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Gauge invariant theories of linear response for strongly correlated superconductors

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We present a diagrammatic theory for determining consistent electromagnetic response functions in strongly correlated fermionic superfluids. While a gauge invariant electromagnetic response is well understood at the BCS level, a treatment of correlations beyond BCS theory requires extending this theoretical formalism. The challenge in such systems is to maintain gauge invariance, while simultaneously incorporating additional self-energy terms arising from strong correlation effects. Central to our approach is the application of the Ward-Takahashi identity, which introduces collective mode contributions in the response functions and guarantees that the f-sum rule is satisfied. We outline a powerful method which determines these collective modes in the presence of correlation effects and in a manner compatible with gauge invariance. Since this method is based on fundamental aspects of quantum field theory, the underlying principles are broadly applicable to strongly correlated superfluids. As an illustration of the technique, we apply it to a simple class of theoretical models that contain a frequency-independent order parameter. These models include BCS-BEC crossover theories of the ultra cold Fermi gases, along with models specifically associated with the high-T_c cuprates. Finally, as an alternative approach, we contrast with the path integral formalism. Here, the calculation of gauge invariant response appears more straightforward. However, the collective modes introduced are those of strict BCS theory, without any modification from additional correlations. As the path integral simultaneously addresses electrodynamics and thermodynamics, we emphasize that it should be subjected to a consistency test beyond gauge invariance, namely that of the compressibility sum-rule. We show how this sum-rule fails in the conventional path integral approach.

I. INTRODUCTION

There has been a recent focus in the literature on strongly correlated superconductors and superfluids. This interest has arisen in two different contexts, via ultra cold atomic Fermi gases[1,2] and via high-T_c superconductors[3–6]. A major challenge in studying these two different systems is to arrive at correct expressions for the electromagnetic (EM) properties, such as the superfluid density and the density-density correlation function, which characterize superconductors and superfluids.

In strict BCS theory there are two different conventional techniques for addressing electromagnetic response while ensuring gauge invariance: the path integral[7–9] and the Ward-Takahashi identity[10]. The first of these methods depends on the derivation of a generating functional while the second depends on the form of the diagrammatic self energy. This body of work has enabled a complete understanding of the gauge invariant electromagnetic response at the BCS level. It does not, however, answer the important questions about how to incorporate stronger correlation effects.

Studies of high-T_c superconductors, which necessarily require a beyond-BCS formalism, are better suited to the Ward-Takahashi based approach. These studies focus on different models for the self energy associated with a normal state that includes pairing, known as the pseudogap phase[3–6]. This correlation contribution to the self energy has been extensively characterized[11] above the transition temperature T_c. In the superfluid phase, presumably one adds to this normal state self energy[3,6] an additional BCS self energy contribution. The challenge in studying strongly correlated superfluids, however, is ensuring gauge invariance. This means that the self-consistent collective modes, compatible with gauge invariance, must be properly included. In this paper we show that a diagrammatic self energy and the gap equation provide all the ingredients required to unambiguously establish the exact electromagnetic response at all temperatures. Our main goals are:

(i) To show how to arrive at the exact gauge invariant electromagnetic response of strongly correlated superfluids. This is based upon an implementation of the Ward-Takahashi identity given an arbitrary diagrammatic scalar self energy.

(ii) To provide a powerful method for obtaining the collective modes in a gauge invariant manner for strongly correlated superfluids. This is based on the gap equation, and the results derived above in (i).

The electrodynamics of superconductors is also widely addressed via the path integral approach[7–9] which requires the introduction of Gaussian-level (beyond saddle point) fluctuations. Incorporating gauge invariance is relatively straightforward, which is in large part due to the fact that the collective modes that enter at this level and beyond are those of strict BCS theory[12]. We shall later revisit this conventional calculation of response functions at the strict BCS level, while simultaneously considering thermodynamics. We find there is a serious shortcoming that has not previously been identified in the literature. This arises from an inconsistency between electrodynamics and thermodynamics, which is manifested as a failure of the compressibility sum-rule.

Our emphasis here is not on a critique of previous work since, quite generally, in the literature the focus has been on either the thermodynamics[2] or the electrodynamics[7–9], but not on both simultaneously. Nevertheless, the violation of the compressibility sum-rule is a serious shortcoming. The source of this sum-rule violation comes from the fact that the BCS-level electrodynamics is derived by incorporating beyond-BCS Gaussian fluctuations. This would seem to require that we also include Gaussian fluctuations in the number equation. However, this in fact leads to the failure of the compressibility sum-rule. A detailed discussion of how to implement consistency between electrodynamics and thermodynamics is presented elsewhere[12].

It is crucial when studying transport phenomena to ensure that all conservation laws, such as energy, momentum, and charge, are satisfied[13,14]. In particular, ensuring gauge invariance, and thus charge conservation, in a superconductor has long been a problem of great importance[10,15–18]. The key insight in the challenge of preserving gauge invariance, even in the presence of a Meissner effect, was the inclusion of long-wavelength collective excitations[15,19]. Following this initial insight, a more diagrammatic approach, based on the establishment of gauge invariance in quantum electrodynamics, was developed by Nambu[10]. Nambu's method of establishing a gauge invariant electromagnetic response was to set up a gauge invariant vertex at the same level of approximation as the self energy. He then showed that this leads to a full vertex that satisfies the Ward-Takahashi identity (WTI), a condition equivalent to gauge invariance[20].

A modern understanding of the role of gauge invariance in a superconductor is best understood from this field theoretic point of view: collective modes are excitations which restore gauge invariance. In the language of quantum field theory they can be interpreted as the Nambu-Goldstone bosons arising from spontaneous symmetry breaking in the condensed phase. Strictly speaking, in a superconductor or superfluid local gauge invariance is never broken[21]. Quite generally, the impossibility of breaking local gauge invariance without explicit gauge fixing, at least for abelian gauge fields, was proved early on by Elitzur[22]. Rather, due to the presence of a condensate, global phase invariance is spontaneously broken. In the case of a neutral order parameter the excitation spectrum contains a gapless mode, which corresponds to the collectives modes discussed throughout this paper. For a charged order parameter the Goldstone modes couple to the longitudinal degrees of freedom of the gauge field, and are gapped out.

In going beyond the BCS theory of superconductivity it is essential that charge conservation is maintained in any approximation scheme. We note that the Kadanoff-Baym approach[13,14], extended below T_c, provides a sufficient, but not necessary, condition for a theory to satisfy macroscopic conservation laws. It is inapplicable to the strongly-correlated theories considered in this paper, where the self energy is not derivable from a Luttinger-Ward functional of the full Green's function. Nonetheless, the exact gauge invariant EM response can be obtained using the diagrammatic formalism based on points (i) - the WTI and (ii) - self-consistent collective modes, discussed on the previous page.

In particular, above the transition temperature, Refs.[23,24] implemented the WTI for a number of different exotic normal phases, which led to a consistent framework for computing all vertex corrections. The challenge in the present paper is then to extend this body of work and formulate a gauge invariant theory below the transition temperature. In this context, Ref.[25] used the WTI to formulate a gauge invariant response for a specific BCS-BEC approximation valid at all temperatures. This theory accounted for non-condensed fermionic pairs by adding a t-matrix self energy to the standard BCS self energy. Inspired by this work, in this paper we will use the WTI to study a broader class of theories, addressed in the context of high T_c superconductors and atomic Fermi superfluids, which are based on an extension of a BCS-based self energy.

Within these approaches we go beyond the pioneering work of Nambu and show by extending the method of Ref.[25], that both the full vertex and the collective modes can be explicitly derived for a class of strongly correlated superfluids. In particular we derive closed form expressions for the response functions. Theories which belong to this class include the work of Refs.[23,24] along with additional theories such as that proposed in Ref.[3], Ref.[4] and Refs.[26–28].

II. CORRELATION EFFECTS BEYOND BCS THEORY: WARD-TAKAHASHI IDENTITY

A. Kubo formulae

The goal of this section of the paper is to address correlations which go beyond the mean-field BCS theory and, making use of Kubo formulae, arrive at properly gauge invariant linear response functions. We begin by summarizing the Kubo formalism for a many-body theory of interacting fermions. In what follows we shall primarily be concerned with neutral superfluids. Incorporating Coulomb effects can be done through the random phase approximation (RPA) formalism[8], once the exact response functions are obtained for the neutral system.

As is common in the literature, and nicely discussed in Ref.[29], we introduce a fictitious vector potential, A^μ, to establish gauge invariance in neutral superfluids. We emphasize that A^μ is a non-dynamical external field, incorporated here merely to derive the EM response kernel in the context of linear response.

In the presence of a weak and externally applied EM field, with four-vector potential $A^\mu = (\phi, \mathbf{A})$, the four-current density $J^\mu = (\rho, \mathbf{J})$ is given by

$$J^\mu(q) = K^{\mu\nu}(q) A_\nu(q), \tag{2.1}$$

where $q = (i\Omega_m, \mathbf{q})$ is a four-momentum, with a bosonic Matsubara frequency $i\Omega_m$. The quantity $K^{\mu\nu}$ is the EM response kernel, which is of principal interest here. Charge conservation ($q_\mu J^\mu = 0$) implies that the response kernel $K^{\mu\nu}$ must satisfy the condition $q_\mu K^{\mu\nu} = 0$. The satisfaction of this condition is what we will mean by a gauge invariant many-body theory. Once the EM response is obtained, A^μ is set to zero[8].

The response kernel $K^{\mu\nu}$ can be written in a general form as[30]

$$K^{\mu\nu}(q) = 2\sum_k G(k_+)\Gamma^\mu(k_+,k_-)G(k_-)\gamma^\nu(k_-,k_+) + \frac{n}{m}\delta^{\mu\nu}(1-\delta_{0\mu}), \tag{2.2}$$

where the full and bare vertices are $\Gamma^\mu(k_+,k_-)$, $\gamma^\mu(k_+,k_-)$ respectively, and $k_\pm \equiv k \pm q/2$ is the incoming (+) or outgoing (−) momenta of a vertex. The particle number is n and m denotes the fermion mass. The full Green's function is denoted by $G(k)$, which we define in terms of the bare Green's function, $G_0^{-1}(k) = i\omega - \xi_\mathbf{k}$, in Eq. (2.4). Here the single particle dispersion is $\xi_\mathbf{k} = k^2/2m - \mu$, where μ is the chemical potential. Throughout we set $\hbar = e = 1$.

It is useful to briefly comment about our notation. Rather than begin with the Nambu representation of a matrix self energy[31], it is more convenient to turn directly to the Gorkov representation of the Green's function which has two components: $G(k)$ and the Gorkov F-function $F_{\rm sc}(k)$, associated with the order parameter.

We now introduce a framework that encapsulates both BCS theory and stronger correlations beyond BCS theory. The general principles we outline (in particular, in the next two sections) should be applicable to more complicated theories such as Eliashberg theory[32], where, because of the dynamics of the order parameter, the establishment of a gauge invariant EM response may be prohibitively difficult to implement. Nevertheless, one can argue that without such a theory of EM response, it is difficult to distinguish between the physics of the normal-Eliashberg[33] and superfluid phases.

To understand what is meant by these correlation effects, here we consider a correlated (scalar) self energy $\Sigma_{\rm corr}(k)$. In order to simultaneously describe a wide variety of theories, we define the partially dressed Green's function

$$\left(G_0^\alpha\right)^{-1}(k) = G_0^{-1}(k) - \alpha\Sigma_{\rm corr}(k). \tag{2.3}$$

This depends on the strong correlation contribution to the self energy $\Sigma_{\rm corr}$ for $\alpha = 1$, and does not include strong correlation effects for $\alpha = 0$. The fermionic Green's function is then given by Dyson's equation

$$G^{-1}(k) = G_0^{-1}(k) - \Sigma(k), \tag{2.4}$$

where the self energy consists of two terms:

$$\Sigma(k) = \Sigma_{\rm corr}(k) - |\Delta_{\rm sc}|^2 G_0^\alpha(-k), \tag{2.5}$$

for a superconducting order parameter $\Delta_{\rm sc}$.

