation functions, since they exhibit here a light cone due to ballistically propagating excitations in the prethermal state [see Sec. III C and Figs. 1 and 2]. Another difference between the open and closed quantum system is that for the latter, entanglement entropy of the unitarily evolving state is a good measure of quantum correlations, and a recent study showed how the aging exponent entering here, also controls the scaling of the entanglement spectrum¹²².

As a future perspective, the study of the destabilization of the prethermal regime and the resulting crossover towards the full thermalization represents an interesting question as well as a technical challenge. In fact, as discussed in Sec. V C, the inclusion of the effect of two-loop diagrams becomes intractable within the usual perturbative techniques because of the appearance of secular terms, and therefore more elaborate resummation schemes are required. Moreover, the study of aging in fermionic systems represents an important issue, also because the classical limit of these systems is far from trivial, and therefore their genuinely quantum nature is expected to bear completely new features.

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Appendix A: Functional derivation of Green's functions

In this Appendix we show how to derive the Green's functions $G_{0K,0R}$ in Eqs. (14a) and (14b) from the Gaussian part of the action (43), (44) considered in Sec. II E. In order to do this, one introduces additional terms in the Keldysh action, in which the various fields are coupled linearly to some external sources $\mathbf{j}_{c,q}$ and $\mathbf{j}_{0c,0q}$. Accordingly, any correlation function can be obtained by differentiating the generating function

$$\mathcal{Z}_K[\{\mathbf{j}\}] = \int D[\phi_c, \phi_q, \phi_{0c}, \phi_{0q}] e^{iS} = e^{W[\{\mathbf{j}\}]}. \quad (\text{A1})$$

where $\{\mathbf{j}\} \equiv (\mathbf{j}_c, \mathbf{j}_q, \mathbf{j}_{0c}, \mathbf{j}_{0q})$. If one were interested in calculating correlation functions which involve $\dot{\phi}_{0c}$ or $\dot{\phi}_{0q}$, additional sources should be coupled to these fields. It turns out that the intermediate results of the actual calculation of W do depend on whether one integrates out first ϕ_c or ϕ_q , as shown below; however, the final result is the same.

First let us integrate out ϕ_q : in order to do this, we integrate by parts the time derivatives in the bulk action in Eq. (44), as

$$\begin{aligned} S_K = & i \int_{\mathbf{k}} \frac{\omega_{0k}}{2} [\phi_{0c}^2 \tanh(\beta\omega_{0k}/2) + \phi_{0q}^2 \coth(\beta\omega_{0k}/2)] \\ & - i \int_{\mathbf{k}} [\mathbf{j}_{0c} \phi_{0q} + \mathbf{j}_{0q} \phi_{0c} - i\phi_{0q} \dot{\phi}_{0c}] \\ & + \int_{\mathbf{k}} \int_0^\infty dt [-\phi_q (\ddot{\phi}_c + \epsilon \dot{\phi}_c + \omega_k^2 \phi_c - \mathbf{j}_c) + \mathbf{j}_q \phi_c]. \end{aligned} \quad (\text{A2})$$

where we set $\Omega_{0c} = \Omega_{0q} = \Omega_0$ for the sake of simplicity and we defined $\mathbf{j}_{0c} \phi_{0q} = \mathbf{j}_{0c} \cdot \phi_{0q}$. Note that the integration by parts generates an additional term $-i\phi_{0q} \dot{\phi}_{0c}$ in the part of the action located at $t = 0$ and that we added to S_K an infinitesimal term proportional to $\epsilon > 0$. This term contains the operator $\phi_q \dot{\phi}_c$, and therefore it can be regarded as an infinitesimal dissipation¹⁰⁰. The reason for including it is twofold, as it regularizes the Keldysh functional while ensuring causality. Notice also that such a term breaks the exchange symmetry $\phi_c \leftrightarrow \phi_q$ of the bulk action, which holds for $\mathbf{j}_c = \mathbf{j}_q = 0$. The integration of ϕ_q generates a functional delta function the support of which is on the solution $\bar{\phi}_c$ of the equation

$$(\partial_t^2 + \epsilon \partial_t + \omega_k^2) \bar{\phi}_c = \mathbf{j}_c, \quad (\text{A3})$$

i.e., on

$$\begin{aligned} \bar{\phi}_c(t) = & \phi_{0c} e^{-\epsilon t} \cos(\omega_k t) + e^{-\epsilon t} \dot{\phi}_{0c} \frac{\sin(\omega_k t)}{\omega_k} \\ & + \int_0^t dt' \frac{e^{-\epsilon t} \sin[\omega_k(t-t')]}{\omega_k} \mathbf{j}_c(t'), \end{aligned} \quad (\text{A4})$$

where $\phi_{0c} \equiv \phi_c(t=0)$ and $\dot{\phi}_{0c} \equiv \dot{\phi}_c(t=0)$. Note that in Eq. (A4) one can safely take the limit $\epsilon \rightarrow 0$. Then, we expand ϕ_c around $\bar{\phi}_c$ and we integrate out the Gaussian fluctuations of $(\phi_c - \bar{\phi}_c)$. Finally, we integrate out the remaining fields ϕ_{0c} , ϕ_{0q} and ϕ_c , finding

$$\begin{aligned} W[\{\mathbf{j}\}] = & \int_{\mathbf{k}} \left\{ \frac{1}{2} \mathbf{j}_{0q}^2 iG_{0K}(k, 0, 0) \right. \\ & - \int_t [\mathbf{j}_{0q} \mathbf{j}_q(t) G_{0K}(k, t, 0) + \mathbf{j}_{0c} \mathbf{j}_q(t) G_{0R}(k, t, 0)] \\ & - \int_{t,t'} \left[\mathbf{j}_q(t) \mathbf{j}_c(t') iG_{0R}(k, t, t') \right. \\ & \left. \left. + \frac{1}{2} \mathbf{j}_q(t) \mathbf{j}_q(t') iG_{0K}(k, t, t') \right] \right\}, \end{aligned} \quad (\text{A5})$$

