Absolute stability and spatiotemporal long-range order in Floquet systems

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Absolute Stability and Spatiotemporal Long-Range Order in Floquet systems

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Recent work has shown that a variety of novel phases of matter arise in periodically driven Floquet systems. Among these are many-body localized phases which spontaneously break global symmetries and exhibit novel multiplets of Floquet eigenstates separated by quantized quasienergies. Here we show that these properties are stable to *all* weak local deformations of the underlying Floquet drives—including those that explicitly break the defining symmetries—and that the models considered until now occupy sub-manifolds within these larger "absolutely stable" phases. While these absolutely stable phases have no explicit global symmetries, they spontaneously break Hamiltonian dependent *emergent* symmetries, and thus continue to exhibit the novel multiplet structure. The multiplet structure in turn encodes characteristic oscillations of the emergent order parameter at multiples of the fundamental period. Altogether these phases exhibit a form of simultaneous long-range order in space and time which is new to quantum systems. We describe how this spatiotemporal order can be detected in experiments involving quenches from a broad class of initial states.

I. INTRODUCTION

The elucidation of phase structure is a major theme in condensed matter physics and statistical mechanics. An early paradigm for doing so, associated most with Landau, characterizes phases through the spontaneous breaking of global symmetries present in the microscopic Hamiltonian *i.e*, phases are either paramagnetic, or spontaneously symmetry broken (SSB). In modern parlance, the phases obtained thereby are symmetry protected since their distinctions are erased if the symmetries are not present microscopically. More recently, it has been found that this characterization is too coarse — not all paramagnetic phases should be considered identical. Indeed, there exist paramagnetic symmetry protected topological (SPT) phases which do not break any symmetries, but which nevertheless cannot be adiabatically connected to one another in the presence of the protecting global symmetry[1]. Remarkably, we now know of other phases, such as those with topological order, which do not even require a global symmetry and are *absolutely stable*—their ground state (and sometimes even low temperature) properties are stable to arbitrary weak local perturbations[2–4]. Equally remarkably, there are also examples of systems whose entire many body spectrum displays some absolutely stable property, namely many body localized[5–10] (MBL) systems which robustly exhibit a full set of emergent local conserved quantities[11–18]. One can also combine MBL with the above quantum orders to obtain MBL phases in which individual highly excited eigenstates show SSB, SPT, or topological order[19–24].

The ideas above assume time translation invariance (TTI) or energy conservation since they involve describing the eigensystem of a time independent many body Hamiltonian. What happens if we relax this constraint, considering instead time dependent Hamiltonians $H(t)$? Generically, it is expected that an interacting, driven many-body system absorbs energy indefinitely and approaches a dynamic approximation to the infinite temperature equilibrium state. However, for Floquet systems with periodic time dependence $H(t+T) = H(t)$, this fate can be avoided in the presence of sufficiently strong disorder (or in the absence of interactions[27–37]) as was shown in recent work extending the physics of MBL to Floquet systems[38–42]. This in turn allowed phases to be defined for MBL-Floquet systems[25] via a generalization of the idea of eigenstate order first discussed for undriven MBL systems. In very recent work, a classification was given for phases that either preserve[43–45] or spontaneously break[26] unitary global symmetries[46].

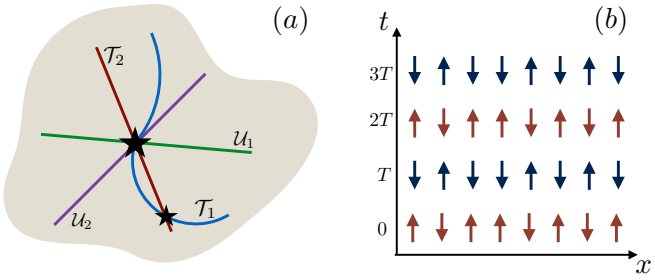

FIG. 1. (Color online): (a) Schematic depiction of the manifold of Floquet unitaries that are absolutely stable and characterized by Hamiltonian dependent emergent symmetries (grey area). Special sub-manifolds (colored lines) within the absolutely stable manifold are characterized by Hamiltonian *independent* unitary ($\mathcal{U}_i$) and antiunitary ($\mathcal{T}_i$) symmetries. Special models (black stars) can lie at the intersection of several sub-manifolds with exact symmetries. As an example, the πSG model defined in Refs. 25 and 26 is absolutely stable and possesses the Ising unitary symmetry P and an antiunitary symmetry $\mathcal{T} = KP$ where K is complex conjugation. (b) Schematic depiction of the spatiotemporal long-range order found in absolutely stable phases—the order looks "antiferromagnetic" in time and glassy in space.

In the present paper we build on the latter work and show that a subset of the SSB phases identified therein are stable to arbitrary weak local perturbations, including those that explicitly break any of the defining global symmetries. Thus this subset is *absolutely* stable—a re-

markable outcome for a driven system. The apparent puzzle that SSB phases can be stable absent Hamiltonian independent symmetries is resolved elegantly: at general points in these absolutely stable phases, the drives (in the infinite volume limit) are characterized by a set of Hamiltonian *dependent* emergent unitary and antiunitary symmetries. *Ex post facto*, we see that the symmetric models in Refs. 25 and 26 live in lower dimensional submanifolds (characterized by Hamiltonian independent symmetries) of a much higher dimensional absolutely stable phase — we sketch the resulting structure in Fig. 1. This analysis uncovers a much richer symmetry structure than the global unitary symmetries used in previous work.

Strikingly, the out of equilibrium dynamics in these phases exhibits sharp universal signatures associated with oscillations of an emergent order parameter; these generalize the multiple period oscillations uncovered in previous work[25,26] on symmetric drives. For example, we show that starting from arbitrary short range correlated initial states, the late time states show sharp oscillations of generic local operators at multiples of the fundamental period. This particular dynamical feature is a great boon to a future experimental detection of these phases as experimentalists are required neither to fine tune the Hamiltonian nor the starting state to observe a sharp signature!

These longer periods raise the question of whether they should be thought of as representing spontaneous breaking of yet another symmetry—that of time translations by a period of the drive[47]. We note that the idea that time translations might be analyzed in this fashion was first mooted by Wilczek[48] for time independent Hamiltonians; there is, however, now a proof[49] that such "time crystals" do not exist for undriven systems in equilibrium. We analyze this question further and find that strictly speaking *all* MBL systems, driven or undriven, exhibit some eigenstate correlations characteristic of temporal *glasses*—an aperiodic breaking of time translation invariance (TTI). For the Floquet broken symmetry phases however, the long distance correlations simultaneously exhibit spin glass order in space and multiple period oscillation in time. These lead to the characteristic space-time snapshot illustrated for the simplest such phase in Fig. 1(b). Evidently the system exhibits spatiotemporal[50] long-range order in both space and time. The modulation in time, which is antiferromagnetic, does indeed break time translation symmetry but it preserves the combination of a translation and emergent Ising reversal. We note that a similar spatiotemporal order—now ferromagnetic in space—was previously exhibited in the large N Floquet theory[51] and discussed in the terminology of a lack of synchronization with the drive.

We note that the discovery of these absolutely stable Floquet phases can also be viewed as the realization that while a Hamiltonian that lacks *any* symmetries (inclusive of time translation invariance) exhibits only a trivial phase, introducing discrete time translation invariance *alone* is sufficient to introduce a non-trivial phase structure. This would appear to be the minimum symmetry condition for this purpose.

