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Stability of Suspended Graphene under Casimir Force

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We consider a graphene sheet suspended above a conducting surface. Treating graphene as an elastic membrane subjected to Casimir force, we study its stability against sagging towards the conductor. There exists a critical elevation at the edges below which the central part of the suspended graphene nucleates a trunk that sinks under the action of the Casimir force. The dependence of the critical elevation on temperature, dimensions, and the elastic stress applied to the graphene sheet is computed.

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Graphene is a remarkable allotrope of carbon in the form of a honeycomb lattice [1]. This 2D material is expected to mark a major breakthrough in the future of technology due to its unique mechanical, thermal, and electronic properties [2, 3]. Micro-electromechanical systems that involve suspended graphene should take into account interaction of graphene with surrounding elements. One important source of such interactions is Casimir force [4–7]. It has been intensively studied in recent years in application to graphene heterostructures [8–17].

In the conventional approach to Casimir interactions one considers forces between two surfaces of fixed geometry. Here we take a different approach. We treat suspended graphene as an elastic membrane and consider its deformation due to Casimir forces. Micromechanical studies of elastic membranes have a long history [18]. They have been recently revived in application to graphene [19–24]. In this paper we show that there exists a critical separation from the underlying conductor below which a suspended graphene becomes unstable against sagging towards the conductor, see Fig. 1.

A 2D elastic membrane is described by the energy [18, 19]

$$H_m = \frac{1}{2}\int d^2r\left[\kappa(\boldsymbol{\nabla}^2 h)^2 + \lambda u_{\alpha\alpha}^2 + 2\mu u_{\alpha\beta}^2\right], \quad (1)$$

where $\kappa(T)$ is the flexural stiffness constant, λ and μ are Lamé elastic coefficients, $h(\mathbf{r})$ is the flexural deformation perpendicular to the plane of the membrane, $\mathbf{u}(\mathbf{r})$ is the displacement field in the plane of the membrane, and

$$u_{\alpha\beta} = \frac{1}{2}\left(\partial_\alpha u_\beta + \partial_\beta u_\alpha + \partial_\alpha h \partial_\beta h\right) \quad (2)$$

is the strain tensor. The stress tensor is given by

$$\sigma_{\alpha\beta} = \frac{\delta H_m}{\delta(\partial_\beta u_\alpha)} = \lambda u_{\gamma\gamma}\delta_{\alpha\beta} + 2\mu u_{\alpha\beta}. \quad (3)$$

The Euler equations for $\mathbf{u}$ and h are

$$\lambda\partial_\alpha u_{\beta\beta} + 2\mu\partial_\beta u_{\alpha\beta} = 0 \quad (4)$$
$$\kappa\nabla^2\nabla^2 h - \lambda\partial_\alpha(u_{\beta\beta}\partial_\alpha h) - 2\mu\partial_\alpha(u_{\alpha\beta}\partial_\beta h) = 0. \quad (5)$$

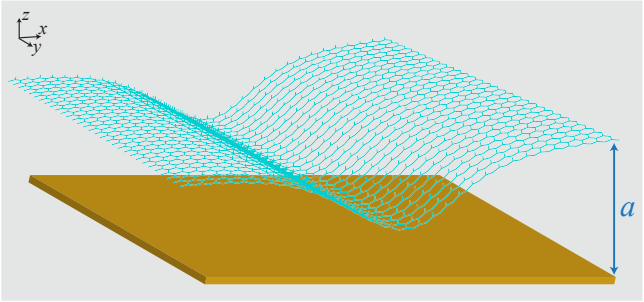

FIG. 1: (Color online) Suspended graphene attracted to a solid surface by Casimir force.

In principle, these equations are non-linear on h. This leads to a plethora of well-known anharmonic effects in graphene, see, e.g., Ref. 24 and references therein. However, in a typical experiment with a suspended graphene sheet it is stretched in the x-direction and held by the edges running in the y-direction. In this case the translational invariance along the y-axis allows one to consider the extremal solutions of Eqs. (4) and (5) that depend on the coordinate x only, that is, the elastic problem becomes one-dimensional. This reduces the equations to

$$\partial_x u_{xx} = 0, \quad \partial_x u_{yx} = 0 \quad (6)$$
$$\kappa\partial_x^4 h - (\lambda + 2\mu)\partial_x(u_{xx}\partial_x h) = 0, \quad (7)$$

rendering constant values of u_{xx} and u_{yx} strains. The stress σ in the x-direction generates the strain $u_{xx} \equiv s = \sigma/(\lambda + 2\mu)$. The equation for $h(x)$ then becomes

$$\kappa\partial_x^4 h - \sigma\partial_x^2 h = 0. \quad (8)$$

This linearization of the theory based upon non-linear Eq. (5) in the presence of the stress is well known [19]. It is completely analogous to the case of transversal modes of a massive string: When a tension along the string is applied, one ends up with a linear wave equation for the transversal displacement, whereas without tension the

anharmonic terms should be taken into account [25]. Eq. (8) can be derived from the effective energy of the membrane

$$H_{\rm eff} = \int d^2r \left[\frac{\kappa}{2}(\partial_x^2 h)^2 + \frac{\sigma}{2}(\partial_x h)^2\right] \qquad (9)$$

that we will use below.

