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Spurious Long-range Entanglement and Replica Correlation Length

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Topological entanglement entropy has been regarded as a smoking-gun signature of topological order in two dimensions, capturing the total quantum dimension of the topological particle content. An extrapolation method on cylinders has been used frequently to measure the topological entanglement entropy. Here, we show that a class of short-range entangled 2D states, when put on an infinite cylinder of circumference L, exhibits the entanglement Rényi entropy of any integer index $\alpha \geq 2$ that obeys $S_\alpha = aL - \gamma$ where $a, \gamma > 0$. Under the extrapolation method, the subleading term γ would be identified as the topological entanglement entropy, which is spurious. A nonzero γ is always present if the 2D state reduces to a certain symmetry-protected topological 1D state, upon disentangling spins that are far from the entanglement cut. The internal symmetry that stabilizes $\gamma > 0$ is not necessarily a symmetry of the 2D state, but should be present after the disentangling reduction. If the symmetry is absent, γ decays exponentially in L with a characteristic length, termed as a replica correlation length, which can be *arbitrarily large* compared to the two-point correlation length of the 2D state. We propose a simple numerical procedure to measure the replica correlation length through replica correlation functions. We also calculate the replica correlation functions for representative wave functions of abelian discrete gauge theories and the double semion theory in 2D, to show that they decay abruptly to zero. This supports a conjecture that the replica correlation length being small implies that the subleading term from the extrapolation method determines the total quantum dimension.

I. INTRODUCTION

Topologically ordered states are nontrivial gapped states, which are beyond Landau's symmetry breaking paradigm.[1] Prototypical examples include gapped spin liquid states and fractional quantum Hall states. These states exhibit robust ground state degeneracy depending on the topology of the system, and fractionalized excitations.[2,3] Even more intriguing is its potential of being a fault-tolerant quantum information processing platform.[4,5] Recently, long-range entanglement has been appreciated to understand the topological order, as topologically nontrivial states cannot be connected to a product state by local unitary transformations.[6–9]

Detecting topological order, however, has still been a challenge. The ground state degeneracy and the fractional quantum numbers of the excitations are difficult to measure even in the numerical calculations, let alone experimental situations. Instead, the so-called topological entanglement entropy (TEE)[10–15] is being recognized as an important quantity especially in numerics, for the purpose of distinguishing topologically ordered states from topologically trivial states.

It is believed that the bipartite entanglement entropy of the ground state of a gapped system in two spatial dimensions obeys an "area" law with a constant subleading correction. Specifically, the entanglement entropy for a disk of circumference L is given by

$$S = \alpha L - \gamma + \cdots \tag{1}$$

where α is a model-specific non-universal coefficient, γ is a sub-leading correction, and the ellipses represent terms that vanish in the large L limit. This constant correction γ is the universal TEE of the state. It is shown that $\gamma = \log \mathcal{D}$, where $\mathcal{D}$ is the so-called total quantum dimension of the system, determined by the anyon content of the topological order that the state represents. Roughly speaking, $\mathcal{D}$ counts the types of fractionalized particles. Since $\mathcal{D} > 1$ implies the state supports fractionalized excitations, γ is regarded as a smoking-gun signature of 2D topological order.

Remark that the definition of γ through Eq. (1) is inherently ambiguous; it depends on fine details of a regularization scheme for the calculation of S. On a lattice, the circumference L and therefore the subleading term γ vary according to how one counts the number of sites along the boundary of the disk; for example, the circumference of a rectangle that encloses $L \times L$ sites may be counted as $(L+2)^2 - L^2 = 4L+4$ or $L^2 - (L-2)^2 = 4L-4$. To resolve this ambiguity by eliminating the boundary term, Ref. 12 and 13 take a linear combination of entanglement entropies for various regions. Concretely, Ref. 12 proposes the following combination.

$$\gamma = S_{AB} + S_{BC} + S_{CA} - S_A - S_B - S_C - S_{ABC} \tag{2}$$

where the subscripts refer to the regions specified in Fig. 1. In fact, it is this linear combination that enables one to argue that γ is a robust quantity under small changes in the Hamiltonian and the region sizes; the combination contains an equal number of terms of opposite signs for each subregion so that a small change in any subregion may be canceled overall. We will refer to this proposal as the *Kitaev-Preskill prescription* hereafter. Note that it is not too important at long distances whether one uses von Neumann entropy or Rényi entropy.[14,15]

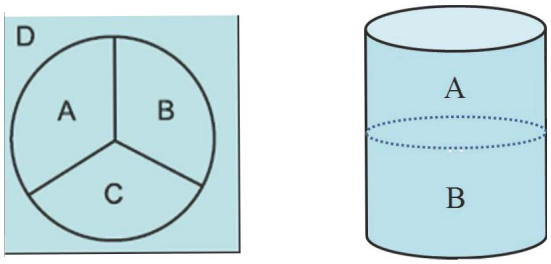


FIG. 1. Two methods of extracting TEE. Left: Kitaev-Preskill prescription divides the system into four parts and extract TEE by using (2). Right: DMRG calculations put the system on an infinite cylinder and divide the system into two parts, then calculate the entanglement entropy $S(L)$ between the two parts for different circumferences L of the cylinder. Fitting the results into (1), TEE is identified as $-S(L=0)$.

Despite of its conceptual importance, the Kitaev-Preskill prescription of extracting TEE is not of great practical use because it requires each partition be much larger than the correlation length of the system.[16] This is very challenging especially in density-matrix-renormalization-group (DMRG) methods. Alternatively, by exploiting the fact that the DMRG algorithm systematically produces minimally entangled states, it is proposed that one can simply put the system on infinite cylinders with various circumferences L (see Fig. 1), and extrapolate the data using (1) to read off γ.[17] The mentioned ambiguity of defining the circumference of a disk on lattices, does not apply here since the circumference of a cylinder is simply well-defined. We will refer to this method as the *cylinder extrapolation method*. An important advantage of this method is that one can regard the region size as large as L, rather than some small fraction of L, and hence can expect finite size effects to be very small. This method might be useful in light of recent experimental developments as there are proposals to measure entanglement entropies, and experiments have already been performed on simple cold atom systems.[18,19]

However, the cylinder extrapolation method seems to yield inconsistent results. By applying the cylinder extrapolation method to the J_1-J_2 antiferromagnetic Heisenberg model on a square lattice, it is found that $S(L=0) \simeq -\ln 2$ in a certain parameter regime,[20] and thus the ground state is identified as a topologically ordered spin liquid. However, this result was later objected by another DMRG study from an independent group, revealing a plaquette valence-bond order.[21] Furthermore, Ref. 22 studies the Heisenberg model on the honeycomb lattice, and reports $S(L=0) \simeq -\ln 2$ by the cylinder extrapolation method, but observes a plaquette valence-bond order, which leads to a suspicion of finite size effects on $S(L=0) < 0$. These numerical results question the validity of the cylinder extrapolation method.

In this paper, we point out one scenario under which the cylinder extrapolation method can be proven to be *invalid*. While we do not attempt to address specific reasons behind the discrepancy between Ref. 20 and 21, we will give a sufficient condition for topologically *trivial* 2D states, under which the cylinder extrapolation method must give a nonvanishing sub-leading term for any Rényi entropy calculations, leading to a spurious TEE.

We start with a general observation that when a two-dimensional state is topologically trivial, the entanglement computation in Fig. 1(b) reduces to that of a one-dimensional state with respect to an extensive bipartition. We show that whenever the derived one-dimensional state exhibits a symmetry-protected topological (SPT) order under a product group, which we define precisely below, then the cylinder extrapolation method must output a nonzero sub-leading term.

Under generic perturbations that break the symmetry of the reduced one-dimensional states in our examples, the sub-leading term is suppressed exponentially in the system size. Furthermore, if one applies the Kitaev-Preskill prescription in the bulk, one still obtains a value that is consistent with the total quantum dimension of the underlying topological particle content. Hence, it is improper to say that the notion of topological entanglement entropy is invalidated. Rather, our examples make it clear that one has to be careful in interpreting results from the cylinder extrapolation method.

To decide when the results from the cylinder extrapolation method can be trusted, we consider a length scale, termed *replica correlation length* ξ_α. The ratio L/ξ_α determines the magnitude the sub-leading term in the cylinder extrapolation method. This replica correlation length may be arbitrarily large compared to the usual correlation length of the 2D state. The usual correlation length of a state ρ is the decay rate of (the connected part of) a two-point function $\mathrm{Tr}(\rho OO')$ as a function of the distance between local operators O and O'. In contrast, the replica correlation length is the decay rate of a two-point replica correlation function $\mathrm{Tr}(\rho\mathcal{O}_1O_1'\rho O_2O_2')/\mathrm{Tr}(\rho^2)$ where the unprimed operators and primed operators are far separated.

At the first glance on its definition, the replica correlation length may seem difficult to calculate. We propose a relatively simple way of measuring the replica correlation function in numerics, as a natural extension of the measurement of Rényi entropies using swap operations.[23] We find that a 2D cluster state (graph state[24]), which is topologically trivial, has an infinite replica correlation length, while certain representative wave functions of the $\mathbb{Z}_N$ gauge theory and the double-semion theory have replica correlation length zero. We conjecture that the γ value from the cylinder extrapolation method is given by the total quantum dimension whenever the replica correlation length is small.

Some of our examples can be adapted to three or higher dimensions[25] and give similar effects. It would also be interesting to consider thermal states.[26–28] We will comment on these in the discussion section.

The rest of the paper is organized as follows. In Sec. II, we study the 2D cluster state on a triangular lattice, an

exactly soluble model of topologically trivial 2D states, and show that the sub-leading term of the entanglement Rényi entropy calculated by the cylinder extrapolation method is nonzero. This example illustrates important points and serves as a warm-up for the more general consideration. In Sec. III, we study the Rényi entropies of a class of topologically trivial 2D states and show that the sub-leading term is strictly negative, if it is calculated using the cylinder extrapolation method. In particular, we will attribute the nonzero sub-leading term to a SPT order under a product group of the derived one-dimensional state. In Sec. IV, we consider generic non-symmetric states and introduce the replica correlation length, that is responsible for the non-vanishing sub-leading term of entanglement entropy. We calculate the replica correlation length to show that it is infinite for the 2D cluster state on triangular lattice, while it is zero for some ideal wave functions of the $\mathbb{Z}_N$ gauge theories and double-semion theory. We conclude in Sec. VI with discussion on higher dimensions and thermal states. Appendices include further considerations. In App. A, we provide an example where the sub-leading term calculated by the cylinder extrapolation method oscillates with the system size. In App. B, we discuss lattice symmetries and time reversal symmetries, and find that these symmetries are not responsible for a robust non-vanishing sub-leading term by the cylinder extrapolation method. App. C contains calculation of the mutual information of the cluster state at nonzero temperature.

II. EXAMPLE: 2D CLUSTER STATE ON A TRIANGULAR LATTICE

In this section we study an example of topologically trivial 2D state that nevertheless exhibits nonvanishing sub-leading term of entanglement entropy under the cylinder extrapolation method. We start with a triangular lattice with one spin-1/2 (qubit) $\{|0\rangle, |1\rangle\}$ per lattice site, governed by a Hamiltonian

$$H_0 = -\sum_j \sigma_j^x \tag{3}$$

The ground state $|\psi_0\rangle$ of H_0 is a product state and there is no entanglement. Clearly, this state is topologically trivial. Next, we apply to $|\psi_0\rangle$ a layer of local unitary transformations U_{jk} for each pair $\langle jk\rangle$ of nearest neighbor qubits. The two-qubit unitary $U = U_{jk} = U_{kj}$ is most conveniently defined in a basis where σ^z is diagonal:

$$U|a\rangle|b\rangle = \begin{cases} -|1\rangle|1\rangle & \text{if } a = b = 1, \\ +|a\rangle|b\rangle & \text{otherwise.} \end{cases} \tag{4}$$

Since U_{jk} are simultaneously diagonal in a basis, they commute with each other

$$U_{jk}U_{j'k'} = U_{j'k'}U_{jk}. \tag{5}$$

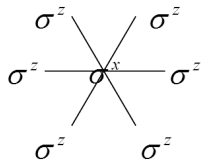


FIG. 2. A 7-spin interaction in the summation of Eq. (8), which is a product of the σ^x operator on the center site and the σ^z operators on the sites that surround it.

