Dielectric function and plasmons in graphene: A self-consistent-field calculation within a Markovian master equation formalism

F. Karimi, A. H. Davoody, and I. Knezevic

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Dielectric function and plasmons in graphene: A self-consistent-field calculation within a Markovian master-equation formalism

F. Karimi, A. H. Davoody, and I. Knezevic
Department of Electrical and Computer Engineering, University of Wisconsin-Madison, Madison, Wisconsin 53706, USA

We introduce a method for calculating the dielectric function of nanostructures with an arbitrary band dispersion and Bloch wave functions. The linear response of a dissipative electronic system to an external electromagnetic field is calculated by a self-consistent-field approach within a Markovian master-equation formalism (SCF-MMEF) coupled with full-wave electromagnetic equations. The SCF-MMEF accurately accounts for several concurrent scattering mechanisms. The method captures interband electron-hole-pair generation, as well as the interband and intraband electron scattering with phonons and impurities. We employ the SCF-MMEF to calculate the dielectric function, complex conductivity, and loss function for supported graphene. From the loss-function maximum, we obtain plasmon dispersion and propagation length for different substrate types [non-polar diamondlike carbon (DLC) and polar SiO$_2$ and hBN], impurity densities, carrier densities, and temperatures. Plasmons on the two polar substrates are suppressed below the highest surface phonon energy, while the spectrum is broad on the nonpolar DLC. Plasmon propagation lengths are comparable on polar and nonpolar substrates and are on the order of tens of nanometers, considerably shorter than previously reported. They improve with fewer impurities, at lower temperatures, and at higher carrier densities.

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I. INTRODUCTION

Surface plasmon-polaritons, often referred to as plasmons for brevity, are hybrid excitations that can arise from the interaction of the electron plasma in good conductors with external electromagnetic fields. Plasmonics, the field of study of plasmon dynamics, has attracted considerable interest in recent years as a promising path towards the miniaturization of nanophotonics. Conventionally, noble metals are considered as plasmonic materials, with diverse applications such as integrated photonic systems, magneto-photonic structures, metamaterials and cloaking, biosensing, and photovoltaic devices. In the visible to near-infrared spectrum, however, at THz to mid-infrared frequencies, which have various applications in information and communication, biology, chemical and biological sensing, homeland security, and spectroscopy, metal plasmonic materials are quite lossy. Therefore, there is considerable interest in low-loss plasmonic materials at these frequencies.

Graphene, the two-dimensional allotrope of carbon, is a promising plasmonic material. Its resonance typically falls in the THz to mid-infrared range and it shows significantly different screening properties and collective excitations than the quasi-2D systems with parabolic electron dispersion. Moreover, graphene’s carrier density is easily tunable by an external gate voltage, which enables electrostatic control of its electronic and optical properties.

Plasmons in graphene have been experimentally excited and visualized by several methods, such as the electron energy-loss spectroscopy (EELS) and Fourier-transform infrared spectroscopy (FTIR) and scanning near-field optical microscopy (SNOM). Theoretical studies have also been performed: the momentum-independent graphene conductivity was calculated within the local phase approximation used along with finite-element and transfer matrix methods in order to solve the electromagnetic equations for the plasmon modes. However, plasmons are in the nonretarded regime at mid-infrared frequencies, so the momentum dependence of the optical response should be considered. The Lindhard dielectric function $\varepsilon(q,\omega)$ has a dependence on both frequency and wave vector. It captures interband transitions due to the electromagnetic field and is based on the random-phase approximation (RPA). The RPA can also explain the coupling between graphene plasmons and surface optical (SO) phonons on polar substrates, a phenomenon that has been captured experimentally.

Graphene plasmon modes have much shorter wave lengths than light with the same frequencies, and their propagation length is very sensitive to the damping pathways, such as intrinsic phonons, ionized impurities, and SO phonons on polar substrates. The damping of graphene plasmons has been calculated based on the Mermin-Lindhard (ML) dielectric function and its simplified version, the Drude dielectric function. The ML dielectric function stems from Mermin’s derivation of the Lindhard expression based on a master equation and within the relaxation-time approximation (RTA), with a single effective relaxation time accounting for all scattering. However, the ML dielectric function was derived for electronic systems with a parabolic band dispersion and is also only valid when intraband dissipation mechanisms dominate. In the case of nanostructures with nonparabolic dispersions, densely
spaced energy subbands, or in the presence of efficient in-
terband dissipation mechanisms (e.g., optical phonons),
the ML dielectric function provides an incomplete pic-
ture.

With multiple concurrent scattering mechanisms, it is
common to extract an averaged relaxation time corre-
sponding to each mechanism separately and then cal-
culate the total relaxation time based on Mattheissen’s
rule.\textsuperscript{[33,40]} However, Mattheissen’s rule technically holds
only when all mechanisms have the same relaxation time
vs energy dependence.\textsuperscript{[60]} Also, employing a single energy-
dependent relaxation time is generally not a good ap-
proximation for systems with pronounced non-Coulomb
scattering mechanisms\textsuperscript{[60,63]} such as phonons in nonpo-
lar materials; indeed, the use of a single relaxation time
has been shown not to accurately capture the loss mech-
anisms in suspended and, to a lower degree, supported
graphene.\textsuperscript{[63]} What is needed is an accurate (and, ide-
ally, computationally inexpensive) theoretical approach
that can treat the interaction of light with charge car-
dies in graphene (and in related nanomaterials) in the
presence of both interband and intraband transitions due
to multiple competing scattering mechanisms, where the
transition rates can have pronounced and widely differing
dependencies on both carrier energy and momentum.