Before proceeding it is useful to discuss the relative magnitude of the two terms contributing to Eq. (8) and Eq. (9). From their structure it is clear that the $(\partial_x h)^2$ term dominates over the $(\partial_x^2 h)^2$ term at curvature radii exceeding $r_c = \sqrt{\kappa/\sigma}$. The typical value of the flexural stiffness constant for graphene is of order $\kappa \sim 1$ eV. The Lamé coefficients are in the ballpark of 10^2 J/m^2 and the typical elastic strain for a suspended graphene sheet is $s \sim 0.001 - 0.01$. This gives $\sigma \sim 0.1 - 1$ J/m^2. Consequently, the critical curvature corresponds to $r_c \sim 1$ nm. We, therefore, conclude that for all effects involving curvature radii in the excess of 1 nm the energy of a suspended graphene sheet is dominated by the second term in Eq. (9).

We shall assume that the graphene sheet is suspended above a flat surface of the perfect conductor and will approximate the energy of the Casimir attraction per unit area of the graphene sheet at a distance a from the conductor by [10]

$$f = -\beta\hbar c\left(\frac{1}{a^3} + \frac{1}{a_t a^2}\right), \qquad (10)$$

where

$$\beta = \frac{\alpha N}{128\pi}\left[\ln\left(1 + \frac{8}{\alpha N\pi}\right) + \frac{1}{2}\right] = 0.0003606235 \qquad (11)$$

and

$$a_t(T) = \frac{16\pi\beta}{\xi(3)}\frac{\hbar c}{k_B T} = 0.01508\frac{\hbar c}{k_B T} \qquad (12)$$

with $\alpha = (4\pi\epsilon_0)^{-1}e^2/(\hbar c) = 1/137.036$ being the fine-structure constant, $N = 4$ being the number of fermion species for graphene, and $\zeta(3) = 1.20205$ being the value of Riemann zeta function $\zeta(x)$ at $x = 3$. The first term in Eq. (10) is due to quantum fluctuations of the electromagnetic field, while the second term is due to thermal fluctuations. The crossover from the low-temperature regime with $f \propto 1/a^3$ to the high-temperature regime with $f \propto 1/a^2$ on increasing separation occurs at $a = a_t$. At $T = 300$ K the crossover takes place at $a = 115$ nm, while at $T = 4$ K it occurs at $a = 8.63\,\mu$m.

One observation is in order related to the range of the applicability of Eq. (10), which is often omitted in literature. In the original approach developed by Casimir the force arises from the quantized oscillations of the electromagnetic field in a confined space [4]. A more general formula for the force was later obtained by Lifshitz in a macroscopic approach that studies electrostatic interaction between two bodies in terms of their susceptibilities [5]. For two ideal conductors the two results coincide. In the Casimir approach the assumption that the electromagnetic radiation is confined between conducting surfaces breaks down at frequencies that are higher than the plasma frequency. Similarly, the Lifshitz approach based upon macroscopic susceptibilities breaks down at frequencies that exceed the absorption resonances of the materials. The corresponding cut-off frequency, ω_c, does not explicitly enter Eq. (10). However, it is present there indirectly. Indeed, at $T = 0$ the dominant contribution to the force comes from the photon wave vectors [7] $k \sim 1/(4a)$. The requirement that photon energies satisfy $E < E_c = \hbar\omega_c$ then translates into the lower limit on the separation, $a > \hbar c/(4E_c)$. At, e.g., $E_c \sim 1$ eV this gives $a > 50$ nm. In line with that estimate the experiments on Casimir force performed down to $a \sim 0.1\,\mu$m agree well with theoretical formulas. However, the lower separations would require not only a significantly greater experimental effort but also a much more involved theory that takes into account the structure of the materials at the atomic level.