For convenience we assume that $\Delta_{\rm sc}$ is frequency independent, as contrasted with Eliashberg theory. Equivalently, $\Sigma(k) = \Sigma_{\rm corr}(k) + \Sigma_{\rm sc}(k)$, where $\Sigma_{\rm sc}(k) = -|\Delta_{\rm sc}|^2 G_0^\alpha(-k)$ is the scalar superconducting self energy contribution. Note that, when $\alpha = 0$ the partially dressed Green's function $G_0^\alpha(k)$ reduces to the bare Green's function $G_0(k)$, so that there are no correlation effects incorporated in the superconducting self energy term $\Sigma_{\rm sc}(k)$. However, for $\alpha = 1$ the partially dressed Green's function $G_0^\alpha(k)$ does depend on the strong correlations, via Eq. (2.3), and thus gives rise to correlation effects present in the superconducting self energy $\Sigma_{\rm sc}(k)$.

Finally, the gap equation can be written[3,6] as $1 - g\sum_k G_0^\alpha(-k)G(k) = 0$. Multiplying both sides of this equation by $\Delta_{\rm sc}$, we obtain

$$\Delta_{\rm sc}/g = \sum_k \Delta_{\rm sc} G_0^\alpha(-k)G(k) \equiv \sum_k F_{\rm sc}(k). \tag{2.6}$$

In this expression the Gorkov F-function $F_{\rm sc}(k)$ has dependence on $\Sigma_{\rm corr}(k)$ via $G_0^\alpha(k)$ and $G(k)$, and there is also implicit dependence on α through $G_0^\alpha(k)$.

This represents a fairly generic class of strongly correlated superfluid systems. When $\Sigma_{\rm corr} = 0$ the system reverts to conventional BCS theory. Thus, the challenge is to include the correlation effects associated with the self energy $\Sigma_{\rm corr}$. Models of this sort are associated with the work of Yang, Rice, and Zhang[3], and also with the work of Refs.[26–28], who address BCS-BEC crossover effects via a t-matrix theory. Also belonging to this class is an alternate t-matrix theory of BCS-BEC crossover[6,25], which, in contrast to the work of Ref.[26], is more directly associated with a BCS-based ground state.

B. The Ward-Takahashi identity

In order to derive the gauge invariant EM response, we now apply the Ward-Takahashi identity (WTI). For a quantum field theory with a $U(1)$ gauge symmetry the WTI is an exact relation between the many-body vertex function that appears in correlation functions and the self energy which enters in the Green's function. Moreover, as shown in Appendix (D), given a full vertex that satisfies the WTI, the f-sum-rule is satisfied and thus charge is conserved.

Given the bare Green's function $G_0(k)$, and the full Green's function $G(k)$, the WTI constrains the full vertex $\Gamma^\mu(k_+,k_-)$ so that it satisfies[20]

$$\begin{aligned} q_\mu\Gamma^\mu(k_+,k_-) &= G^{-1}(k_+) - G^{-1}(k_-), \\ &= q_\mu\gamma^\mu(k_+,k_-) + \Sigma(k_-) - \Sigma(k_+). \end{aligned} \tag{2.7}$$

The bare WTI, $q_\mu\gamma^\mu(k_+,k_-) = G_0^{-1}(k_+) - G_0^{-1}(k_-)$, is satisfied for a bare vertex $\gamma^\mu(k_+,k_-) = (1, \mathbf{k}/m)$. Therefore, given a self energy $\Sigma(k)$, the above equation provides a constraint which can be used to determine the full vertex.

The WTI is equivalent to self-consistent perturbation theory, and allows one to compute the exact n-loop full vertex, given any n-loop self energy. If the self energy depends on the full Green's function, then applying the WTI leads to an integral equation for the full vertex of the Bethe-Salpeter form[34]. However, if the self energy depends on only a finite number of bare or partially dressed Green's functions, then this integral equation terminates, and the full vertex can be obtained exactly. This is the situation with regard to the strong correlation approaches we consider in Secs. (II D 1-II D 2) of this paper.

We now turn to the superconducting case. For a superconductor, where gauge invariance is "spontaneously broken", the presence of a condensate below the transition temperature leads to a more complicated formulation of the WTI. Imposing gauge invariance in the presence of a condensate requires excitations known as collective modes. The explicit form of the collective modes, however, must be derived from the gap equation[25].

The Ward-Takahashi identity is equivalent to requiring that the full vertex be obtained by performing all possible vertex insertions into the self energy[10]. Below the transition temperature, however, we must account for the effect of an external (non-dynamical) vector potential A_μ on the self-consistency condition (Eq. (2.6)). This necessitates the introduction of collective mode vertices $\Pi^\mu(q)$, $\bar{\Pi}^\mu(q)$ in the full vertex, which are inserted into every location of the condensate terms $\Delta_{\rm sc}$, $\Delta^*_{\rm sc}$, respectively. In the next section we discuss these collective mode vertices in greater detail. As shown in Appendix (A), performing all vertex insertions into the self energy of Eq. (2.5), and using Eq. (2.7), then gives the full vertex:

$$\begin{aligned}\Gamma^\mu(k_+,k_-) &= \gamma^\mu(k_+,k_-) + \Lambda^\mu(k_+,k_-)\\ &- \Delta^*_{\rm sc}\Pi^\mu(q)G_0^\alpha(-k_-) - \Delta_{\rm sc}\bar{\Pi}^\mu(q)G_0^\alpha(-k_+)\\ &- |\Delta_{\rm sc}|^2 G_0^\alpha(-k_-)G_0^\alpha(-k_+)\times\\ &\quad[\gamma^\mu(-k_-,-k_+) + \alpha\Lambda^\mu(-k_-,-k_+)]. \end{aligned} \tag{2.8}$$

Here we have introduced the vertex correction $\Lambda^\mu(k_+,k_-)$, which relates to the correlated self energy contribution and satisfies $q_\mu\Lambda^\mu(k_+,k_-) = \Sigma_{\rm corr}(k_-) - \Sigma_{\rm corr}(k_+)$. The collective mode vertices in this expression are (as yet) unknowns which satisfy $q_\mu\Pi^\mu(q) = 2\Delta_{\rm sc}$, $q_\mu\bar{\Pi}^\mu(q) = -2\Delta^*_{\rm sc}$. However, by ensuring that these collective mode vertices are consistent with the gap equation, a unique expression for them can be obtained[25]. This will be outlined in the next section. Using these relations, along with the bare WTI, one can check explicitly that this full vertex satisfies the full WTI in Eq. (2.7).

By way of comparison, we note that the full vertex in Eq. (2.8) is analogous to the BCS full vertex, but with the mapping $\gamma^\mu \to \gamma^\mu + \alpha\Lambda^\mu, G_0 \to G_0^\alpha$. The many-body effect of the correlation term $\Sigma_{\rm corr}$ (in the partially dressed Green function G_0^α) is therefore to modify both the bare vertex and the single particle Green's function appearing in the superconducting part of the full vertex. The expression in Eq. (2.8) is completely general, given a self energy of the form in Eq. (2.5). In what follows we will illustrate how to compute the full vertex, and corresponding response kernel, for some examples of strongly correlated superfluids.

Two important limiting cases of the full vertex in Eq. (2.8) can be checked against known results. When $\Sigma_{\rm corr} = 0$, then $\Lambda^\mu = 0$, and the full vertex reduces to the known strict BCS case[18]. If we set $\Delta_{\rm sc} = 0$, then the full vertex also reduces to the known full vertex in the exotic normal state[23,24].

C. Collective mode vertices

The challenge in studying strongly correlated superfluids, at all temperatures, is to treat the collective modes in a manner compatible with gauge invariance. In this section we implement a powerful method of obtaining the expressions for the collective mode vertices $\Pi^\mu(q)$, $\bar{\Pi}^\mu(q)$, which applies even in the presence of correlation effects. Gauge invariance alone requires that $q_\mu\Pi^\mu(q) = 2\Delta_{\rm sc}$, $q_\mu\bar{\Pi}^\mu(q) = -2\Delta^*_{\rm sc}$. The gap equation imposes a self-consistency condition on both vertices which we will use in order to determine the explicit form of these vertices. This gap equation is written in Eq. (2.6) and in what follows we also consider the conjugate gap equation.

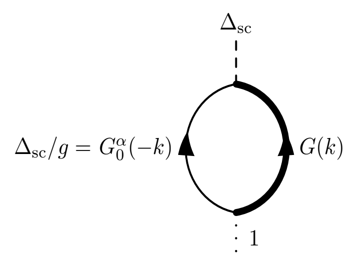

FIG. 1. Feynman diagram for the gap equation $\Delta_{\rm sc}/g = \Delta_{\rm sc}\sum_k G_0^\alpha(-k)G(k)$.

In Fig. (1) the gap equation is expressed as a Feynman diagram. Diagrammatically, the collective mode vertices are obtained by performing all possible vertex insertions into the gap equation. In Fig. (1) there are three possible vertex insertions: (1) at the $\Delta_{\rm sc}$ location one can insert $\Pi^\mu(q)$, (2) at the full Green function $G(k)$ location one can insert the full vertex $\Gamma^\mu(k_+,k_-)$, (3) at the partially dressed Green function $G_0^\alpha(-k)$ location one can insert the partially dressed vertex $\gamma^\mu(-k_-,-k_+) + \alpha\Lambda^\mu(-k_-,-k_+)$. After performing these vertex insertions we obtain the equation in Fig. (2) expressed in terms of Feynman diagrams.

Mathematically, Fig. (2) implies that the collective

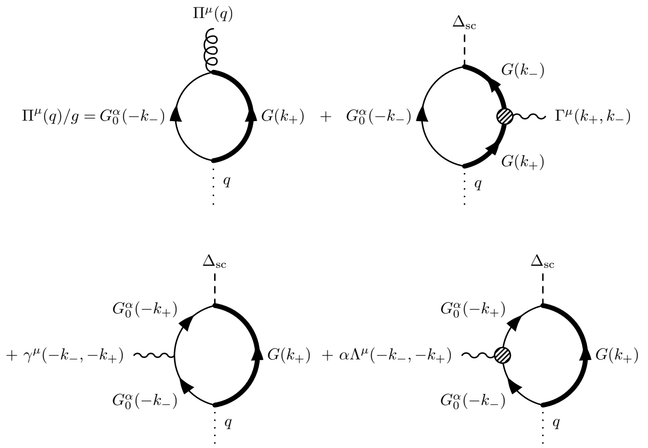

FIG. 2. Self consistent equation for the collective modes after performing all possible vertex insertions into the gap equation.

mode vertices must satisfy the following equation

$$\begin{aligned}\Pi^\mu(q)/g &= \Pi^\mu(q)\sum_k G_0^\alpha(-k_-)G(k_+)\\ &+\Delta_{sc}\sum_k G_0^\alpha(-k_-)G(k_+)\Gamma^\mu(k_+,k_-)G(k_-)\\ &+\Delta_{sc}\sum_k\Big(G_0^\alpha(-k_-)G_0^\alpha(-k_+)G(k_+)\\ &\qquad\times[\gamma^\mu(k_+,k_-)+\Lambda^\mu(k_+,k_-)]\Big).\end{aligned}\tag{2.9}$$

Notice that the full vertex $\Gamma^\mu(k_+,k_-)$ appears in this expression. The full vertex was already determined in Eq. (2.8) using the Ward-Takahashi identity. Therefore if we insert the expression for the full vertex, which contains the collective mode vertices, into Eq. (2.9) (and its conjugate), then Eq. (2.9) (and its conjugate) becomes a self-consistent set of equations for the collective mode vertices Π^μ and $\bar{\Pi}^\mu$. The solution to this self-consistent set of linear equations will uniquely determine the collective mode vertices.