In the rest of the paper we describe these results in more detail. We begin with the simplest example of a SSB phase that is absolutely stable—this is the Ising π spin-glass or πSG first described in Refs. 25 and 26. In Sec. II we establish its absolute stability and analyze its emergent symmetries, correlations and characteristic spectral features within the paradigm of eigenstate order. Next, in Sec. III we study the nature of dynamical correlations in the πSG in individual eigenstates and starting from generic short ranged entangled states, and discuss why the πSG should be identified as a Floquet space-time crystal. We then discuss the catalog of other absolutely stable Floquet phases in Sec. IV, and show how some Floquet SPT phases exhibit time crystallinity at their boundaries. We end with some concluding remarks in Sec. V.

Before proceeding we note that a recent paper by Else, Bauer and Nayak[52] studies one of the submanifolds of our primary example of an absolutely stable phase, the πSG and identifies it as a pure time crystal on the grounds that the drives break the unitary Ising symmetry. Our work clarifies that the order in the πSG and its cousins is always spatiotemporal and never purely temporal. Indeed the specific submanifold studied in[52] turns out to be protected by an antiunitary symmetry (see Eq. (7)) and thus exhibits spatial order in a particularly transparent form as we discuss below.

II. THE π SPIN GLASS: ABSOLUTE STABILITY AND EMERGENT SYMMETRIES

We consider systems with time periodic local Hamiltonians $H(t) = H(t+T)$. The Floquet unitary is the time evolution operator for one period $U(T) \equiv \mathcal{T} e^{-i\int_0^T dt H(t)}$. The Floquet eigenstates $|\alpha\rangle$ of $U(T)$ have eigenvalues $e^{-i\epsilon_\alpha T}$, where ϵ_α are the quasienergies defined modulo $2\pi/T$. Indeed, the Floquet eigensystem in phases with special forms of eigenstate order/quasienergy spectral pairing will form a central part of our discussion.

A. Properties of the πSG phase

Refs. 25, 26, and 43 discussed various SSB/SPT phases with Floquet eigenstate order, but not all of these phases are absolutely stable to arbitrary perturbations. In this work, our canonical example of an absolutely stable Floquet phase will be the π spin-glass (πSG) phase[25]. A concrete model Floquet unitary in this phase in 1d is

$$U_{f0} = P_x \exp[-i \sum_{r=1}^{L-1} J_r \sigma_r^z \sigma_{r+1}^z]; \quad P_x = \prod_r \sigma_r^x, \qquad (1)$$

where L is the system size, the σ_r^α for $\alpha = \{x, y, z\}$ are Pauli spin 1/2 degrees of freedom on site r, $P \equiv P_x$ is the global Ising parity symmetry ($P_{y,z}$ analogously defined), and the J_r's are random couplings drawn uniformly from $[\overline{J} - \delta J, \overline{J} + \delta J]$. We note several properties of this model, some of which were deduced in previous work[25,26]:

1. U_{f0} commutes with the unitary symmetry P. Defining anti-unitary operators $\mathcal{T}_\alpha = P_\alpha K$ where K is complex conjugation, U_{f0} also has $\mathcal{T} \equiv \mathcal{T}_x$ symmetry: $\mathcal{T} U_{f0} \mathcal{T}^{-1} = U_{f0}^{-1}$. It similarly has $\mathcal{T}_{y,z}$ symmetry for systems with an even number of sites[53]. Thus, this model lies at the intersection of several special submanifolds with Hamiltonian independent symmetries (Fig. 1) and is extremely robust to a large class of perturbations which preserve *some* exact symmetry. Note that the anti-unitary symmetries $\mathcal{T}$ are a combination of K and a spatial Ising flip.

2. The eigenspectrum of U_{f0} can be found by noting that all the domain wall operators $D_r \equiv \sigma_r^z \sigma_{r+1}^z$ commute with P_x, U_{f0} and with one another. Thus, the eigenstates look like symmetric/antisymmetric global superposition states (also called cat states) of the form

$$|\pm\rangle \sim |\{d_r\}, p = \pm 1\rangle = \frac{1}{\sqrt{2}} |\{\sigma_r^z\}\rangle \pm \frac{1}{\sqrt{2}} |\{\overline{\sigma_r^z}\}\rangle \,,$$

   where $\{\sigma_r^z\} = \{\uparrow\downarrow\downarrow \cdots \uparrow\}$ labels a frozen spin-glass configuration of z spins (and hence the domain wall expectation values d_r), $\{\overline{\sigma_r^z}\}$ is its spin-flipped partner, and $p = \pm 1$ is the Ising parity eigenvalue of the eigenstates.

3. The eigenstates above have corresponding unitary eigenvalues $u(d, p) = p e^{-i \sum_{r=1}^{L-1} J_r d_r}$. Note that the opposite parity cat-state partners have unitary eigenvalues differing by a minus sign $u(d, -1) = -u(d, -1)$ and hence quasienergies differing by π/T. We refer to this phenomenon as a π spectral pairing of cat states.

4. The Floquet eigenstates exhibit long range connected correlations (LRO) and spin glass[54,55] (SG) order in σ_i^z, but show no long-range order in σ_i^x and σ_i^y.

5. The order parameter for the πSG model oscillates with frequency π/T or period $2T$, as indicated by the stroboscopic equation of motion $\sigma_r^z(nT) = (-1)^n \sigma_r^z$[25,26]. This follows directly from the fact that σ_r^z anticommutes with U_{f0}. While $\langle \sigma_r^z(nT) \rangle = 0$ in the Floquet eigenstates, the observable shows a periodic time dependence with period $2T$ in short-range correlated states of the form $|\{\sigma_r^z\}\rangle \sim |+\rangle + |-\rangle$. On the other hand, the σ^x and σ^y operators do not show period $2T$ oscillations.

B. Absolute stability and emergent symmetries

How robust are the above properties to perturbations of the form $H(t) \to H(t) + \lambda V(t)$? Numerical results have already demonstrated the stability of U_{f0} to weak Ising[25] symmetric perturbations. We will provide evidence that this phase is, in fact, absolutely stable to *all* generic weak perturbations — we will define dressed spin operators (Floquet l-bits) for the perturbed system and show that it displays *emergent* symmetries with the same effect on eigenspectrum properties as the exact Ising symmetry.

The first step in the argument is to observe that the stability of the localization of the unperturbed unitary to arbitrary weak local perturbations (for sufficiently strong disorder) is itself not a consequence of symmetries. More technically, call the corresponding perturbed Floquet unitary $U_{f\lambda}$ where λ is the strength of the perturbation. We expect that the stability of localization implies the existence of a family of local unitaries[56] $\mathcal{V}_\lambda$ which relate the eigenvectors of U_{f0} to those of $U_{f\lambda}$ for λ in some non-vanishing range[26,39,41,52]. Note that the locality of such a unitary is a subtle business outside of the very strongly localized region due to proliferating resonances and Griffiths effects[21,57].