We will be interested in a situation when the size of the suspended graphene sheet is large compared to its distance from the underlying surface. The curvature of the graphene sheet will be small so that the expression (10) for f derived for a flat graphene must be approximately valid locally. In this case the total energy can be approximated by $H = H_{\rm eff} + \int d^2r f$, that is

$$H = \frac{1}{2}\int d^2r\left[\kappa(\partial_x^2 h)^2 + \sigma(\partial_x h)^2\right] - \beta\hbar c\int d^2r\left[\frac{1}{(a-h)^3} + \frac{1}{a_t(a-h)^2}\right], \qquad (13)$$

where a is a fixed distance from the conductor at the edges and $h(x)$ describes the sagging profile (with $h = 0$ at the edges),

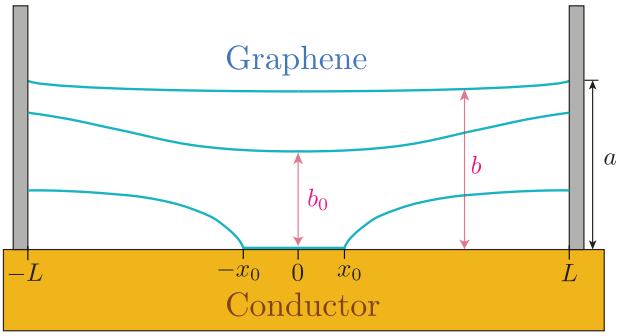

FIG. 2: (Color online) Casimir-driven sagging and instability of a rectangular graphene sheet against attachment to the underlying surface.

The energy (13) is minimized by $h(\mathbf{r})$ satisfying the Euler equation

$$\partial_x^2 \frac{\delta H}{\delta \partial_x^2 h} - \partial_x \frac{\delta H}{\delta \partial_x h} + \frac{\delta H}{\delta h} = 0 \tag{14}$$

which gives

$$\left(\kappa \partial_x^4 - \sigma \partial_x^2\right) h = \beta \hbar c \left[\frac{3}{(a-h)^4} + \frac{2}{a_t (a-h)^3}\right]. \tag{15}$$

Equilibrium sagging profile of a suspended graphene sheet is determined by three factors: The gain in the Casimir energy, the loss in the elastic energy, and the condition $h = 0$ at the boundary. As we shall see below, there is a critical separation at which graphene becomes unstable against attaching to the underlying surface, see Fig. 2. The curvature radius of a suspended graphene will always greatly exceed $r_c \sim 1$ nm. This allows one to drop the first term in the left-hand side of Eq. (15). In terms of dimensionless variables

$$\bar{x} = \frac{x}{a}, \quad \bar{\partial}_x = a\partial_x, \quad \bar{h} = \frac{h}{a} \tag{16}$$

the resulting equation is

$$\bar{\partial}_x^2 \bar{h} = -\frac{\gamma}{(1-\bar{h})^4} - \frac{\delta_t'}{(1-\bar{h})^3}, \tag{17}$$

where

$$\gamma(a) = 3\beta \frac{\hbar c}{\sigma a^3}, \qquad \delta_t(T) = 2\beta \frac{\hbar c}{\sigma a^2 a_t}. \tag{18}$$

In the low-temperature limit, $a \ll a_t$, Eq. (17) reduces to

$$\bar{\partial}_x^2 \bar{b} = \frac{\gamma}{\bar{b}^4} \tag{19}$$

where $\bar{b} = 1 - \bar{h}$ is the distance from the conductor in the units of a. The first integral of Eq. (19) is

$$\left(\frac{d\bar{b}}{d\bar{x}}\right)^2 = \frac{2\gamma}{3}\left(\frac{1}{\bar{b}_0^3} - \frac{1}{\bar{b}^3}\right), \tag{20}$$

where $\bar{b}_0 < 1$ is the minimal separation of graphene from the conductor at $\bar{x} = 0$. We consider a rectangular graphene sheet of length $2L$. The value of $\bar{b}_0$ must follow from the boundary conditions $\bar{b}(\bar{x} = \pm L/a) = 1$ for the sheet clipped at $x = \pm L$.

The minimal separation at the edges, a_c, before suspended graphene sheet becomes unstable under the action of Cassimir force (see Fig. 2) can be estimated from the following argument. The boundary condition, $\bar{b} = 1$ at $x = \pm L$, provides $\left(d\bar{b}/d\bar{x}\right)^2 \sim (1-\bar{b}_0)^2 (a/L)^2 \sim (2\gamma/3)\left(\bar{b}_0^{-3} - 1\right)$, that is, $(2\gamma/3)\,(L/a)^2 \sim \bar{b}_0^3 (1 - \bar{b}_0)/(1 + \bar{b}_0 + \bar{b}_0^2) < \bar{b}_0 < 1$, which gives $a > a_c \sim 2^{1/5}\,(\beta\hbar c/\sigma)^{1/5}\, L^{2/5}$.