Inserting the full vertex into Eq. (2.9), and doing the same analysis for the conjugate gap equation, then gives the following two self-consistent equations for the collective mode vertices:

$$\begin{aligned}\Pi^\mu(q)/g &= \Pi^\mu(q)\sum_k G(k_+)G_0^\alpha(-k_-)\left[1-\Delta_{sc}^*F_{sc}(k_-)\right]\\ &-\bar{\Pi}^\mu(q)\sum_k F_{sc}(k_+)F_{sc}(k_-)\\ &+\sum_k\left[\gamma^\mu(k_+,k_-)+\Lambda^\mu(k_+,k_-)\right]G(k_+)F_{sc}(k_-)\\ &+\sum_k\Big([\gamma^\mu(-k_-,-k_+)+\alpha\Lambda^\mu(-k_-,-k_+)]\times\\ &\qquad F_{sc}(k_+)G_0^\alpha(-k_-)\left[1-\Delta_{sc}^*F_{sc}(k_-)\right]\Big).\end{aligned}\tag{2.10}$$

$$\begin{aligned}\bar{\Pi}^\mu(q)/g &= \bar{\Pi}^\mu(q)\sum_k G(k_-)G_0^\alpha(-k_+)\left[1-\Delta_{sc}F_{sc}^*(k_+)\right]\\ &-\Pi^\mu(q)\sum_k F_{sc}^*(k_+)F_{sc}^*(k_-)\\ &+\sum_k\left[\gamma^\mu(k_+,k_-)+\Lambda^\mu(k_+,k_-)\right]F_{sc}^*(k_+)G(k_-)\\ &+\sum_k\Big([\gamma^\mu(-k_-,-k_+)+\alpha\Lambda^\mu(-k_-,-k_+)]\times\\ &\qquad G_0^\alpha(-k_+)F_{sc}^*(k_-)\left[1-\Delta_{sc}F_{sc}^*(k_+)\right]\Big).\end{aligned}\tag{2.11}$$

This is conveniently expressed as a matrix equation if we define the two-point correlation functions

$$\begin{aligned}Q_{+-}(q) &= 1/g-\sum_k G(k_+)G_0^\alpha(-k_-)\left[1-\Delta_{sc}^*F_{sc}(k_-)\right],\\ Q_{++}(q) &= \sum_k F_{sc}(k_+)F_{sc}(k_-),\\ P_+^\mu(q) &= \sum_k\left[\gamma^\mu(k_+,k_-)+\Lambda^\mu(k_+,k_-)\right]G(k_+)F_{sc}(k_-)\\ &+\sum_k\Big([\gamma^\mu(-k_-,-k_+)+\alpha\Lambda^\mu(-k_-,-k_+)]\times\\ &\qquad F_{sc}(k_+)G_0^\alpha(-k_-)\left[1-\Delta_{sc}^*F_{sc}(k_-)\right]\Big),\end{aligned}\tag{2.12}$$

and $Q_{-+}(q)=Q_{+-}^*(q)$, $Q_{--}(q)=Q_{++}^*(q)$, $\Delta_{sc}^*P_+^\mu(q)=\Delta_{sc}P_-^\mu(-q)$. To connect to the literature, we define an alternative set of of two-point correlation functions Q_{ab} and $Q^{a\mu}$, where $a,b=1,2$ through, $Q_{11}=Q_{+-}+Q_{-+}+Q_{++}+Q_{--}$, $Q_{22}=Q_{+-}+Q_{-+}-Q_{++}-Q_{--}$, $Q_{12}=i(Q_{+-}-Q_{-+}+Q_{--}-Q_{++})$, $Q_{21}=-i(Q_{+-}-Q_{-+}+Q_{++}-Q_{--})$, and $Q^{1\mu}=-\left(P_+^\mu+P_-^\mu\right)$, $Q^{2\mu}=-i(P_-^\mu-P_+^\mu)$. Similarly, we define the collective mode vertices $\Pi_{1,2}^\mu(q)$ through $\Pi^\mu(q)=\Pi_1^\mu(q)+i\Pi_2^\mu(q)$, $\bar{\Pi}^\mu(q)=\Pi_1^\mu(q)-i\Pi_2^\mu(q)$. This amounts to a change of basis from a complex to a real and imaginary parameterization. From Eq. (2.10) and Eq. (2.11), these vertices satisfy the relation

$$\begin{pmatrix}\Pi_1^\mu\\ \Pi_2^\mu\end{pmatrix}=-\begin{pmatrix}Q_{11} & Q_{12}\\ Q_{21} & Q_{22}\end{pmatrix}^{-1}\begin{pmatrix}Q^{1\mu}\\ Q^{2\mu}\end{pmatrix}.\tag{2.13}$$

The form of these collective mode vertices is structurally similar to the BCS case[18,25], and in the strict BCS limit they agree with the literature[18]. The matrix Q_{ab} can be interpreted as a propagator for bosonic degrees of freedom. However, the explicit response functions entering on the right hand side of Eq. (2.13) are modified due to the presence of the self energy Σ_{corr}.

In Appendix (B) we verify that the collective mode vertices $\Pi^\mu(q)$ and $\bar{\Pi}^\mu(q)$ satisfy the gauge invariant conditions $q_\mu\Pi^\mu(q)=2\Delta_{sc}$, $q_\mu\bar{\Pi}^\mu(q)=-2\Delta_{sc}^*$, which was assumed in their definitions.

D. Vertex correction Λ^μ

We can now summarize the central results of this paper, and repeat key equations. The full electromagnetic response kernel can generically be written as

$$K^{\mu\nu}(q) = 2\sum_k G(k_+)\Gamma^\mu(k_+,k_-)G(k_-)\gamma^\nu(k_-,k_+) + \frac{n}{m}\delta^{\mu\nu}(1-\delta_{0\mu}), \qquad (2.2)$$

where the full vertex

$$\begin{aligned}\Gamma^\mu(k_+,k_-) &= \gamma^\mu(k_+,k_-) + \Lambda^\mu(k_+,k_-)\\ &- \Delta^*_{\rm sc}\Pi^\mu(q)G_0^\alpha(-k_-) - \Delta_{\rm sc}\bar{\Pi}^\mu(q)G_0^\alpha(-k_+)\\ &- |\Delta_{\rm sc}|^2 G_0^\alpha(-k_-)G_0^\alpha(-k_+)\times\\ &\quad[\gamma^\mu(-k_-,-k_+) + \alpha\Lambda^\mu(-k_-,-k_+)], \end{aligned} \qquad (2.8)$$

contains contributions due to both the collective mode vertices Π^μ and $\bar{\Pi}^\mu$ (computed in Eq. (2.13)) and the vertex contribution Λ^μ arising from the self energy $\Sigma_{\rm corr}$.

The techniques described above are sufficient to calculate the gauge invariant response function for a large class of theories. All that is required to derive the full gauge invariant electromagnetic response is to arrive at a form of Λ^μ. This vertex depends on the details of the correlation self energy $\Sigma_{\rm corr}$, so we must consider it on a case by case basis. We now consider three relevant examples from the literature.

1. Pairing pseudogap

The first type of strong correlations we study is that proposed in Ref.[5] at a phenomenological level and in Ref.[6] from a more microscopic perspective. In Ref.[35] an early attempt to address how collective modes are affected by these pseudogap effects was performed. This model is based on a BCS like self energy but with a normal state gap $\Delta_{\rm pg}$. For this model, which we call the "pairing pseudogap approximation", $\alpha = 0$ in Eq. (2.3), and the correlated self energy in Eq. (2.5) is given by

$$\Sigma_{\rm corr}(k) = -\Delta^2_{\rm pg}G_0(-k). \qquad (2.14)$$

The pairing gap $\Delta_{\rm pg}$ is non-zero in the range of temperatures $T^* > T_c > 0$, where T^* is the mean-field transition temperature ($\Delta_{\rm pg}(T^*=0)$). At a more microscopic level[6] $\Delta_{\rm pg}$ is to be associated with non-condensed (finite momentum) pairs and is distinct from the superconducting order parameter $\Delta_{\rm sc}$ which corresponds to a condensate of pairs at zero net momentum.

Unlike the order parameter $\Delta_{\rm sc}$, the gap $\Delta_{\rm pg}$ does not fluctuate in the presence of A_μ. Nevertheless, its inclusion in the self energy will lead to a vertex correction. Using this form of $\Sigma_{\rm corr}(k)$, along with the definition $q_\mu\Lambda^\mu(k_+,k_-) = \Sigma_{\rm corr}(k_-) - \Sigma_{\rm corr}(k_+)$, we obtain

$$\Lambda^\mu(k_+,k_-) = \Delta^2_{\rm pg}G_0(-k_-)\gamma^\mu(-k_-,-k_+)G_0(-k_+). \qquad (2.15)$$

Inserting this expression into Eq. (2.8), along with $\alpha = 0$, then gives the full superconducting vertex in the pseudogap approximation.

Note that the pseudogap self energy is an approximation of a theory with $\alpha = 0$ and $\Sigma_{\rm corr}(k) = \sum_l t_{\rm pg}(l)G_0(l-k)$, where $t_{\rm pg}(l)$ is a t-matrix. This theory was considered in Ref.[25], and the vertex Λ^μ was calculated exactly. The exact $t_{\rm pg}$ depends on the full Green's function, so the exact Λ^μ will itself depend on the full vertex Γ^μ, and thus a self-consistent integral equation will arise for Γ^μ. In the pairing pseudogap approximation, $\Delta_{\rm pg}$ is constructed such that it contains no external momentum. Thus no vertex insertions into the gap are possible in Λ^μ, resulting in the above condition that $\Delta_{\rm pg}$ does not fluctuate with A_μ.

2. YRZ model

As a second model we consider a phenomenological self energy developed for the high-T_c superconductors and associated with Yang, Rice, and Zhang[3]. This is known as the YRZ model. For the YRZ model, in Eq. (2.3) and Eq. (2.5) one sets $\alpha = 1$ and

$$\Sigma_{\rm corr}(k) = -\Delta^2_{\rm pg}G_0(-k). \qquad (2.16)$$

Since $\Sigma_{\rm corr}(k)$ is the same as in the pairing pseudogap approximation, in the YRZ model we also obtain

$$\Lambda^\mu(k_+,k_-) = \Delta^2_{\rm pg}G_0(-k_-)\gamma^\mu(-k_-,-k_+)G_0(-k_+). \qquad (2.17)$$

Inserting this vertex correction into Eq. (2.8), along with $\alpha = 1$, then gives the full superconducting vertex in the YRZ model. In the normal state, this full vertex, along with the response kernel in Eq. (2.2), is in agreement with the results obtained in Ref.[23]. Here we have extended this work to the superconducting case.

3. Particle-only t-matrix

A third and final model was introduced by Strinati and collaborators using a generalized t-matrix[26–28]. In this model the self energy is obtained from Eq. (2.3) and Eq. (2.5) by setting $\alpha = 1$ and

$$\Sigma_{\rm corr}(k) = \sum_l t(l)G(l-k). \qquad (2.18)$$

Here G is the full Green's function, $t(l)$ is a t-matrix, the details of which are presented in Appendix (C 3). In Ref.[28], the authors propose "good candidates" for the response function Feynman diagrams. Here we emphasize that the WTI provides a direct procedure to determine not just good candidates but the exact full vertex, given in Eq. (2.8), which is manifestly gauge invariant.