Assuming that a low depth $\mathcal{V}_\lambda$ exists, it relates the new eigenvectors of $U_{f\lambda}$ denoted $|\alpha\rangle_\lambda$ to the eigenvectors of U_{f0} via

$$|\alpha\rangle_\lambda = \mathcal{V}_\lambda |\{d_r\}, p\rangle.$$

The new quasienergies are similarly denoted as ϵ_λ^α. These local unitaries allow us to define a set of dressed, exponentially localized operators $\tau_{r,\lambda}$ (analogous to the l-bits[11,12,14–16] in static MBL systems) together with a dressed parity operator P^λ via

$$\begin{aligned} \tau_{r,\lambda}^\beta &= \mathcal{V}_\lambda \sigma_r^\beta \mathcal{V}_\lambda^\dagger \\ P^\lambda &= \prod_r \tau_r^x. \end{aligned} \tag{2}$$

We will often suppress the explicit λ dependence of $\tau_{r,\lambda}^\alpha$ for brevity and $\beta = x, y, z$. Defining (local) dressed domain wall operators as $D_r^\lambda \equiv \tau_r^z \tau_{r+1}^z$, we get

$$\begin{aligned} D_r^\lambda |\alpha\rangle_\lambda &= \mathcal{V}_\lambda (\sigma_r^z \sigma_{r+1}^z) |\{d_r\}, p\rangle = d_r |\alpha\rangle_\lambda, \\ P^\lambda |\alpha\rangle_\lambda &= \mathcal{V}_\lambda P |\{d_r\}, p\rangle = p |\alpha\rangle_\lambda. \end{aligned} \tag{3}$$

Thus, the perturbed eigenstates are also eigenstates of the dressed operators D_r^λ and P^λ which means these operators commute with $U_{f\lambda}$, and we can rewrite $|\alpha\rangle_\lambda$ more suggestively as $|\{\tau_r^z\}, p = \pm 1\rangle$ using the same notation as before. By definition, τ_r^z anticommutes with P^λ. Further we show in App. A that it also anticommutes with $U_{f\lambda}$ in the large system limit

$$[\tau_r^z, U_{f\lambda}]_+ = O(e^{-cL}) \xrightarrow{L \to \infty} 0 \,, \tag{4}$$

using only the assumptions of locality and continuity. This implies that the Floquet eigenvalues are odd in

p. Together with the previous statements about the commutation properties of P^λ and D^λ with $U_{f\lambda}$, it is easy to show that the unitary eigenvalues take the form $u_\lambda(\{d_r\}, p) = pe^{-if(\{d\})}$. Re-expressing the eigenvalue dependence on conserved quantities in operator language gives

$$U_{f\lambda} = P^\lambda e^{-if(\{D_r^\lambda\})}, \tag{5}$$

where f is a functional of D^λ, or equivalently an even functional of the τ_r^z's. One can moreover argue that f can be chosen to be local, using the fact that the Floquet unitary itself is low depth[26,43]. Thus, f generically takes the form

$$f(\{D^\lambda\}) = \sum_{ij} J_{ij}\tau_i^z\tau_j^z + \sum_{ijkl} J_{ijkl}\tau_i^z\tau_j^z\tau_k^z\tau_l^z + \cdots$$

where the couplings $J_{ij} \sim e^{-|i-j|/\xi}$ decay exponentially with distance reflecting the locality of the unitary.

Written this way, the Floquet unitary (5) clearly has a $\mathbb{Z}_2$ symmetry P^λ — although we say it is emergent because P^λ, in general, depends on the details of the underlying Hamiltonian. $U_{f\lambda}$ similarly has an emergent antiunitary symmetry $\mathcal{T}^\lambda \equiv P^\lambda K^\lambda$ where K^λ is complex conjugation defined with respect to the τ^α. Note that Eq. (5) takes much the same functional form as the model unitary Eq. (1), and correspondingly its eigenstates exhibit long-range order in the dressed order parameter τ_r^z (associated with spontaneous breaking of P_λ), and short range order in $\tau_r^{x,y}$. The statements about π spectral pairing and the temporal dependence of observables (in particular $\tau^z(nT) = (-1)^n\tau^z(0)$) also follow directly[58].

Finally, we note that Refs 25 and 43 also defined a 0SG phase with the model unitary $\exp[-i\sum_{i=1}^{L-1} J_i\sigma_i^z\sigma_{i+1}^z]$. Like the πSG, this is also a phase with long-range SSB Ising order, but one in which the cat states are degenerate instead of being separated by π/T. If we generically perturb about this drive, we must begin with Floquet eigenstates that explicitly break the Ising symmetry in order for the change of basis unitary $\mathcal{V}_\lambda$ to be local. Implicitly this requires us to work in the infinite volume limit directly. In this case, one can show that τ^z commutes (rather than anticommutes) with the Floquet unitary, and one can readily use this to split the degeneracy between the Floquet eigenstates, rendering this phase unstable to arbitrary perturbations. By contrast, in the πSG phase, the cat states are π split and therefore non-degenerate — a fact which is essential to the stability of the SSB order to arbitrary perturbations.

C. Long range order and numerics

We now numerically check for the predicted π spectral pairing in a perturbed model of the form

$$U_{f\lambda} = P\exp[-i\sum_{r=1}^{L-1} J_r\sigma_r^z\sigma_{r+1}^z - i\lambda\sum_{r=1}^{L} h_r^x\sigma_r^x + h_r^y\sigma_r^y + h_r^z\sigma_r^z] \tag{6}$$

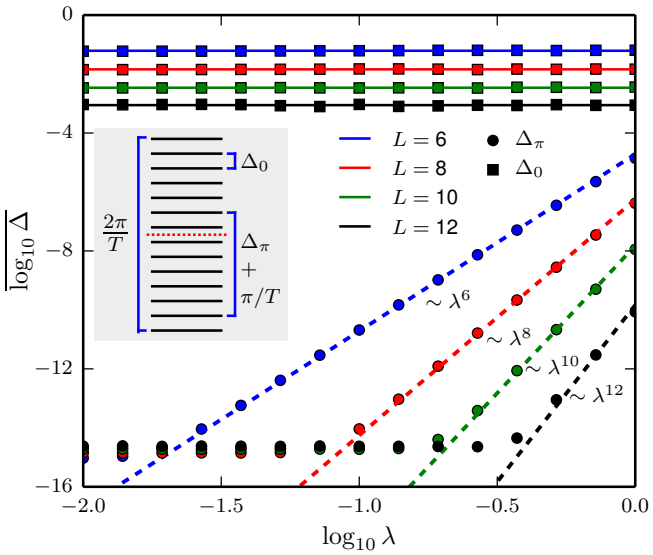

FIG. 2. (Color online): Disorder and eigenstate averaged spectral gaps for the generically perturbed model (6) without any P and $\mathcal{T}$ symmetries plotted as a function of the perturbation strength λ and system size L. The nearest-neighbor quasienergy gap Δ_0 shows no λ dependence but decreases exponentially with L. On the other hand Δ_π which measures the spectral pairing of even-odd parity states scales as λ^L (fits to this form superimposed). Thus, there is a window of λs for which $\Delta_\pi \ll \Delta_0$ and the system exhibits robust spectral pairing in the $L \to \infty$ limit. Gaps smaller than $\sim 10^{-14}$ are below numerical precision, thus the initial λ independent trend in the Δ_π data for larger L. (inset): Cartoon of the quasienergy spectrum illustrating the definitions of Δ_0 and Δ_π.

The fields $J_r, h_r^{x,y,z}$ are drawn randomly and uniformly with $\overline{J_r} = 1, \delta J_r = 0.5$, $\overline{h_r^x} = \delta h_r^x = 0.1$, $\overline{h_r^y} = \delta h_r^y = 0.15$, $\overline{h_r^z} = \delta h_r^z = 0.45$ and the notation $\overline{x}, \delta x$ means that x is drawn from $[\overline{x} - \delta x, \overline{x} + \delta x]$. The perturbation breaks all the unitary and anti-unitary symmetries present in the original U_{f0} model. To check for spectral pairing, we define the nearest neighbor gap between the perturbed quasienergies as $\Delta_0^i = \epsilon_{i+1}^\lambda - \epsilon_i^\lambda$ and the π gap as $\Delta_\pi^i = \epsilon_{i+\mathcal{N}/2}^\lambda - \epsilon_i^\lambda - \pi/T$ where $\mathcal{N} = 2^L$ is the Hilbert space dimension, and where the second equation follows from the fact that the quasienergy bandwidth is π/T and we expect states halfway across the spectrum to be paired at π/T (See Fig. 2 (inset) for an illustration of these definitions). The system shows spectral pairing at π if there is a range of λ's for which $\Delta_\pi \ll \Delta_0$ as $L \to \infty$. Fig. 2 shows the mean Δ_π and Δ_0 log-averaged over eigenstates and several disorder realizations for different λ's and L's. We see that $\overline{\Delta_\pi} \sim \lambda^L$ whereas $\overline{\Delta_0} \sim e^{-sL}$ where $s \sim \log(2)$ is a λ independent entropy density. Thus, we can get robust pairing in the window $|\log\lambda| > s$.