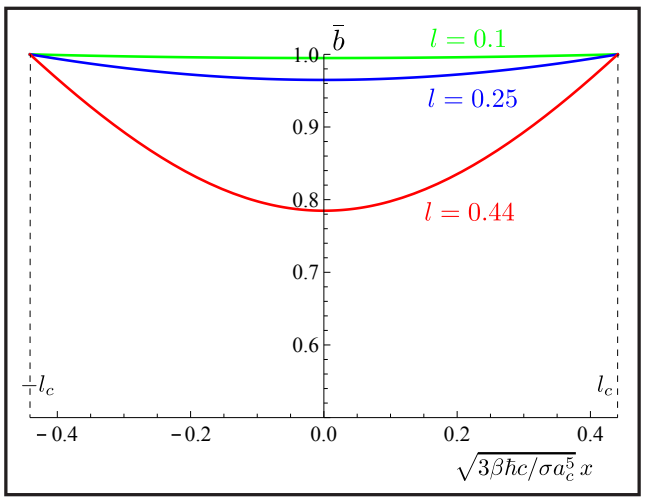

FIG. 3: (Color online) Profile of a sagging graphene due to Casimir force for three values of parameter $l = \gamma^{1/2} L/a$. At $l = 0.1$ the effect of Casimir attraction is weak, while at $l = 0.44$ the graphene sheet is close to the critical separation below which it becomes unstable against sagging all the way down to the conductor.

This qualitative analysis is confirmed by numerical solution of Eq. (20) illustrated in Fig. 3. It shows that the strength of the Casimir effect is determined by the parameter $l = \gamma^{1/2} L/a = \sqrt{(3\beta\hbar c)/(a^5\sigma)}\, L$. The smooth sagging profile shown in Fig. 3 exists at $l < l_c = 0.441$. At $l > l_c$ it is unstable against nucleation of a sinking trunk in the central part of the graphene sheet. The exact numerical result for the critical separation at the edges in the low-temperature limit reads

$$a_c = 1.73 \left(\frac{\beta\hbar c}{\sigma}\right)^{1/5} L^{2/5}. \tag{21}$$

Note that at the critical separation one has $a_c/L \ll 1$, that is, the graphene sheet is still close to flat in real space (when unrenormalized units of length are used). This justifies the use of Eq. (13) with the Casimir potential derived for a flat graphene layer. Another important observation is that as the separation at the edges a approaches a_c from above, the minimum critical distance b_0 from the center of the graphene sheet to the underlying surface remains finite, $\bar{b}_0 = b_0/a \to 0.765$. For, e.g., $2L \sim 2$ mm one obtains $a_c \sim 0.17\,\mu$m. These value is small compared to a_t at 4 K, which justifies the low-temperature approximation used to find the critical separation in Eq. (21).

In the high temperature-limit, $a \gg a_t$, the first integral of Eq. (17) is

$$\left(\frac{d\bar{b}}{d\bar{x}}\right)^2 = \delta_t \left(\frac{1}{\bar{b}_0^2} - \frac{1}{\bar{b}^2}\right). \tag{22}$$

The solution is

$$\bar{b}(\bar{x}) = \bar{b}_0 \sqrt{1 + \delta_t \bar{x}^2/\bar{b}_0^4}. \quad (23)$$

The boundary conditions $\bar{b}(\pm L/a) = 1$ give the following expression for $\bar{b}_0$

$$\bar{b}_0^2 = \frac{1}{2}\left[1 + \sqrt{1 - 4\delta_t (L/a)^2}\right]. \quad (24)$$

The stability of the solution requires $4\delta_t'(L/a)^2 < 1$, which translates into

$$a > a_c = \left[\frac{\zeta(3) k_{\rm B} T}{2\pi\sigma}\right]^{1/4} L^{1/2}. \quad (25)$$

As a approaches a_c from above, the distance from the graphene sheet to the underlying surface at the center approaches $b_0 = a/\sqrt{2} \approx 0.707a$, which is comparable to $b_0 \approx 0.765a$ in the low-T limit. At $a < a_c$ graphene is unstable against its central part developing a trunk that sinks towards the underlying surface, see Fig. 2. Choosing $2L \gg 2$ mm one obtains from Eq. (25) $a_c \sim 0.17\,\mu$m $> a_t = 0.11\,\mu$m at $T = 300$ K. Notice the week dependence of a_c on the elastic strain and temperature. The latter explains why the values of the critical separation in the low and high temperature regimes are similar.

In Conclusion, by treating a suspended graphene sheet as an elastic membrane we have studied its sagging profile due to the Casimir attraction to the underlying conductor. Critical separation at which graphene becomes unstable against sagging all the way down has been computed as a function of temperature, the size of the graphene sheet, and the elastic stress applied to it. While our model ignores certain effects such as, e.g., corrections to the Casimir force by weak bending of the graphene sheet, it provides a reasonable estimate of the critical separation and can serve as the first approximation to the stability problem.

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