The challenge here is in determining the exact vertex correction $\Lambda^\mu(k_+,k_-)$. This is more complicated than in the previous two cases. Nevertheless, following the procedure outlined above, the vertex correction due to this self energy can be obtained by performing all possible vertex insertions into all internal lines. That is, by inserting all possible vertices into both the Green's function and into the t-matrix. In Sec. (C 3) we explicitly derive the vertex correction Λ^μ for the self energy appearing in Eq. (2.18). We should note that the authors of this body of work do not presume a self-consistent gap equation, such as that appearing in Eq. (2.6), and such as we have assumed in arriving at Eq. (2.13). Rather, they fix the order parameter to be the same as in BCS theory, and add additional correlations to the number equation only.

In summary, this section has shown how to derive a gauge invariant full vertex for a generic self energy of the form in Eq. (2.5). Using the Ward-Takahashi identity there is an exact procedure to determine the full vertex. Moreover there is an analogous procedure to determine the collective modes and thus maintain gauge invariance. The resulting Feynman diagrams, which are shown in Fig. (3) of Appendix (A), are then completely determined.

III. ALTERNATIVE SCHEME TO WARD-TAKAHASHI: PATH INTEGRAL

A. Gauge invariant electrodynamics

A large class of theories in the literature derive the gauge invariant electromagnetic response using a path integral approach[7–9]. We now connect, when possible, the above results using the Ward-Takahashi identity to the EM response as calculated in the path integral literature. Here we will include both amplitude and phase fluctuations of the order parameter[1,2]. This is in contrast to previous studies[7–9] which incorporate only phase fluctuations. We introduce these amplitude fluctuations in order to address the compressibility sum-rule.

The inverse Nambu Green's function is $\mathcal{G}^{-1} = \mathcal{G}_0^{-1} - \Sigma$, where $\mathcal{G}_0^{-1} = i\omega - \xi_{\mathbf{k}}\tau_3$ and the self energy is $\Sigma = -\Delta(x)\tau_+ - \Delta^*(x)\tau_-$. The Nambu Pauli matrices are $\tau_{1,2,3}$, which define the raising and lowering operators $\tau_\pm = \frac{1}{2}(\tau_1 \pm i\tau_2)$. We begin with the action functional in terms of the Hubbard-Stratonovich field Δ[1]:

$$S[\Delta^*, \Delta, A^\mu] = -\text{Tr}\,\ln\left[-\mathcal{G}^{-1}\right] + \int dx\, \frac{|\Delta(x)|^2}{g}, \quad (3.1)$$

and following convention, the trace Tr represents a trace over both Nambu and position indices. As in Sec. (II), here we consider only neutral superfluids. In Eq. (3.1) we have introduced a non-dynamical external field A^μ through minimal-coupling to obtain the EM response; at the end of the calculation $A^\mu \to 0$. (Note that, the minimal-coupling is through the electric charge, e, and not the charge of the neutral superfluid. As is conventional in the literature[29], A^μ is viewed as a fictitious vector potential.) We now follow the literature and perform the saddle point expansion. To lowest order the effective action is $S_{\text{eff}}[\Delta^*, \Delta, A^\mu] = S_{\text{mf}}[\Delta^*_{\text{mf}}, \Delta_{\text{mf}}]$, where the mean-field (mf) action is

$$S_{\text{mf}}[\Delta^*_{\text{mf}}, \Delta_{\text{mf}}] = -\text{Tr}\,\ln\left[-\mathcal{G}_{\text{mf}}^{-1}\right] + \int dx\, \frac{|\Delta_{\text{mf}}|^2}{g}, \quad (3.2)$$

and the inverse mean-field Nambu Green's function is $\mathcal{G}_{\text{mf}}^{-1} = \mathcal{G}_0^{-1} - \Sigma[\Delta(x) \to \Delta_{\text{mf}}]$. The BCS gap equation then follows upon setting $\delta S_{\text{mf}}[\Delta^*_{\text{mf}}, \Delta_{\text{mf}}]/\delta\Delta^*_{\text{mf}} = 0$. It is straightforward to see that the resulting response kernel is not gauge invariant.

We now calculate the gauge invariant EM response kernel $K^{\mu\nu}$. In order to implement gauge invariance, the conventional literature introduces fluctuations $\eta(x)$ about the mean-field value of the order parameter Δ_{mf}, expressing $\Delta(x) = \Delta_{\text{mf}} + \eta(x)$. (In Sec. (II), $\Delta_{\text{sc}} \equiv \Delta_{\text{mf}}$ for strict BCS theory.) Expanding the action functional to second order in $\eta(x)$ gives $S[\Delta^*, \Delta, A^\mu] \approx S_{\text{mf}}[\Delta^*_{\text{mf}}, \Delta_{\text{mf}}] + S^{(2)}[\eta^*, \eta, A^\mu]$. To calculate $S^{(2)}[\eta^*, \eta, A^\mu]$, we first consider fluctuations of the Green's function about the mean-field solution:

$$\mathcal{G}^{-1} - \mathcal{G}_{\text{mf}}^{-1} = -\delta\Gamma - \Sigma_\eta, \quad (3.3)$$

where $\delta\Gamma = \Gamma_1 + \Gamma_2$, with $\Gamma_1 = \gamma_\mu A^\mu$, $\Gamma_2 = (\mathbf{A}^2/2m)\tau_3$, is a vector potential fluctuation and $\Sigma_\eta = \Sigma[\Delta(x) \to \eta(x)]$ is a gap fluctuation. Expanding to second order in η and A_μ, the second order action functional is

$$\begin{aligned} &S^{(2)}[\eta^*, \eta, A^\mu] \\ &= \frac{1}{2}\sum_q \left[A_\mu(q)K^{\mu\nu}_{0,\text{mf}}(q)A_\nu(-q) + \eta_a(q)Q^{ab}_{\text{mf}}(q)\eta_b(-q)\right] \\ &- \frac{1}{2}\sum_q \left[A_\mu(q)Q^{\mu b}_{\text{mf}}(q)\eta_b(-q) + \eta_a(q)Q^{a\nu}_{\text{mf}}(q)A_\nu(-q)\right]. \end{aligned}$$

In this expression we write $\eta(x) = \eta_1(x) - i\eta_2(x)$ with $\eta_1(x), \eta_2(x) \in \mathbb{R}$. This decomposes the fluctuations into their (Cartesian) real and imaginary parts, which amounts to an amplitude and phase decomposition. Since we keep the saddle point condition at the mean-field level, an explicit amplitude and phase decomposition, in polar coordinates, will lead to the same electromagnetic response. (If one uses a different saddle point condition, not relevant to this work, then issues associated with the use of either a Cartesian or polar decomposition may arise[2].) Even within this framework, we shall point out an inconsistency within the conventional path integral formalism in failing to satisfy the compressibility sum-rule.

To complete the calculation, we transform to momentum space, $k = (i\omega_n, \mathbf{k})$ and $q = (i\Omega_m, \mathbf{q})$, where $i\omega_n$ $(i\Omega_m)$ is a fermionic (bosonic) Matsubara frequency. If we denote the trace over Nambu indices by tr, then the "bubble" response kernel is $K^{\mu\nu}_{0,\text{mf}}(q) = \text{tr}\sum_k \mathcal{G}_{\text{mf}}(k_+)\gamma^\mu(k_+, k_-)\mathcal{G}_{\text{mf}}(k_-)\gamma^\nu(k_-, k_+) + \frac{n}{m}\delta^{\mu\nu}(1 - \delta_{\mu 0})$ and the two-point response function $Q^{ab}_{\text{mf}}(q) = \frac{2}{g}\delta_{ab} + \text{tr}\sum_k \mathcal{G}_{\text{mf}}(k_+)\tau_a\mathcal{G}_{\text{mf}}(k_-)\tau_b$ can be viewed as a bosonic propagator. We also have $Q^{\mu a}_{\rm mf}(q) = {\rm tr}\sum_k \mathcal{G}_{\rm mf}(k_+)\gamma^\mu(k_+,k_-)\mathcal{G}_{\rm mf}(k_-)\tau_a$, and $Q^{b\nu}_{\rm mf}(q)$ has $(\mu,a)\leftrightarrow(b,\nu)$. These mean-field response functions are equivalent to previous results in the literature[18]. They are also equivalent to the response functions which appear in Eq. (2.13) for a theory with only a strict BCS self energy.

After integrating out the η field, the beyond-mean-field effective action contribution is given by

$$S_{\rm eff} - S_{\rm mf} = \sum_q A_\mu(q) K^{\mu\nu}_{\rm mf}(q) A_\nu(-q) + \frac{1}{2}{\rm Tr}\ \ln\left[Q^{ab}_{\rm mf}(q)\right]. \quad (3.4)$$

Thus the fluctuation action decomposes into two separate terms. The second term in the fluctuation action provides a contribution to thermodynamics arising from Gaussian fluctuations. This form of the Gaussian fluctuation part of the action is equivalent to the standard results in the literature[2]. The first term is the gauge invariant EM response kernel, with both amplitude and phase fluctuations of the order parameter included, defined by $K^{\mu\nu}_{\rm mf}(q) = K^{\mu\nu}_{0,{\rm mf}}(q) - \sum_{a,b} Q^{\mu a}_{\rm mf}(q)\left[Q^{ab}_{\rm mf}(q)\right]^{-1} Q^{b\nu}_{\rm mf}(-q)$. If we expand the response kernel appearing in Eq. (3.4), then we obtain[17,18]:

$$K^{\mu\nu}_{\rm mf} = K^{\mu\nu}_{0,{\rm mf}} - \frac{Q_{11}Q^{\mu 2}_{\rm mf}Q^{2\nu}_{\rm mf} + Q_{22}Q^{\mu 1}_{\rm mf}Q^{1\nu}_{\rm mf} - Q_{12}Q^{\mu 1}_{\rm mf}Q^{2\nu}_{\rm mf} - Q_{21}Q^{\mu 2}_{\rm mf}Q^{1\nu}_{\rm mf}}{Q_{11}Q_{22} - Q_{12}Q_{21}}. \quad (3.5)$$

In Ref.[18] it is proved that the response kernel in Eq. (3.5) is both gauge invariant $q_\mu K^{\mu\nu}_{\rm mf}(q) = 0$, and charge conserving $K^{\mu\nu}_{\rm mf}(q) q_\nu = 0$. References[17,18] used a matrix linear response formalism known as "consistent fluctuation of the order parameter". Our derivation, however, is based on the path integral.

B. Inconsistency with the compressibility sum-rule

We now turn to the implications of the two contributions to the action in Eq. (3.4). Here we focus on the compressibility sum-rule, which provides an important consistency check on the path integral approach[12]. The explicit form of the compressibility sum-rule is[36]:

$$\lim_{\mathbf{q}\to 0}\left[K^{00}(\omega = 0, \mathbf{q})\right] = -\frac{\partial n}{\partial \mu}. \quad (3.6)$$

Here the real frequency ω is the analytic continuation of the Matsubara frequency $i\Omega_m$, defined by $i\Omega_m = \omega + i\gamma$ with $\gamma \to 0$. This sum-rule shows how to associate the electromagnetic contributions to the action with their counterpart contributions to the thermodynamic response. The compressibility, $\kappa = n^{-2}(\partial n/\partial\mu)$, is then related to the density response via Eq. (3.6).