Hence, there is no ambiguity in the formula

$$\mathbf{U} = \prod_{\langle jk\rangle} U_{jk}, \tag{6}$$

and we define a state $|\psi\rangle = \mathbf{U}|\psi_0\rangle$, which is called the cluster state. Using the identity

$$U_{jk}(\sigma_j^x \otimes I)U_{jk}^\dagger = \sigma_j^x \otimes \sigma_k^z, \tag{7}$$

we see that the cluster state is a ground state of a Hamiltonian

$$H = \mathbf{U}H_0\mathbf{U}^\dagger = -\sum_j \left(\sigma_j^x \prod_{k:\langle jk\rangle} \sigma_k^z\right). \tag{8}$$

Graphically, each term in the summation of the new Hamiltonian (8) is the 7-spin interaction as shown in Fig. 2.

A. Entanglement entropy on a cylinder

Since we obtain the cluster state from a product state by a small depth quantum circuit, the cluster state is also topologically trivial. One may expect its bipartite entanglement entropy has a vanishing sub-leading term. This is indeed the case if we use the Kitaev-Preskill prescription to extract the sub-leading term. Now let us examine it using the cylinder extrapolation method. We put our state on an infinite cylinder by imposing a periodic boundary condition along one of three directions parallel to any side of a triangle. If the circumference is L, the number of bond cuts is $2L$. We will compute the entanglement entropy between the two sides A, B divided by this circumference (see Fig. 3).

Even though we started with a 2D state, the entropy computation reduces to that of a 1D chain with an extensive bipartition. To see this, recall that the entanglement entropy is invariant under any unitary that acts exclusively on either side of the bipartition:

$$S = S(\rho_A) = S(U_A\rho_A U_A^\dagger) = S(\rho_B) = S(U_B\rho_B U_B^\dagger) \tag{9}$$

where the subscript A or B denotes the region on which the operator is supported. In particular, we can choose U_A to be the product of all U_{jk} where the *edge* $\langle jk\rangle$ belongs to A. Since $U_{jk}^2 = I$, this amounts to disentangling

$|\psi\rangle$ on the region A. A similar disentangling unitary can be applied on B. What is left is a zig-zag 1D chain that straddles two regions along the cut, and some completely disentangled qubits in the product state. See Fig. 4. It remains to compute the entanglement entropy of the 1D chain $|\psi_1\rangle$, which is a ground state of

$$H_1 = -\sum_{j=1}^{L} \sigma^x_{j,A}\sigma^z_{j-1,B}\sigma^z_{j,B} - \sum_{j=1}^{L} \sigma^x_{j,B}\sigma^z_{j,A}\sigma^z_{j+1,A}$$
$$= -\sum_{k=1}^{2L} \sigma^z_{k-1}\sigma^x_k\sigma^z_{k+1} \qquad (10)$$

where a periodic boundary condition is imposed, i.e., $j = L+1$ site is equal to $j = 1$ site. Note that since the Hamiltonian is commuting and the ground state is non-degenerate, any correlation function is identically zero beyond distance 2, and hence the correlation length vanishes.

One might guess that the entanglement entropy of $|\psi_1\rangle$ is just proportional to the number of bond cuts, because the entangling unitary U_{jk} makes the product state $|+\rangle_j |+\rangle_k$ into a maximally entangled state, where $|+\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$. However, a more careful and direct computation reveals that

$$S(A) = (\log 2)L - \log 2. \qquad (11)$$

This computation can be done by exploiting the fact that the Hamiltonian H_1 consists of commuting tensor products of Pauli matrices. It is known that the eigenvalue spectrum of the reduced density matrix ρ_A for any set of qubits A consists of a single nonzero value (flat entanglement spectrum), and the number of nonzero eigenvalues is always 2^k for some integer $k \geq 0$. The exponent k depends only on the number of operators P that are products of Pauli matrices and stabilize the state, $P|\psi\rangle = |\psi\rangle$; more precisely, k equals the number of qubits in the region A minus the logarithm of the order of the stabilizer group $\mathcal{G}_A$ supported on A,

$$k = |A| - \log_2 |\mathcal{G}_A|. \qquad (12)$$

See Section V A for a simple proof. In the present example, there are L qubits in A, and there is a single non-identity stabilizer $\prod_{j=1}^{L} \sigma^x_{j,A}$ supported on A, and hence Eq. (11) follows. From Eq. (11), one may mistakenly conclude that $\gamma = \log 2$ from the cylinder extrapolation method and that the original 2D state is topologically ordered.

In the next section, we explain why such a topologically trivial state can give rise to a nonzero sub-leading term in the entanglement entropy calculated by the cylinder extrapolation method. We will attribute the nonzero sub-leading correction term $\gamma = \log 2$ to the nontriviality of the state $|\psi_1\rangle$ as a SPT order under symmetry $\mathbb{Z}_2 \times \mathbb{Z}_2$, where the first $\mathbb{Z}_2$ factor acts by σ^x on the red side of the entanglement cut, and the second $\mathbb{Z}_2$ factor by σ^x on the green side of Fig. 4.

B. Reduction to 1D state and its symmetries

Before proceeding, we remark that for any state constructed from a product state by a small-depth quantum circuit the entanglement entropy calculation reduces to that of a 1D chain with an extensive partition. Therefore, our consideration of 1D chains is appropriate and general to study the bipartite entanglement entropy of 2D topologically trivial states. The proof of this remark is simple. One can remove all entangling unitaries except for those near the cut without changing the entanglement. The unitaries that cannot be removed are supported on a strip whose width is proportional to the number of layers in the quantum circuit. Hence, one finally arrives at a quasi-1D system with a bipartition *along* the extended direction. Note that this simple argument proves the area law of entanglement entropy for such 2D states.

We also emphasize that the symmetry of the resulting 1D state is *not necessarily* the symmetry of the original 2D state. The disentangling deformations have no reason to obey any symmetry of the 2D state; their role is purely to transform the 2D state to a 1D chain immersed in a product state background. This means that even if the resulting 1D state is symmetric or close to symmetric, this symmetry is not necessarily visible in the 2D state. As an example, one can consider the deformation of the 2D triangular lattice cluster state in the form discussed

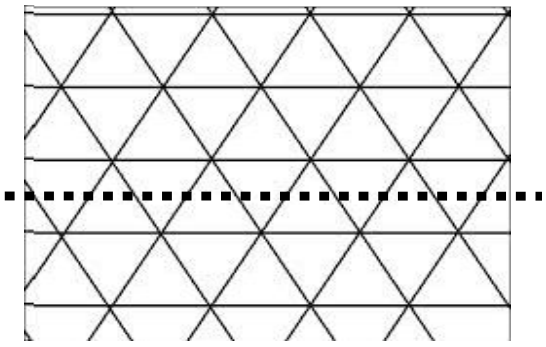


FIG. 3. Triangular lattice with an entanglement cut parallel to one of the sides of a triangle. The upper region is A, and the lower is B.

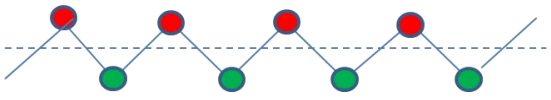


FIG. 4. The reduced 1D chain of the 2D cluster state. The red circles represent qubits in region A and the green circles represent qubits in region B.

in Sec. IV. One could deform the bonds arbitrarily except those that are crossed by the entanglement cut such that there is no on-site symmetry. After the disentangling transformations for the entanglement entropy evaluation, one can find $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry of the 1D chain.

Because the symmetry of the reduced 1D state is obscure from the *original* 2D system, we will introduce a replica correlation length in Sec. IV, V below. It is distinct from the usual correlation length but can be checked directly for the original 2D system without investigating a hidden 1D state.

C. Bravyi's example

Bravyi has considered the cluster state on a circular zig-zag chain on a plane, where an entanglement cut is chosen such that exactly half of the chain is inside the cut.[29] The rest of the plane is assumed to be occupied by qubits in a trivial product state. Dividing the disk into three circular sectors (see Figure 5), and taking the Kitaev-Preskill combination, one will find that $\log 2$ remains. This is, as far as we know, the first example in which Kitaev-Preskill combination can be nonzero for a topologically trivial state. Levin-Wen combination gives no different answer. The state is highly inhomogeneous, and the partition must be introduced very carefully. In contrast, our example is manifestly translation-invariant, and more relevant to current DMRG methods. Bravyi's example and ours are of course closely related: If we cap off one end of the cylinder to turn it into a topological plain, then our zig-zag chain and partition becomes those of Bravyi's.

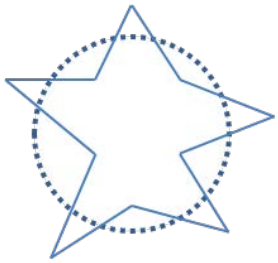


FIG. 5. Bravyi's example. There is a qubit on each vertex of the zig-zag chain, and the dashed line represents an entanglement cut that divides the chain into two halves. If the chain is in the cluster state of Eq. (10), the entanglement entropy is equal to $\frac{1}{2}$#(bond cuts) - 1 in the units of $\log 2$.

III. A SUFFICIENT CONDITION FOR A NONZERO SUB-LEADING TERM OF ENTANGLEMENT ENTROPY UNDER THE CYLINDER EXTRAPOLATION METHOD

In this section, we will provide a sufficient condition for a class of topologically trivial 2D states under which these states nevertheless give a nonzero sub-leading term of entanglement entropy under the cylinder extrapolation method. Since the bipartite entanglement entropy of any topologically trivial 2D states is identical to that of a reduced 1D chain under an extensive bipartition, this amounts to find a condition for such 1D chains to have a nonvanishing sub-leading term in the entanglement entropy.

We will show for any nontrivial 1D SPT under a product group symmetry $G_1 \times G_2$ (defined more precisely below), there must be a nonzero negative sub-leading term to the entanglement entropy with respect to a bipartition where the partition $i = 1, 2$ includes all degrees of freedom acted on by G_i. In Fig. 4, for example, G_1 acts on the red sites and G_2 acts on the green sites. Here the entanglement entropy is measured by the Rényi entropy

$$S_\alpha(\rho) = \frac{1}{1-\alpha} \log \mathrm{Tr}(\rho^\alpha), \tag{13}$$

and our state is assumed to have a matrix-product state (MPS) representation with a finite virtual bond dimension. We restrict ourselves to Rényi entropy of integer indices $\alpha = 2, 3, 4, \ldots$. The von Neumann entropy ($S_{\alpha\to 1}$) will not be treated explicitly.

A. Nontrivial SPT

First, we specify what a nontrivial SPT under a product group is. Recall that 1D SPTs are classified by the second group cohomology $H^2(G; U(1))$, which enumerates all equivalence classes of factor systems

$$\omega : G \times G \to U(1) \tag{14}$$

obeying the cocycle condition

$$\frac{\omega(b,c)\omega(a,bc)}{\omega(ab,c)\omega(a,b)} = 1 \quad \text{for all } a, b, c \in G \tag{15}$$

up to *exact cycles* defined by $\delta\lambda(a,b) := \lambda(a)\lambda(b)/\lambda(ab)$ for some function $\lambda : G \to U(1)$.[30,31]

Now, suppose the symmetry group is $G = G_1 \times G_2$, and each component acts on different physical qudits. Graphically, G_1 acts on the red sites in Fig. 4 and G_2 acts on the green sites there. We say *an SPT state under a product group is nontrivial* if its associated factor system ω admits

$$\Omega := \frac{\omega(a,b)}{\omega(b,a)} \neq 1 \quad \text{for some } a \in G_1, b \in G_2. \tag{16}$$

Note that Ω is independent of the cohomology representative ω since multiplying ω by an exact cycle does not change Ω. Since any factor system gives rise to a projective representation V, $\Omega \neq 1$ means that a commuting pair of elements a, b of G are represented by a pair of non-commuting unitaries:

$$V_a V_b = \frac{1}{\omega(a,b)} V_{ab} = \frac{1}{\omega(a,b)} V_{ba} = \frac{\omega(b,a)}{\omega(a,b)} V_b V_a \neq V_b V_a \tag{17}$$

In the matrix product state (MPS) representation, this projective representation appears in how the symmetry action is implemented on the virtual level. That is, if the local tensor for a translation-invariant SPT state is given by

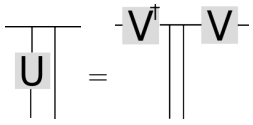
$$\mathbb{M} = \sum_{p,v,w} M^{(p)}_{vw} |p\rangle \otimes |v\rangle \langle w| , \tag{18}$$

the state is acted on by the symmetry such that

$$\sum_{p'} (U_g)^{p}{}_{p'} M^{(p')} = \eta_g V_g^\dagger M^{(p)} V_g \tag{19}$$

for any $g \in G$, where η_g is a phase factor.[32] Thus, if the SPT is nontrivial, the commuting symmetry actions are lifted to non-commuting unitaries on the virtual level.