In this paper, we present a method for calculat-
ing the dielectric function of a dissipative electronic
system with an arbitrary band dispersion and Bloch
wave functions by a self-consistent-field approach within
an open-system Markovian master-equation formalism
(SCF-MMEF) coupled with full-wave electromagnetic
equations (Sec. II). We derive a generalized Markovian
master equation\textsuperscript{[225,32]} of the Lindblad form (i.e., conserv-
ing the positivity of the density matrix), which includes
the interaction of the electronic system with an external
electromagnetic field (to first order) and with a dissipa-
tive environment (to second order). We solve for the time
evolution of coherences and calculate the induced charge
density as a function of the self-consistent field within
linear response. Based on the electrodynamic relation
between the induced charge density and the induced po-
tential, we obtain the expressions for the polarization,
dielectric function, and conductivity (Sec. II). Numeri-
cal implementation is achieved with the aid of Brillouin-
zone discretization, and the resulting matrix equations
are readily solved using modern linear solvers (Sec. II C).

We use the SCF-MMEF to calculate the graphene di-
electric function, complex conductivity, and loss function
(Sec. II). The complex conductivity agrees well with ex-
periment and its DC limit shows a well-known sublinear-
ity at high carrier densities. At low carrier densities, the
screening length depends on the impurity density, which
is a phenomenon that the ML approach cannot capture
(Sec. II A). The plasmon modes and their propagation
length are obtained from the loss-function maximum
(Sec. II C). We did our calculations for three substrates:
diamondlike carbon (DLC)\textsuperscript{[40,67]} as a non-polar substrate,
and SiO\textsubscript{2}\textsuperscript{[64,74–76]} and hBN\textsuperscript{[71–73]} as polar substrates, with the
effect of SO phonons captured via a weak Fröhlich term.
We investigate the effects of substrate type, substrate im-
purity density, carrier density, and temperature on the
dielectric function and plasmon characteristics. Plasmon
dispersion is fairly insensitive to varying impurity den-
sity or temperature, but is pushed towards shorter wave
vectors with increasing carrier density. Plasmons are
strongly suppressed on polar substrates below the sub-
strate optical-phonon energies. Plasmon damping – in-
versely proportional to the number of wave lengths that
a plasmon propagates before dying out – worsens with
more pronounced scattering, such as when the impurity
density or temperature are increased. However, damping
drops with increasing carrier density, benefiting plasmon
propagation. Overall, the plasmon propagation length in
absolute units improves on substrates with fewer impu-
rities, and is better at low temperatures and at higher
carrier densities. Plasmon propagation lengths on polar
and nonpolar substrates are comparable and in the tens
of nanometers, an order of magnitude lower than previ-
ously reported.\textsuperscript{[33,40]} We conclude with Sec. IV.

II. THEORY

In this section, we present a detailed derivation of the
SCF-MMEF in order to calculate the dielectric function
and the conductivity of graphene, a two-dimensional ma-
terial. We note that the SCF-MMEF is quite general
and can be applied to electronic systems of arbitrary di-
menisonality (see Appendix A for quasi-one-dimensional
materials).

Single-layer graphene has a density-independent
Wigner-Seitz radius \( r_s < 1 \) (\( r_s \approx 0.5 \) for graphene on
SiO\textsubscript{2}\textsuperscript{[53,72]}), owing to linear electron dispersion.
Therefore, it can be considered a weakly interacting system in which
the random-phase approximation (RPA, the equivalent of the SCF approx-
mation we use here) is valid. Intuitively, the question is whether the carriers in graphene are
effective at screening. For graphene on a substrate, the
affirmative answer is supported by experimental obser-
vations of Drude-like behavior of the electronic system in
frequency-dependent conductivity measurement.\textsuperscript{[64,74–76]}\textsuperscript{[10]}
This holds even at low carrier densities, as electrons and
holes form puddles, so the carrier density locally ex-
ceeds the impurity density.\textsuperscript{[53,72]} (We note that, in bilayer
graphene and at lower carrier densities, the SCF/RPA ap-
proximation becomes more difficult to justify\textsuperscript{[76,79]}.)

A. The self-consistent-field approximation

The dielectric function describes how the electron sys-
tem screens a perturbing potential \( V_{\text{ext}} \). The system re-
response results in an induced potential \( V_{\text{ind}} \) that stems
from electron-electron interactions. The two give rise to
a combined self-consistent field, \( V_{\text{SCF}} = V_{\text{ind}} + V_{\text{ext}} \).
We assume the graphene sheet is in the \( xy \)-plane (\( z = 0 \)). The induced charge density \( n \) will be of the form

\[
n(r, z, t) = n_s(r, t)\delta(z).
\]

\( \delta(\cdot) \) denotes the Dirac delta function, \( r \) is a vector in the \( xy \)-plane, and \( n_s \) is the sheet density. Then, by taking a temporal Fourier transform (\( \omega \) is the angular frequency) and a two-dimensional spatial Fourier transform over \( r \) (\( q \) is the in-plane wave vector), the inhomogeneous wave equation for the induced potential is given by

\[
\left[ \frac{\partial}{\partial z^2} + \left( iQ \right)^2 \right] V_{\text{ind}}(q, z, \omega) = -\frac{e^2}{\varepsilon_b(\omega)\varepsilon_0} n_s(q, \omega)\delta(z),
\]

where \( \varepsilon_b(\omega) \equiv \frac{1+\varepsilon_r(\omega)}{2} \) represents the background lattice dielectric function, while \( \varepsilon_r(\omega) \) is the complex dielectric function of the substrate (Appendix B) and \((iQ)^2 = \frac{\varepsilon_r(\omega)\varepsilon_0^2}{\varepsilon_0} - q \cdot q \). In general, the Fourier transform of an arbitrary function \( f(r, z, t) \) is defined as \( \mathcal{F}\{f(r, z, t)\} = A^{-1} \int f(r, z, t)e^{-iqr}e^{iQz}drdt \), where \( A \) is the sample area.

By employing the Green’s function analysis, the solution of Eq. \([3]\) becomes

\[
V_{\text{ind}}(q, z = 0, \omega) = -\frac{e^2}{\varepsilon_b(\omega)\varepsilon_0} \frac{n_s(q, \omega)}{2Q}.
\]

Here, we assumed that the potential does not vary significantly across the graphene sheet. To simplify the notation, from now on we drop the \( z \) argument, and remember that all the wave vectors are in the plane of graphene.