where $\int_t = \int_0^\infty dt$ and G_{0R}, G_{0K} are those defined in Eqs. (14a) and (14b) and indeed they can be obtained by taking suitable functional derivatives of this $W[\{\mathbf{j}\}]$, which establish the connection between these functions $G_{0K,0R}$ and the correlation functions of the fields. Consider now the case in which ϕ_c and ϕ_q are integrated in the reversed order: we rewrite S_K as

$$\begin{aligned} S_K = & i \int_{\mathbf{k}} \frac{\omega_{0k}}{2} [\phi_{0c}^2 \tanh(\beta\omega_{0k}/2) + \phi_{0q}^2 \coth(\beta\omega_{0k}/2)] \\ & - i \int_{\mathbf{k}} [\mathbf{j}_{0c} \phi_{0q} + \mathbf{j}_{0q} \phi_{0c} - i\phi_{0c} \dot{\phi}_{0q}] \\ & + \int_{\mathbf{k}} \int_0^\infty dt [-\phi_c (\ddot{\phi}_q - \epsilon \dot{\phi}_q + \omega_k^2 \phi_q - \mathbf{j}_q) + \mathbf{j}_c \phi_q], \end{aligned} \quad (\text{A6})$$

where the additional term generated in the action located at $t = 0$ is different from the one in Eq. (A2). Note that, as in Eq. (A2) we introduced an infinitesimal term with $\epsilon > 0$. The integration of ϕ_c generates a delta function the support of which is on the solution of the equation:

$$(\partial_t^2 - \epsilon \partial_t + \omega_k^2) \bar{\phi}_q = \mathbf{j}_q, \quad (\text{A7})$$

i.e., on

$$\begin{aligned} \bar{\phi}_q(t) = & \phi_{0q} e^{\epsilon t} \cos(\omega_k t) + \dot{\phi}_{0q} e^{\epsilon t} \frac{\sin(\omega_k t)}{\omega_k} \\ & + \int_0^t dt' \frac{\sin[\omega_k(t-t')]}{\omega_k} \mathbf{j}_q(t'), \end{aligned} \quad (\text{A8})$$

Note that, differently from Eq. (A4), the infinitesimal term $\propto \epsilon$ generates an exponential factor which makes $\bar{\phi}_q$ grow exponentially in time. In order to avoid this

unphysical divergence, we have to impose $\bar{\phi}_q(t \rightarrow +\infty) = 0$, which fixes the value of ϕ_{0q} and $\dot{\phi}_{0q}$ to

$$\phi_{0q} = \int_0^{+\infty} dt \frac{\sin(\omega_k t)}{\omega_k} e^{-\epsilon t} \mathbf{j}_q(t), \quad (\text{A9})$$

$$\dot{\phi}_{0q} = - \int_0^{+\infty} dt \cos(\omega_k t) e^{-\epsilon t} \mathbf{j}_q(t), \quad (\text{A10})$$

from which Eq. (A8) can be rewritten as

$$\bar{\phi}_q(t) = - \int_t^{+\infty} dt' \frac{\sin[\omega_k(t-t')]}{\omega_k} e^{\epsilon(t-t')} \mathbf{j}_q(t'). \quad (\text{A11})$$

Due to the fact that $t' > t$, the limit $\epsilon \rightarrow 0$ can now be safely taken in this last expression. Finally, we expand ϕ_q around $\bar{\phi}_q$ and integrate out the Gaussian fluctuations, we replace the values of ϕ_{0q} and $\dot{\phi}_{0q}$ with Eqs. (A9) and (A10) in the surface action and we eventually integrate out ϕ_{0c} . The final result is the same W as the one given in Eq. (A5).

Appendix B: Relationship between G_K and G_R in a deep quench

In this Appendix we show that Eq. (49), which connect G_K and G_R for a deep quench, is valid to all orders in perturbation theory if the Keldysh self-energy Σ_K defined below vanishes. This is the case, e.g., in the limit $N \rightarrow \infty$ of the present model, discussed in Ref. 59.

To show this, we recall that the Dyson equation¹⁰⁰ for the post-quench propagator G of a theory in the presence of interaction can be written symbolically as

$$(G_0^{-1} - \Sigma) G = 1, \quad (\text{B1})$$

in terms of the post-quench Gaussian propagator G_0 and of the self-energy Σ of the system after the quench. In turn, G and Σ for a Keldysh action have the structure

$$G = \begin{pmatrix} G_K & G_R \\ G_A & 0 \end{pmatrix}, \quad \Sigma = \begin{pmatrix} 0 & \Sigma_A \\ \Sigma_R & \Sigma_K \end{pmatrix}, \quad (\text{B2})$$

and analogous for the Gaussian propagator. The functions Σ_K and Σ_R are, respectively, the Keldysh and retarded self-energies, which fully encode the effect of the interaction $u_{c,q}$. Equation (B1) is actually a shorthand notation for a set of integral equations of the form (we drop the dependence on the momentum variables \mathbf{k} since the equations are local in \mathbf{k})

$$\begin{aligned} G_K(t, t') = & G_{0K}(t, t') + \int_{t_1, t_2} G_{0R}(t, t_1) \Sigma_K(t_1, t_2) G_R(t', t_2) \\ & + \int_{t_1, t_2} G_{0R}(t, t_1) \Sigma_R(t_2, t_1) G_K(t_2, t') \\ & + \int_{t_1, t_2} G_{0K}(t, t_1) \Sigma_R(t_1, t_2) G_R(t', t_2), \end{aligned} \quad (\text{B3})$$

and

$$\begin{aligned} G_R(t, t') = & G_{0R}(t, t') + \int_{t_1, t_2} G_{0R}(t_1, t') \Sigma_R(t_1, t_2) G_R(t, t_2) \\ = & G_{0R}(t, t') + \int_{t_1, t_2} G_R(t_1, t') \Sigma_R(t_1, t_2) G_{0R}(t, t_2). \end{aligned} \quad (\text{B4})$$