Eq. (19) is most conveniently expressed in diagrams. If U is a matrix representing the symmetry action of G_1, it is



(20)

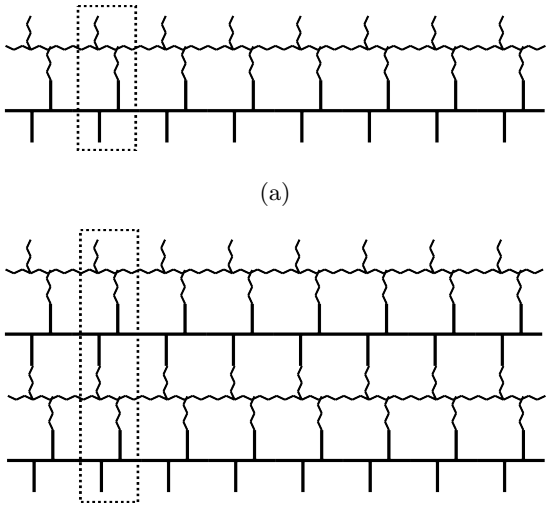
where we have drawn two lines for the physical qudit, to emphasize that we assumed two components of the symmetry group act on distinct physical qudits. We have omitted the phase factor η_g. If U were representing G_2, the box of U on the left-hand side would have been on the second vertical line. Typically in literature, a box is inserted at the intersection of lines to signify a tensor $\mathbb{M}$, but we omitted it.

B. Transfer matrix

Consider a 1D chain on a ring. Suppose there are L physical sites with two physical qudits per site, and the two qudits on each site will be acted on by G_1 and G_2, respectively. Tracing out one qudit for every site, we have a reduced density matrix ρ. In a diagram, ρ is depicted as Figure 6(a). An integral power of the reduced density matrix can also be represented by a diagram. For example, ρ^2 is depicted in Figure 6(b).

For Rényi entropy computations, we need to evaluate $\mathrm{Tr}(\rho^\alpha)$ for a positive integer α. This amounts to contracting all top vertical bonds with the bottom vertical bonds in Figure 6. It is instructive to look at ρ^2. Due to the 1D structure, it is useful to analyze *transfer matrix* $\mathbb{T}$ defined by the left-most diagram in Figure 7. For each integer Rényi index α, there is a transfer matrix $\mathbb{T}_\alpha$, which is independent of system size L. Note that the rows and columns of the reduced density matrix ρ are indexed by the physical qudits, whereas those of the transfer matrix are indexed by the virtual bonds. We have a trivial yet useful identity:

$$\mathrm{Tr}(\rho^\alpha) = \mathrm{Tr}(\mathbb{T}_\alpha^L). \tag{21}$$



(a)

(b)

FIG. 6. (a) ρ obtained by tracing out one physical qudits for every site. (b) ρ^2. Connected bonds are contracted. Wiggling lines represent complex conjugation. The horizontal virtual bonds are contracted due to the periodic boundary condition, which is not drawn. $\mathrm{Tr}(\rho)$ and $\mathrm{Tr}(\rho^2)$ are computed by contracting the upper vertical wiggling bonds with the lower straight ones. For this purpose, it is enough and more efficient to consider the transfer matrix designated by the dotted rectangle; see Eq. (21).

If the eigenvalues of $\mathbb{T}$ are $\{\lambda_i\}$, then $\mathrm{Tr}(\mathbb{T}^L) = \sum_{i=1}^{D^{2\alpha}} \lambda_i^L$, where $D^{2\alpha}$ is the size of the matrix $\mathbb{T}$.

There is no guarantee that λ_i are all positive; indeed, in Appendix A we give an example whose nonzero eigenvalues of $\mathbb{T}_2$ are $\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}$. In this case, the entanglement entropy is

$$S_2(L) = L \log 2 - \begin{cases} \log 4 & \text{if } L \text{ is even,} \\ \log 2 & \text{otherwise.} \end{cases} \tag{22}$$

Summarizing, the Rényi entropy for L sites is

$$S_\alpha = \frac{\log(1/\lambda_1^{(\alpha)})}{\alpha - 1} L - \frac{\log m}{\alpha - 1} + \cdots \tag{23}$$

where $\lambda_1^{(\alpha)}$ is the largest eigenvalue of $\mathbb{T}_\alpha$, which is necessarily real positive, $m = m(L)$ is an integer which may depend on L, and $\cdots$ represents vanishing terms in the large L limit, The number m is usually the degeneracy of the largest eigenvalue of $\mathbb{T}$.

C. Degeneracy of the transfer matrix

We now use the nontrivial SPT to show that the transfer matrix $\mathbb{T}_\alpha$ has degeneracy $m \geq q^{\alpha-1}$ for some integer $q > 1$. The degeneracy bound is uniform to every eigen-

value. This implies that

$$\gamma_\alpha = \frac{\log m}{\alpha - 1} \geq \log q > 0. \tag{24}$$

To this end, we rewrite the symmetry lifting Eq. (20) as in Figure 7(a), by which we define the unitaries V and W up to phase factors. The diagrams in Figure 7(b) follows at once. The nontrivial SPT implies that

$$WV = \Omega VW, \quad \Omega = \exp(2\pi ip/q) \neq 1 \tag{25}$$

where p and $q > 1$ are coprime.

Let us define $X_j = W^*_{2j-1} \otimes W_{2j}$ and $Y_j = V_{2j} \otimes V^*_{2j+1}$, where we have indexed the virtual bonds from the top to the bottom by integers modulo 2α. The index j takes values $1, \ldots, \alpha$. The symmetries X_j, Y_j of $\mathbb{T}$ form an algebra obeying the following commutation relations.

$$\begin{aligned} X_j Y_j &= \Omega Y_j X_j \quad (j = 1, \ldots, \alpha) \\ X_{j+1} Y_j &= \Omega^{-1} Y_j X_{j+1} \quad (X_{\alpha+1} = X_1). \end{aligned} \tag{26}$$

All other commutators among X_i, Y_j are vanishing. They can be rearranged as follows to determine a minimal representation. Since $X_1 X_2 \cdots X_\alpha$ and $Y_1 Y_2 \cdots Y_\alpha$ are in the center of the algebra, we take out X_α and Y_α from the generating set, and do not consider them any more.

$$\begin{aligned} Z_j &:= X_1 X_2 \cdots X_j \quad \text{for } j = 1, \ldots, \alpha - 1 \\ Z_j Y_j &= \Omega Y_j Z_j \\ [Z_j, Y_{j'}] &= 0 \quad \text{if } 1 \leq j \neq j' \leq \alpha - 1 \end{aligned} \tag{27}$$

Since Z_j, Y_j generate the same algebra as X_j, Y_j do, it is clear that the minimal representation of the symmetry algebra generated by X_i, Y_i has dimension $q^{\alpha-1}$ where q is the multiplicative order of Ω. It follows from the nontrivial SPT assumption that $q > 1$. We have proved Eq. (24) for any integer $\alpha > 1$. Therefore, we have found the promised condition that if the reduced 1D chain is a nontrivial SPT under a product group, the cylinder extrapolation method will give a nonzero sub-leading term of entanglement entropy.

Note that we have not used all physical symmetry elements. Sufficient is only one pair of commuting physical symmetry operators that are lifted to non-commuting virtual unitaries. The actual degeneracy may be even larger.

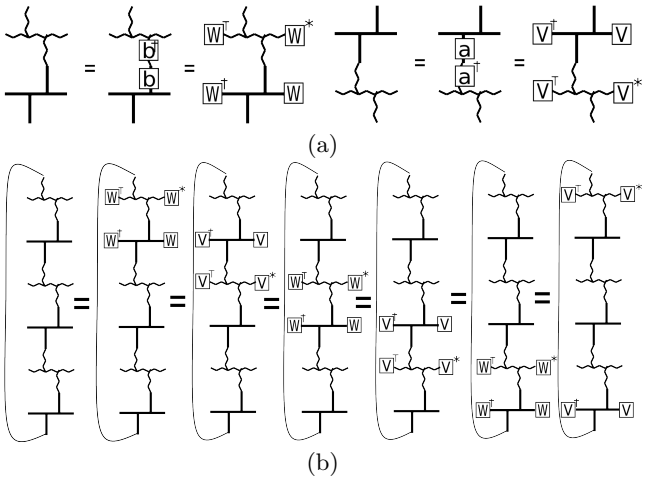


FIG. 7. Transfer matrix and its symmetry. The left-most diagram of (b) represents the transfer matrix $\mathbb{T}_3$ for ρ^3. The ensuing equalities are direct consequences of the symmetry lifted to the virtual level. This implies that the transfer matrix $\mathbb{T}_\alpha$ has degeneracy $q^{\alpha-1}$ for some integer $q > 1$.

D. Cluster state

Let us apply our general analysis to the previous example of the cluster state. It turns out that the 1D cluster state has $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry under which it is a nontrivial SPT, and the nonzero γ will be a consequence of this. This example is the simplest possible. We will need to construct an MPS representation, and identify the projective symmetry on the virtual level.

To find an MPS representation, we write the wave function of the 1D cluster state in σ^z basis using Eq. (6).

$$\langle \cdots s_{j-1} s_j s_{j+1} \cdots | \psi \rangle = (-1)^{\sum_j s_j s_{j+1}} / \sqrt{2^{2L}} \tag{28}$$

where $s_j = 0, 1$ and $2L$ is the number of qubits. It is therefore sufficient for the local tensor A_j at site j to take value -1 if $s_j = s_{j+1} = 1$ and $+1$ otherwise. The following tensor satisfies this condition.

$$\mathbf{M} = \sum_{s=0,1} \frac{(-1)^{ss'}}{\sqrt{2}} |s\rangle \otimes |s\rangle \langle s'| \tag{29}$$

The physical qubit at site j is synchronized with the left virtual bond, and thus s' is the state of the physical qubit at site $j + 1$ upon contracting the virtual bonds. The local tensor $\mathbf{M}$ correctly describes the cluster state, but the symmetry action will not be on-site. So, we block two neighboring physical qubits as one super-site and write the MPS representation as

$$\mathbb{M} = \sum_{s_L, s_R = 0,1; s' = 0,1} \frac{(-1)^{s_L s_R + s_R s'}}{2} |s_L s_R\rangle \otimes |s_L\rangle \langle s'| \tag{30}$$

It is easier to determine the symmetry once we rewrite $\mathbb{M}$ as a collection of matrices.

$$\begin{aligned} \mathbb{M}^{(++)} &= \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, & \mathbb{M}^{(+-)} &= \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \\ \mathbb{M}^{(-+)} &= \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, & \mathbb{M}^{(--)} &= \frac{1}{2} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \end{aligned} \tag{31}$$

where $|\pm\rangle = (|0\rangle \pm |1\rangle)/\sqrt{2}$. Notice if there was no $\mathbb{M}^{(++)}$, the remaining three tensors form an MPS description of

the AKLT state.[33] The action of σ^x's on every other physical qubit forms a group $G = \mathbb{Z}_2 \times \mathbb{Z}_2$, where the first component $\mathbb{Z}_2$ is implemented as

$$\mathbb{M}^{(+\star)} \leftrightarrow \mathbb{M}^{(+\star)}, \quad \mathbb{M}^{(-\star)} \leftrightarrow -\mathbb{M}^{(-\star)}, \tag{32}$$

and the second $\mathbb{Z}_2$ is implemented as

$$\mathbb{M}^{(\star+)} \leftrightarrow \mathbb{M}^{(\star+)}, \quad \mathbb{M}^{(\star-)} \leftrightarrow -\mathbb{M}^{(\star-)}. \tag{33}$$

These transformations can be enacted by conjugations by σ^x and σ^z. The conjugations on the virtual level does not change the state at all, and therefore G is a symmetry of the cluster state.