The dielectric function of the graphene sheet is defined as \( \varepsilon(q, \omega) = V_{\text{ext}}(q, \omega)/V_{\text{SCF}}(q, \omega) \). Now, by assuming \( Q \approx |q| = q \) as a valid approximation in the non-retarded regime of interest, and using Eq. \([3]\), the dielectric function may be written as

\[
\varepsilon(q, \omega) = 1 + \frac{e^2}{\varepsilon_b(\omega)\varepsilon_0} \frac{1}{2q} n_s(q, \omega) V_{\text{SCF}}(q, \omega),
\]

where we have introduced the surface polarization, \( P_s(q, \omega) = \frac{n_s(q, \omega)}{V_{\text{SCF}}(q, \omega)} \). Also, we can derive an expression for the conductivity of the graphene sheet. The continuity equation in the frequency and momentum domain reads

\[
\omega \epsilon n_s(q, \omega) = q \cdot J_s(q, \omega),
\]

where \( J_s(q, \omega) \) is the surface current density. \( J_s(q, \omega) = \sigma_s(q, \omega) E_{\text{SCF}}(q, z = 0, \omega) \), \( \sigma_s \) being the sheet conductivity and \( -ie\epsilon E_{\text{SCF}} = -iqV_{\text{SCF}} \). Now, the graphene sheet conductivity is written as

\[
\sigma_s(q, \omega) = \frac{-ie\omega}{q^2} P_s(q, \omega).
\]

The surface polarization \( P_s(q, \omega) \), needed to obtain \( \varepsilon \) and \( \sigma \), carries the information about electron interactions in the material and is calculated next, based on a Markovian master-equation formalism.

**B. Markovian master-equation formalism**

We consider a quantum-mechanical electronic system that interacts with an environment. Assuming \( H_e \) to be the unperturbed Hamiltonian of free electrons in a lattice, its eigenkets and eigenenergies are represented by \( |k| \) and \( \epsilon_{kl} \), respectively. \( k \) is the in-plane electron wave vector and \( l \) denotes the band index. The spatial representation of the single-particle Bloch wave functions (BWF) corresponding to \( |k| \) has the form of

\[
(r, z|k|) = \frac{1}{\sqrt{N_{uc}}} e^{iK \cdot r} u_{kl}(r, z).
\]

\( N_{uc} \) is the number of unit cells in a finite-sized graphene sample. The induced charge density in the second quantization form is given by

\[
n(r, \omega) = -\frac{1}{N_{uc}} \sum_{\mathbf{k}, \mathbf{q}, l, l'} u_{k+q, l}^* u_{k, l} e^{-i\mathbf{q} \cdot \mathbf{r}} \langle c_{k, l}^\dagger c_{k+q, l'} \rangle.
\]

\( c^\dagger \) and \( c \) are the electron creation and destruction operators, respectively. \( \langle \Omega \rangle \equiv \langle \mathcal{T} \rangle \langle \Omega \mathcal{T} \rangle \) is the expectation value of an electronic operator \( \Omega \) and \( \mathcal{T} \) is the many-body statistical operator (also referred to as the many-body density matrix). Taking the Fourier transform in the \( xy \)-plane, and integrating over the \( z \) direction yield the induced surface charge density

\[
n_s(q, \omega) = -\frac{1}{A} \sum_{\mathbf{k}, \mathbf{q}, l, l'} \langle c_{k, l}^\dagger c_{k+q, l'} \rangle (|k| + \mathbf{q})
\]

where \( A \) is the area of the graphene sheet. We defined the following overlap integrals over a unit cell as

\[
\langle \mathbf{k} + \mathbf{q} \mathbf{l}' | \mathbf{k} \mathbf{l} \rangle = \int_{\mathbf{uc}} d^3 r u_{k+q, l'}^*(r') u_{k, l}(r').
\]

Equation \( [9] \) shows that the expectation value of coherences, \( \langle c_{k, l}^\dagger c_{k+q, l'} \rangle \), is required to calculate the induced charge density. The time dependence of the expectation value of coherences is calculated through a quantum master equation.

The total Hamiltonian of an open electronic system within the SCF approximation is

\[
\mathcal{H}(t) = \mathcal{H}_e + \mathcal{V}_{\text{SCF}}(t) + \mathcal{H}_{\text{col}} + \mathcal{H}_{\text{ph}}.
\]

\( \mathcal{H}_{\text{ph}} \) denotes the free Hamiltonian of the phonon bath. \( \mathcal{H}_{\text{col}} \), where subscript “col” stands for collisions, is the sum of the interaction Hamiltonian of electrons with...
phonons ($\mathbb{H}_{\text{e-ph}}$) and the interaction Hamiltonian of electrons with ionized impurities ($\mathbb{H}_{\text{e-ii}}$). $\mathbb{V}_{\text{SCF}}(t)$, $\mathbb{H}_{\text{e-ph}}$, and $\mathbb{H}_{\text{e-ii}}$ have the forms

$$\mathbb{V}_{\text{SCF}}(t) = \sum_{k,l,l'} (k + q' | \mathbb{V}_{\text{SCF}}(t) | kl') c_{k+q'}^l c_{kl},$$

$$\mathbb{H}_{\text{e-ph}} = \sum_{kq,l'} \mathcal{M}_{\text{ph}}(q) (k + q' | kl') c_{k+q'}^l c_{kl} (b_q + b_q^\dagger),$$

$$\mathbb{H}_{\text{e-ii}} = \sum_{kq,l'} \mathcal{M}_{\text{ii}}(q) (k + q' | kl') c_{k+q'}^l c_{kl}.$$

(12)

$b$ and $b^\dagger$ are the phonon creation and destruction operators, respectively. The interaction Hamiltonian can be written as

$$\mathbb{H}_{\text{col}} = \sum_{\alpha} \mathbb{A}_{\alpha} \otimes \mathbb{B}_{\alpha},$$

(13)

where $\alpha$ is the set of variables $\{kq, l'l'g\}$, and $g$ determines the scattering mechanism. $\mathbb{A}$ operates on the electron and $\mathbb{B}$ on the phonon subspace. For ionized-impurity scattering $\mathbb{B}$ is equal to unity operator.