In the last equality we have expressed the equation in an equivalent form in which G_R and G_{0R} are interchanged, which corresponds to a different order in the resummation of the one-particle irreducible diagrams contributing to Σ . In order to simplify the notation, we do not indicate here the two indices of the components of the fields involved in these $G_{R,K}$, $G_{0R,0K}$ and Σ_R , as they are all diagonal in component space. In order to prove Eq. (49), we assume that $\Sigma_K = 0$ and that, within the Gaussian approximation, G_{0K} and G_{0R} are related via

$$iG_{0K}(t, t') = \Omega_0 G_{0R}(t, 0) G_{0R}(t', 0), \quad (\text{B5})$$

which holds for the case we are interested in, see Eqs. (14a) and (14b). Then, we prove that the function $iG_K^T(t, t') \equiv \Omega_0 G_R(t, 0) G_R(t', 0)$ satisfies the same equation as G_K , and therefore they have to be equal, i.e., $iG_K(t, t') = iG_K^T(t, t') = \Omega_0 G_R(t, 0) G_R(t', 0)$, since the Dyson equation is linear. In fact, by using Eq. (B4), one can rewrite $G_K^T(t, t')$ as:

$$\begin{aligned} iG_K^T(t, t') = & \Omega_0 G_{0R}(t, 0) G_{0R}(t', 0) \\ & + \Omega_0 \int_{t_1, t_2} G_{0R}(t, 0) G_{0R}(t_1, 0) \Sigma_R(t_1, t_2) G_R(t', t_2) \\ & + \Omega_0 \int_{t_1, t_2} G_R(t', 0) G_R(t_1, 0) \Sigma_R(t_1, t_2) G_{0R}(t, t_2). \end{aligned} \quad (\text{B6})$$

The relation (B5) between G_{0K} and G_{0R} implies, after an exchange of the integration variables t_1 and t_2 in the last integral of the previous equation, that

$$\begin{aligned} G_K^T(t, t') = & G_{0K}(t, t') \\ & + \int_{t_1, t_2} G_{0K}(t, t_1) \Sigma_R(t_1, t_2) G_R(t', t_2) \\ & + \int_{t_1, t_2} G_{0R}(t, t_1) \Sigma_R(t_2, t_1) G_K^T(t', t_2), \end{aligned} \quad (\text{B7})$$

which is exactly Eq. (B3) satisfied by G_K under the assumption that $\Sigma_K = 0$.

An alternative derivation of the same result is the following: Thinking of Ω_0 as a perturbative parameter, one can now include perturbatively its effects in Σ . In particular, Σ_K then accounts for the Gaussian term $\propto \Omega_0$ while G_0 includes only G_{0R} , which is not proportional to Ω_0 , while $G_{0K} = 0$ — see Eqs. (14a) and (14b). Accordingly, Eq. (B1) implies the integral equation

$$G_K(t, t') = \int_{t_1, t_2} G_R(t, t_1) G_R(t', t_2) \Sigma_K(t_1, t_2), \quad (\text{B8})$$

which holds regardless of whether interactions are present or not. Note that, in the absence of interactions, Eq. (B8) implies

$$\Sigma_K^0(t, t') = -i\Omega_0\delta(t - t')\delta(t), \quad (\text{B9})$$

because of Eq. (B5), where Σ_K^0 can be thought of as a Keldysh “self-energy” at the zeroth order in the interactions, but which, in fact, simply accounts for the initial distribution after the quench, and it emerges naturally from discretizing the Keldysh action in time and from accounting for the initial density matrix¹⁰⁰. Under the assumption that the interaction introduced by $u_c \neq 0$ does not affect Σ_K (a condition which we expressed in the derivation above as $\Sigma_K = 0$) — which therefore equals its Gaussian value Σ_K^0 — one obtains immediately Eq. (49).

Appendix C: Renormalization of G_R in momentum space at initial times

In this section we show how a logarithmic dependence on the large cutoff Λ appears in the retarded Green’s function $G_R(q, t, t')$ in perturbation theory at one-loop when the earlier time t' is set at the temporal boundary, i.e., with $t' = 0$. Let us focus on Eq. (57), which expresses the perturbative correction δG_R to G_R , and assume that the time-independent part B_0 of the tadpole $i\mathcal{T}$ (see Eq. (51)) has been reabsorbed in a proper redefinition of the parameter r , according to Eq. (59). Due to the causal structure of G_R , the domain of integration in Eq. (57) is effectively restricted to the interval $t' \leq \tau \leq t$ and therefore the integral is finite for $\Lambda \rightarrow +\infty$ as long as $t' \neq 0$, because the only possible source of singularity associated with $B(\tau) \sim \tau^{-2}$ in Eq. (54) is located at $\tau = 0$, see Eq. (56). On the other hand, if $t' = 0$, Eq. (57) renders, at the critical point $r = r_c$,