In relation to Sec. (II), we note that satisfying the WTI does not imply the compressibility sum-rule is satisfied[18,37]. In the appropriate limits, the WTI gives a constraint on the vector component of the full vertex: $\lim_{\mathbf{q}\to 0}\left[\mathbf{q}\cdot\mathbf{\Gamma}(k_+,k_-)|_{\omega=0}\right] = -\lim_{\mathbf{q}\to 0}\left[\left(G^{-1}(k_+) - G^{-1}(k_-)\right)|_{\omega=0}\right]$, whereas the compressibility sum rule is $\lim_{\mathbf{q}\to 0}\left[\Gamma^0(k_+,k_-)|_{\omega=0}\right] = 1 - \partial\Sigma(k)/\partial\mu$, which is a constraint on the time component of the full vertex. Note that, the order in which the limits are taken is crucial: first the frequency ω is set to zero, and then the momentum $\mathbf{q}\to 0$. Thus the WTI is not a sufficient condition to ensure satisfaction of the compressibility sum-rule. In the path integral formalism, however, both the electrodynamics and thermodynamics arise from the same action, and so Eq. (3.6) is an important constraint relating the two responses.

The relationship in Eq. (3.6) is also particularly useful in characterizing various orders of approximation within the path integral scheme. This is because at the heart of the path integral is a close connection between electrodynamics and thermodynamics. With the inclusion of amplitude fluctuations, which are essential for this sum-rule, we can now test the compressibility sum-rule within the standard path integral formalism in the literature.

Note that, this sum-rule depends on the number equation. Consistency would seem to require that we include Gaussian fluctuations $n_{\rm fl} = -\beta^{-1}\partial S_{\rm fl}[\Delta^*_{\rm mf},\Delta_{\rm mf}]/\partial\mu$ to the number equation coming from the second line in Eq. (3.4). This is, in fact, incorrect and points to an underlying inconsistency. Instead, we will show the proper calculation level for thermodynamics is that of pure mean-field, giving a mean-field particle number

$$n_{\rm mf} = -\frac{1}{\beta}\frac{\partial S_{\rm mf}[\Delta^*_{\rm mf},\Delta_{\rm mf}]}{\partial\mu} = 2\sum_k G(k). \quad (3.7)$$

Taking the derivative of the mean-field number equation with respect to μ gives

$$\frac{\partial n_{\rm mf}}{\partial \mu} = -2\sum_k \left[G^2(k) - F^2(k) + 2G(k)F(k)\frac{\partial \Delta_{\rm mf}}{\partial \mu}\right], \tag{3.8}$$

where we henceforth take $\Delta_{\rm mf} = \Delta^*_{\rm mf}$ for convenience. Here we define the single particle Green's function in terms of the Nambu Green's function by $G(k) = (\mathcal{G}_{\rm mf}(k))_{11} = -(\mathcal{G}_{\rm mf}(-k))_{22}$, and the Gorkov F-function is similarly $F(k) = \Delta_{\rm mf} G(k) G_0(-k) = (\mathcal{G}_{\rm mf}(k))_{12} = (\mathcal{G}^*_{\rm mf}(k))_{21}$. The fluctuation of the mean-field gap with respect to the chemical potential, $\partial\Delta_{\rm mf}/\partial\mu$, can be found using the BCS gap equation

$$\mathrm{GAP}[\Delta_{\rm mf}, \mu] := \frac{\Delta_{\rm mf}}{g} - \sum_k \mathrm{Tr}[\mathcal{G}(k)\tau_-] = 0. \tag{3.9}$$

Since $\Delta_{\rm mf}$ depends on μ, by taking the total derivative with respect to μ, we arrive at the condition

$$\frac{\partial \Delta_{\rm mf}}{\partial \mu} = -\frac{\partial \mathrm{GAP}/\partial\mu}{\partial \mathrm{GAP}/\partial \Delta_{\rm mf}}. \tag{3.10}$$

To see that the compressibility sum-rule is satisfied, notice that $\partial\mathrm{GAP}/\partial\mu = 2\sum_k G(k)F(k)$ and $\partial\mathrm{GAP}/\partial\Delta = 2\sum_k F(k)F(k)$. Therefore, the last term in Eq. (3.8) can be expressed as $2\frac{(\partial\mathrm{GAP}/\partial\mu)^2}{\partial\mathrm{GAP}/\partial\Delta_{\rm mf}}$. Now, in the limit that $\omega = 0, \mathbf{q} \to 0$, the following identifications can be made: $Q^{10}_{\rm mf} = 2\partial\mathrm{GAP}/\partial\mu$, and $Q^{11}_{\rm mf} = 2\partial\mathrm{GAP}/\partial\Delta_{\rm mf}$. By computing the summation over Matsubara frequencies, one also obtains $2\sum_k \left[G^2(k) - F^2(k)\right] = K^{00}_{0,\rm mf}$.

Therefore, using Eq. (3.5), Eq. (3.8) now becomes

$$-\frac{\partial n_{\rm mf}}{\partial\mu} = K^{00}_{0,\rm mf} - \frac{Q^{10}_{\rm mf} Q^{01}_{\rm mf}}{Q_{11}} = K^{00}(0, \mathbf{q} \to 0). \tag{3.11}$$

This demonstrates the expected consistency between $-(\partial n_{\rm mf}/\partial\mu)$ and $K^{00}(0, \mathbf{q} \to 0)$ and proves the compressibility sum-rule at the BCS level[18].

The reason for the need to include amplitude fluctuations in the density-density response can be seen from Eq. (3.8). This equation shows that fluctuations in the gap $(\partial\Delta_{\rm mf}/\partial\mu)$ must be included, and therefore amplitude fluctuations in the gap are necessary in order to satisfy the compressibility sum-rule. If only phase fluctuations are retained, the compressibility sum-rule is violated. For a different context where amplitude fluctuations are important see Ref.[38].

The compressibility sum-rule has only been satisfied by ignoring the Gaussian fluctuations in the number equation. Had these been included, we would obtain $-\partial n/\partial\mu = -\partial n_{\rm mf}/\partial\mu - \partial n_{\rm fl}/\partial\mu \neq K^{00}(0, \mathbf{q} \to 0)$, which violates the compressibility sum-rule[39].

In summary, the path integral formalism, as currently applied in the literature, treats electrodynamics and thermodynamics inconsistently. In this derivation of gauge invariant electrodynamics at the BCS level, beyond BCS fluctuations are necessarily incorporated in thermodynamics. However, these thermodynamic fluctuations should not appear in the number equation if the compressibility sum rule is to be satisfied. The discussion in Sec. (II) provides insights into the resolution to this inconsistency: there gauge invariance is obtained by determining the collective modes that arise due to vertex insertions into the gap equation. This suggests that, within the path integral formalism, one should consider a new saddle point condition in the presence of a nonzero vector potential. More details on this resolution are presented elsewhere[12].

IV. CONCLUSIONS

The goal of this paper was to show how to arrive at a proper gauge invariant description of the electromagnetic response in strongly correlated fermionic superfluids. In this paper correlation effects are represented by "correlated self energy" contributions which appear in addition to the usual superconducting self energy of the condensate. The theoretical approach is sufficiently general to apply to a theory that has (1) a "Nambu-diagonal" diagrammatic self-energy and (2) a frequency independent self-consistent gap equation. Using (1) the Ward-Takahashi identity and (2) insertions on the self-consistent gap equation, the exact gauge-invariant electromagnetic response can then, in principle, be calculated for any theory satisfying these conditions. Adopting a rather generic class of strongly correlated models (widely used for the high temperature superconductors and ultra cold gases) we are able to give exact expressions for the electromagnetic response. This method, which obtains expressions for all vertex corrections and collective modes in a manner compatible with the f-sum-rule, is an important tool for studying strongly correlated superfluids and superconductors.

For comparison we also discuss an alternative tool which builds on the path integral approach. With few exceptions this scheme has been used to address the BCS-level response, i.e., in the absence of stronger correlations. In contrast to approaches which build on the Ward-Takahashi identity, here gauge invariance and the f-sum-rule are relatively straightforward to ensure. What is more complicated is to arrive at consistency with the compressibility sum-rule. This sum-rule relates electrodynamics and thermodynamics and provides a natural test of the path integral scheme, since the two are simultaneously calculated. We show that in the conventional path integral literature for the gauge invariant electrodynamics at the BCS level, the compressibility sum-rule is violated.

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Appendix A: Obtaining the full vertex using the Ward-Takahashi identity

Here we show how to apply the Ward-Takahashi identity to obtain the gauge invariant full vertex for a given self energy. If we define the partially dressed Green's function $G_0^\alpha(k)$ by

$$\left(G_0^\alpha\right)^{-1}(k) = G_0^{-1}(k) - \alpha\Sigma_{\rm corr}(k), \tag{A.1}$$

where $\Sigma_{\rm corr}$ is a self energy describing strong correlations, then the class of self-energies considered in the main text are of the form

$$\Sigma(k) = \Sigma_{\rm corr}(k) - |\Delta_{\rm sc}|^2 G_0^\alpha(-k). \tag{A.2}$$

The second term in this expression represents the superconducting self energy $\Sigma_{\rm sc}(k) = -|\Delta_{\rm sc}^2|G_0^\alpha(-k)$. For convenience we treat $\Delta_{\rm sc}$ and $\Delta_{\rm sc}^*$ as independent degrees of freedom. This will be important in the next section, but for now it is not essential. Writing the self energy in this form shows that the second term is a BCS-like self energy, but with the bare Green's function G_0 replaced by the partially dressed Green's function G_0^α. Strict BCS theory is obtained by setting $\Sigma_{\rm corr} = 0$. The three models that we will consider are the pairing pseudogap approximation[5,6], the Yang, Rice, and Zhang (YRZ) model[3], and the t-matrix model of Ref.[26]. For the pairing pseudogap approximation, $\alpha = 0, \Sigma_{\rm corr}(k) = -\Delta_{\rm pg}^2 G_0(-k)$, for the YRZ model $\alpha = 1, \Sigma_{\rm corr}(k) = -\Delta_{\rm pg}^2 G_0(-k)$, and for the t-matrix model $\alpha = 1, \Sigma_{\rm corr}(k) = \sum_l t(l)G(l-k)$.

The bare Ward-Takahashi identity is $q_\mu\gamma^\mu(k_+,k_-) = G_0^{-1}(k_+) - G_0^{-1}(k_-)$. Using this, it follows that the Ward-Takahashi identity for the full vertex is[20]

$$\begin{aligned} q_\mu\Gamma^\mu(k_+,k_-) &= G^{-1}(k_+) - G^{-1}(k_-), \\ &= q_\mu\gamma^\mu(k_+,k_-) + \Sigma(k_-) - \Sigma(k_+). \end{aligned} \tag{A.3}$$

As discussed in the main text, both the strong correlation self energy $\Sigma_{\rm corr}$, and the superconducting self energy $\Sigma_{\rm sc}$ give rise to vertex contributions. Hence we write $\Sigma(k) = \Sigma_{\rm corr}(k) + \Sigma_{\rm sc}(k)$ and derive the vertex contributions from both self-energies separately. The strong correlation self energy gives a vertex contribution $\Lambda^\mu(k_+,k_-)$ defined by

$$q_\mu\Lambda^\mu(k_+,k_-) = \Sigma_{\rm corr}(k_-) - \Sigma_{\rm corr}(k_+). \tag{A.4}$$

The general form of this vertex depends on the specific model under consideration. In Sec. (C) we will derive the explicit form of this vertex for three models of interest in the literature. The superconducting vertex is defined by

$$q_\mu\Gamma_{\rm sc}^\mu(k_+,k_-) = \Sigma_{\rm sc}(k_-) - \Sigma_{\rm sc}(k_+). \tag{A.5}$$

Using these definitions, the full vertex is then

$$\Gamma^\mu(k_+,k_-) = \gamma^\mu(k_+,k_-) + \Lambda^\mu(k_+,k_-) + \Gamma_{\rm sc}^\mu(k_+,k_-), \tag{A.6}$$

which can be found from the full Ward-Takahashi identity in Eq. (A.3).