The equation of motion for the statistical operator in the interaction picture (in units of $\hbar = 1$) is

$$\frac{d}{dt} \tilde{\rho}(t) = -i[\mathbb{H}_{\text{int}}(t), \tilde{\rho}(t)],$$

$$\tilde{\rho}(t) = \tilde{\rho}(0) - i \int_0^t [\mathbb{H}_{\text{int}}(t'), \tilde{\rho}(t')] dt'.$$

(14)

The tilde denotes that the operators are in the interaction picture, i.e., $\tilde{\mathcal{O}}(t) = e^{i(\mathbb{H}_{\text{e}} + \mathbb{H}_{\text{ph}}) t} \mathcal{O} e^{-i(\mathbb{H}_{\text{e}} + \mathbb{H}_{\text{ph}}) t}$. The collision Hamiltonian in the interaction picture takes the form

$$\mathbb{H}_{\text{col}}(t) = \sum_{\alpha} e^{-i\Delta_{\alpha} t} \mathbb{A}_{\alpha} \otimes \mathbb{B}_{\alpha}(t),$$

(15)

where $\mathbb{B}_{\alpha}(t) \equiv e^{i\mathbb{H}_{\text{ph}} t' \mathbb{B}_{\alpha} e^{-i\mathbb{H}_{\text{ph}} t}$ and $\Delta_{\alpha} = \epsilon_{k_{\alpha}l_{\alpha}} - \epsilon_{k_{\alpha}+q_{\alpha}l_{\alpha}}$ are used. We assume an uncorrelated initial state of the form $\tilde{\rho}(0) = \tilde{\rho}_e(0) \otimes \tilde{\rho}_{\text{ph}}$. Within the Born approximation, the interaction is assumed weak so the environment is negligibly affected by it and no considerable system-environment correlations arise due to it on the timescales relevant to the open system, so the the total density matrix can be written as a tensor product $\tilde{\rho}(t) = \tilde{\rho}_e(t) \otimes \tilde{\rho}_{\text{ph}}$ at all times. Within the Markov approximation, the system is considered memoryless so the evolution of the density matrix does not depend on its past, but only on its present state. Putting the integral form in Eq. (14) into the right-hand side (RHS) of the differential form, applying the Born and Markov approximations, and taking the trace over the phonon reservoir yield

$$\frac{d}{dt} \tilde{\rho}_e(t) = -i \text{tr}_{\text{ph}} \left\{ \mathbb{H}_{\text{int}}(t), \tilde{\rho}_e(t) \otimes \tilde{\rho}_{\text{ph}} \right\}$$

$$- \int_0^t ds \text{ tr}_{\text{ph}} \left\{ \mathbb{H}_{\text{int}}(t), \mathbb{H}_{\text{int}}(s), \tilde{\rho}_e(t) \otimes \tilde{\rho}_{\text{ph}} \right\}.$$

(16)

The above equation is the Redfield equation. We can substitute $s$ by $t - s$; the new $s$ denotes the time difference from $t$ and, because we expect the integrand to be negligible for large values of $s$, we can let the upper limit of the integral go to infinity:

$$\frac{d}{dt} \tilde{\rho}_e(t) = -i \text{tr}_{\text{ph}} \left\{ \mathbb{H}_{\text{int}}(t), \tilde{\rho}_e(t) \otimes \tilde{\rho}_{\text{ph}} \right\}$$

$$- \int_0^\infty ds \text{ tr}_{\text{ph}} \left\{ \mathbb{H}_{\text{int}}(t), \mathbb{H}_{\text{int}}(t - s), \tilde{\rho}_e(t) \otimes \tilde{\rho}_{\text{ph}} \right\}.$$

(17)

Now, to remove the temporal dependence of interaction Hamiltonians, we switch back to the Schrödinger picture

$$\frac{d}{dt} \rho_e(t) = -i \text{tr}_{\text{ph}} \left\{ \mathbb{H}_{\text{e}}(t), \rho_e(t) \otimes \rho_{\text{ph}} \right\}$$

$$- \int_0^\infty ds \text{ tr}_{\text{ph}} \left\{ \mathbb{H}_{\text{int}}(t), \mathbb{H}_{\text{int}}(t - s), \rho_e(t) \otimes \rho_{\text{ph}} \right\}.$$

(18)