$$\delta G_R(t, 0) = \int_0^{t_2} dt G_{0R}(q, t_2, t) B(t) G_{0R}(q, t, 0) \equiv I_R(t), \quad (\text{C1})$$

in which one cannot take $\Lambda \rightarrow +\infty$ in $B(\tau)$ from the outset, but this can be done only after a convenient subtraction of the singular term $\sim 1/\tau^2$ has been introduced in the integral $I_R(t)$. Since $G_{0R}(q, \tau, 0) = -\tau + O(\tau^2)$ and $G_{0R}(q, t, \tau) = G_R(q, t, 0) + O(\tau)$, a convenient way of writing $I_R(t)$ and of introducing the subtraction is

$$I_R(t) = \int_0^t d\tau [G_{0R}(q, t, \tau) G_{0R}(q, \tau, 0) + G_{0R}(q, t, 0) \tau] B(\tau) - G_{0R}(q, t, 0) \int_0^t d\tau \tau B(\tau), \quad (\text{C2})$$

where the first integral on the r.h.s. is finite for $\Lambda \rightarrow +\infty$, given that the term within square brackets vanishes $\sim \tau^2$ as $\tau \rightarrow 0$ and therefore regularizes the singularity that $B(\tau)$ develops as Λ grows. The second integral,

instead, can be explicitly calculated by using the explicit expression of $B(t)$ in Eq. (54) and it gives

$$\begin{aligned} \int_0^t d\tau \tau B(\tau) &= -\theta_N \int_0^{\Lambda t} d\tau \tau \frac{1 - \tau^2}{(\tau^2 + 1)^2} \\ &= \theta_N \ln(\Lambda t) + \text{finite terms}, \end{aligned} \quad (\text{C3})$$

where θ_N is given in Eq. (55) and the terms which have been omitted are finite as Λt increases. Accordingly, a logarithmic divergence proportional to $G_R(q, t, 0)$ occurs in $I_R(t)$ in the limit $\Lambda \rightarrow +\infty$.

Appendix D: Renormalization of G_K in momentum space

In this section we provide some details of the calculation of the one-loop correction δiG_K to the Keldysh Green’s function iG_K of the model after a deep quench, reported in Eq. (67). As in App. C, the correction involves the tadpole $i\mathcal{T}$, the time-independent part of which can be reabsorbed in a redefinition of the parameter r as in Eq. (59). As a result, Eq. (67) can be rewritten, at the critical point $r = r_c$ (see after Eq. (59)), as

$$\delta iG_K(q, t, t') = -\frac{\Omega_0}{q^2} [I_K(t) \sin(qt') + I_K(t') \sin(qt)], \quad (\text{D1})$$

where the integral I_K is given by

$$\begin{aligned} I_K(t) &= \frac{1}{q} \int_0^t d\tau \sin(q(t - \tau)) B(\tau) \sin(q\tau) \\ &= \frac{1}{q} \int_0^{+\infty} d\tau \sin(q(t - \tau)) B(\tau) \sin(q\tau) \\ &\quad - \frac{1}{q} \int_t^{+\infty} d\tau \sin(q(t - \tau)) B(\tau) \sin(q\tau) \\ &\equiv I_{K1}(t) + I_{K2}(t), \end{aligned} \quad (\text{D2})$$

where, for later convenience, we introduced the two integrals $I_{K1,2}$, which we calculate separately. Given the expression of $B(t)$ in Eq. (54), I_{K1} can be calculated analytically and it reads:

$$\begin{aligned} I_{K1}(t) &= \theta_N \left\{ -\frac{\pi}{2} e^{-q/\Lambda} \cos(qt) - \sin(qt) \times \right. \\ &\quad \left. \times \left[\cosh\left(\frac{q}{\Lambda}\right) \text{Chi}\left(\frac{q}{\Lambda}\right) - \sinh\left(\frac{q}{\Lambda}\right) \text{Shi}\left(\frac{q}{\Lambda}\right) \right] \right\}, \end{aligned} \quad (\text{D3})$$

where we introduced the hyperbolic cosine and sine integral, defined as

$$\text{Chi}(x) = \gamma + \ln x + \int_0^x dt \frac{\cosh t - 1}{t}, \quad (\text{D4})$$

$$\text{Shi}(x) = \int_0^x dt \frac{\sinh t}{t}, \quad (\text{D5})$$

respectively, where $\gamma \simeq 0.577$ is the Euler-Mascheroni constant. The integral $I_{K2}(t)$ in Eq. (D2) can be easily

calculated for $t \gg \Lambda^{-1}$, in which case the function $B(t)$ is approximated as in Eq. (56) and yields

$$I_{K2}(t) = \theta_N \left\{ \sin(qt) \text{Ci}(2qt) + \cos(qt) \left[\frac{\pi}{2} - \text{Si}(2qt) \right] \right\}, \quad (\text{D6})$$

where Ci and Si were introduced in Eq. (62). Summing Eqs. (D3) and (D6) one eventually finds for $I_K(t)$ in Eq. (D2), at the leading order for $q/\Lambda \ll 1$,

$$I_K(t) = -\theta_N \{ \sin(qt) [\ln(q/\Lambda) - \text{Ci}(2qt)] + \cos(qt) \text{Si}(2qt) \}. \quad (\text{D7})$$

This expression can be inserted into Eq. (D1) which, taking into account Eq. (16a) for the Gaussian Keldysh Green's function, gives the explicit form of the correction

$$\delta iG_K(q, t, t') = iG_{0K}(q, t, t') \theta_N [2 \ln(q/\Lambda) - F_K(qt, qt')] + \delta g_K(q, t, t'),$$

where $F_K(x, y)$ is a scaling function defined as:

$$F_K(x, y) = \text{Ci}(2x) + \text{Ci}(2y), \quad (\text{D8})$$

while $\delta g_K(q, t, t')$ contains only oscillating and finite corrections:

$$\delta g_K(q, t, t') = \theta_N \frac{\Omega_0}{q^2} [\sin(qt) \cos(qt') \text{Si}(2qt') + \sin(qt') \cos(qt) \text{Si}(2qt)]. \quad (\text{D9})$$