Having shown how the robustness of the πSG phase is associated with spontaneously broken emergent symmetries and long-range order in the τ^z variables, we can now

ask what effect this long-range order has on correlations in the physical σ^α degrees of freedom. Generically we expect the expansion of the physical spins in terms of l-bits to have some components which are diagonal and odd in τ^z, for example $\sigma_r^\alpha = c^\alpha \tau_r^z + \cdots$. As a result $\sigma_r^{\alpha=x,y,z}$ are all expected to have long range connected correlation functions, as well as a component exhibiting $2T$ periodic stroboscopic oscillations. These predictions agree with our numerical results Fig. 3 and Fig. 4 respectively.

On the other hand, when we perturb U_{f0} in a manner that respects an explicit symmetry like P or $\mathcal{T}$, the resulting models reside in a special submanifold of the absolutely stable phase. The presence of the exact symmetries constrains the form of the dressed τ^α operators and leads to concrete predictions about the order in and temporal dependence of different operators. For example, it was argued[26] that when the perturbation $\lambda V(t)$ is such that $U_{f\lambda}$ continues to have Ising symmetry, $\mathcal{V}_\lambda$ can be chosen to commute with P. As a result, $P^\lambda = P$, and $\sigma^{y,z}$ are odd under P^λ whereas σ^x is even under P^λ. This means an operator expansion of σ_r^x in terms of the dressed τ_r^α operators can only involve even combinations of τ: $\sigma_r^z = \alpha_1 \tau_r^x + \beta_2 \tau_r^z \tau_{r+1}^z + \ldots$. Hence the *connected* correlation functions of σ_{rs}^x should decay exponentially with $|r-s|$, and this operator is not expected to have robust period $2T$ oscillations. On the other hand $\sigma_r^{y,z}$ will generically exhibit both long range connected correlations as well as period $2T$ oscillations.

Similarly we can pick perturbations for which $U_{f,\lambda}$ respects antiunitary symmetries like $\mathcal{T} = PK$, i.e., for which $\mathcal{T} U_{f,\lambda} \mathcal{T} = U_{f,\lambda}^{-1}$. As an example, the model studied in Ref. 52 resembles Eq. (6) with $h^y = 0$, so has the effect of perturbing Eq. (1) by $\lambda V \sim h_r^z \sigma_r^z + h_r^x \sigma_r^x$. With this choice of V it is straightforward to verify that the corresponding $U_{f,\lambda}$ respects $\mathcal{T}$ symmetry

$$\begin{aligned}
&\mathcal{T} U_{f,\lambda} \mathcal{T}^{-1} = (PK) U_{f\lambda} (PK)^\dagger \\
&= (PK) P \exp[-i \sum_{r=1}^{L-1} J_r \sigma_r^z \sigma_{r+1}^z + h_r^z \sigma_r^z + h_r^x \sigma_r^x] (PK)^\dagger \\
&= \exp[i \sum_{r=1}^{L-1} J_r \sigma_r^z \sigma_{r+1}^z + h_r^z \sigma_r^z + h_r^x \sigma_r^x] P \\
&= U_{f,\lambda}^{-1} . \qquad (7)
\end{aligned}$$

In this case, we can pick the change of basis matrix $\mathcal{V}_\lambda$ to commute with $\mathcal{T}$ (see App. B) which implies that τ^x, τ^y, τ^z are even, even, and odd respectively under $\mathcal{T}$. In turn, the operator expansions of $\sigma^{x,y}$ can only contain terms with even numbers of τ^zs in their expansions. Hence neither should exhibit protected π/T oscillations, nor should they have long range connected correlations as demonstrated in Fig. 3. This accounts for the absence of π/T oscillations for $\sigma_r^x(nT), \sigma_r^y(nT)$ in the data presented in Ref. 52.

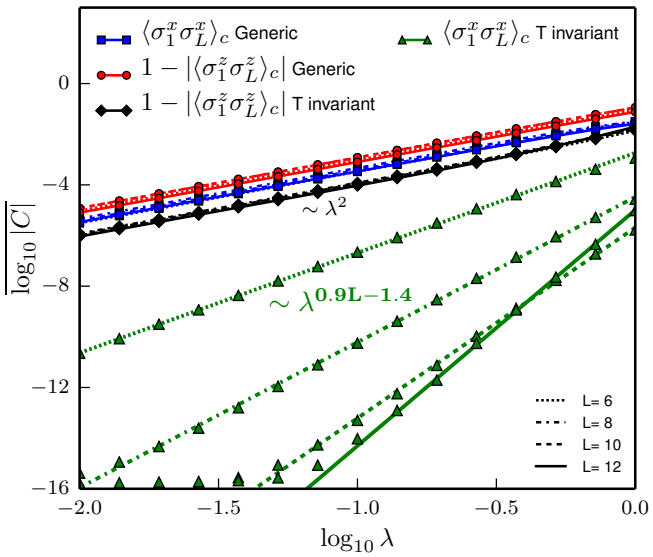

FIG. 3. (Color online): Disorder and eigenstate averaged end-to-end connected correlation functions for $\sigma^{x,z}$ in the "generic" model (6) with no $P, \mathcal{T}$ symmetries (blue squares, red circles) and a model[52] with $\mathcal{T}$ symmetry obtained by setting $h^y = 0$ in (6) (black diamonds, green triangles). As discussed in the text, the generic model shows long-range order for both operators which is signaled here by correlations scaling as λ^2 independent of system size. On the other hand, in the model with $\mathcal{T}$ symmetry, only σ^z shows long-range order while the σ^x correlator scales as $\lambda^{f(L)}$ where $f(L) \sim 0.9L - 1.4$ (fits shown) and thus vanishes in the $L \to \infty$ limit. This is to be expected from symmetry constraints. The σ^y correlators (not shown here) also display long-range order in the generic model but not in the $\mathcal{T}$ symmetric model.

III. THE π SPIN GLASS: SPATIOTEMPORAL LONG RANGE ORDER

We have already discussed above that at general points in the absolutely stable πSG phase the emergent order parameter operators, τ_i^z, change sign every period. Prima facie, this implies the spatiotemporal order sketched in Fig. 1b: spin glass order in space and antiferromagnetic order in time.

The aim of this section is to more sharply characterize this spatiotemporal order. As the πSG is a localized phase, unlike in the equilibrium context, there is not an obviously correct set of correlations one should examine to detect said order. We propose to examine the time dependent one and two point correlation functions of local operators in two families of states. The first are the Floquet eigenstates which are the basis of the eigenstate order paradigm of phase structure in Floquet systems. The second are the late time states reached by time evolving from general initial states; these are particularly relevant to experiments where the preparation of Floquet eigenstates is not feasible.

A. Eigenstate correlations and response

We start by considering Floquet eigenstates for the πSG. All single time operators $\langle O(t)\rangle$ in these are strictly periodic with period T—this is the analog of the time independence of single time operators in Hamiltonian eigenstates and hence the temporal component of the order is invisible to such operators. The invisibility of temporal order in the $\langle O(t)\rangle$ is analogous to the invisibility of Ising symmetry breaking in one point expectations of spatially local Ising-odd operators in globally Ising symmetric states. From this perspective[49] it follows that to detect temporal order we must either (a) examine a two time function of some operator or (b) explicitly add an infinitesimal field that selects the desired temporal order (much as we would examine long-range order in two-point functions of Ising-odd variables and/or add an infinitesimal Ising symmetry breaking term to detect spontaneously broken Ising symmetry).