In order to obtain the time evolution of $\langle \epsilon_{k_{\alpha}l_{\alpha}}^\dagger \epsilon_{k_{\alpha}+q_{\alpha}l_{\alpha}} \rangle$, which we need to calculate the charge density in Eq. (9), we multiply Eq. (18) by $\epsilon_{k_{\alpha}l_{\alpha}}^\dagger \epsilon_{k_{\alpha}+q_{\alpha}l_{\alpha}}$, and take a trace over the electron subspace. Also, we replace $\mathbb{H}_{\text{int}}(t)$ by $\mathbb{V}_{\text{SCF}}(t) + \mathbb{H}_{\text{e-ii}}(t) + \mathbb{H}_{\text{e-ph}}(t)$. Because we are seeking linear response, we keep the linear terms of $\mathbb{V}_{\text{SCF}}(t)$, and neglect the higher-order terms. As a result, we can approximate $\mathbb{V}_{\text{SCF}}(t) \rho_e(t) \approx \mathbb{V}_{\text{SCF}}(t) \rho_e(0)$. $\rho_e(0)$ denotes the initial density matrix which is the density matrix of the unperturbed system. Thus, Eq. (18) can be rewritten as
\[
\frac{d\langle c_{\mathbf{k}l}^\dagger c_{\mathbf{k}l+\mathbf{q}q'} \rangle}{dt} = -\text{itr}_{\mathbf{e}}\left\{ \left[ \mathbb{H}_{\mathbf{e}}, \rho_{\mathbf{e}}(t) \right] c_{\mathbf{k}l}^\dagger c_{\mathbf{k}l+\mathbf{q}q'} \right\} - \text{itr}_{\mathbf{r}}\left\{ \text{tr}_{\mathbf{ph}}\left\{ \mathbb{H}_{\mathbf{int}}(t), \rho_{\mathbf{e}}(0) \otimes \rho_{\mathbf{ph}} \right\} c_{\mathbf{k}l}^\dagger c_{\mathbf{k}l+\mathbf{q}q'} \right\} \\
- \int_0^\infty ds \text{tr}_{\mathbf{e}}\left\{ \text{tr}_{\mathbf{ph}}\left\{ \left[ \mathbb{H}_{\mathbf{ph}}, \left[ \mathbb{H}_{\mathbf{ph}}, \rho_{\mathbf{e}}(t) \otimes \rho_{\mathbf{ph}} \right] \right] + [\mathbb{H}_{\mathbf{e}}-\mathbb{i}\omega, \rho_{\mathbf{e}}(t) \otimes \rho_{\mathbf{ph}}] \right\} c_{\mathbf{k}l}^\dagger c_{\mathbf{k}l+\mathbf{q}q'} \right\} \\
- \int_0^\infty ds \text{tr}_{\mathbf{e}}\left\{ \text{tr}_{\mathbf{ph}}\left\{ \left[ \mathbb{H}_{\mathbf{ph}}, \left[ \mathbb{H}_{\mathbf{ph}}, \rho_{\mathbf{e}}(t) \otimes \rho_{\mathbf{ph}} \right] \right] + [\mathbb{H}_{\mathbf{e}}-\mathbb{i}\omega, \rho_{\mathbf{e}}(t) \otimes \rho_{\mathbf{ph}}] \right\} c_{\mathbf{k}l}^\dagger c_{\mathbf{k}l+\mathbf{q}q'} \right\} \\
- \int_0^\infty ds \text{tr}_{\mathbf{e}}\left\{ \text{tr}_{\mathbf{ph}}\left\{ \left[ \mathbb{H}_{\mathbf{ph}}, \left[ \mathbb{H}_{\mathbf{ph}}, \rho_{\mathbf{e}}(t) \otimes \rho_{\mathbf{ph}} \right] \right] + [\mathbb{H}_{\mathbf{e}}-\mathbb{i}\omega, \rho_{\mathbf{e}}(t) \otimes \rho_{\mathbf{ph}}] \right\} c_{\mathbf{k}l}^\dagger c_{\mathbf{k}l+\mathbf{q}q'} \right\} \\
+ \int_0^\infty ds \text{tr}_{\mathbf{e}}\left\{ \text{tr}_{\mathbf{ph}}\left\{ \left[ \mathbb{H}_{\mathbf{ph}}, \left[ \mathbb{H}_{\mathbf{ph}}, \rho_{\mathbf{e}}(t) \otimes \rho_{\mathbf{ph}} \right] \right] + [\mathbb{H}_{\mathbf{e}}-\mathbb{i}\omega, \rho_{\mathbf{e}}(t) \otimes \rho_{\mathbf{ph}}] \right\} c_{\mathbf{k}l}^\dagger c_{\mathbf{k}l+\mathbf{q}q'} \right\} .
\]
\[
\frac{d\langle c_{k}^\dagger c_{k+q'}\rangle}{dt} = -i(\epsilon_{k+q'} - \epsilon_{k})\langle c_{k}^\dagger c_{k+q'}\rangle - i(\mathbf{k} + q'\hat{V}_{\text{SCF}}(t))\langle \mathbf{k}l\rangle f_{kl} - f_{k+q'}
\]
\[
+ \sum_{\alpha, \beta, \pm} \delta_{q_{\alpha}, q_{\beta}, \pm} \pi \delta(\Delta_{\alpha} \pm \omega_{q_{\alpha}q_{\beta}})(N_{q_{\alpha}q_{\beta}} + \frac{1}{2} + \frac{1}{2})M_{\phi, q_{\alpha}q_{\beta}}(q_{\beta})M_{\phi, q_{\alpha}q_{\beta}}^*(q_{\alpha})(\mathbf{k} + q_{\alpha}l'_{\beta})\langle \mathbf{k}l\rangle (\mathbf{k}l_{\alpha} + q_{\alpha}l'_{\alpha})
\]
\[
\times tr \left\{ \left( c_{k+q'\alpha}^\dagger c_{k+q'\alpha} - c_{k+q'\beta}^\dagger c_{k+q'\beta} - c_{k+q'\alpha}^\dagger c_{k+q'\alpha} - c_{k+q'\beta}^\dagger c_{k+q'\beta} \rho_{\mathbf{t}}(t) \right) c_{k+q'\alpha}^\dagger c_{k+q'\alpha} \right\}
\]
\[
+ \sum_{\alpha, \beta, \pm} \delta_{q_{\alpha}, q_{\beta}, \pm} \pi \delta(\Delta_{\alpha} \pm \omega_{q_{\alpha}q_{\beta}})(N_{q_{\alpha}q_{\beta}} + \frac{1}{2} + \frac{1}{2})M_{\phi, q_{\alpha}q_{\beta}}(q_{\beta})M_{\phi, q_{\alpha}q_{\beta}}^*(q_{\alpha})(\mathbf{k} + q_{\alpha}l'_{\beta})\langle \mathbf{k}l\rangle (\mathbf{k}l_{\alpha} + q_{\alpha}l'_{\alpha})
\]
\[
\times tr \left\{ \left( c_{k+q'\alpha}^\dagger c_{k+q'\alpha} - \rho_{\mathbf{t}}(t) c_{k+q'\alpha}^\dagger c_{k+q'\alpha} \right) c_{k+q'\alpha}^\dagger c_{k+q'\alpha} \right\}
\]
\[
+ \sum_{\alpha, \beta} \delta_{q_{\alpha}, q_{\beta}} \pi \delta(\Delta_{\alpha})M_{ll}(q_{\alpha})M_{ll}^*(q_{\alpha})(\mathbf{k} + q_{\beta}l'_{\beta})\langle \mathbf{k}l\rangle (\mathbf{k}l_{\alpha} + q_{\alpha}l'_{\alpha})
\]
\[
\times tr \left\{ \left( c_{k+q'\alpha}^\dagger c_{k+q'\alpha} - \rho_{\mathbf{t}}(t) c_{k+q'\alpha}^\dagger c_{k+q'\alpha} \right) c_{k+q'\alpha}^\dagger c_{k+q'\alpha} \right\}
\]
\[
+ \sum_{\alpha, \beta} \delta_{q_{\alpha}, q_{\beta}} \pi \delta(\Delta_{\alpha})M_{ll}(q_{\alpha})M_{ll}^*(q_{\alpha})(\mathbf{k} + q_{\beta}l'_{\beta})\langle \mathbf{k}l\rangle (\mathbf{k}l_{\alpha} + q_{\alpha}l'_{\alpha})
\]
\[
\times tr \left\{ \left( c_{k+q'\alpha}^\dagger c_{k+q'\alpha} - \rho_{\mathbf{t}}(t) c_{k+q'\alpha}^\dagger c_{k+q'\alpha} \right) c_{k+q'\alpha}^\dagger c_{k+q'\alpha} \right\}
\]
\[
+ \sum_{\alpha, \beta} \delta_{q_{\alpha}, q_{\beta}} \pi \delta(\Delta_{\alpha})\langle c_{k}^\dagger c_{k+q'} \rangle c_{k+q'} \langle c_{k+q'}^\dagger c_{k} \rangle - \rho_{\mathbf{t}}(t) \langle c_{k}^\dagger c_{k+q'} \rangle c_{k+q'} \langle c_{k+q'}^\dagger c_{k} \rangle \right\}.
\]