Appendix E: Corrections to Green's functions in real space

In this Appendix we analyze the one-loop perturbative corrections to $G_{R,K}$ in real space and time, in order to investigate how their structure is modified by fluctuations. The starting point is Eq. (27) which we rewrite for convenience

$$G_{R,K}(x, t, t') = \frac{1}{x^{1-\epsilon/2} (2\pi)^{2-\epsilon/2}} \int_0^\infty dq q^{2-\epsilon/2} \times J_{1-\epsilon/2}(qx) f(q/\Lambda) G_{R,K}(q, t, t'), \quad (\text{E1})$$

in which we introduced the cut-off function $f(x)$, as specified above Eq. (32). The first correction $\delta G_{R,K}^\epsilon$ to the Gaussian $G_{0R,0K}$ comes from the first-order expansion of this expression in powers of ϵ , which involves an expansion of $J_{1-\epsilon/2}$, given by¹²³

$$\partial_\nu J_\nu(y)|_{\nu=1} = \frac{\pi}{2} Y_1(y) + \frac{J_0(y)}{y}, \quad (\text{E2})$$

where $Y_1(y)$ is the Bessel function of the second kind. The above expansion does not produce any logarithmic divergence. For the response function, using the Gaussian retarded Green's function G_{0R} in Eq. (16a), we may

write,

$$\begin{aligned} \delta G_R^\epsilon(x, t, t') = & -\frac{1}{x^{1-\epsilon/2} (2\pi)^2} \int_0^\infty dq q^{1-\epsilon/2} J_1(qx) \sin(q(t-t')) f(q/\Lambda) \\ & + \frac{1}{x (2\pi)^2} \int_0^\infty dq q J_1(qx) \sin(q(t-t')) f(q/\Lambda). \end{aligned} \quad (\text{E3})$$

Due to the oscillatory nature of the integrands, the two integrals above provide a non-vanishing contribution only if the oscillations from the asymptotic form of the Bessel function $J_1(qx)$ as a function of q are in phase with the oscillations of $\sin(q(t-t'))$, i.e., only for $x = t - t'$, in which case one finds

$$\begin{aligned} \delta G_R^\epsilon(x = t - t') \simeq & -\frac{1}{x^{3-\epsilon} (2\pi)^2} \int_0^{\Lambda x} dy \frac{y^{1-\epsilon/2}}{\sqrt{y}} \\ & + \frac{1}{x^3 (2\pi)^2} \int_0^{\Lambda x} dy \frac{y}{\sqrt{y}} \propto \frac{x^{\epsilon/2} - 1}{x^{3/2}}, \end{aligned} \quad (\text{E4})$$

where we used the fact that $f(y)$ sharply drops to zero for $y \gtrsim 1$. Expanding this expression in ϵ , we obtain $\delta G_R^\epsilon = (\epsilon/2) (\ln x) G_{0R}$.

Concerning the correction δG_K^ϵ to G_K , we use again the argument that the term coming from the expansion of $J_{1-\epsilon/2}$ does not generate any logarithmic divergence, and therefore we focus on the remaining terms; for simplicity, as in Sec. IID, we consider on the case of equal times, i.e., of $G_K(x, t, t)$ for which Eqs. (E1) and (16b) give

$$\begin{aligned} i\delta G_K^\epsilon(x, t, t) = & \frac{\Omega_0}{2x^{1-\epsilon/2} (2\pi)^2} \int_0^\infty dq q^{-\epsilon/2} \\ & \times J_1(qx) [1 - \cos(2qt)] f(q/\Lambda) \\ & - \frac{\Omega_0}{2x (2\pi)^2} \int_0^\infty dq J_1(qx) [1 - \cos(2qt)] f(q/\Lambda). \end{aligned} \quad (\text{E5})$$

On the light cone $x = 2t$, the oscillations of $J_1(qx)$ in the integrand as a function of q are in phase with those of $1 - \cos(2qt)$ and therefore, after the change of variable $y = qx$,

$$\begin{aligned} i\delta G_K^\epsilon(x = 2t, t, t) \simeq & \frac{\Omega_0}{2x^{2-\epsilon} (2\pi)^2} \int_0^{\Lambda x} dy \frac{y^{-\epsilon/2}}{\sqrt{y}} \\ & - \frac{\Omega_0}{2x^2 (2\pi)^2} \int_0^{\Lambda x} dy \frac{1}{\sqrt{y}} \propto \frac{x^{\epsilon/2} - 1}{x^{3/2}}. \end{aligned} \quad (\text{E6})$$

Accordingly, on the light cone,

$$\delta G_K^\epsilon(x = 2t, t, t) = (\epsilon/2) (\ln x) G_{0K}. \quad (\text{E7})$$

Inside the light cone, i.e., for $x \ll 2t$, $\cos(2qt)$ may be neglected as it is out of phase with the asymptotic oscillations of J_1 , which actually provides an UV cutoff to the remaining integral:

$$\begin{aligned} i\delta G_K^\epsilon(x \ll 2t, t, t) \simeq & \frac{\Omega_0}{2x^{2-\epsilon} (2\pi)^2} \int_0^\infty dy y^{-\epsilon/2} J_1(y) \\ & - \frac{\Omega_0}{2x^2 (2\pi)^2} \int_0^\infty dy J_1(y) \propto \frac{x^\epsilon - 1}{x^2}. \end{aligned} \quad (\text{E8})$$