We begin with (a) and examine time-dependent correlators

$$\begin{aligned} C_\alpha(nT;r,s) &\equiv \langle\alpha|O_r(nT)O_s|\alpha\rangle \\ &= \sum_\beta e^{-inT(\epsilon_\alpha-\epsilon_\beta)}\langle\alpha|O_r|\beta\rangle\langle\beta|O_s|\alpha\rangle \end{aligned} \quad (8)$$

of operators $O_{r/s}$ localized near sites r,s in the Floquet eigenstates $|\alpha\rangle = |\{d\},\pm\rangle_\lambda$ (see Sec. II for notation). The operator expansion of $O_{r/s}$ in the τ^α basis will generically contain terms that are odd combinations of τ^zs. In the πSG phase, these have matrix elements between $|\alpha\rangle$ and its parity flipped partner and thus $C_\alpha(nT)$ generically has a frequency π/T component. In addition, the off-diagonal terms in the operator expansion involving $\tau^{\{x,y\}}$ will make local domain wall excitations near sites r/s. Now a crucial point: if r,s are held a fixed distance apart in the infinite volume limit, then $C_\alpha(nT)$ breaks TTI for *any* MBL-Floquet system. The reason is that one can crudely view a Floquet MBL system as a set of weakly interacting localized modes (the effective domain wall operators in this case) each with their own local spectra. As in the simplest case of 2-level systems whose physics is that of Rabi oscillations, these local subsystems (which are excited by $\tau^{x/y}$) exhibit response at frequencies incommensurate with the driving frequency. The presence of these incommensurate frequencies means $C_\alpha(nT)$ in *all MBL-Floquet systems always look glassy*, although for the πSG there is generically also a quantized response at π/T.

This short distance temporal glassiness however goes away when we examine long distances in space by placing the operators arbitrarily far apart in an infinite system, *i.e.*, by taking $\lim_{L\to\infty}$ before examining the limit $|r-s|\to\infty$. Since the operator expansions of $O_{r/s}$ are exponentially localized near sites r/s, the off-diagonal terms in the expansion of O_r which create domain-wall excitations near site r *cannot be annihilated* by the action of O_s in the limit $|r-s|\to\infty$ under the assumption of locality.

Thus, the only terms that contribute to $C_\alpha(nT;r,s)$ in this limit are diagonal in τ^zs. Terms odd in τ^z give a response at π/T while the even terms give a response at frequency 0. Thus we can write

$$C_\alpha(nT;r,s) \sim c_0(r;\alpha)c_0(s;\alpha) + c_1(r;\alpha)c_1(s;\alpha)(-1)^n$$

where the second piece reflects the spatiotemporal order of the odd τ^z terms, as well as the connected part of the correlation function. The dependence of the coefficients on r, s and α has been made explicit to emphasize the glassy nature of the order in space. This establishes a connection between the long range spatial order in the eigenstates and the period $2T$ temporal order.

The above analysis can be complemented by taking the approach (b) and adding to $H(t)$ a "staggered field" in time of the form $\epsilon\sum_n(-1)^nV\delta(t-nT)$, where V is odd and diagonal in τ^z. Now consider time-dependent expectation values of generic local operators O_r (which have a projection on odd τ^z terms) in the Floquet eigenstates $|\alpha\rangle_\epsilon$ for the new period $2T$ unitary which can be reshuffled to the form $U_{f,\epsilon}(2T) = e^{-i2\epsilon V}U_{f,0}^2$. This problem looks like the classic Ising symmetry breaking problem. At $\epsilon=0$, $U_{f,\epsilon}(2T) = U_{f,0}^2$ has two degenerate states in the infinite volume limit. If V breaks the symmetry between two members of the doublet then

$$\lim_{\epsilon\to 0}\lim_{L\to\infty}{}_\epsilon\langle\alpha|O_r(nT)|\alpha\rangle_\epsilon = b_0(r;\alpha) + b_1(r;\alpha)(-1)^n$$

since the perturbed period $2T$ eigenstates $|\alpha\rangle_\epsilon$ just look like product states of τ^z in this limit and are thus superpositions of the opposite parity eigenstates of $U_{f\lambda}$. On the other hand, the opposite order of limits gives $\lim_{L\to\infty}\lim_{\epsilon\to 0}{}_\epsilon\langle\alpha|O_r(nT)|\alpha\rangle_\epsilon = b_0(r;\alpha)$. We emphasize that the measures discussed here are *eigenstate* measures. If averaged over all eigenstates the signatures vanish.

B. Quenches from general initial states

We now turn to the question of evolution from more general initial states rather than eigenstates. This is experimentally important, and more particularly so because the Floquet eigenstates for the πSG are macroscopic superpositions and thus hard to prepare. For concreteness, consider starting from a short-range correlated state like a product state of the physical spins. In the following we will adapt the analysis of dephasing in quenches in MBL systems[14,19]. We will assume that the starting state exhibits a non-zero expectation value for the order parameter, i.e. $\langle\psi_0|\tau_i^z|\psi_0\rangle \neq 0$; if it does not the temporal features will be entirely absent. For simplicity we will only discuss one point functions as they are already non-trivial in this setting and the generalization is straightforward.

In a finite size system, τ^z only anticommutes with the Floquet unitary up to exponentially small in L correc-

tions (4), which in turn introduce corrections to the equation of motion: $\tau^z(nT) = (-1)^n\tau^z(0) + O(e^{-L})$. This leads to exponentially small shifts in the spectral pairing at π/T which varies randomly between pairs of eigenstates. Ignoring these shifts for times $1 \ll t \ll O(e^{+L})$, one can readily show that for any finite system the one point functions will generically show glassy behavior with incommensurate Fourier peaks along with an additional peak at π/T; see Fig. 4 for an illustration. More precisely, the logarithmic in time dephasing of correlations in MBL systems[11,14] can be used to show that the correlators will show aperiodic behavior stemming from these additional Fourier peaks with a power law envelope $\sim t^{-b}$, where $b > 0$ depends on the localization length[14]. *Thus, finite systems at large but not exponentially large times look like time-glasses with an additional quantized response at* $\omega = \pi/T$. However, if one waits a time $t \sim e^L$ that is long enough to (i) resolve the exponentially small many-body level spacings and (ii) to resolve the shifts in the spectral pairing away from π/T, both the peak at π/T and the extra incommensurate peaks almost entirely decay away due to usual dephasing mechanisms leaving behind aperiodic oscillations with a magnitude of $O(e^{-L})$. It is worth reminding the reader that the precise details of the time dependence will reflect the choice of the starting state and disorder realization.

We can formalize the above in two non-commuting limits: (a) $\lim_{t\to\infty}\lim_{L\to\infty}$ and (b) $\lim_{L\to\infty}\lim_{t\to\infty}$. While (a) characterizes the "intrinsic" quench dynamics of this phase, experiments will only have access to limit (b). In (b) the late time aperiodic oscillations with envelope $O(e^{-L})$ discussed above also go away, and the one-point functions are constants. In (a) we never reach times of $O(e^L)$ and instead observe persistent oscillations with period $2T$ out to $t \to \infty$ with all additional incommensurate oscillations decaying away as a power of time.