It is worth remembering that \( \Delta_{\alpha} = \epsilon_{k+q_{\alpha}l_{\alpha}} - \epsilon_{k+q_{\alpha}l'_{\alpha}} \). The Markovian master equation (26) is of the Lindblad form and preserves the positivity of the density matrix, removing terms would violate this important constraint. By employing Eq. (20b), changing variables, and seeking a time-harmonic solution \( \frac{\partial}{\partial t} \rightarrow -i\omega \), the equation of motion for \( \langle c_{k}^\dagger c_{k+q'} \rangle \) becomes

\[
- i(\epsilon_{k+q'} - \epsilon_{k})\langle c_{k}^\dagger c_{k+q'}\rangle
\]
\[
= - i(f_{kl} - f_{k+q'})\langle \mathbf{k} + q' | \hat{V}_{\text{SCF}}(\omega) | \mathbf{k}l \rangle
\]
\[
+ \sum_{k' mm' g, \pm} \pi \delta(\epsilon_{k' + q'm'} - \epsilon_{k'm}) \left( \mp \Delta M_{q_{\alpha}q_{\beta}} \right) \langle \mathbf{k} + q' | k' | qm' \rangle \langle k + q' | \mathbf{k} | km \rangle \langle c_{k' m}^\dagger c_{k + q'm'} \rangle
\]
\[
+ \sum_{k' mm' g, \pm} \pi \delta(\epsilon_{k' - q'm'} - \epsilon_{k'm}) \left( \mp \Delta M_{q_{\alpha}q_{\beta}} \right) \langle \mathbf{k} + q' | k' | qm' \rangle \langle k + q' | \mathbf{k} | km \rangle \langle c_{k' m}^\dagger c_{k - q'm'} \rangle
\]
\[
- \sum_{k' mm' g, \pm} \pi \delta(\epsilon_{k' + q'm'} - \epsilon_{k'm}) \left( \mp \Delta M_{q_{\alpha}q_{\beta}} \right) \langle \mathbf{k} + q' | k' | qm' \rangle \langle k + q' | \mathbf{k} | km \rangle \langle c_{k' m}^\dagger c_{k + q'm'} \rangle.
\]

To simplify the notation, we defined the scattering weights \( \Delta M_{q_{\alpha}q_{\beta}} \), which correspond to the absorption and emission processes, respectively. Also, \( \Delta M_{q_{\alpha}q_{\beta}} \equiv M_{q_{\alpha}q_{\beta}} - M_{q_{\alpha}q_{\beta}}^* \). Details for each scattering mechanism are provided in Appendix C. It should be mentioned that the first sum is negligible, because the Dirac delta function in it implicitly forces a momentum conservation and, only insignificant number of transitions may satisfy both the energy conservation and the momentum conservation.

Equation (27) is the central equation of this paper. It captures the temporal variation of the coherences due to a harmonic field, and carries information about dissipation mechanisms, as well as the band structure and the Bloch-wave overlaps.
C. Numerical implementation of the SCF-MMEF for graphene

We now solve Eq. (27) for graphene. For simplicity, we use the first-nearest-neighbor tight binding with carbon $p_z$ orbitals to obtain the band structure and Bloch waves. The overlap integrals are $(k|k'|) \approx \frac{1}{\sqrt{2}} (1 + i l' \epsilon_{kk'})$, where $\epsilon_{kk'} = \arg((k'_x + ik'_y)(k_x - ik_y))$. In evaluating the energy terms in Eq. (27), the energy dispersion near the Dirac point is approximated as linear and isotropic $\epsilon_{kl} = l v_F |k|$, where $l = \pm 1$ denotes the valence (-1) and conduction (+1) bands.