We now derive the explicit form of $\Gamma_{\rm sc}^\mu$. The superconducting vertex contributions are most easily found by defining the collective mode vertices $\Pi^\mu(q)$ and $\bar{\Pi}^\mu(q)$ such that $q_\mu\Pi^\mu(q) = 2\Delta_{\rm sc}$, $q_\mu\bar{\Pi}^\mu(q) = -2\Delta_{\rm sc}^*$. For now, these will be left as a definition, but the explicit form of Π^μ, $\bar{\Pi}^\mu$, along with the contraction identities, are derived in Sec. (II C) and Appendix (B), respectively. Using the superconducting self energy given in Eq. (A.2), we then have

$$\Sigma_{\rm sc}(k_-) - \Sigma_{\rm sc}(k_+) = -\Delta_{\rm sc}^* q_\mu\Pi^\mu(q)G_0^\alpha(-k_-) - \Delta_{\rm sc}q_\mu\bar{\Pi}^\mu(q)G_0^\alpha(-k_+) - |\Delta_{\rm sc}|^2\left[G_0^\alpha(-k_+) - G_0^\alpha(-k_-)\right]. \tag{A.7}$$

The difference of the two partially dressed Green's functions is

$$\begin{aligned} G_0^\alpha(-k_+) - G_0^\alpha(-k_-) &= G_0^\alpha(-k_-)\left[G_0^{-1}(-k_-) - G_0^{-1}(-k_+) + \alpha\left(\Sigma_{\rm corr}(-k_+) - \Sigma_{\rm corr}(-k_-)\right)\right]G_0^\alpha(-k_+), \\ &= G_0^\alpha(-k_-)\left[q_\mu\gamma^\mu(-k_-,-k_+) + \alpha q_\mu\Lambda^\mu(-k_-,-k_+)\right]G_0^\alpha(-k_+). \end{aligned} \tag{A.8}$$

In the second line we have used both the bare Ward-Takahashi identity, as well as the definition of the Λ^μ vertex. Substituting Eq. (A.7) and Eq. (A.8) into Eq. (A.5) then gives the superconducting vertex:

$$\Gamma^\mu_{\rm sc}(k_+,k_-) = -\Delta^*_{\rm sc}\Pi^\mu(q)G_0^\alpha(-k_-) - \Delta_{\rm sc}\bar{\Pi}^\mu(q)G_0^\alpha(-k_+) \\ -|\Delta_{\rm sc}|^2 G_0^\alpha(-k_-)\left[\gamma^\mu(-k_-,-k_+) + \alpha\Lambda^\mu(-k_-,-k_+)\right]G_0^\alpha(-k_+). \tag{A.9}$$

This then produces the exact gauge invariant full vertex given in Eq. (2.8) of the main text:

$$\Gamma^\mu(k_+,k_-) = \gamma^\mu(k_+,k_-) + \Lambda^\mu(k_+,k_-) - \Delta^*_{\rm sc}\Pi^\mu(q)G_0^\alpha(-k_-) - \Delta_{\rm sc}\bar{\Pi}^\mu(q)G_0^\alpha(-k_+) \\ - |\Delta_{\rm sc}|^2 G_0^\alpha(-k_-)[\gamma^\mu(-k_-,-k_+) + \alpha\Lambda^\mu(-k_-,-k_+)]G_0^\alpha(-k_+). \tag{A.10}$$

From the above expression it is clear that if $\Sigma_{\rm corr} = 0 \Rightarrow \Lambda^\mu = 0$, then the full vertex reduces to the BCS full vertex[18,25]. Similarly if $\Delta_{\rm sc} = 0$, then the full vertex reduces to the paired normal state vertex[23,24]. In order to uniquely determine the full vertex, the collective mode vertices $\Pi^\mu(q)$, $\bar{\Pi}^\mu(q)$ and the vertex correction $\Lambda^\mu(k_+,k_-)$ must be determined. The Feynman diagrams for the full response function are given in Fig. (3).

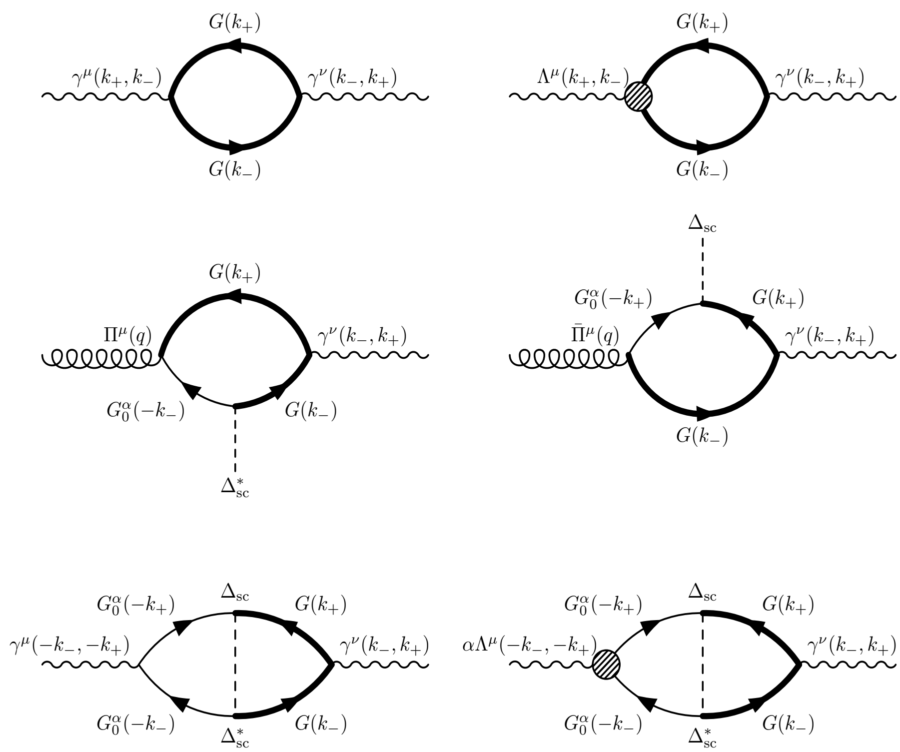

FIG. 3. Feynman diagrams for the two particle response function $P^{\mu\nu}(q) = 2\sum_k G(k_+)\Gamma^\mu(k_+,k_-)G(k_-)\gamma^\nu(k_-,k_+)$ given a self energy of the form in Eq. (A.2). The order of appearance of the diagrams from left to right and top to bottom corresponds directly to the order of appearance of terms in Eq. (A.10). The pseudogap approximation corresponds to $\alpha = 0, \Sigma_{\rm corr}(k) = -\Delta^2_{\rm pg}G_0(-k)$, for the YRZ model $\alpha = 1, \Sigma_{\rm corr}(k) = -\Delta^2_{\rm pg}G_0(-k)$, and for the t-matrix model $\alpha = 1, \Sigma_{\rm corr}(k) = \sum_l t(l)G(l-k)$.

Appendix B: Collective mode vertices

In this section verify that the collective mode vertices $\Pi^\mu(q)$ and $\bar{\Pi}^\mu(q)$ satisfy the gauge invariant conditions $q_\mu\Pi^\mu(q) = 2\Delta_{\rm sc}$, $q_\mu\bar{\Pi}^\mu(q) = -2\Delta^*_{\rm sc}$, which was assumed in their definitions. These vertices are conveniently expressed as a matrix equation if we define the two-point correlation functions

$$Q_{+-}(q) = 1/g - \sum_k G(k_+)G_0^\alpha(-k_-)\left[1 - \Delta_{\rm sc}^* F_{\rm sc}(k_-)\right], \tag{B.1}$$

$$Q_{-+}(q) = 1/g - \sum_k G(k_-)G_0^\alpha(-k_+)\left[1 - \Delta_{\rm sc} F_{\rm sc}^*(k_+)\right], \tag{B.2}$$

$$Q_{++}(q) = \sum_k F_{\rm sc}(k_+)F_{\rm sc}(k_-) = Q_{--}^*(q), \tag{B.3}$$

$$\begin{aligned} P_+^\mu(q) &= \sum_k \left[\gamma^\mu(k_+,k_-) + \Lambda^\mu(k_+,k_-)\right] G(k_+)F_{\rm sc}(k_-) \\ &+ \sum_k \left[\gamma^\mu(-k_-,-k_+) + \alpha\Lambda^\mu(-k_-,-k_+)\right] F_{\rm sc}(k_+)G_0^\alpha(-k_-)\left[1 - \Delta_{\rm sc}^* F_{\rm sc}(k_-)\right], \end{aligned} \tag{B.4}$$

$$\begin{aligned} P_-^\mu(q) &= \sum_k \left[\gamma^\mu(k_+,k_-) + \Lambda^\mu(k_+,k_-)\right] F_{\rm sc}^*(k_+)G(k_-) \\ &+ \sum_k \left[\gamma^\mu(-k_-,-k_+) + \alpha\Lambda^\mu(-k_-,-k_+)\right] G_0^\alpha(-k_+)F_{\rm sc}^*(k_-)\left[1 - \Delta_{\rm sc} F_{\rm sc}^*(k_+)\right]. \end{aligned} \tag{B.5}$$

From Eqs. (2.10-2.11) of the main text, the collective modes can then be written as

$$\begin{pmatrix} \Pi^\mu \\ \bar{\Pi}^\mu \end{pmatrix} = \begin{pmatrix} Q_{+-} & Q_{++} \\ Q_{--} & Q_{-+} \end{pmatrix}^{-1} \begin{pmatrix} P_+^\mu \\ P_-^\mu \end{pmatrix}. \tag{B.6}$$

We now contract each side of Eq. (B.6) with q_μ. In order to calculate the right-hand side, we calculate the contraction $q_\mu P_\pm^\mu(q)$:

$$\begin{aligned} q_\mu P_+^\mu(q) &= q_\mu \sum_k \left[\gamma^\mu(k_+,k_-) + \Lambda^\mu(k_+,k_-)\right] G(k_+)F_{\rm sc}(k_-) \\ &+ q_\mu \sum_k \left[\gamma^\mu(-k_-,-k_+) + \alpha\Lambda^\mu(-k_-,-k_+)\right] F_{\rm sc}(k_+)G_0^\alpha(-k_-)[1 - \Delta_{\rm sc}^* F_{\rm sc}(k_-)]. \end{aligned} \tag{B.7}$$

Explicit calculation shows that both lines have the same value, so that

$$\begin{aligned} q_\mu P_+^\mu(q) &= 2\left[\Delta_{\rm sc}\left(1/g - \sum_k G(k_+)G_0^\alpha(-k_-)[1 - \Delta_{\rm sc}^* F_{\rm sc}(k_-)]\right) - \Delta_{\rm sc}^* \sum_k F_{\rm sc}(k_-)F_{\rm sc}(k_+)\right], \\ &= 2\left(\Delta_{\rm sc}Q_{+-} - \Delta_{\rm sc}^* Q_{++}\right). \end{aligned} \tag{B.8}$$