Thus, the intrinsic dynamical response of this phase is characterized by a single quantized Fourier peak at $\omega = \pi/T$ which goes along with formally exact spectral pairing at π/T and LRO in τ^z. In this limit, the late time state exhibits a precisely doubled period for every single realization of disorder and combined space-time measurements would lead precisely to the kind of snapshot sketched in Fig. 1b. More concretely, state-of-the-art experiments in ultracold atoms[59–62] have convincingly demonstrated that a fingerprint of the initial state persists to asymptotically late times in the MBL phase. In a generalized experimental setup probing the πSG phase in the MBL Floquet problem[63], the persistence of the starting fingerprint would measure localization and spatial spin glass order, while oscillations in time would measure the temporal response at π/T. We also note that a recent experiment demonstrated signatures of MBL in two dimensions[62] and, more generally, we expect our considerations to apply in all dimensions where MBL exists[64].

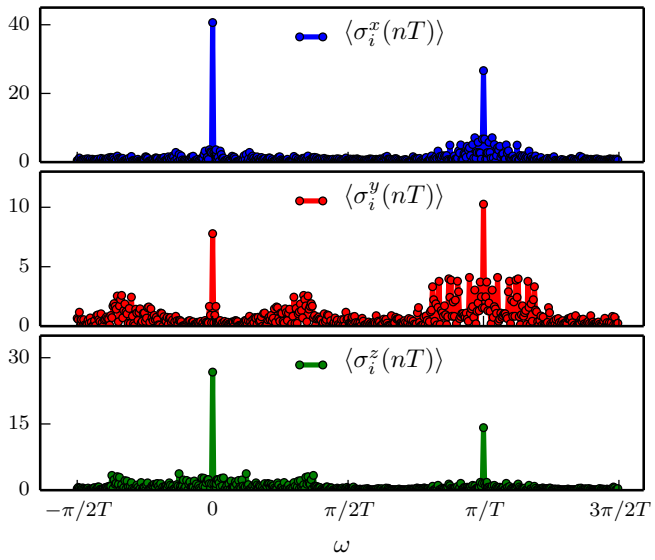

FIG. 4. (Color online): Fourier transform over time window $\Delta t = 500T$ of one point time-dependent expectation values $\langle\psi_0|\sigma^{\{x,y,z\}}(nT)|\psi_0\rangle$ in the "generically" perturbed model (6). The initial state $|\psi_0\rangle$ is a product state with physical spins σ^α randomly pointing on the Bloch sphere and uncorrelated from site to site. As discussed in the text, the response looks "glassy" with several incommensurate Fourier peaks in the addition to the peak at π/T, although we expect these to decay away in the $L \to \infty, T \to \infty$ limit. Data is shown for a single disorder realization in a system of length $L = 10$.

C. Comments

In the above discussion we have considered two settings, that of Floquet eigenstates and of late time states stemming from quenches. It is useful to contrast our findings with their analogs for general MBL phases (Floquet or undriven), and for ETH obeying phases (focussing on the undriven case, as the Floquet version has trivial infinite temperature correlations). We find that unequal time correlations in eigenstates generically break TTI in all MBL phases, which thus generically look glassy. By contrast similar correlations in ETH systems do *not* generically break TTI. In the πSG we find that eigenstate correlations specifically designed to pick out the order parameter dynamics are "antiferromagnetic" in the time domain and thus break TTI while they are "ferromagnetic" for the 0SG and thus do not. Turning now to the late time states coming from quenches, in MBL phases these are initial state dependent while in ETH phases these are not. Hence if we look for TTI breaking via these late time states we do not observer it in all ETH phases as well as MBL phases except the πSG (and its relatives which we discuss in the next section). We remind the reader though that in the πSG we need to quench from states that exhibit a macroscopic expectation value for the order parameter. All in all we conclude that the πSG exhibits a distinct and novel pattern of spa-

tiotemporal order that is new to quantum systems.

IV. GENERALIZATIONS

Here we list a number of generalizations of the πSG phase. Ref. 26 presented a family of models with an explicit global symmetry group G which exhibit eigenstate long-range order, protected spectral pairing and temporal crystallinity. First we note that, much like the πSG, many of these models are absolutely stable to local perturbations, even those that break the global symmetry G. We then explain why bosonic SPT Floquet drives[43–45,65] are not stable to the inclusion of symmetry breaking perturbations, although in the presence of the protecting symmetry they exhibit time crystallinity at their edges.

A. $\mathbb{Z}_n$ and non-abelian models:

Consider first models with global $\mathbb{Z}_n$ symmetry[26,66]. There are n possible phases with completely spontaneously broken symmetry[26], labelled by $k = 0, 1, \ldots n-1$. The eigenvectors of the corresponding unitary are the $\mathbb{Z}_n$ equivalents of cat states i.e., macroscopic superpositions of n spin configurations. In cases with $k \neq 0$, and in the presence of $\mathbb{Z}_n$ symmetry, the spectrum consists of multiplets of n cat states appearing in n/g distinct groups each with degeneracy $g \equiv \gcd(n, k)$. The n/g distinct groups are split by quasienergy multiples of $2\pi g/nT$. As for the πSG, some of these statements survive even when $\mathbb{Z}_n$ symmetry is explicitly broken. In particular, while the g fold degeneracy for each group of cat states can readily be broken, it remains the case that each eigenstate is paired in a multiplet of n/g related cat states, separated by quasienergy $2\pi g/nT$. A similar statement holds for the non-abelian models in Ref. 26. These more general drives have an explicit unitary non-abelian symmetry G, and are classified by an element of the center of the group $z \in Z(G)$. Let q denote the order of z. The spectrum consists of q groups of G/q degenerate cat-like states, and the q groups are separated by quasi-energies which are multiples of $2\pi/qT$. The $|G|/q$ degeneracy at each quasienergy can once again be lifted using symmetry breaking perturbations, but each eigenstate is still paired with q cat state partners, split by quasienergy multiples of $2\pi/qT$.

B. Stability of SPTs and boundary time crystallinity

While the πSG phase is absolutely stable, similar Floquet generalizations of bosonic SPT phases[26] are not. Before showing this, let us first note that some Floquet SPTs spontaneously break TTI at their boundaries. This boundary TTI breaking is not tied to bulk LRO and the

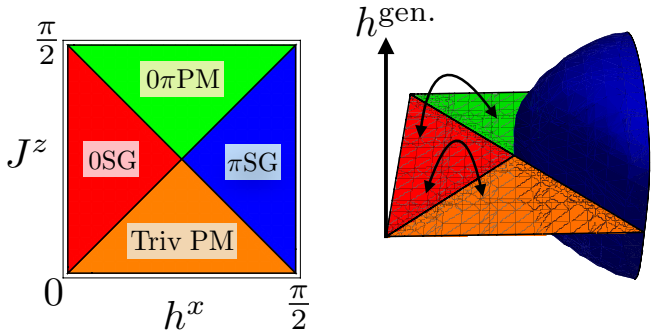

FIG. 5. (Color online): (left) Phase diagram for the MBL Ising symmetric drives presented in Refs. 25 and 26 showing the 0SG and πSG phases which are long-range ordered and spontaneously break Ising symmetry, as well as the 0πPM and trivial paramagnetic phases which have no LRO. The 0πPM is an SPT with non-trivial edge modes and can spontaneously break time translation symmetry on its edges. (right): On perturbing with generic Ising symmetry breaking fields h^{gen}, only the πSG is absolutely stable and continues into a phase with LRO and an emergent symmetry. The other three phases can be continuously connected to the trivial MBL paramagnet in the presence of h^{gen}.