The Brillouin zone is gridded up in the polar coordinates, with $N_k$ points in the radial direction and $N_\theta$ in the azimuthal direction. With the aid of this discretization, Eq. (27) is written in the matrix form and solved numerically for $\mathcal{X}$:

\[
\mathcal{F}(k'l') = \sum_{k} \frac{A}{2N_\theta} \frac{|k' + q'|}{v_F} \left[ m_{k-k',g}^\pm + \Delta m_{k-k',g} f_{kq} \right] |k + q'| |k' + qm| |k' + qm| |k + qm| |k + qm|,
\]

(28)

Each pair $(q, \omega)$ results in its own Eq. (28). The set of variables $\{k|l\}$ corresponds to a position in the column $\mathcal{X}$. The matrices and vectors in Eq. (28) are defined as

\[
\mathcal{E}(k'l')(k'm'm') = \delta(k'l') \left( \epsilon_{kl} - \epsilon_{k + ql'} + \omega \right),
\]

(29a)

\[
\mathcal{X}(k'l') = \langle c_{k'q} c_{k+ql'} \rangle,
\]

(29b)

\[
\mathcal{F}(k'l') = (f_{kl} - f_{k+ql'}) (k + ql' | kl),
\]

(29c)

Equation (28) is solved for $\mathcal{X}$ for every $(q, \omega)$. With the typical values we used ($N_\theta = 12$ or 14 and $N_k$ between 100 and 200) the square matrices in Eq. (28) are between $4800 \times 4800$ to $11200 \times 11200$ in size.

We introduce a vector $\mathcal{C}$ as $\mathcal{C}_{(k'l')} = (k|l + ql')$, which helps calculate surface polarization as

\[
P_s(q, \omega) = - \frac{e}{A} \mathcal{C}^T \mathcal{X}.
\]

(30)

$P_s$ is then used in Eqs. (3) and (4) to calculate the dielectric function and the surface conductivity, respectively. For a quasi-one dimensional system, Eq. (27) can be written in a matrix representation similar to Eq. (28); see Appendix A for details.

III. RESULTS

A. Complex conductivity of graphene

From the polarization, Eq. (30), the electrical and optical properties of the graphene sheet are calculated. In Fig. 1 we show the frequency dependence of the real part of the graphene AC conductivity, $\sigma(\omega)$, as calculated via the SCF-MMEF. The calculation shows excellent agreement with experimental results. The symbols are the experimental results for graphene on SiO$_2$, for two samples: (circles) before annealing ($n = 9.35 \times 10^{12}$ cm$^{-2}$, i.e., $\epsilon_F = 354$ meV) and (stars) after annealing ($n = 2.28 \times 10^{12}$ cm$^{-2}$, i.e., $\epsilon_F = 170$ meV). In order to reproduce the measurements, the only variable parameter is the sheet impurity density. We assume that impurities are spread uniformly across a sheet placed 4Å below the
interface of graphene and the substrate. The sheet impurity densities of $N_i = 1.6 \times 10^{12}$ cm$^{-2}$ (before annealing) and $N_i = 0.6 \times 10^{12}$ cm$^{-2}$ (after annealing) yield the best agreement with experimental data. The obtained impurity density for the after-annealing case is close to the one obtained by the EMC/FDTD/MD method with clustered impurities. At high enough frequencies (almost twice the Fermi frequency), it can be seen in Fig. 1 that the interband conductivity emerges, as expected.

We also calculated the mobility and the corresponding relaxation time for each case, based on the SCF-MMEF, and used them in the Mermin-Lindhard (ML) model. The results are plotted in Fig. 1 (dashed lines) for comparison. Even with an appropriate relaxation time, the ML model slightly underestimates the conductivity at lower frequencies.

In Fig. 2 (a), the real part of the DC conductivity of graphene, $\sigma_{DC}$, is plotted as a function of carrier density for three different values of the impurity density. As expected from experiments, for low impurity density and high carrier density, the sublinear relation between $\sigma_{DC}$ and carrier density becomes pronounced. The RTA provides a qualitative explanation for this behavior. Within the RTA, $\sigma_{DC} \approx e^2 \ell / (\pi h)$, and the relaxation time due to intrinsic phonon scattering is inversely proportional to the Fermi energy; therefore at high-enough carrier densities (equivalently at high Fermi levels) phonon scattering dominates over ionized-impurity scattering and $\sigma_{DC}$ gradually becomes less dependent on carrier density. However, these results are obtained for the case of uniformly distributed impurities, whereas it is known that the impurity distribution in realistic samples is nonuniform. The nonuniformity causes the formation of electron-hole puddles at low carrier densities, which significantly affects carrier transport but is not captured in our model. A realistic model for nonuniform distribution of impurities is provided in Ref. 44.

We also calculate the screening wave number, defined as $q_s = \lim_{q \to 0} q \left[ \varepsilon(\omega = 0, q) - 1 \right]$. Figure 2(b) shows that, for carrier densities comparable to or lower than the impurity density, the screening wave number falls off. However, the screening wave number calculated via the ML model is unable to show this phenomenon and is instead independent of the impurity density.

### B. Dielectric function of graphene

In order to study plasmons in graphene, the dielectric function $\varepsilon(q, \omega)$, Eq. (4), should be calculated. In the case of graphene on a polar substrate, it has been shown that plasmons and SO phonons couple with each other in Si-like structures. In order to take this phenomenon into account, Eq. (4) should be modified for polar substrates as

$$
\varepsilon(q, \omega) = 1 + \frac{e^2}{\varepsilon_0(\omega)} \left( \frac{P(q, \omega)}{2q} - \sum_j \tilde{\varepsilon} e^{-q_d \omega_{SO,j}} D_0^j(\omega) \right).
$$

where $\omega_{SO,j}$ is the $j$th SO phonon mode, and $\tilde{\varepsilon} = \varepsilon_{\alpha\alpha} - \varepsilon_{\alpha\beta} \varepsilon^{-1} \varepsilon_{\beta\alpha}$, $\varepsilon_{\alpha\alpha}$ and $\varepsilon_0^j$ are the high-frequency
and low-frequency permittivity of the substrate, respectively. \( d \) is the distance between the graphene sheet and the substrate, which in our calculations is assumed to be 4 Å. In the above equation, \( D_j^0(\omega) \) is the free phonon Green’s function

\[
D_j^0(\omega) = \frac{2\omega_{SO,j}}{(\omega + i\tau_j^{-1})^2 - \omega_{SO,j}^2},
\]

where \( \tau_j \) is the relaxation time corresponding to the \( j \)th SO phonon mode. Further information on the surface phonon modes is provided in Appendix [13].