Similarly, since $\Delta_{\rm sc}^* P_+^\mu(q) = \Delta_{\rm sc} P_-^\mu(-q)$, we also find $q_\mu P_-^\mu(q) = -\left(q_\mu P_+^\mu(q)\right)^*$. The contractions of the collective mode vertices are then

$$\begin{pmatrix} q_\mu\Pi^\mu \\ q_\mu\bar{\Pi}^\mu \end{pmatrix} = \begin{pmatrix} Q_{+-} & Q_{++} \\ Q_{--} & Q_{-+} \end{pmatrix}^{-1} \begin{pmatrix} 2\left(\Delta_{\rm sc}Q_{+-} - \Delta_{\rm sc}^* Q_{++}\right) \\ -2\left(\Delta_{\rm sc}^* Q_{-+} - \Delta_{\rm sc} Q_{--}\right) \end{pmatrix} = \begin{pmatrix} 2\Delta_{\rm sc} \\ -2\Delta_{\rm sc}^* \end{pmatrix}. \tag{B.9}$$

This confirms that, for all q, we have the desired relations

$$q_\mu \Pi^\mu(q) = 2\Delta_{\rm sc}, \quad q_\mu \bar{\Pi}^\mu(q) = -2\Delta_{\rm sc}^*. \tag{B.10}$$

Finally, we now show that at $q = 0$ the gap equation is consistent with the poles of the collective mode vertices. These poles are given by the solution of $\det(Q_{ab}) = Q_{+-}Q_{-+} - Q_{++}Q_{--} = 0$, which arises when taking the matrix inverse of Eq. (B.6). Let $q = 0$, and suppose $\Delta_{\rm sc} = \Delta_{\rm sc}^*$, then the poles occur when $Q_{+-} - Q_{++} = 0$. Using the expressions in Eqs. (B.1-B.3), and the definition of $F_{\rm sc}(k)$, this reduces to

$$1 - g\sum_k G_0^\alpha(-k)G(k) = 0, \tag{B.11}$$

which is the expected gap equation.

In summary, we have obtained the collective mode vertices, and thus obtained the gauge invariant full vertex. The next section determines the form of the vertex Λ^μ for three example cases of $\Sigma_{\rm corr}$.

Appendix C: Specific examples for the Λ^μ vertex

1. Pairing pseudogap approximation

In the pairing pseudogap approximation[5,6], $\Sigma_{\rm corr}(k) = -\Delta_{\rm pg}^2 G_0(-k)$, which implies that

$$\begin{aligned} q_\mu \Lambda^\mu(k_+, k_-) &= \Sigma_{\rm corr}(k_-) - \Sigma_{\rm corr}(k_+), \\ &= \Delta_{\rm pg}^2 G_0(-k_-) q_\mu \gamma^\mu(-k_-, -k_+) G_0(-k_+). \end{aligned} \tag{C.1}$$

Thus, it follows that

$$\Lambda^\mu(k_+, k_-) = \Delta_{\rm pg}^2 G_0(-k_-) \gamma^\mu(-k_-, -k_+) G_0(-k_+). \tag{C.2}$$

2. YRZ

In the YRZ model[3], $\Sigma_{\rm corr}(k) = -\Delta_{\rm pg}^2 G_0(-k)$. Thus, as in the case for the pseudogap approximation, we obtain

$$\Lambda^\mu(k_+, k_-) = \Delta_{\rm pg}^2 G_0(-k_-) \gamma^\mu(-k_-, -k_+) G_0(-k_+). \tag{C.3}$$

3. Particle-only t-matrix

In the t-matrix model of Ref.[26], $\Sigma_{\rm corr}(k) = \sum_l t(l) G(l-k) = \sum_l t(l+k) G(l)$. (In Ref.[26] the gap is fixed to be the BCS gap, arising from a BCS Gorkov F-function. However, in the number equation additional correlations are added via adding $\Sigma_{\rm corr}$ to the BCS self-energy. In order to implement a consistent framework, based on Ref.[26], we take the self energy to be $\Sigma = \Sigma_{\rm corr} + \Sigma_{\rm sc}$, where $\Sigma_{\rm sc} = -|\Delta_{\rm sc}|^2 G_0^\alpha(k)$, and $\alpha = 1$. The gap is thus set to be $\Delta_{\rm sc}$ and the additional correlations are present in the number equation.) Here $G(k)$ is the full Green's function, where we define

$$\begin{aligned} G^{-1}(k) &= G_0^{-1}(k) - \Sigma(k), \\ F(k) &= \Delta_{\rm sc} G_0^\alpha(-k) G(k). \end{aligned} \tag{C.4}$$

The self energy is $\Sigma(k) = \Sigma_{\rm corr}(k) - |\Delta_{\rm sc}|^2 G_0^\alpha(-k)$, where $\Delta_{\rm sc}$ is the superconducting gap and $\alpha = 1$ in $G_0^\alpha(k)$, defined in Eq. (2.3) of the main text. The Gorkov F-function satisfies $F(k) = F(-k)$.

The inverse t-matrix $t^{-1}(l)$ in the formalism of Ref.[26] is given by

$$t^{-1}(l) = \chi_{11}(l) - \chi_{22}^{-1}(l) \chi_{12}(l) \chi_{21}(l), \tag{C.5}$$

where the susceptibilities are given by

$$\begin{aligned} \chi_{11}(l) &= \frac{1}{g} - \sum_m G(l+m) G(-m), \\ \chi_{12}(l) &= \sum_m F(l+m) F^*(-m), \end{aligned} \tag{C.6}$$

where $1/g$ is the standard s-wave interaction[26]. Note that $\chi_{22}(l) = \chi_{11}(-l)$, and because $F(m)$ is even, $\chi_{12}(l) = \chi_{21}^*(l) = \chi_{21}(l)$.

The vertex correction Λ^μ for this model is defined by Eq. (A.4). We now proceed to evaluate the right hand side of Eq. (A.4). From the definition of $\Sigma_{\rm corr}$, it follows that

$$\begin{aligned} \Sigma_{\rm corr}(k_-) - \Sigma_{\rm corr}(k_+) = 2 &\sum_l G(l) t(l+k_+) t(l+k_-) \left(t^{-1}(l+k_+) - t^{-1}(l+k_-) \right) \\ &- \sum_l t(l) G(l-k_+) G(l-k_-) \left(G^{-1}(l-k_+) - G^{-1}(l-k_-) \right). \end{aligned} \tag{C.7}$$

The full Green's function obeys the Ward-Takahashi identity, which defines the full vertex Γ^μ:

$$q_\mu \Gamma^\mu(k_+, k_-) = G^{-1}(k_+) - G^{-1}(k_-). \tag{C.8}$$

Equivalently, Γ^μ is given by Eq. (A.10). Thus, we now have

$$\begin{aligned}\Sigma_{\text{corr}}(k_-) - \Sigma_{\text{corr}}(k_+) = 2\sum_l G(l)t(l+k_+)t(l+k_-)\left(t^{-1}(l+k_+) - t^{-1}(l+k_-)\right) \\ + \sum_l t(l)G(l-k_-)q_\mu\Gamma^\mu(l-k_-, l-k_+)G(l-k_+).\end{aligned} \tag{C.9}$$

From the t-matrix definition in Eq. (C.5), the difference of the two inverse t-matrices is

$$\begin{aligned}t^{-1}(l+k_+) - t^{-1}(l+k_-) = \chi_{11}(l+k_+) - \chi_{11}(l+k_-) \\ + \chi_{22}^{-1}(l+k_-)\chi_{12}(l+k_-)\chi_{21}(l+k_-) - \chi_{22}^{-1}(l+k_+)\chi_{12}(l+k_+)\chi_{21}(l+k_+).\end{aligned} \tag{C.10}$$

For the first line of this expression we can use the Ward-Takahashi identity in Eq. (C.8) to obtain

$$\chi_{11}(l+k_+) - \chi_{11}(l+k_-) = \sum_m G(-m)G(l+m+k_+)q_\mu\Gamma^\mu(l+m+k_+, l+m+k_-)G(l+m+k_-). \tag{C.11}$$

It remains to compute the difference term in the second line of Eq. (C.10). To do this, first note that

$$\begin{aligned}&\chi_{22}^{-1}(l+k_-)\chi_{12}(l+k_-)\chi_{21}(l+k_-) - \chi_{22}^{-1}(l+k_+)\chi_{12}(l+k_+)\chi_{21}(l+k_+) \\ &= \Big\{[\chi_{22}(l+k_+) - \chi_{22}(l+k_-)]\chi_{12}(l+k_-)\chi_{21}(l+k_-) \\ &\quad + [\chi_{12}(l+k_-) - \chi_{12}(l+k_+)]\chi_{22}(l+k_-)\chi_{21}(l+k_-) \\ &\quad + [\chi_{21}(l+k_-) - \chi_{21}(l+k_+)]\chi_{22}(l+k_-)\chi_{12}(l+k_+)\Big\}(\chi_{22}(l+k_-)\chi_{22}(l+k_+))^{-1}.\end{aligned} \tag{C.12}$$

This form simplifies the problem to computing the vertex insertions into both χ_{12}, χ_{21}, and χ_{22} individually, and then summing the result. Since $\chi_{22}(k) = \chi_{11}(-k)$, we can use the result in Eq. (C.11) to obtain

$$\chi_{22}(l+k_+) - \chi_{22}(l+k_-) = -\sum_m G(m)G(-l-m-k_-)q_\mu\Gamma^\mu(-l-m-k_-, -l-m-k_+)G(-l-m-k_+). \tag{C.13}$$

We now study the χ_{12} difference term in the third line of Eq. (C.12). This difference amounts to performing all possible vertex insertions into χ_{12}. If we write $\chi_{12}(k) = |\Delta_{\text{sc}}|^2 \sum_m G_0^\alpha(-m-k)G(m+k)G_0^\alpha(m)G(-m)$, then it is clear that there are six possible positions for vertex insertions; two full vertices can be inserted into the full Green's functions, two bare vertices can be inserted into the bare Green's functions, and two collective mode vertices can be inserted into the fluctuating gap Δ_{sc} or Δ_{sc}^*. For convenience, in this section only we define $\gamma_\alpha^\mu(k_+, k_-) = \gamma^\mu(k_+, k_-) + \alpha\Lambda^\mu(k_+, k_-)$. Performing all these vertex insertions then gives the following result:

$$\begin{aligned}&2(\chi_{12}(l+k_+) - \chi_{12}(l+k_-)) \\ &= q_\mu\bar{\Pi}^\mu(q)\sum_m G_0^\alpha(m+q)G(-m)F(m+l+k_+) + q_\mu\Pi^\mu(q)\sum_m G_0^\alpha(-m-l-k_-)G(m+l+k_+)F^*(m) \\ &\quad + \sum_m F(m+l+k_+)\left[G_0^\alpha(m+q)q_\mu\gamma_\alpha^\mu(m+q, m)F^*(m) - G(-m)q_\mu\Gamma^\mu(-m, -m-q)F^*(m+q)\right] \\ &\quad + \sum_m F^*(m)[G_0^\alpha(-m-l-k_-)q_\mu\gamma_\alpha^\mu(-m-l-k_-, -m-l-k_+)F(m+l+k_+) \\ &\quad - G(m+l+k_+)q_\mu\Gamma^\mu(m+l+k_+, m+l+k_-)F(m+l+k_-)].\end{aligned} \tag{C.14}$$