It is evident now that the symmetry we just identified is in accordance with Eq. (19) with η_g being trivial. The commuting symmetry action G on the physical level is lifted to a noncommuting symmetry $D_4 = \langle \sigma^x, \sigma^z \rangle$ on the virtual level. In fact, it is known $H^2(\mathbb{Z}_2 \times \mathbb{Z}_2; U(1)) = \mathbb{Z}_2$, and this representation is precisely a projective representation of the group $\mathbb{Z}_2 \times \mathbb{Z}_2$ and the cluster state is a nontrivial SPT associated with it. The phase factor of Eq. (25) is $\Omega = -1$, and the result of the previous section implies that $\gamma \geq \log 2$. Indeed, we have $\gamma = \log 2$ in Eq. (11).

IV. GENERIC BEHAVIOR OF γ: REPLICA CORRELATION LENGTH

In the previous section we attributed the subleading term γ of the entanglement entropy to the degeneracy of the transfer matrix for Rényi entropies. Generically when there is no degeneracy, the entanglement entropy would be

$$S_\alpha = \frac{|\log \lambda_1|}{\alpha - 1} L + O(r^L), \quad r = \left|\frac{\lambda_2}{\lambda_1}\right| = e^{-1/\xi_\alpha} < 1. \tag{34}$$

where λ_2 is the second largest eigenvalue of the transfer matrix. The subleading term converges to zero with an characteristic length scale ξ_α. This indicates that without a symmetry protection the subleading term γ should be zero for long chains. However, it is important to remark that ξ_α has little to do with the correlation length ξ. Indeed, being the ground state of a commuting Hamiltonian, the cluster state has the correlation length $\xi = 0$. Nonetheless, there is degeneracy in the transfer matrix for Rényi entropy, which means that $\xi_\alpha = \infty$ for any $\alpha = 2, 3, 4, \ldots$.

To understand the generic finite size effect more closely, we consider a deformed cluster state $|\theta\rangle$ specified by an angle $\theta \in [0, \pi]$. The deformation is achieved by replacing the -1 with $e^{i\theta}$ in the MPS representation Eq. (30). This amounts to transforming the state $\bigotimes_i |+\rangle$ by a two-qubit unitary $U(\theta) = \mathrm{diag}(1, 1, 1, e^{i\theta})$ instead of U in Eq. (4). It is anyway a transformation by a small-depth quantum circuit from a state of no correlation, the resulting state $|\theta\rangle$ has correlation length identically zero. Clearly, $\theta = \pi$ reproduces the previous cluster state.

The tensor network in Figure 6 for $|\theta\rangle$ can be explicitly evaluated, although the computation becomes more complicated as α is increased. The eigenvalues $\lambda_{1,2,\ldots}$ of the transfer matrix $\mathbb{T}_2$ for ρ^2 can be easily computed by a computer algebra system. The result is that

$$\begin{aligned} \lambda_1 &= \frac{3 + y^2 + (y+1)\sqrt{(y-1)^2 + 4}}{8} \\ \lambda_2 &= \frac{3 + y^2 - (y+1)\sqrt{(y-1)^2 + 4}}{8} \\ \lambda_3 &= \frac{y^2 - 1}{4}, \quad \text{where } y = \cos\theta \end{aligned} \tag{35}$$

and all other eigenvalues vanish for any θ. The largest eigenvalue λ_1 is non-degenerate unless $\theta = \pi$, at which the symmetry $G = \mathbb{Z}_2 \times \mathbb{Z}_2$ is restored. The $\theta = 0$ point is also G-symmetric; however, the symmetry is lifted to an abelian virtual symmetry, and hence the state is a trivial SPT. The ratio of the second largest eigenvalue to λ_1 can be any value between 0 and 1. In other words, the length scale $\xi_{\alpha=2}(\theta)$ interpolates from 0 to ∞ continuously, while the correlation length ξ is held at zero.

Kitaev and Preskill[12] gave an argument that the subleading term γ can be robustly defined by taking a linear combination of entanglement entropies. There, it was essentially used that a small change in a region A far from a region B leaves the following combination invariant:

$$\Delta S(A) - \Delta S(A \cup B) \simeq 0.$$

This is false when $A \cup B$ happen to include exactly a half of a nontrivial 1D SPT chain as in Bravyi's example in Sec. II C. From our consideration, generically, the distance between the region B and the region at which the change occurs should be compared to the replica length scale ξ_α of Eq. (34), not to the usual correlation length. It should be made clear that we did not prove that a small ξ_α implies that the cylinder extrapolation method or Kitaev-Preskill prescription gives the total quantum dimension of the topological particle content. Rather, we showed a short correlation length does not imply that TEE that results from the cylinder extrapolation method gives the total quantum dimension. We gave an evidence for the conjecture that a short replica correlation length would imply the validity of the cylinder extrapolation method.

V. REPLICA CORRELATION FUNCTIONS

We can actually probe the replica correlation length ξ_α by a *replica correlation function* on the original 2D state, without reducing it into a 1D chain. Let $\psi = |\psi\rangle\langle\psi|$ denote the density matrix of a state. Given a two copies of a state $\psi^{\otimes 2}$, and a bipartition $A \sqcup B$ of the system, we are formally provided with four subsystems A_1, B_1, A_2, B_2. Define $\mathcal{F}_A$ to be the swap operator between A_1 and A_2. If $\psi_B = \mathrm{Tr}_A[\psi]$ is the reduced density matrix for B, it

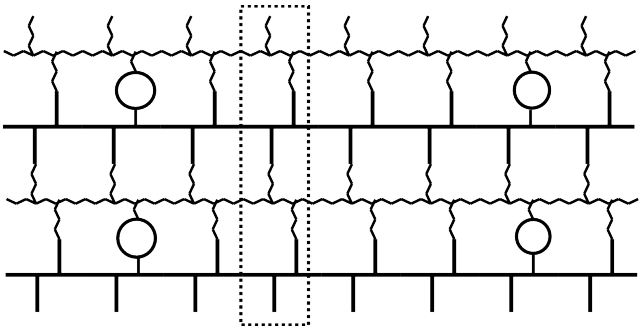


FIG. 8. Measuring replica correlation functions. Observables are inserted in the circles. Even if the global state ρ_{AB} has a short correlation length, the positive semi-definite operator ρ_A^2, treated as a normalized state $\rho_A^2/\operatorname{Tr}(\rho_A^2)$, may have much longer correlation length. The latter length scale, which we call as the replica correlation length, can be simply measured in numerical calculations, and is the relevant length scale for the subleading term in the entanglement entropy.

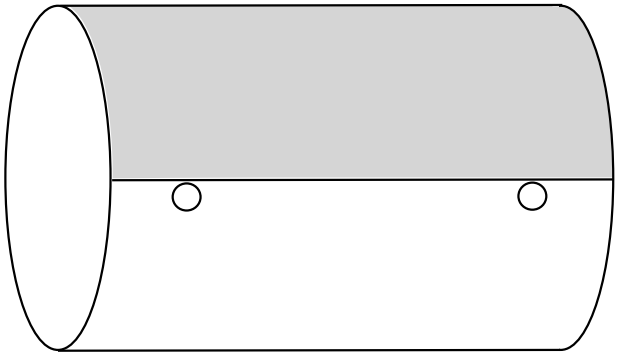


FIG. 9. Replica correlation function calculation. One prepares two copies of the state, and apply the swap operator on the shaded region; insert observables in the circles, and compute the overlap with the original, unswapped state. The overlap is generally exponentially small in the boundary length of the shaded region, but after normalization this reveals the replica correlation length.

holds that

$$\begin{aligned}\operatorname{Tr}[\psi^{\otimes 2}\mathcal{F}_A(O_{B_1}\otimes O_{B_2})] &= \operatorname{Tr}[\psi_B O_{B_1}\psi_B O_{B_2}]\\ &=: \operatorname{Tr}[\psi_B^2]\langle \mathbf{O}_B\rangle_{\alpha=2} \end{aligned} \tag{36}$$

for an arbitrary observable $\mathbf{O}_B = O_{B_1}\otimes O_{B_2}$ on the subsystem B. By Eq. (36), we have defined $\langle \mathbf{O}_B\rangle_{\alpha=2}$. For an observable $\mathbf{O}_B = O_{B_1}\otimes O_{B_2}$, we define a *replica* correlation function

$$\begin{aligned}&\operatorname{Cor}_{\alpha=2}(\mathbf{O}_B)(x) \\ &:= \langle \mathbf{O}_B(i=0)\mathbf{O}_B(i=L)\rangle_{\alpha=2}\\ &\quad - \langle \mathbf{O}_B(i=0)\rangle_{\alpha=2}\langle \mathbf{O}_B(i=L)\rangle_{\alpha=2}.\end{aligned} \tag{37}$$

The slowest possible decay of $\operatorname{Cor}_{\alpha=2}$ is determined by the length scale ξ_2 of Eq. (34), and this is actually achievable.

We prove this claim by an example. We will show the cluster state $|\theta=\pi\rangle$ has a nonzero constant replica correlation function that does not decay at all as a function of the distance between observables.

In addition, we will calculate the replica correlation functions for the $\mathbb{Z}_N$ gauge theory ground state in two dimensions, and the double semion ground state also in two dimensions. We find that the replica correlation function decays abruptly; the replica correlation length is zero. The purpose of this computation is to show that genuinely topologically ordered phases that give nonzero γ do have representative wave functions with fast decaying replica correlation functions. We expect that every Levin-Wen wave function[34] would have a small replica correlation length. This strongly suggests that the replica correlation function *can* be used to determine when one should rely on the γ value obtained from, e.g., DMRG computation.

In a DMRG calculation for a (isotropic) 2D state, we propose a measurement of $\xi_{\alpha=2}$ by the following steps. One prepares two copies of the state, and inserts the swap operator on the half strip of the state. Then, one measures the correlation function for a pair of point observables inserted near the region where swap operator is applied. See Fig. 9.

A. Flat entanglement spectrum

Before we delve into the replica correlation function calculations, we review a technique and expression for special reduced density matrices that are proportional to projectors.[35,36] This includes the cluster state and the ground states of $\mathbb{Z}_N$ gauge theory that are eigenstates of string operators. The technique here will be used crucially in later calculations. We follow Proposition 9 and Corollary 10 of Ref. 37.

Define $N\times N$ matrices as

$$X_N = \sum_{j\in\mathbb{Z}_N} |j+1\rangle\langle j|\,, \quad Z_N = \sum_{j\in\mathbb{Z}_N} e^{2\pi i j/N}|j\rangle\langle j|\,. \tag{38}$$

Any product $P = X_N^n Z_N^m$ of these matrices has a property that

$$\operatorname{Tr} P = \begin{cases} N & \text{if } n=m=0\in\mathbb{Z}_N,\\ 0 & \text{otherwise.}\end{cases} \tag{39}$$

Now, consider any multiplicative group $\mathcal{G}$ generated by tensor products of these matrices together with the phase factor $e^{2\pi i/N}$ on $(\mathbb{C}^N)^{\otimes n}$. Examples are the multiplicative group generated by the term of the Hamiltonian of the cluster state, and of the $\mathbb{Z}_N$ gauge theory as in (49) below. If $\mathcal{G}$ is abelian, there exists a common eigenstate $|\psi\rangle\in(\mathbb{C}^N)^{\otimes n}$ of all elements of $\mathcal{G}$.

Suppose that

- there is a unique eigenstate $|\psi\rangle$ of eigenvalue $+1$ for all $g\in\mathcal{G}$.