Accordingly, inside the light cone,

$$\delta G_K^\epsilon(x \ll 2t, t, t) = \epsilon(\ln x)G_{0K}. \quad (\text{E9})$$

Concerning, instead, the interaction corrections $\delta G_{R,K}^u$ at one-loop (see Eq. (77)), one essentially needs to Fourier transform the corresponding expressions of $\delta G_{R,K}$ obtained in Sec. III A, which can be calculated directly for $d = 4$:

$$\delta G_{R,K}^u(x, t, t') = \frac{1}{4\pi^2 x} \int_0^\Lambda dq q^2 J_1(qx) \delta G_{R,K}(q, t, t'). \quad (\text{E10})$$

Using Eq. (60) for δG_R , we obtain Eq. (79) in the main text. Similarly, using the one-loop interaction correction to G_K given in Eq. (68), we obtain Eq. (84). Extracting logarithmic corrections from δG_R^u in Eq. (79) is straightforward and this is discussed in the main text, see Sec. III C. Here, instead, we focus on δG_K^u and outline how to extract such corrections from the corresponding expression in Eq. (84). At long times $x \ll 2t$,

$$\begin{aligned} i\delta G_K^u(x \ll 2t, t, t) &\simeq 2\theta_N \frac{\Omega_0}{8\pi^2 x^2} \int_0^\infty dy J_1(y) \ln\left(\frac{y}{\Lambda x}\right) \\ &\simeq -2\theta_N \ln(\Lambda x) G_{0K} + 2\theta_N \frac{\Omega_0}{8\pi^2 x^2} (-\gamma + \ln 2), \end{aligned} \quad (\text{E11})$$

where we have used the identity¹²³ $\int_0^\infty dy J_1(y) \ln(y) = -\gamma + \ln 2$, where $\gamma \simeq 0.577$ is the Euler-Mascheroni constant. On the light cone $x = 2t$, instead, it is convenient to rewrite Eq. (84) as

$$i\delta G_K^u(x = 2t) = -2\theta_N \ln(\Lambda x) G_{0K} + 2\theta_N \frac{\Omega_0}{8\pi^2 x^2} k(\Lambda x) \quad (\text{E12})$$

with

$$k(x) = \int_0^x dy J_1(y) (1 - \cos y) [-\text{Ci}(y) + \ln y], \quad (\text{E13})$$

where we have dropped the $\sin(x)\text{Si}(x)$ term, as it does not provide a logarithmic correction. Now integrating Eq. (E13) by parts, one finds

$$\begin{aligned} k(x) &= [-\text{Ci}(x) + \ln x] \int_0^x dy J_1(y) (1 - \cos y) \\ &\quad - \int_0^x dy \frac{1 - \cos y}{y} \int_0^y dy' J_1(y') (1 - \cos y'). \end{aligned} \quad (\text{E14})$$

$$\begin{aligned} \delta S_{\text{coh}}^{(2)} &= -u_c^2 \frac{N+8}{N^2} \left(\frac{2}{4!}\right)^2 \sum_{j,l=1}^N \int_{1,2} \{ \phi_{cl}^2(1) \phi_{cj}(2) \phi_{qj}(2) iG_{0K}(1,2) G_{0R}(2,1) + \phi_{cl}^2(2) \phi_{cj}(1) \phi_{qj}(1) iG_{0K}(1,2) G_{0R}(1,2) \} \\ &\quad - u_q u_c \left(\frac{2}{4!}\right)^2 \frac{N+8}{N^2} \sum_{j,l=1}^N \int_{1,2} \{ \phi_{ql}^2(1) \phi_{cj}(2) \phi_{qj}(2) iG_{0K}(1,2) G_{0R}(2,1) + \phi_{ql}^2(2) \phi_{cj}(1) \phi_{qj}(1) iG_{0K}(1,2) G_{0R}(1,2) \} \\ &= -u_c^2 \left(\frac{2}{4}\right)^2 \frac{N+8}{N^2} \sum_{j,l=1}^N \int_{1,2} \{ u_c \phi_{cl}^2(1) \phi_{cj}(2) \phi_{qj}(2) + u_q \phi_{ql}^2(1) \phi_{cj}(2) \phi_{qj}(2) \} iG_{0K}(1,2) G_{0R}(2,1), \end{aligned} \quad (\text{F4})$$

Moreover, since $\int_0^y dy' J_1(y') (1 - \cos y') \sim \sqrt{y}$ for $y \rightarrow \infty$, the last term does not give logarithmic corrections. Accordingly, the logarithmic correction comes only from the first term on the r.h.s. of Eq. (E14), which is actually proportional to G_{0K} in real space (see Eq. (30)) and therefore the logarithmic contribution of the second term $\Omega_0 k(\Lambda x \gg 1)/(8\pi^2 x^2) \simeq \ln(\Lambda x) G_{0K}$ on the r.h.s. of Eq. (E12) cancels the one coming from the first term. This means that the interaction term, up to this order in perturbation theory, does not affect the leading algebraic behavior of G_K on the light cone.