phases are correspondingly *unstable* to symmetry breaking perturbations. We illustrate this with the simple example of an Ising Floquet SPT, the so-called 0πPM[25,43]. In fact, the 0πPM and πSG are neighbors on a common Floquet phase diagram[25,26,43] Fig. 5(left) which also contains the 0SG discussed earlier and a trivial MBL paramagnet. A simple Floquet unitary for 0πPM on a system with boundary is[43]

$$U_f = \sigma_1^z \sigma_N^z \exp[-i \sum_{r=2}^{N-1} h_r \sigma_r^x], \tag{9}$$

where the fields h_r are randomly distributed. This model has trivial bulk paramagnetic eigenstate order, but it also has non-trivial Ising odd "pumped charges" $\sigma_{1/L}^z$, using the parlance of Ref. 43. As a consequence, the eigenspectrum exhibits "spectral quadrupling". Labeling the simultaneous eigenvalues of U_f, P by $(u, p = \pm 1)$, it can be shown that states always appear in multiplets of the form $(u, 1), (u, -1), (-u, 1), (-u, -1)$ *i.e.*, there are two groups of degenerate states split by exactly π/T quasienergy—hence the name 0πPM. The π/T quasienergy splitting in πSG was associated with the breaking of TTI, so it is natural to also expect TTI breaking for the 0πPM. Indeed, for the special model Eq. (9), the σ^x edge operators have stroboscopic equations of motion $\sigma_{1,N}^x(nT) = (-1)^n \sigma_{1,N}^x(0)$, with period $2T$. At generic points in the 0πPM phase obtained by perturbing (9) with Ising symmetric perturbations, dressed versions of these edge Pauli operators (and generic edge operators with non-zero projections on the dressed Pauli edge operators) will exhibit period $2T$ oscillations persistent for exponentially long time scales in system size (in the same spirit as Ref. 23).

Indeed, using Ising duality[25], statements about the dynamics of Ising even edge operators in the 0πPM paramagnet directly translate into statements about local bulk operator dynamics in the (Ising symmetric) πSG in Sec. III. We emphasize, however, that for 0πPM generic local bulk operators will not show period doubling in the limit $L \to \infty$.

Despite the non-trivial dynamics in the 0πPM, the spectral pairing properties of this phase (and the more general bosonic Floquet SPT phases discussed in Ref. 43) are unstable to the inclusion of small, generic symmetry breaking perturbations at the boundary. To see how this works in more generality, note that Floquet MBL unitaries can be re-expressed in a certain canonical form[43]

$$U_{f0} = v_L v_R e^{-if}\,, \tag{10}$$

where f is a local MBL Hamiltonian functional of the l-bits in the bulk, and $v_{L,R}$ are unitaries localized at the left/right edges of the system respectively which commute with the bulk l-bits. Note that the model Eq. (9) is a special realization of this more general canonical form. The SPT order of U_{f0} is captured by two pieces of data: (i) The bulk SPT order, which is determined by the classification of f as an undriven Hamiltonian, and (ii) the "pumped charge", characterized by the commutation relations between the $v_{L,R}$ and the global symmetry generators[43]. Note that Eq. (10) can readily be detuned – whilst maintaining locality and unitarity – to a form with trivial pumped charge, e^{-if}, through an interpolating family of unitaries $U_{f\lambda} = e^{-i\lambda \log v_R} e^{-i\lambda \log v_L} U_{f0}$ with λ being tuned from 0 to 1. Note further that if $v_{L,R}$ have non-trivial commutation relations with the global symmetry, this interpolating family of unitaries breaks the global symmetry. It may still occur that f, an MBL Hamiltonian, has a non-trivial SPT classification and therefore e^{-if} has spectral pairing and edge states. However, this SPT order is readily destroyed by perturbing f non-symmetrically as one would perturb an undriven SPT so as to gap out its edge states. This instability of the boundary-TTI breaking SPT phases reiterates our central message that the absolute stability of a TTI breaking phase is intrinsically tied to the coexistence of bulk spatial LRO.

The instability of 0πPM SPT combined with our prior statements on the instability of pairing in the 0SG leads to the picture depicted in Fig. 5(right)—in the presence of generic Ising symmetry breaking perturbations, the four Ising symmetric MBL-Floquet phases are reduced to two: the absolutely stable continuation of the πSG, and a trivial PM. The 0SG and the 0πPM can be continuously connected to the trivial PM without going through a phase transition in the presence of Ising symmetry breaking terms.

We end this section by briefly commenting on the stability of fermionic SPTs. Interacting SPTs protected by fermion parity are more robust. Let us focus on class D[25,28,43] for concreteness. While it is true that edge modes are unstable to fermion parity breaking perturbations, fermion parity is never broken for physical/local Hamiltonians $H(t)$ – hence, in the detuning argument above, $U_{f\lambda}$ is not a truly local unitary for intermediate values of λ when $v_{L,R}$ are fermion parity odd (we say the pumped charge is fermion parity odd[43]). However, as with all of the examples discussed here, the Floquet edge modes can be removed by breaking time translation symmetry.

V. CONCLUDING REMARKS

We have shown the existence of a family of phases of Floquet systems which are absolutely stable—a generic interior point in such a phase is stable to *all* weak local perturbations of its governing unitary. These phases are characterized by emergent, Hamiltonian dependent, abelian global symmetries and spatiotemporal long range order based on these. Submanifolds of these phases exhibit Hamiltonian independent symmetries which can be unitary or anti-unitary. At generic points in these phases, late time states evolved from randomly picked short ranged entangled states exhibit long range order in space and sharp oscillations of the emergent order parameter which can be used to identify the phases.

These Floquet phases join two previously established paradigms for such absolute stability—those of topological order and that of MBL for time independent Hamiltonians—and a comparison between these three is in order. Topological order, exemplified by the Z_2 order of the toric code and its weak local perturbations, is characterized by the absence of symmetry breaking and the presence of emergent gauge fields. Such phases are in a different language quantum liquids with long range (ground state) entanglement[1] which features account intuitively for their absolute stability.

MBL is characterized by a complete set of emergent, Hamiltonian dependent, local integrals of the motion (l-bits) and in its minimal form involves eigenstates that exhibit only short ranged entanglement. Its absolute stability can be attributed to the localization being unrelated to any spatial ordering—it is primarily a dynamical phenomenon. By contrast, broken symmetries are *not* absolutely stable—symmetry induced degeneracies are lifted when symmetries are broken.

It is not hard to believe that one can mix topological order and MBL and still end up with an absolutely stable phase and this was discussed as an example of eigenstate order in Ref. 19. By contrast it is also natural to conclude that MBL and symmetry breaking to not lead to absolute stability and this is also trivially the case. What is therefore striking is that a third ingredient, Floquet periodicity, allows broken symmetries and MBL to combine to yield absolutely stable phases. The resulting phases also exhibit long range entanglement in the form of the cat eigenstates and thus are stabilized by a relative of the mechanism which operates in the case of topological order.

Finally we note that the absolute stability of symmetry broken phases in this paper can be put on a similar footing to the well known absolute stability of topological phases[2]. Recent work[67,68] characterizes pure abelian gauge theories as spontaneously breaking 1-form global symmetries in their deconfined phases. In the presence of matter, the generators for these higher form symmetries are emergent and thus Hamiltonian dependent. For example, in the perturbed 2D toric code, the 1-form symmetries are generated by dressed line operators[4]. More generally, a large class of well known and undriven absolutely stable topologically ordered phases are characterized by spontaneously broken emergent 1-form global symmetries, while the Floquet drives in this work are characterized by emergent global (0-form) symmetries. In a related note, one can consider Floquet unitaries constructed from topologically ordered Hamiltonians, such as the toric code, which toggle states between different topological sectors. Such drives exhibit spatial topological order, do not break any global symmetries, but do break TTI because the Floquet unitary described does not commute with operators which measure the topological sector. Just as the cat states are split by π/T quasi-energy in the πSG, different topological sectors are split by π/T in this topological example. It is somewhat a matter of taste whether these should be identified as Floquet time crystals.