Then, $\mathcal{G}$ cannot contain any pure scalar element $\eta \neq 1$ because such a scalar can only have eigenvalue that is not 1. The projector onto $|\psi\rangle$ can be written as

$$|\psi\rangle\langle\psi| = \Pi_{\mathcal{G}} := \frac{1}{|\mathcal{G}|}\sum_{g\in\mathcal{G}} g. \tag{40}$$

Because of (39), any non-identity element of $\mathcal{G}$ has zero trace. Hence, taking the trace on both sides we see

$$1 = \frac{1}{|\mathcal{G}|}\operatorname{Tr}(I) = \frac{N^n}{|\mathcal{G}|}; \tag{41}$$

that is, the order of the group must be the full dimension of the Hilbert space $(\mathbb{C}^N)^{\otimes n}$.

Now, divide the system into two subsystems A and $B = A^c$ so that $(\mathbb{C}^N)^{\otimes n} = (\mathbb{C}^N)^{\otimes|A|}\otimes(\mathbb{C}^N)^{\otimes|B|}$. Tracing out A from the density matrix $|\psi\rangle\langle\psi|$, we obtain the reduced density matrix for B:

$$\rho_B = \operatorname{Tr}_A \Pi_{\mathcal{G}} = \frac{1}{N^{|A|+|B|}}\sum_{g\in\mathcal{G}}\operatorname{Tr}_A(g) \tag{42}$$

Again due to (39), $\operatorname{Tr}_A(g)$ is zero unless g acts on A by the identity (supported on B), in which case $\operatorname{Tr}_A(g) = (g|_B)N^{|A|}$. The elements of $\mathcal{G}$ that are supported on B form a subgroup $\mathcal{G}_B$, and we can write

$$\rho_B = \frac{N^{|A|}}{N^{|A|+|B|}}\sum_{g\in\mathcal{G}_B} g = \frac{|G_B|}{N^{|B|}}\underbrace{\frac{1}{|G_B|}\sum_{g\in\mathcal{G}_B} g}_{\Pi_{\mathcal{G}_B}}. \tag{43}$$

This implies that ρ_B is proportional to the projector $\Pi_{\mathcal{G}_B}$ of rank $N^{|B|}/|\mathcal{G}_B|$. In other words, the entanglement spectrum (the eigenvalues of ρ_B) is flat and the entanglement entropy (von Neumann or Rényi) is

$$S(B) = |B|\log N - \log|\mathcal{G}_B|. \tag{44}$$

This expression for the entropy is also derived in Ref. 10 and 38.

B. Cluster state

In this subsection we calculate the replica correlation length of the 2D cluster state and its deformed cousins, by considering the behavior of the replica correlation functions. We will find observables that achieve the slowest possible decay of replica correlation functions. Since a local observable is mapped to a local observable under finite depth quantum circuits (local unitaries), we may study the replica correlation function after simplifying the state by local unitaries. This means that we can focus on the 1D cluster state that is reduced from the 2D cluster state.

Consider the cluster state $|\theta = \pi\rangle$ on $2L$ spins with the periodic boundary condition. As before, let B be the region that contains every other spin, in total of L spins. The reduced density matrix ψ_B has a flat eigenvalue spectrum; ψ_B is proportional to a projector.

$$\psi_B = \frac{1}{2^L}\left(I + \prod_{i\in B}\sigma_i^x\right) \tag{45}$$

where the number of nonzero eigenvalues is $M = 2^{L-1}$. See the previous subsection V A for a derivation.

Let $\mathbf{O}_B(i) = \sigma_i^z\otimes\sigma_i^z$ be an observable for two copies of the state. The normalization factor $\operatorname{Tr}[\psi_B^2]$ in Eq. (36) is equal to $1/M$. Thus,

$$\begin{aligned}&\langle\mathbf{O}_B(0)\mathbf{O}_B(i)\rangle_{\alpha=2}\\ &= M\operatorname{Tr}[\psi_B\sigma_0^z\sigma_i^z\psi_B\sigma_0^z\sigma_i^z]\\ &= M\operatorname{Tr}[\psi_B\sigma_0^z\sigma_i^z\sigma_0^z\sigma_i^z\psi_B]\\ &= M\operatorname{Tr}[\psi_B\psi_B]\\ &= 1\end{aligned} \tag{46}$$

where the second equality is because $\sigma_0^z\sigma_i^z$ commutes with ψ_B. On the other hand,

$$\begin{aligned}&\langle\mathbf{O}_B(j)\rangle_{\alpha=2}\\ &= M\operatorname{Tr}[\psi_B\sigma_j^z\psi_B\sigma_j^z]\\ &= 0\end{aligned} \tag{47}$$

because $\psi_B\sigma_j^z\psi_B = 0$. Therefore, the replica correlation function reads

$$\operatorname{Cor}_{\alpha=2}(\mathbf{O}_B(0),\mathbf{O}_B(x)) = 1 \tag{48}$$

independent of the separation x.

For generic values of θ, we numerically checked that replica correlation functions of generic observables on the state $|\theta\rangle$ decay according to the finite replica correlation length calculated from (35). We emphasize once again that the usual correlation length ξ measured by

$$\langle\theta|K_1K_2|\theta\rangle - \langle\theta|K_1|\theta\rangle\langle\theta|K_2|\theta\rangle$$

is identically zero for any observables K_1 and K_2 for the state $|\theta\rangle$ for any θ.

C. $\mathbb{Z}_N$ gauge theory

The replica correlation length/function is introduced to pick up a fine detail of a state, and therefore we calculate it for a particular ground state of an exactly soluble model of the $\mathbb{Z}_N$ gauge theory in the deconfined phase. Unlike the 2D cluster state, abelian discrete gauge theory ground states, as well as the double semion state of the next subsection, requires deep quantum circuit to disentangle,[6,8,9] so we are forced to work with the 2D state directly in the replica correlation functions. The lattice of the model is not too important, but we consider the square lattice in two dimensions. The Hamiltonian is

sum of star terms (gauge transformation), and plaquette terms (flux):

$$H_{\mathbb{Z}_N} = -\sum_s X_{s,\text{east}} X_{s,\text{north}} X^\dagger_{s,\text{west}} X^\dagger_{s,\text{south}} - \sum_p Z_{p,\text{south}} Z_{p,\text{east}} Z^\dagger_{p,\text{north}} Z^\dagger_{p,\text{west}}. \quad (49)$$

where s denotes a site (vertex) and p denotes a plaquette (face), and $X = X_N$, $Z = Z_N$ of (38). The ground state is an equal amplitude superposition of "loop" configurations, where the loops come in N types and obey the group law of $\mathbb{Z}_N$.

When put on a thin torus as in Fig. 9, the Hamiltonian $H_{\mathbb{Z}_N}$ has an N^2-fold degenerate ground space. As the DMRG algorithm is biased to states with minimal entanglement across the circumferential cut,[17] we consider the state $|\psi\rangle$ that has $+1$ eigenvalue of the Z-type string operator and $+1$ of X-type string operator along the circumference (the shortest nontrivial loop).

For this state $|\psi\rangle$, which is the unique common $(+1)$-eigenvector of a commuting set of tensor products of X and Z matrices, the entanglement spectrum for any bipartition is flat, and the reduced density matrix ψ_B is proportional to a projector

$$\Pi = \frac{1}{|\mathcal{G}_B|} \sum_{g \in \mathcal{G}_B} g. \quad (50)$$

Here, the group $\mathcal{G}_B$ is a subgroup of the multiplicative group $\mathcal{G}$ that is generated by all Hamiltonian terms.[39] $\mathcal{G}_B$ consists of all elements of $\mathcal{G}$ that is supported on B. (See the previous subsection V A.)

In fact, $\mathcal{G}_B$ is generated precisely by the Hamiltonian terms supported on B. To see this, suppose $g \in \mathcal{G}_B$. Since $\mathcal{G}$ is abelian, the operator g can be written as a product of closed Z-loop operators and closed X-loop operators. These loop operators are contractible and supported on B, so each X- or Z-loop can be deformed to vanish by multiplying the smallest loop operators, which is exactly the Hamiltonian terms on B. This implies that g is a product of the Hamiltonian terms on B. $\mathcal{G}_B$ having a local generating set is an important difference from the cluster state, for which the density matrix (45) is the sum over a group with *no local* generators.

Note that since Π is a sum of all elements of a group, we see

$$\Pi g = \Pi = g\Pi \quad (51)$$

for any element $g \in \mathcal{G}_B$. We claim that for any operator O there exists $\tilde{O}$ of the *same* support such that

$$\Pi O \Pi = \Pi \tilde{O} \Pi, \quad \text{and} \quad [\Pi, \tilde{O}] = 0. \quad (52)$$

To construct $\tilde{O}$, let $\mathcal{G}_i \le \mathcal{G}_B$ be the group generated by the Hamiltonian terms that overlap the site (or region) i on which O is supported. Let

$$\tilde{O} = \frac{1}{|\mathcal{G}_i|} \sum_{g \in \mathcal{G}_i} g O g^{-1}. \quad (53)$$

The support of $\tilde{O}$ is the same as that of O because every element $g \in \mathcal{G}_i$ is a tensor product unitary operator. Any tensor component of g that acts outside of the support of O cancels in the expression gOg^{-1}. Hence, every summand gOg^{-1} has the same support as O, so is the sum $\tilde{O}$. Note that $\tilde{O}$ is zero if, e.g., O anti-commutes with some $g \in \mathcal{G}_i$.

Next, we verify (52).

$$\begin{aligned} \Pi \tilde{O} \Pi &= \frac{1}{|\mathcal{G}_i|} \sum_{g \in \mathcal{G}_i} \Pi g O g^{-1} \Pi \\ &= \frac{1}{|\mathcal{G}_i|} \sum_{g \in \mathcal{G}_i} \Pi O \Pi \\ &= \Pi O \Pi, \end{aligned} \quad (54)$$

where we used (51). In addition, $g\tilde{O}g^{-1} = \tilde{O}$ if a Hamiltonian term $g \in \mathcal{G}_i$ overlaps i by (53). If $h \in \mathcal{G}_B \setminus \mathcal{G}_i$ is a Hamiltonian term that does not overlap i, then h commutes with $g \in \mathcal{G}_i$ because $\mathcal{G}_B$ is abelian, and h also commutes with O trivially, so h commutes with $\tilde{O}$. Since $\mathcal{G}_B$ is generated by Hamiltonian terms, $\tilde{O}$ commutes with every element of $\mathcal{G}_B$.[40] Since the projector Π is the sum of all group elements of $\mathcal{G}_B$, it follows that

$$[\Pi, \tilde{O}] = 0. \quad (55)$$

Moreover, if O and O' are far separated so that no generator of $\mathcal{G}_i$ or $\mathcal{G}_{i'}$ overlaps with both O and O', then

$$\begin{aligned} \tilde{O}\tilde{O}' &= \frac{1}{|\mathcal{G}_i| \cdot |\mathcal{G}_{i'}|} \sum_{g \in \mathcal{G}_i} \sum_{h \in \mathcal{G}_{i'}} (gOg^{-1})(hO'h^{-1}) \\ &= \frac{1}{|\mathcal{G}_i| \cdot |\mathcal{G}_{i'}|} \sum_{g \in \mathcal{G}_i} \sum_{h \in \mathcal{G}_{i'}} ghOO'(gh)^{-1} \\ &= \frac{1}{|\mathcal{G}_i \times \mathcal{G}_{i'}|} \sum_{g \in \mathcal{G}_i \times \mathcal{G}_{i'}} gOO'g^{-1} \\ &= \widetilde{OO'}. \end{aligned} \quad (56)$$

Consider arbitrary observables $\mathbf{O} = \sum_a P_a \otimes Q_a$ near a site i and $\mathbf{O}' = \sum_b P'_b \otimes Q'_b$ near a site i' on the two copies of the state. Suppose i and i' are sufficiently separated, say by 5 lattice spacing, so that no term in the Hamiltonian $H_{\mathbb{Z}_N}$ overlaps simultaneously with $\mathbf{O}$ and $\mathbf{O}'$. The normalization factor of (36) is given by $M = \text{Tr}[\psi_B^2]^{-1} = \text{Tr}[\Pi] = |\mathcal{G}_B|$. Using tilde operators, the replica correlation function becomes a usual correla-

tion function:

$$\begin{aligned}
&\langle \mathbf{OO}'\rangle_2 - \langle \mathbf{O}\rangle_2\langle \mathbf{O}'\rangle_2 \\
&= |\mathcal{G}_B|^{-1} \sum_{a,b} \mathrm{Tr}[\Pi P_a P'_b \Pi Q_a Q'_b] \\
&\quad - |\mathcal{G}_B|^{-2} \sum_{a,b} \mathrm{Tr}[\Pi P_a \Pi Q_a]\, \mathrm{Tr}[\Pi P'_b \Pi Q'_b] \\
&= |\mathcal{G}_B|^{-1} \sum_{a,b} \mathrm{Tr}[\Pi \tilde{P}_a \tilde{P}'_b \tilde{Q}_a \tilde{Q}'_b] \\
&\quad - |\mathcal{G}_B|^{-2} \sum_{a,b} \mathrm{Tr}[\Pi \tilde{P}_a \tilde{Q}_a]\, \mathrm{Tr}[\Pi \tilde{P}'_b \tilde{Q}'_b] \\
&= \mathrm{Cor}\left(\sum_a \tilde{P}_a \tilde{Q}_a, \sum_b \tilde{P}'_b \tilde{Q}'_b \right) \\
&= 0
\end{aligned} \tag{57}$$

where in the second equality $\widetilde{P_a P'_b} = \tilde{P}_a \tilde{P}'_b$ is because they act on separated spins, and the last equality is because $H_{\mathbb{Z}_N}$ is commuting with locally indistinguishable ground space.