Here we have introduced the collective mode vertices $\Pi^\mu(q), \bar{\Pi}^\mu(q)$, which satisfy $q_\mu\Pi^\mu(q) = 2\Delta_{\text{sc}}, q_\mu\bar{\Pi}^\mu(q) = -2\Delta_{\text{sc}}^*$. These are the collective mode vertices discussed in Sec. (II C). Since $\chi_{12} = \chi_{21}$, the same result derived above holds for χ_{21}. We can now combine all the previous results from this subsection and define the following vertices

$$v_{11}^{\mu}(l+k_+,l+k_-) = \sum_m G(-m)G(l+m+k_+)\Gamma^{\mu}(l+m+k_+,l+m+k_-)G(l+m+k_-) + \sum_m G(l+m+k_+)G(-m)\Gamma^{\mu}(-m,-m-q)G(-m-q). \tag{C.15}$$

$$v_{22}^{\mu}(l+k_+,l+k_-) = -\frac{\chi_{12}(l+k_-)\chi_{21}(l+k_-)}{\chi_{22}(l+k_-)\chi_{22}(l+k_+)} \times \left\{\sum_m G(m)G(-l-m-k_-)\Gamma^{\mu}(-l-m-k_-,-l-m-k_+)G(-l-m-k_+) + \sum_m G(-l-m-k_-)G(m)\Gamma^{\mu}(m,m-q)G(m-q)\right\}, \tag{C.16}$$

$$v_{12}^{\mu}(l+k_+,l+k_-) = -\frac{\chi_{21}(l+k_-)}{\chi_{22}(l+k_+)}\Big\{\bar{\Pi}^{\mu}(q)\sum_m G_0^{\alpha}(m+q)G(-m)F(m+l+k_+) + \Pi^{\mu}(q)\sum_m G_0^{\alpha}(-m-l-k_-)G(m+l+k_+)F^*(m) + \sum_m F(m+l+k_+)\left[G_0^{\alpha}(m+q)\gamma_{\alpha}^{\mu}(m+q,m)F^*(m) - G(-m)\Gamma^{\mu}(-m,-m-q)F^*(m+q)\right] + \sum_m F^*(m)[G_0^{\alpha}(-m-l-k_-)\gamma_{\alpha}^{\mu}(-m-l-k_-,-m-l-k_+)F(m+l+k_+) - G(m+l+k_+)\Gamma^{\mu}(m+l+k_+,m+l+k_-)F(m+l+k_-)]\Big\}. \tag{C.17}$$

$$v_{21}^{\mu}(l+k_+,l+k_-) = \frac{\chi_{12}(l+k_+)}{\chi_{21}(l+k_-)}v_{12}^{\mu}(l+k_+,l+k_-). \tag{C.18}$$

Using the definitions of these vertices, along with Eq. (C.7), finally gives the vertex $\Lambda^{\mu}(k_+,k_-)$ for the t-matrix model of Ref.[26]

$$\Lambda^{\mu}(k_+,k_-) = \sum_l t(l)G(l-k_-)\Gamma^{\mu}(l-k_-,l-k_+)G(l-k_+) + \sum_l G(l)t(l+k_+)\Big[v_{11}^{\mu}(l+k_+,l+k_-) + v_{12}^{\mu}(l+k_+,l+k_-) + v_{21}^{\mu}(l+k_+,l+k_-) + v_{22}^{\mu}(l+k_+,l+k_-)\Big]t(l+k_-). \tag{C.19}$$

It can be shown that this vertex does indeed satisfy $q_{\mu}\Lambda^{\mu}(k_+,k_-) = \Sigma_{\rm corr}(k_-) - \Sigma_{\rm corr}(k_+)$. Diagrammatically, the first line in this expression is a Maki-Thompson (MT) diagram. The first term in the parentheses of the second line represents two identical Aslamazov-Larkin (AL) diagrams[25]. Similarly the fourth term in parentheses is similar to two identical Aslamazov-Larkin diagrams. The second and third terms in parentheses are additional diagrams which must be retained in order to satisfy the gauge invariant condition $q_{\mu}\Lambda^{\mu}(k_+,k_-) = \Sigma_{\rm corr}(k_-) - \Sigma_{\rm corr}(k_+)$.

Appendix D: f-sum rule and longitudinal sum rule

In this section we show that, given bare and full vertices that satisfy the Ward-Takahashi identity, the density-density and current-current response functions satisfy the f and longitudinal sum rules, respectively.

The exact response function is constructed from a two point correlation function containing one full vertex, $\Gamma^{\mu}(k_+,k_-) = (\Gamma^0(k_+,k_-),\mathbf{\Gamma}(k_+,k_-))$, and one bare vertex $\gamma^{\nu}(k_+,k_-) = (\gamma^0(k_+,k_-),\boldsymbol{\gamma}(k_+,k_-))$:

$$P^{\mu\nu}(q) = 2\sum_k G(k_+)\Gamma^{\mu}(k_+,k_-)G(k_-)\gamma^{\nu}(k_-,k_+). \tag{D.1}$$

To show consistency with sum rules, we use the Ward-Takahashi identity (as in Eq. (A.3))

$$q_{\mu}\Gamma^{\mu}(k_+,k_-) = G^{-1}(k_+) - G^{-1}(k_-). \tag{D.2}$$

Contracting the response function with q_μ, and using the Ward-Takahashi identity, we then have

$$\begin{aligned} q_\mu P^{\mu\nu}(q) &= 2\sum_k G(k_+)[G^{-1}(k_+) - G^{-1}(k_-)]G(k_-)\gamma^\nu(k_-, k_+), \\ &= 2\sum_k G(k)[\gamma^\nu(k, k+q) - \gamma^\nu(k-q, k)]. \end{aligned} \tag{D.3}$$

The $\nu = 0$ component of the bare vertex is equal to one, so that

$$q_\mu P^{\mu 0}(q) = 0 \tag{D.4}$$

On the other hand, the spatial components $\nu = j \in \{1, 2, 3\}$ are

$$\begin{aligned} q_\mu P^{\mu j}(q) &= 2\sum_k G(k)[\boldsymbol{\gamma}(k, k+q) - \boldsymbol{\gamma}(k-q, k)], \\ &= \frac{n}{m}\mathbf{q}. \end{aligned} \tag{D.5}$$

Here we used the fact that $\boldsymbol{\gamma}(k, k+q) - \boldsymbol{\gamma}(k-q, k) = \mathbf{q}/m$ is independent of $\mathbf{k}$, and $2\sum_k G(k) = 2\sum_{\mathbf{k}} n_{\mathbf{k}} = n$.

In terms of components, and real frequencies, these equations become

$$\omega P^{00}(\omega, \mathbf{q}) - \mathbf{q}\cdot\mathbf{P}^{i0}(\omega, \mathbf{q}) = 0, \tag{D.6}$$

$$\omega\mathbf{P}^{0j}(\omega, \mathbf{q}) - \mathbf{q}\cdot\overleftrightarrow{P}^{ij}(\omega, \mathbf{q}) = \frac{n}{m}\mathbf{q}. \tag{D.7}$$

Setting $\omega = 0$ and then operating with $-\mathbf{q}$ (on the right) in Eq. (D.7) gives

$$\mathbf{q}\cdot\overleftrightarrow{P}^{ij}(0, \mathbf{q})\cdot\mathbf{q} = -\frac{n}{m}\mathbf{q}\cdot\mathbf{q}. \tag{D.8}$$

Now use the identity Im $P^{i0}(\omega, \mathbf{q}) = -$Im $P^{0i}(-\omega, -\mathbf{q})$ and Eq. (D.6), Eq. (D.7) to solve for Im P^{00} in terms of Im $\overleftrightarrow{P}^{ij}$. Applying the Kramers-Kronig relations and Eq. (D.8) then gives

$$\begin{aligned} \int\frac{d\omega}{\pi}(-\omega\mathrm{Im}\ P^{00}(\omega, \mathbf{q})) &= \int\frac{d\omega}{\pi}\left(-\frac{\mathbf{q}\cdot\mathrm{Im}\ \overleftrightarrow{P}^{ij}(\omega, \mathbf{q})\cdot\mathbf{q}}{\omega}\right) \\ &= -\mathbf{q}\cdot\mathrm{Re}\ \overleftrightarrow{P}^{ij}(0, \mathbf{q})\cdot\mathbf{q} \\ &= \frac{n}{m}\mathbf{q}\cdot\mathbf{q}. \end{aligned} \tag{D.9}$$

The density-density and current-current response functions are respectively defined by $\chi_{\rho\rho}(q) \equiv P^{00}(q)$, $\overleftrightarrow{\chi}_{JJ}(q) \equiv P^{ij}(q), i, j \in \{1, 2, 3\}$. The f-sum rule is then

$$\int\frac{d\omega}{\pi}(-\omega\mathrm{Im}\ \chi_{\rho\rho}(\omega, \mathbf{q})) = \frac{n}{m}\mathbf{q}\cdot\mathbf{q}. \tag{D.10}$$

Similarly the longitudinal sum rule is

$$\int\frac{d\omega}{\pi}\left(-\frac{\mathbf{q}\cdot\mathrm{Im}\ \overleftrightarrow{\chi}_{JJ}(\omega, \mathbf{q})\cdot\mathbf{q}}{\omega}\right) = \frac{n}{m}\mathbf{q}\cdot\mathbf{q}. \tag{D.11}$$

Therefore, provided the full vertex satisfies the Ward-Takahashi identity, both sum rules will hold exactly for all $\mathbf{q}$ in this continuum limit.

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[31] For comparison to other notation in the literature, we note that the Green's function is occasionally[40] written as $G^{-1}(k) = G_0^{-1}(k) - \Sigma^{\rm nor}(k) - W(k)$, where $W(k) = -\Sigma^{\rm ano}(k)^2[G_0^{-1}(-k) - \Sigma^{\rm nor}(-k)]^{-1}$. In our notation, and for $\alpha = 1$, this corresponds to $\Sigma^{\rm ano}(k) \equiv \Delta_{\rm sc}$, $\Sigma^{\rm nor}(k) \equiv \Sigma_{\rm corr}(k)$, $W(k) \equiv \Sigma_{\rm sc}(k)$, $G_0^{-1}(-k) - \Sigma^{\rm nor}(-k) \equiv G_0^{\alpha}(-k)$. In Nambu notation $\Delta_{\rm sc}$ is the off diagonal component of the Nambu matrix self energy. For $\alpha = 1$, $\Sigma_{\rm corr}$ is the diagonal component of the Nambu matrix self energy. The advantage of the scalar self energy notation is that it treats both the $\alpha = 0$ and $\alpha = 1$ notations simultaneously, with the modification of only the single parameter α.
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[39] We note that the failure to satisfy the compressibility sum-rule is not a consequence of an inconsistency between the number equation and the Green's function[41]. Indeed, the number equation is $n = n_{\rm mf} + n_{\rm fl}$, where $n_{\rm mf} = -\partial\Omega_{\rm mf}/\partial\mu = {\rm Tr}\mathcal{G}_{\rm mf}$ and a fluctuation term $n_{\rm fl} = -\partial\Omega_{\rm fl}/\partial\mu$. If one calculates $\mathcal{G} = \mathcal{G}_{\rm mf} + \mathcal{G}_{\rm fl}$ at the fluctuation level, then $n = n_{\rm mf} + n_{\rm fl} = {\rm Tr}(\mathcal{G}_{\rm mf} + \mathcal{G}_{\rm fl})$ is obtained. Taking two derivatives of $n = n_{\rm mf} + n_{\rm fl}$, with respect to μ, the result is $-\partial n/\partial\mu = \partial^2\Omega_{\rm mf}/\partial\mu^2 + \partial^2\Omega_{\rm fl}/\partial\mu^2 = K_{00}(\omega = 0, \mathbf{q} \to 0) + \partial^2\Omega_{\rm fl}/\partial\mu^2 \neq K_{00}(\omega = 0, \mathbf{q} \to 0)$. As stated in the main text, gauge invariance (equivalently charge conservation) is not a sufficient condition to ensure that the compressibility sum-rule is satisfied.
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