Appendix F: Wilson's RG: one-loop corrections

As discussed in Sec. V, in the Wilson's RG the effective action for the slow fields can be obtained by integrating out the fast fields. After a rescaling of space and time coordinates and fields, one obtains the recursion relations for the parameters of the (effective) action, as summarized in Eqs. (115). In order to highlight the emergence of a non-trivial fixed point, we will consider the ϵ -expansion around $d_c = 4$, with $\epsilon = d_c - d$. The first-order correction to the original (bare) action of the slow fields is

$$\delta S^{(1)} = -u_c \int_{\mathbf{x},t} \phi_c \cdot \phi_q I_1(t), \quad (\text{F1})$$

where, exploiting the spatial translational invariance of the Keldysh Green's function,

$$I_1(t) = \frac{N+2}{12N} iG_{0K}^>(\mathbf{x} = 0, t, t). \quad (\text{F2})$$

The second-order correction, instead, is given by:

$$\delta S^{(2)} = i \frac{1}{2} \langle S_{\text{int}}^2 \rangle_c^c \quad (\text{F3})$$

where S_{int} is given in Eq. (113), while the superscript c indicates that only connected diagrams have to be considered. $\delta S^{(2)}$ contains corrections $\delta S_{\text{coh}}^{(2)}$ and $\delta S_{\text{diss}}^{(2)}$ to both the coherent and dissipative vertices, respectively, and is non-local both in time and space. Let us focus first on the coherent vertices; denoting, for simplicity, $n \equiv (\mathbf{x}_n, t_n)$, the corrections read:

where in the last equality we simply relabelled the variables in the second integral. In order to retain only the local part of the resulting interaction, we expand the field $\phi_{c,q}(1)$ around point 2 as

$$\phi_{c,q}(\mathbf{z}_1) \simeq \phi_{c,q}(\mathbf{z}_2) + (\mathbf{z}_1 - \mathbf{z}_2) \cdot \frac{\partial \phi_{c,q}(\mathbf{z}_2)}{\partial \mathbf{z}_2} + \dots \quad (\text{F5})$$

with $\mathbf{z}_i \equiv (\mathbf{x}_i, t_i)$, and we retain only the first term of the expansion $\phi_{c,q}(2) \equiv \phi_{c,q}(\mathbf{z}_2)$, as derivatives of higher order are expected to be irrelevant. Accordingly, the correction to the local vertices is

$$\delta S_{\text{coh}}^{(2)} = -\frac{2u_c}{4!N} \int_{x,t} [u_c (\phi_q \cdot \phi_c) \phi_c^2 + u_q (\phi_c \cdot \phi_q) \phi_q^2] I_2(t), \quad (\text{F6})$$

where

$$I_2(t) = \frac{N+8}{6N} \int_{\mathbf{x}', t'} iG_{0K}(\mathbf{x}', t, t') G_{0R}(\mathbf{x}', t - t'), \quad (\text{F7})$$

where the spatial translational invariance of $G_{R,K}$ has been used. Finally, assuming for simplicity $T = 0$ in the initial state, by using Eqs. (14a) and (14b), the integrals I_1 and I_2 read:

$$\begin{aligned} I_1 &= \frac{N+2}{12N} \int_{>} \frac{d^d k}{(2\pi)^d} iG_{0K}(k, t, t) \\ &\simeq \frac{N+2}{12N} a_d \frac{d\Lambda}{\Lambda} \Lambda^d iG_{0K}(\Lambda, t, t) \\ &= \frac{d\Lambda}{\Lambda} \frac{N+2}{12N} a_d \frac{\Lambda^d}{\omega_\Lambda} [K_+ + K_- \cos(2\omega_\Lambda t)]; \end{aligned} \quad (\text{F8})$$

$$\begin{aligned} I_2 &= \frac{N+8}{6N} \int_0^t dt' \int_{>} \frac{d^d k}{(2\pi)^d} iG_{0K}(k, t, t') G_{0R}(k, t - t') \\ &\simeq \frac{N+8}{6N} a_d \frac{d\Lambda}{\Lambda} \Lambda^d \int_0^t dt' iG_{0K}(\Lambda, t, t') G_{0R}(\Lambda, t - t') \\ &= -\frac{N+8}{24N} a_d \frac{d\Lambda}{\Lambda} \frac{\Lambda^d}{\omega_\Lambda^3} \{K_- 2\omega_\Lambda t \sin(2\omega_\Lambda t) \\ &\quad + (K_+ - K_-)[1 - \cos(2\omega_\Lambda t)]\}, \end{aligned} \quad (\text{F9})$$

where $\int_{>} \equiv \int_{\Lambda - d\Lambda \leq |\mathbf{k}| \leq \Lambda}$ and K_\pm are given in Eq. (118) and their argument is calculated for $k = \Lambda$. Finally, the generated dissipative vertices $\propto |\phi_c|^2 |\phi_q|^2$ and $\propto (\phi_c \cdot \phi_q)^2$ due to the interaction vertices $\propto u_c^2$ and $\propto u_c u_q$ can be calculated analogously. For the latter we find,

$$\begin{aligned} \delta S_{\text{diss}}^{(2)} \Big|_{u_c u_q} &= -i \left(\frac{2}{4!N} \right)^2 \sum_{jklm=1}^N u_c u_q \times \\ &\int_{1,2} [2\phi_{qj}^<(1) \phi_{cj}^>(1) \phi_{cl}^>(1) \phi_{cl}^<(1) + \phi_{qj}^<(1) \phi_{cj}^<(1) \phi_{cl}^>(1) \\ &\quad [2\phi_{qk}^>(2) \phi_{ck}^<(2) \phi_{qm}^>(2) \phi_{qm}^<(2) + \phi_{qk}^<(2) \phi_{ck}^<(2) \phi_{qm}^>(2)]] , \end{aligned} \quad (\text{F10})$$

where, for convenience, we reinstated the explicit indication of the fast and the slow components of the involved

fields, with the understanding that one has eventually to integrate out the fast components, as discussed in Sec. V. The previous equation is a sum of four terms, where, for instance, the first one is:

$$\begin{aligned} &4 \left(\frac{2}{4!N} \right)^2 u_c u_q \int_{1,2} G_{0R}^2(1, 2) \\ &\times \left\{ [\phi_c^<(2) \cdot \phi_q^<(2)]^2 + [\phi_c^<(2)]^2 [\phi_q^<(2)]^2 \right\}. \end{aligned} \quad (\text{F11})$$