The replica correlation function is not identically zero when the site i and i' are close. This is simple. The replica correlation function becomes the usual correlation function when $\mathbf{O} = O \otimes I$. Consider $O = X_{s,\mathrm{east}} X_{s,\mathrm{north}} + h.c.$ and $O' = X^\dagger_{s,\mathrm{west}} X^\dagger_{s,\mathrm{south}} + h.c.$ where s is in the interior of B. Then, $\mathrm{Tr}[OO'\psi_B] = 2$, but $\mathrm{Tr}[O\psi_B] = \mathrm{Tr}[O'\psi_B] = 0$, so $\mathrm{Cor}(O, O') = 2$.

In conclusion, we have shown that the replica correlation function for $H_{\mathbb{Z}_N}$ is not identically zero but decays to zero after separation distance 5. Therefore, the replica correlation length is zero. This is in contrast to the cluster state calculation where the replica correlation function is nonzero and does not decay at all.

D. Double semion model

Using the similar techinque as in the previous subsection, we will calculate the replica correlation function for a version of double semion model on the honeycomb lattice,[34,41] and find that it decays abruptly in the same ways as for the $\mathbb{Z}_N$ gauge theory.

The lattice consists of 2 spin-$\frac{1}{2}$'s at each edge. The two spins on an edge is going to be (energetically) "synchronized". The Hamiltonian is

$$\begin{aligned}
H_{DS} = &- \sum_p \underbrace{- \prod_{e,e' \in p} \sigma^x_e \sigma^x_{e'} \prod_{e \in \partial p} \sqrt{\sigma^z_e}}_{g_p} + h.c. \\
&- \sum_v \underbrace{\prod_{e \in v} \sigma^z_e}_{g_v} - \sum_e \underbrace{\sigma^z_e \sigma^z_{e'}}_{g_e}.
\end{aligned} \tag{58}$$

The first term g_p has the minus sign, and is defined for every hexagon p where $e, e' \in p$ means all 12 spins on the plaquette, and $e \in \partial p$ means the 6 spins on the legs of the plaquette that are immediate neighbors of the plaquette. The second term is defined for every trivalent vertex v, and $e \in v$ means the 3 spins that are immediate neighbors of v. The third term is defined for every edge that contains two spins e and e'; this ferromagnatic term "synchronizes" the two spins on the edge. Our Hamiltonian (58) is slightly different from those in Ref. 34 and 41, but splitting the edge spin into two spins has appeared in Ref. 13. We are considering the split version because of the simplicity of reduced density matrix expression.

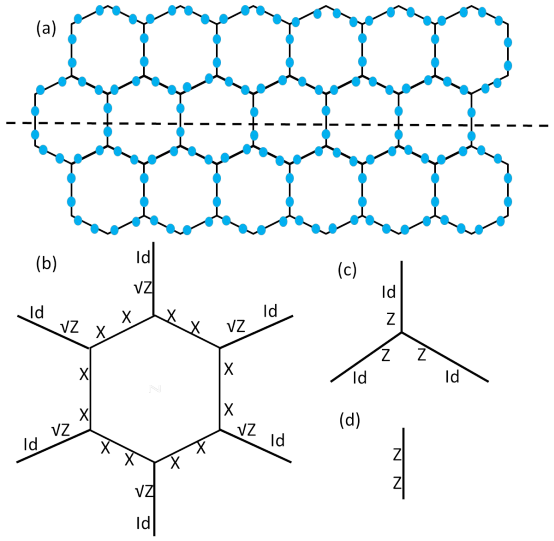


FIG. 10. Double-semion model defined in (58). (a) shows the lattice configuration and the location of the degrees of freedom. Each edge of the hexagon accommodates two spins, denoted by the blue dots on the edge. The dashed line is the entanglement cut. (b-d) pictorially show the three terms in the Hamiltonian for a plaquette, vertex and edge. The symbol near a spin denotes the operator acting on this spin, where X means σ_x, Z means σ_z, $\sqrt{Z} = \mathrm{diag}(1, i)$, and Id is the identity matrix. The total Hamiltonian is the summation over all plaquettes, vertices and edges, with appropriate sign factors defined in (58).

The plaquette terms g_p commute with the vertex terms g_v and edge terms g_e, but they do not commute among themselves; however, they do commute on the constrained subspace where every vertex and edge term takes $+1$ eigenvalue ($g_v = g_e = 1$). This is equivalent to the following. Let $\mathcal{G}^P$ denote the nonabelian multiplicative group generated by all g_p's, and let $\mathcal{G}^Z$ denote the abelian multiplicative group generated by all g_v's and g_e's. Also, let $\mathcal{G} = \mathcal{G}^P \mathcal{G}^Z$ denote the group generated by all terms in the Hamiltonian. Then,

$$\begin{aligned}
&gzg^{-1}z^{-1} = 1 \quad \forall g \in \mathcal{G}^P, z \in \mathcal{G}^Z \\
&ghg^{-1}h^{-1} \in \mathcal{G}^Z \quad \forall g, h \in \mathcal{G}^P.
\end{aligned} \tag{59}$$

(See also Ref. 42.)

The ground space of H_{DS} on a torus is four-fold degenerate, and the minimally entangled states are eigenstates

of string operators along a shortest topologically nontrivial loop. The string operators whose end points, if open, corresponds to semions are not too simple (Eq. 9 of Ref. 41), but satisfies an important property that the partial trace is zero. Let us fix the entanglement cut that passes an array of plaquettes along a straight line that is orthogonal to an edge. This entanglement cut passes in between the two spins on the intersecting edges, which is considered in Ref. 13. The purpose of this special cut is to have

$$\mathrm{Tr}_A[g] = 0 \text{ if } g \in \mathcal{G}^P \text{ overlaps with } A \tag{60}$$

since any such g has a σ^x or σ^z tensor component in addition to $\sqrt{\sigma^z}$ within A. Then, it follows by the same reasoning as in Sec. V A that

$$\psi_B = \mathrm{Tr}_A[\psi] \propto \sum_{g\in\mathcal{G}_B} g \tag{61}$$

where $\mathcal{G}_B \le \mathcal{G} = \mathcal{G}^P\mathcal{G}^Z$, consists of the elements supported on B.[43] The entanglement spectrum is flat.

The subgroup $\mathcal{G}_B$ is generated by the local Hamiltonian terms supported on B. The reason is similar to that for the previous $\mathbb{Z}_N$ theory. Observe that $\sigma^x\sqrt{\sigma^z}\sigma^x = i(\sqrt{\sigma^z})^\dagger$. Hence, any element $g \in \mathcal{G} = \mathcal{G}^P\mathcal{G}^Z$ can be written as a product of $\prod_i \sigma_i^x$ and $\prod_i \sqrt{\sigma_i^z}^{n_i}$ up to an overall phase factor. The first factor $\prod_i \sigma_i^x$ has to form a closed loop since it arises from g_p terms. The closed loop of σ^x must be entirely contained in B, and we can eliminate it by multiplying g_p operators on B to g. Therefore, it suffices for us to show that any "diagonal" element $z \in \mathcal{G}$ (a product of $\sqrt{\sigma^z}$) supported on B, is given by a product of g_v and g_e on B. Note that any diagonal element of $\mathcal{G}^P$ arises from g_p^2, and g_p^2 belongs to $\mathcal{G}^Z$.[42] The group $\mathcal{G}^Z$ can be viewed as the group of null-homologous $\mathbb{Z}_2$-loops on the triangular lattice (the dual lattice of the honeycomb lattice). Therefore, if $z \in \mathcal{G}^Z$ is supported on B, then z is a product of g_v and g_e on B. This implies that $g \in \mathcal{G}$ supported on B is a product of Hamiltonian terms on B up to a phase factor. The phase factor must be 1 because the group $\mathcal{G}$ does not contain any nontrivial phase factor. This completes the reasoning.

As in the previous section, we can turn any operator O supported on B into $\tilde{O}$ such that

$$\begin{aligned} \mathrm{support}(O) &= \mathrm{support}(\tilde{O}),\\ \psi_B O\psi_B &= \psi_B\tilde{O}\psi_B,\\ [\tilde{O},\psi_B] &= 0,\\ \widetilde{OO'} &= \tilde{O}\tilde{O}' \quad \text{if separated.} \end{aligned} \tag{62}$$

Let $\mathcal{G}_i^P$ be the group generated by g_p that overlaps i on which O is supported, and let $\mathcal{G}_i^Z$ be the group generated by g_e and g_v on spins where $\mathcal{G}_i^P$ is supported. The choice of group $\mathcal{G}_i^Z$ is to have

$$\begin{aligned} &gzg^{-1}z^{-1} = 1 \quad \forall g \in \mathcal{G}^P, z \in \mathcal{G}_i^Z\\ &ghg^{-1}h^{-1} \in \mathcal{G}_i^Z \quad \forall g \in \mathcal{G}^P, h \in \mathcal{G}_i^P. \end{aligned} \tag{63}$$

Now, let $\mathcal{G}_i = \mathcal{G}_i^P\mathcal{G}_i^Z$, and define $\tilde{O}$ by the formula (53). $\tilde{O}$ has the same support as O is because $\mathcal{G}_i$ is a group of tensor product unitary operators. That $\psi_B O\psi_B = \psi_B\tilde{O}\psi_B$ follows from a similar equation as (54) since ψ_B for the double semion state is also a sum over a group $\mathcal{G}_B \supset \mathcal{G}_i$. As for $[\tilde{O},\psi_B] = 0$, we see $h\tilde{O}h^{-1} = \tilde{O}$ if $h \in \mathcal{G}_i^P$ is one of the plaquette terms g_P that overlap i, by definition of $\tilde{O}$. If $k \in \mathcal{G}^P \setminus \mathcal{G}_i^P$ is a plaquette term that does not overlap i, then

$$\begin{aligned} k\tilde{O}k^{-1} &= \sum_{g\in\mathcal{G}_i} kgOg^{-1}k^{-1}\\ &= \sum_{g\in\mathcal{G}_i} gk(k^{-1}g^{-1}kg)O(k^{-1}g^{-1}kg)^{-1}(gk)^{-1}\\ &= \sum_{g\in\mathcal{G}_i} g(k^{-1}g^{-1}kg)O(k^{-1}g^{-1}kg)^{-1}g^{-1}\\ &= \sum_{g\in\mathcal{G}_i} gOg^{-1}\\ &= \tilde{O} \end{aligned} \tag{64}$$

where in the third equality we used (63) and that $[k,O] = 0$, and in the fourth equality we redefined the dummy variable g since $(k^{-1}g^{-1}kg) \in \mathcal{G}_i$. If z is one of g_e or g_v terms, then a similar calculation, simpler than (64), shows $z\tilde{O}z^{-1} = \tilde{O}$. As we have shown that ψ_B is a sum over a group generated by the terms of H_{DS}, the proof that $[\psi_B,\tilde{O}] = 0$ is complete. The last property $\widetilde{OO'} = \tilde{O}\tilde{O}'$ follows from similar equations as (56).