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Appendix A: $\tau^z_{r,\lambda}$ either commutes or anti-commutes with $U_{f\lambda}$

To prove this assertion, we will use only the locality of the $\mathcal{V}_\lambda, U_{f\lambda}$. First note that we can express a product of any two τ^z_λ operators as a product of l-bits $\tau^z_{r,\lambda}\tau^z_{s,\lambda} = \prod_r^{s-1} D^\lambda_r$. This compound operator commutes with with $U_{f,\lambda}$ because the D^λ_r do , i.e.,

$$U_{f,\lambda}\tau^z_{r,\lambda}\tau^z_{s,\lambda}U^\dagger_{f,\lambda} = \tau^z_{r,\lambda}\tau^z_{s,\lambda} \tag{A1}$$

However note that the unitaries defined as

$$\theta_r \equiv \tau^z_{r,\lambda}U_{f,\lambda}\tau^z_{r,\lambda}U^\dagger_{f,\lambda} \tag{A2}$$

$$\theta_s \equiv \tau^z_{s,\lambda}U_{f,\lambda}\tau^z_{s,\lambda}U^\dagger_{f,\lambda} \tag{A3}$$

are local to r, s respectively. This follows from two observations. First $\tau^z_{r,\lambda}$ is local to r because $\mathcal{V}_\lambda$ is assumed low depth. Second, $U_{f,\lambda}\tau^z_{r,\lambda}U^\dagger_{f,\lambda}$ is local to r because $\tau^z_{r,\lambda}$ is, and $U_{f,\lambda}$ is low depth (being the finite time ordered exponent of a bounded local Hamiltonian). Plugging Eq. (A2) and Eq. (A3) into Eq. (A1) gives

$$U_{f,\lambda}\tau^z_{r,\lambda}\tau^z_{s,\lambda}U^\dagger_{f,\lambda} = \tau^z_{r,\lambda}\theta_r\theta_s^{-1}\tau^z_{s,\lambda} = \tau^z_{r,\lambda}\tau^z_{s,\lambda} \tag{A4}$$

implying that

$$\theta_{r\lambda} = \theta_{s\lambda} \tag{A5}$$

despite the fact that $\theta_{r\lambda}, \theta_{s\lambda}$ are exponentially localized to potentially distant sites r, s – in particular we could say choose $|r-s| = L/2$ to be of order the system size. The implication is then that, up to exponentially small corrections in system size, $\theta_{r,s\lambda}$ are pure phases. The corrections take the form $Ce^{-L/\xi}$, where C, ξ do not depend on the system size, and only depends on the details of $\mathcal{V}_\lambda, U_{f,\lambda}$ (such as their depth, which is assumed to be finite). The fact $(\tau^z_{r,\lambda})^2 = 1$ and $\theta_{r\lambda}$ approximately a pure phase implies $\theta^2_{r\lambda} = 1 + \epsilon$ where ϵ is a correction of the form $ce^{-L/\xi}$ and $c = O(1)$. This shows that

$$\theta_{r\lambda} = \pm 1\,. \tag{A6}$$

to the same degree of a approximation. Supposing we know that $\theta_{r0} = -1$ exactly – as is the case for the fixed point πSG model Eq. (1). If $\mathcal{V}_\lambda, U_{f\lambda}$ is a continuous family of unitaries it follows by continuity that $\theta_{r\lambda} = -1$ in the large system limit, for all applicable λ.

Appendix B: Symmetries and the $\mathcal{V}_\lambda$ unitaries

Here we argue that diagonalizing unitaries $\mathcal{V}_\lambda$ for families of unitaries $U_{f\lambda}$ respecting a fixed symmetry (e.g., Ising parity or time reversal) and exhibiting absolutely stable long ranged order, can themselves be chosen to commute with the fixed symmetry. For concreteness, focus on a system with an anti-unitary symmetry $\mathcal{T}$ with $\mathcal{T}^2 = 1$ – the unitary symmetry case goes through similarly. Thus we consider a family of unitaries $U_{f\lambda}$ obeying $\mathcal{T}U_{f\lambda}\mathcal{T}U_{f\lambda} = 1$, with U_{f0} given by Eq. (1). Note first that the spectrum of U_{f0} generically has no degeneracies. Assuming the same is true of $U_{f\lambda}$ for now, consider the action of $\mathcal{T}$ on eigenstates. As $\mathcal{T}U_{f\lambda}\mathcal{T}U_{f\lambda} = 1$, it follows that $U_{f\lambda}\mathcal{T} \mid \{d\}, p\rangle_\lambda = u_{d,p,\lambda}\mathcal{T} \mid \{d\}, p\rangle_\lambda$. Hence $\mathcal{T}$ preserves eigenstates of $U_{f\lambda}$. As the eigenstates are non-degenerate it follows that

$$\mathcal{T} \mid \{d\}, p\rangle_\lambda = e^{i\theta_{d,p}} \mid \{d\}, p\rangle_\lambda \tag{B1}$$

for some state dependent phase $e^{i\theta_{d,p}}$. Eq. (B1) immediately implies $d_{\lambda,r} = \mathcal{T}d_{\lambda,r}\mathcal{T}$ and $P^\lambda = \mathcal{T}P^\lambda\mathcal{T}$ which we can rewrite as

$$\mathcal{V}_\lambda d_r \mathcal{V}_\lambda^{-1} = \mathcal{V}_{\lambda,\mathcal{T}} d_r \mathcal{V}_{\lambda,\mathcal{T}}^{-1}$$
$$\mathcal{V}_\lambda P \mathcal{V}_\lambda^{-1} = \mathcal{V}_{\lambda,\mathcal{T}} P \mathcal{V}_{\lambda,\mathcal{T}}^{-1}$$

where $\mathcal{V}_{\lambda,\mathcal{T}} \equiv \mathcal{T}\mathcal{V}_\lambda\mathcal{T}^{-1}$, and d_r, P are the undressed domain wall and parity operators. The upshot is that the unitary

$$\mathcal{Q}_\lambda \equiv \mathcal{V}_\lambda^{-1}\mathcal{V}_{\lambda,\mathcal{T}} \tag{B2}$$

commutes with the commuting set of operators $\{d_r\}, P$. As these operators uniquely label a complete basis, Q_λ is completely diagonal in $\{d_r\}, P$. In other words it can be expressed as

$$\mathcal{Q}_\lambda = e^{-iq_\lambda(d_r,P)} \tag{B3}$$

for some real functional q_λ of the labels. In fact, using locality arguments similar to those in Sec. A (and in the appendix to Ref. 43) we find

$$\mathcal{Q}_\lambda = P^a e^{-is_\lambda(\{d\})} \tag{B4}$$

up to exponentially small corrections in system size, where $a = 0, 1$, and s is a local functional of domain walls. We can use continuity of $\mathcal{V}_\lambda$ again to argue moreover that $a = 0$. Therefore we have shown that $\mathcal{V}_{\lambda,\mathcal{T}} = \mathcal{V}_\lambda\mathcal{Q}_\lambda$. We now use this result to construct a new change of basis matrix which is invariant under time reversal. We define a new change of basis unitary $\mathcal{W}_\lambda \equiv \mathcal{V}_\lambda e^{-is_\lambda(\{d\})/2}$. $\mathcal{W}_\lambda$ indeed achieves the desired local change of basis, but is also time reversal invariant. We henceforth redefine $\mid \{d\}, p\rangle_\lambda \equiv \mathcal{W}_\lambda \mid \{d\}, p\rangle$. The operators $d_{\lambda,r}, P^\lambda$ are unaffected by this change in convention.