All the remaining terms can be analogously evaluated. Summing all terms, including the $u_c u_q$ and u_c^2 vertices, one obtains

$$\begin{aligned} \delta S_{\text{diss}}^{(2)} &= -i \frac{2u_c}{144N^2} \int_{1,2} [u_q G_{0R}^2(1, 2) + u_c G_{0K}^2(1, 2)] \\ &\times \left\{ (N+6) \left(\phi_c^<(1) \cdot \phi_q^<(1) \right)^2 + 2 |\phi_c^<(1)|^2 |\phi_q^<(1)|^2 \right\}. \end{aligned} \quad (\text{F12})$$

Performing the integrations, one obtains,

$$\begin{aligned} \delta S_{\text{diss}}^{(2)} &= -iu_c \frac{d\Lambda}{\Lambda} \frac{a_d}{144N^2} \frac{\Lambda^d}{\omega_\Lambda^2} [u_q - u_c (K_+^2 + K_-^2)] \int_1 t_1 \\ &\times \left\{ (N+6) [\phi_c^<(1) \cdot \phi_q^<(1)]^2 + 2 [\phi_c^<(1)]^2 [\phi_q^<(1)]^2 \right\} \end{aligned} \quad (\text{F13})$$

where we neglected the oscillating terms coming from the integration of $G_{R,K}$ in time. The coefficients K_\pm are given in Eq. (118) and their argument is calculated for $k = \Lambda$.

Appendix G: Four-point function at one-loop

In this Appendix we calculate explicitly the one-loop contribution to the four-point function introduced in Sec. VI. As anticipated in the main text, among the many terms generated by the repeated use of the Wick's theorem, we need to consider only those which renormalize the vertex. In particular, several terms which arise in the perturbative expansion actually renormalize the Green's functions rather than the interaction vertex. Those we are interested in can be written as

$$\begin{aligned} I_2 &= -\frac{1}{2} \left(\frac{2u_c}{4!N} \right)^2 \int_{1',2'} \langle \phi_{cj}(1) \phi_{ql}(2) \phi_{qm}(3) \phi_{qn}(4) \\ &\times [\phi_q(1') \cdot \phi_c(1')] \phi_c^2(1') [\phi_q(2') \cdot \phi_c(2') \phi_c^2(2')]_0 \\ &= \frac{2u_c^2}{4!N} \frac{N+8}{6} \int_{1',2'} [\langle \phi_{cj}(1) \phi_{ql}(2) \phi_{qm}(3) \phi_{qn}(4) \\ &\times [\phi_c(1') \cdot \phi_q(1')] \phi_c^2(2')]_0 G_{0K}(1', 2') G_{0R}(1', 2')], \end{aligned} \quad (\text{G1})$$

where $\langle \dots \rangle_0$ denotes the average over the Gaussian part of the action (43), while in the last equality we relabelled

the integration variables. In order to select only the local contribution to the vertex, we expand the fields contracted with the external legs around the point indicated as α (first term) and we retain only the zeroth order term (cf. Eq. (F5)). Accordingly, we have

$$\begin{aligned}
I_2 &= \frac{2u_c^2}{4!N} \frac{N+8}{6} \int_{1'} \langle \phi_{cj}(1) \phi_{ql}(2) \phi_{qm}(3) \phi_{qn}(4) (\phi_q \cdot \phi_c) \phi_c^2 \rangle_0 \\
&\times \int_{2'} G_{0K}(1', 2') G_{0R}(1', 2') \\
&= \frac{N+8}{36N} F_{jlmn} u_c^2 \int_{1'} G_{0R}(1, 1') G_{0R}(1', 2) G_{0R}(1', 3) \\
&\times G_{0R}(1', 4) \int_{2'} G_{0K}(1', 2') G_{0R}(1', 2') \\
&\equiv \frac{-iu_c}{6N} F_{jlmn} \int_{1'} G_{0R}(1, 1') G_{0R}(1', 2) G_{0R}(1', 3) G_{0R}(1', 4) J \\
&\quad (G2)
\end{aligned}$$

where F_{jlmn} is defined below Eq. (137), while the last integral J can be easily calculated:

$$\begin{aligned}
J &= \frac{N+8}{6N} u_c \int_{\mathbf{x}} \int_0^t dt' iG_{0K}(\mathbf{x}, t, t') G_{0R}(\mathbf{x}, t - t') \\
&= \frac{N+8}{6N} u_c \int_{\mathbf{q}} \int_0^t dt' iG_{0K}(q, t, t') G_{0R}(q, t - t') e^{-2q/\Lambda} \\
&= -\frac{N+8}{48N} u_c \Omega_{0q} a_4 \left\{ -\frac{(\Lambda t)^2}{1 + (\Lambda t)^2} + \ln \left[1 + (\Lambda t)^2 \right] \right\}, \\
&\quad (G3)
\end{aligned}$$

where $e^{-2q/\Lambda}$ implements the UV regularization of the integral over momenta k , and in the last equality we set $d = 4$. Finally, taking the limit $\Lambda t \gg 1$, we find the logarithmic dependence on Λ

$$J = -\frac{N+8}{24N} a_4 u_c \Omega_{0q} \ln(\Lambda t), \quad (G4)$$

reported in Eq. (140).

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