In conclusion, Eq. (57) holds without any modification using (62) for the double semion model under our bipartition. Therefore, the replica correlation function reduces to a usual correlation function, and the replica correlation length is zero.

VI. DISCUSSION

In this paper we have studied the behavior of the subleading term of the bipartite entanglement entropy of topologically trivial 2D states calculated by the cylinder extrapolation method, and found a sufficient condition under which a topologically trivial state will give a nonvanishing sub-leading term under this method. In particular, we showed the bipartite entanglement entropy of a such 2D state can be reduced to that of a 1D chain under an extensive bipartition. If this 1D chain is in a nontrivial SPT state under a product group $G = G_1 \times G_2$, where G_1 and G_2 act exclusively on the two sides of the bipartition, then a nonvanishing sub-leading term appears in the cylinder extrapolation method.

Our result does not necessarily invalidate the notion of topological entanglement entropy. In fact, the examples in this paper that are translation-invariant yield the correct total quantum dimension of the topological phase under the Kitaev-Preskill or Levin-Wen prescription. Rather, our finding makes it clear that the cylinder extrapolation method may give a different answer

than the Kitaev-Preskill or Levin-Wen prescription in the bulk.

Notice the above condition requires the state be in a nontrivial SPT state of G in the way we described above. This requirement, where the nontriviality of the state is protected by G_1 and G_2 simultaneously, and G_1 and G_2 act exclusively on the two sides of the bipartition, is stronger than the general condition of 1D SPT based on group cohomology.[30,31] A nontrivial SPT in the general sense can be protected by G_1 or G_2 alone, but this is not sufficient to yield a nonvanishing sub-leading term from the cylinder extrapolation method.

We have introduced the replica correlation length/function. In Sec. V we gave an operational meaning to it and demonstrated that it can be determined numerically. Though we only discussed $\alpha = 2$ replica correlation length/function, it is straightforward to consider $\alpha > 2$ cases by considering cyclic permutation operators instead of the swap operator. Our result suggests a conjecture that the cylinder extrapolation method on minimally entangled states yields the total quantum dimension of the topological particle content if the replica correlation length is small compared to the system size. All our examples of the SPT states under a product group, the $\mathbb{Z}_N$ lattice gauge theory, and the double semion model should be read as evidences in favor of this conjecture.

Analogues of TEE for the ground states of gapped Hamiltonians in 3 or higher dimensions have been proposed.[25] Ref. 25 studies various solid torus geometries and identifies multitude of TEEs that are associated with Betti numbers of the region for which entanglement entropy is calculated. Our examples can be generalized to this setting using graph states[24], and indeed modifies the subleading constant term of the entanglement entropy. If one tries a (hyper-)cylinder extrapolation method to read off the subleading term, then our ideas here give translation-invariant states with a modified subleading term. As remarked before, our examples in this higher dimensional generalization will be fine-tuned, but the length scale where the finite size effect is relevant can be arbitrarily larger than usual correlation lengths.

Besides, it is natural to consider topological entropy for thermal states. An immediate problem is that the entropy of the reduced density matrix of a thermal state obeys a volume law. This is easily overcome by using mutual information,[26] which obeys an area law at any nonzero temperature.[44] In 2D, while every known *ground* state with a nonzero TEE requires a local unitary transformation (quantum circuit) of large depth (linear in system size) in order to be transformed to a product state, every Gibbs state at any nonzero temperature of any commuting Hamiltonian can be transformed by a quantum circuit of small depth (logarithmic in system size) into a Gibbs state of a classical Hamiltonian.[28] This is consistent to the calculation of topological entropy of the 2D $\mathbb{Z}_2$ gauge theory at nonzero temperature where the subleading term of mutual information is shown to vanish.[26]

In 3D, the entropies on solid torus of $\mathbb{Z}_2$ lattice gauge theory at nonzero temperature have been calculated.[27] It is observed that at low nonzero temperature, a nonzero subleading term survives when a certain linear combinations of mutual information is used to cancel extensive parts. But Ref. 28 also shows that the Gibbs state of this model at nonzero temperature can be connected to the Gibbs state of a classical Hamiltonian by a small-depth quantum circuit. (See also Ref. 45.) Therefore, on the contrary to the 2D case, this value being nonzero is not related to topological order in the sense of generating quantum circuits of large depth. This means that if we accept the complexity of the generating quantum circuit as the definition of topological order for thermal states, then we should conclude that the subleading term of mutual information does not give an 'order parameter' for topological order for thermal states in 3 or higher dimensions.

At nonzero temperature, it appears that our examples only give a contribution that is exponentially small in the system size to the subleading term (although the length scale of our effect is still different from the usual correlation length). The mutual information of the 2D cluster state Eq. (8) at finite inverse temperature β with respect to the bipartition in Fig. 3 is

$$I(A:B) = 2L(\log 2 + \mathcal{O}((1-t)\log(1-t))) - \mathcal{O}(t^{2L}) \tag{65}$$

where $t = \tanh\beta$ is close to but smaller than 1. The detail of the calculation can be found in Appendix C.

Finally, we note there exists a notion of localizable entanglement and associated entanglement length,[46] whose divergence is connected to a string order parameter.[47] It is shown that a subclass of our nontrivial 1D SPT can be used as a perfect quantum repeater.[48] However, it remains unclear how the entanglement length is related to our replica correlation length ξ_α. A technical difference is that in our definition ξ_α carries an (artificial) index α, as it is defined by the eigenvalues of the transfer matrix $\mathbb{T}_\alpha$ for ρ^α.

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Appendix A: Intertwined chains of cluster state

We present examples where the sub-leading term γ has a system-size-dependent oscillation. Recall the local ten-

sor for the cluster state Eq. (29):

$$M^{(0)} = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 0 & 0\end{pmatrix}, \quad M^{(1)} = \frac{1}{\sqrt{2}}\begin{pmatrix}0 & 0\\ 1 & -1\end{pmatrix} \tag{A1}$$

where the superscripts are the physical indices. Define a local tensor $K^{(ab)}$ with two physical qubits per site and bond dimension 4 as

$$K^{(ab)} = \left(M^{(a)} \otimes M^{(b)}\right)\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}. \tag{A2}$$

We trace out the physical qubit a and keep b. One can verify that the transfer matrix for $\alpha = 2$ has four nonzero eigenvalues $\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}$. Thus, the entanglement entropy is

$$S = -\log\left[3\left(\frac{1}{2}\right)^L + \left(-\frac{1}{2}\right)^L\right] \tag{A3}$$

under the periodic boundary condition. We omitted Rényi index α; in fact, this formula is true for any $0 \le \alpha \le \infty$ since the distribution of nonzero Schmidt coefficients (entanglement spectrum) is flat.

This local tensor represents two independent cluster states that are intertwined. When the chain length L is even, there are two symmetries supported entirely on a qubits. But, these symmetries cannot be defined separately when L is odd. The even-odd behavior may be attributed to the extra symmetry when L is even, which are not uniformly on-site.

More generally, one can define an MPS such that the subleading term of the entanglement entropy has periodicity n as a function of chain length L for any positive integer n. Eq. (A2) will corresponds to $n = 2$. The construction is to imagine a helix of n strands, project it into a plane, put a qubit to each outer vertex, and interpret the line between vertices as the bonds of the cluster state. Under periodic boundary condition with length L, the number of distinct strands is given by $\gcd(n, L)$. Still, the translation-invariance of the state is observed. The local tensor can be given as

$$K^{(ab)} = \left(M^{(a)} \otimes I^{\otimes(n-2)} \otimes M^{(b)}\right)\mathcal{C} \tag{A4}$$
$$\mathcal{C} = \text{cyclic rotation of tensor factors.}$$

The entanglement entropy for any Rényi index $0 \le \alpha \le \infty$ is given by

$$\begin{aligned} S &= -\log\left[\sum_{k=1}^{n} c_k \left(\frac{e^{2\pi i k/n}}{2}\right)^L\right] \\ &= L\log 2 - \gcd(L,n)\log 2 \end{aligned} \tag{A5}$$

where c_k are multiplicities of the transfer matrix' eigenvalues. We do not compute c_k from the transfer matrix, but they have to be determined by this formula since $L \mapsto \gcd(L, n)$ is a periodic function. The appearance of gcd is because there are $\gcd(L, n)$ rings of cluster states. This proves an interesting statement that

$$c_k = \frac{1}{n}\sum_{j=1}^{n} e^{2\pi i j k/n} 2^{\gcd(j,n)} \tag{A6}$$

are nonnegative integers for all $k = 1, 2, \ldots, n$.

The examples of this section pose a challenge to unconditionally define the subleading term γ. For example, the following limit

$$\lim_{L\to\infty} S(L) - L(S(L+1) - S(L)) \tag{A7}$$

does not exist for the MPS state in Eq. (A2). Even a Kitaev-Preskill-like combination, canceling off the length (area) contribution, will not converge. One might inevitably have to introduce some perturbation to define γ, as our examples are not generic in the absence of any symmetry.

Appendix B: Time reversal and lattice symmetries

In this appendix, we raise a question whether other symmetries on a 1D chain reduced from a 2D chain can give rise to a robust finite sub-leading term of the entanglement entropy. We consider three kinds of symmetries: the time reversal $\mathcal{T}$, lattice reflection $\mathcal{R}$, and lattice inversion $\mathcal{I}$. For 1D systems the lattice reflection and the inversion may coincide. The reason we are distinguishing the two is that the 1D chain is divided into two parts, upper and lower, across the entanglement cut. Upon the lattice reflection, the upper part and the lower part are not exchanged; however, by the lattice inversion, which amounts to π-rotation about a point, the two parts are exchanged.

The scope of this appendix is restricted to situations where the symmetry group G is $G = \mathcal{T}$, $G = \mathcal{R}$, $G = \mathcal{I}$, $G = \mathcal{T}\times\mathcal{R}$, or $G = \mathcal{T}\times\mathcal{I}$. The argument below does not apply to a situation where $G = \mathcal{R}\times\mathcal{I}$ or $G = \mathcal{T}\times\mathcal{R}\times\mathcal{I}$.

We will find symmetry-respecting deformations of states, after which the entanglement entropy becomes

$$S_{\text{new}} = \alpha L \tag{B1}$$

for some α, or

$$S_{\text{new}} = \text{const.} \ge 0. \tag{B2}$$

This will prove that if the non-positive subleading term $(-\gamma)$ is robust under those symmetries, it must be zero. This is in contrast to the situation where the 1D state form a nontrivial SPT under a product group of internal symmetry, where a strictly negative subleading term $(-\gamma)$ is stabilized by the internal symmetry.

1. Lattice reflection

Recall that our 1D chain has two physical qudits, a_i and b_i, at each site i. The entanglement cut separates the physical qudits so that all a_i qudits are in one partition and all b_i are in another. The lattice reflection is realized as

$$\mathcal{R} : a_i \leftrightarrow a_{-i},\ b_i \leftrightarrow b_{-i} \tag{B3}$$

for all $i = \ldots, -2, -1, 0, 1, 2, \ldots$.

Consider inserting auxiliary qudits c_i in the product state into the chain. Each site is now consisting of a_i, b_i, c_i, and we assume that b_i, c_i belong to the same partition with respect to the entanglement cut. Now we introduce the unitary operator W_i on site i that implements the swap between a_i and c_i (see Fig. 11):

$$W = \sum_{u,v} |v, u\rangle \langle u, v| \tag{B4}$$

The uniform application $\prod_i W_i$ obviously respects the lattice reflection symmetry $\mathcal{R}$. Since c_i were in the product state, they had no entanglement with the rest. Thus, after the swap operation, the entanglement entropy becomes identically zero across the existing cut. One can implement W smoothly since the unitary group is connected. In this way, we have found a smooth deformation of the state such that the entanglement entropy in the final state is simply zero. In particular, we have smoothly changed the subleading term, if any, to zero.

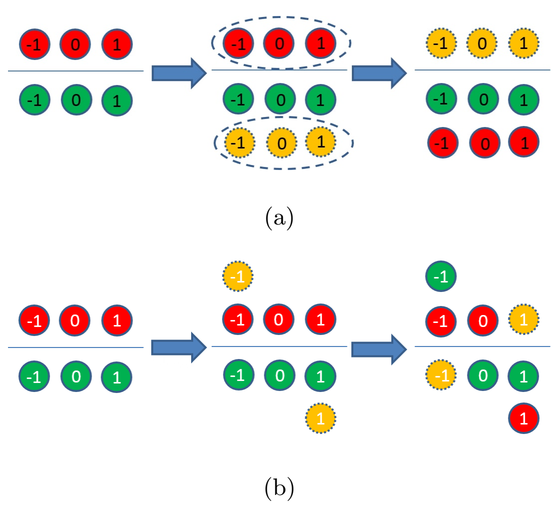

FIG. 11. Deforming the state while respecting the lattice reflection or inversion symmetry. From the original 1D chain which consists of red (a_i) and green (b_i) qudits, one can insert an auxiliary yellow (c_i) qudit. Then one can apply the swap operator that exchanges the red qudits and yellow qudits circled by the dashed ellipses. This swap operation can be implemented continuously without breaking the lattice reflection symmetry. The numbers in each qudits label the positions of the corresponding qudits before the swap operation. It is understood that we have a 1D chain of qudits, although only a few sites are shown here.

2. Lattice inversion without translation

The lattice inversion is implemented as

$$\mathcal{I} : a_i \leftrightarrow b_{-i} \tag{B5}$$

for all $i = \ldots, -1, 0, 1, \ldots$. Similarly as in the previous subsection, by introducing auxiliary qudits in the product state and swap unitary, one can push the physical qudits, expect those at $i = 0$, to one side of the entanglement cut, while respecting the lattice inversion symmetry (see Fig.11). The deformed state can be viewed as a 1D state where the entanglement cut divides the chain into halves of length $L/2$. The entanglement entropy does not depend on the system size (the "area" law of entanglement entropy) and is equal to some positive constant h. If the chain was a nontrivial SPT under this inversion symmetry, such as the Haldane spin-1 chain, then h cannot be made to become zero; if it was trivial SPT, then a smooth deformation such that $h \to 0$ is possible.

If we used Levin-Wen combination to define the subleading term $(-\gamma)$, then

$$\gamma = S_{AB} + S_{BC} - S_{ABC} - S_B \tag{B6}$$

which is nonnegative by the strong subadditivity. (At this point, we should not use the Rényi entropy, but von Neumann) The deformed state clearly gives $\gamma = 0$.

3. Time reversal

We assume that the system consists of spin-J's, and the time reversal is implemented by

$$\mathcal{T} = e^{-i\pi J_y} K \tag{B7}$$

for spin systems in the J_z-basis, where K is the complex conjugation. We will construct a similar deformation as in the previous cases. We insert auxiliary spins in product states, and swap the spins of the original chain with the auxiliary ones. Complication arises from two sources: The first one is that a half-integer spin cannot be time-reversal invariant. This is easily resolved by inserting singlets formed by two spins. The second one is that the swap W and its smooth implementation $W(t)$ must commute with $\mathcal{T}$. To resolve the second one, we will shortly prove that there exists $W_2(t)$ for any J such that

$$W_2(t = 0) = I, \quad W_2(t = \pi) = W \otimes W, \\ [W_2(t), \mathcal{T}] = 0 \text{ for all } t. \tag{B8}$$

Equipped with $W_2 = W_2(t)$, we can deform the state so that the final state has entanglement entropy

$$S = \alpha_J L \tag{B9}$$

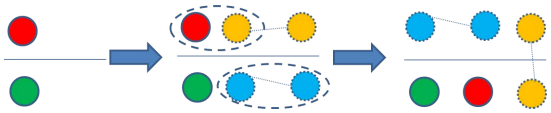

FIG. 12. Deforming state while respecting symmetry. In addition to the red (a_i) and green (b_i) qudits, a pair of time reversal-invariant spin singlets are inserted in each site (the yellow qudits are a'_i and blue qudits are b'_i). The swap unitary is applied to the qudits circled by the dashed ellipses, so that the entanglement across the cut solely arises from the inserted singlet. The swap can be implemented continuously during which the time-reversal symmetry is unbroken. It is understood we have a 1D chain of qudits, although only one site is shown here.

exactly without any subleading term, where $\alpha_J = \log(2J+1)$ is the entanglement entropy of a singlet consisting of two spin-J's. To see this, insert singlets a'_i to the partition where a_i belong, and another set of singlets b'_i to the partition where b_i belong. Note that each of a'_i or b'_i consists of two spins, whereas each of a_i or b_i consists of one spin. Apply W_2 such that the pair of a_i and one auxiliary spin from a'_i is exchanged with the whole singlet b'_i. See Figure 12. The original b_i is not moved at all, and a_i is brought to the partition where b_i belongs. The singlet b'_i is moved to the opposite partition, and the singlet a'_i is now shared between the entanglement partitions. Thus, the entanglement entropy of the deformed state entirely comes from the singlets a'_i, and Eq. (B9) holds.

Remark that this time-reversal invariant deformation respects lattice reflection and translation symmetry, if they were present in the original state. The deformation using W_2 can also be adapted to a situation where there is a lattice inversion symmetry.

We now construct the promised $W_2(t)$. In the basis where J_z is diagonal, we will show that there exists a real orthogonal matrix $W_2(t)$ such that it commutes with $(e^{-i\pi J_y})^{\otimes 4}$, and it smoothly interpolates between the identity and $W \otimes W$. Observe that $R = e^{-i\pi J_y}$ is a real matrix since $J_y = (J_+ - J_-)/2i$ is purely imaginary. Since J_y is hermitian, we have $R^T = R^\dagger = R^{-1}$. Moreover, R^2 is $+1$ for integer spins or -1 for half-integer spins. Therefore, $(R\otimes R)^2 = R^2 \otimes R^2 = 1$. It follows that $R^{\otimes 2}$ is real symmetric with eigenvalues ± 1. The swap matrix W is obviously real symmetric and squares to 1. Since W and $R^{\otimes 2}$ commute, they can be simultaneously diagonalized by a real orthogonal matrix. Likewise, $W^{\otimes 2}$ and $R^{\otimes 4}$ can be simultaneously diagonalized, and there exists a real orthonormal basis $|w = \pm 1, r = \pm 1, k\rangle$ consisting of common eigenvectors of $W^{\otimes 2}$ and $R^{\otimes 4}$, where w, r are the eigenvalue of $W^{\otimes 2}$ and $R^{\otimes 4}$, respectively, and the index k runs from 1 to the degeneracy $k_{w,r}$ of the common eigenspace. We claim that both $k_{-1,+1}$ and $k_{-1,-1}$ are always even. Given this claim, we can construct $W_2(t)$ by

$$W_2(t)|_{\mathrm{span}\{|-1,r,2m-1\rangle,|-1,r,2m\rangle\}} = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \tag{B10}$$

for $m = 1, \ldots, k_{-1,r}/2$, and the identity on $w = +1$ subspace. The constructed $W_2(t)$ is clearly real orthogonal, and commutes with $R^{\otimes 4}$ since it preserves the eigenspaces of $R^{\otimes 4}$. We have $W_2(0) = I$ by definition, and $W_2(\pi) = +1$ on the $w = +1$ subspace and $W_2(\pi) = -1$ on the $w = -1$ subspace; hence, $W_2(\pi) = W^{\otimes 2}$.

It remains to compute the degeneracy $k_{-1,r}$ to show that it is even. Let $\{|a\rangle : a = 1, \ldots, 2J+1\}$ be a complete orthonormal set of eigenvectors of R. These $|a\rangle$ may not be real vectors, but the degeneracy of eigenspaces can be computed with respect to any basis we choose. Then, $(|ab\rangle \pm |ba\rangle)/\sqrt{2}$ are complete common eigenvectors of $R \otimes R$ and the swap W. Then, the (-1)-eigenvectors of $W^{\otimes 2}$ in an eigenspace of $R^{\otimes 4}$ are

$$\frac{(|ab\rangle - |ba\rangle)(|cd\rangle + |dc\rangle)}{2}, \quad \frac{(|ab\rangle + |ba\rangle)(|cd\rangle - |dc\rangle)}{2} \tag{B11}$$

which always come in pairs. The degeneracy is $k_{-1,r} = 2\frac{N(N-1)}{2}\frac{N(N+1)}{2} = \frac{1}{2}N^2(N+1)(N-1)$, an even number for any $N = 2J+1$.

Without introducing W_2, a continuous real implementation of W alone from the identity is not possible. For spin-$\frac{1}{2}$, the swap W has determinant -1, so W belongs to the non-identity component of $O(4)$.

Appendix C: Cylinder extrapolation method on the 2D cluster state at nonzero temperature

Here we calculate the mutual information for the cluster state across the circumferential cut of a cylinder. We consider the geometry of Fig. 3. The mutual information

$$I(A:B) = S(A) + S(B) - S(AB) \tag{C1}$$

is preferred to the entropy $S(A)$ (or $S(B)$) because it obeys an area law even at finite temperatures.[26,44]

The Hamiltonian is given by Eq. (8). By definition of the entropy, the mutual information is invariant under local unitary in either region A, B, and it is oblivious to any tensor product factor. Hence, by the same argument as for the ground state of the cluster state, the mutual information of the 2D cluster state reduces to that of mutual information of the 1D cluster state with the extensive bipartition. The Gibbs state at inverse temperature β is given by

$$\rho_{AB} = \mathcal{Z}^{-1} \prod_j^{2L} \exp\left(\beta \sigma^z_{j-1}\sigma^x_j\sigma^z_{j+1}\right) \tag{C2}$$

$$= \mathcal{Z}^{-1} \prod_j^{2L} \left(I\cosh\beta + \sigma^z_{j-1}\sigma^x_j\sigma^z_{j+1}\sinh\beta\right). \tag{C3}$$

The partition function $\mathcal{Z}$ is equal to that of uncoupled $2L$ spins in a magnetic field,

$$\mathcal{Z} = 2^{2L} \cosh^{2L} \beta, \tag{C4}$$

and the spectrum of ρ_{AB} is the tensor product of $2L$ identical spectra $\{\frac{1}{2}(1 \pm \tanh \beta)\}$. Hence, the von Neumann entropy is

$$S(AB) = 2Lf\left(\frac{1+t}{2}\right) \tag{C5}$$

where $t = \tanh \beta$ and $f(x) = -x \log x - (1-x)\log(1-x)$ is the binary entropy function.

Generalizing the result in Sec. V A, we get the reduced density matrix for A

$$\rho_A = \mathrm{Tr}_B(\rho_{AB}) = \frac{1}{2^L}\left(I + t^L \prod_{j\in A} \sigma_j^x\right), \tag{C6}$$

whose entropy is

$$S(A) = 2^{L-1}\left(\frac{1+t^L}{2^L}\log\frac{2^L}{1+t^L} + \frac{1-t^L}{2^L}\log\frac{2^L}{1-t^L}\right) \tag{C7}$$

$$= L\log 2 - \frac{1}{2}\log(1-t^{2L}) - \frac{t^L}{2}\log\frac{1+t^L}{1-t^L}. \tag{C8}$$

Therefore, the mutual information is

$$I(A:B) = 2L(\log 2 - f((1+t)/2)) - \log(1-t^{2L}) - t^L \log\frac{1+t^L}{1-t^L} \tag{C9}$$

$$= 2\alpha_t L - t^{2L} + O(t^{4L}) \tag{C10}$$

where $\alpha_t = O(t^2)$ for small $t = \tanh\beta$ and $\alpha_t \sim \log 2$ for $t \sim 1$.

In conclusion, at any finite β, the subleading term is exponentially small in L. Note that for any $0 < \beta \leq \infty$, the usual correlation length of the cluster state is zero since it differ from a product Gibbs state by a quantum circuit of depth 2. So, the length scale of the subleading term is greater than the usual correlation length, and is diverging as $\beta \to \infty$.

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