Figure 3 shows the real and imaginary parts of the graphene dielectric function with the carrier density of \( n = 3 \times 10^{12} \text{ cm}^{-2} \) (\( \epsilon_F = 197 \text{ meV} \)) on SiO\(_2\) and DLC. The impurity density of graphene on SiO\(_2\) is \( N_i = 4 \times 10^{11} \text{ cm}^{-2} \), which corresponds to the electron mobility of 2500 cm\(^2\)V\(^{-1}\)s\(^{-1}\) as measured in [22]. The impurity density of graphene on DLC is chosen to be \( N_i = 4.15 \times 10^{11} \text{ cm}^{-2} \), which yields the electron mobility of 3000 cm\(^2\)V\(^{-1}\)s\(^{-1}\), as reported in [22]. The effect of SO phonons can be easily seen by comparing the dielectric function of graphene on DLC and SiO\(_2\) in Fig. 3.

C. Plasmons in graphene

The zeros of the real part of the dielectric function give an estimate of the plasmon dispersion [Fig. 3], but in the case of graphene on a polar substrate this approach becomes impractical. A more accurate method is to seek the maxima of the loss function, which is proportional to \(-3\{\epsilon(q,\omega)\}\) and can be directly measured. As scattering strongly affects the imaginary part of the dielectric function (without scattering, \( \Im(\epsilon(q,\omega)) = 0 \)), the loss function is also a sensitive probe for the role of dissipation. In an ideal, dissipation-free electronic system, the loss function would peak to infinity at plasmon resonances \( (\Re(\epsilon(q,\omega)) = 0) \); in realistic dissipative systems, the loss function has finite peaks at plasmon resonances, which higher peaks corresponding to lower plasmon damping.

Figures 4(a)-(c) show the loss function of graphene on SiO\(_2\) with different carrier densities and different impurity densities. Plasmon modes below the highest SO-phonon mode of the substrate (147 meV) are suppressed. Increasing the impurity density (or, equivalently, decreasing the electron mobility) does not change the plasmon dispersion and only enhances plasmon damping [compare Figs. 4(a) and 4(b)]. However, decreasing the carrier density (or, equivalently, increasing the Fermi level) not only raises plasmon damping, but also pushes the plasmon dispersion towards higher wave vectors, and both phenomena result in a decreased plasmon propagation length [compare Figs. 4(a) and 4(c)].

Overall, for graphene on DLC, all dispersions for different carrier and impurity densities, when presented in scaled quantities \( q/q_F \) and \( \omega/\omega_F \), reduce to a single curve [the red curve in Fig. 4(h)]. This dispersion curve is the same one that would be obtained with scattering-free RPA approaches [33–37]. For graphene on the polar SiO\(_2\) [Fig. 4 panels (a)–(c)], we see the effect of SO phonons on plasmon dispersions. There are four dispersion branches associated with interface plasmon-phonon (IPP) modes [35], the hybrid modes that stem from the coupling of plasmons with SO phonons. The three nearly dispersionless low-energy curves are SO-like [also seen in Fig. 4(h)], while the highest-energy curve is a plasmon-like IPP branch, in line with Refs. [38–40].

What our work captures is the effect of scattering [specifically, of the dominant ionized-impurity scattering (see Appendix C.3)] on the plasmon propagation length (Figs. 4 and 5), which the dissipation-free RPA calculations cannot do. Because of dissipation, the plasmon wave vector is complex and the plasmon propagation length is limited. We can write \( q = q_r + iq_i \), and the propagation length will be \( 2\pi/\omega \). The plasmon dispersion \( \omega_p(q_r) \) is obtained from the loss-function maximum [39, 41]. We write a Taylor expansion of \( \epsilon(q,\omega) \) in the vicinity of \( (q_r,\omega_p(q_r)) \) in terms of \( q - q_r = iq_i \) up to the fourth order. We then re-calculate the loss function based on the expansion; its maximum gives us the imaginary part of the complex wave vector of a plasmon. Figure 4(d) shows
the loss function of the star-marked point on Fig. 4(c) as a function of \( q_i/q_r \), the normalized propagation length, \( q_r/q_i \) is the number of wavelengths a plasmon propagates before dying out, and quantifies plasmon damping; higher normalized propagation length corresponds to lower damping.

In Fig. 5, it can be seen that increasing the carrier density not only increases the normalized propagation length (equivalently decreases the plasmon damping), but also moves the minimum damping to higher frequencies. Changing carrier density, though, pushes the plasmon dispersion to higher wave vectors. But, in general, increasing the carrier density enhances the plasmon propagation length.

Figure 6 shows the plasmon propagation length versus frequency for the carrier density of \( 3 \times 10^{13} \) \( \text{cm}^{-2} \) on three different substrates and at three different temperatures. The complex dielectric function of SiO\(_2\) and hBN is provided in Appendix B. As a nonpolar material with a relative permittivity of 3.2, DLC provides a longer plasmon propagation length at low frequencies than the polar substrates. In contrast, plasmons are highly suppressed for frequencies below the highest surface modes of SiO\(_2\) and hBN. Graphene on DLC shows a much greater normalized propagation length (\( q_r/q_i \), i.e., less plasmon damping (Fig. 6) left). However, because the plasmon wave vector (\( q_r \)) in graphene on DLC is considerably longer [inset to left panel of Fig. 6], the absolute value of the plasmon propagation length (\( \sim q_r^{-1} \)) in graphene-on-DLC is not considerably improved over the two polar substrates.

Finally, plasmon dispersion remains almost independent of temperature (inset to Fig. 6 left), while decreasing temperature reduces plasmon attenuation and shifts the damping minimum to higher frequencies (Fig. 6 left).
FIG. 6. (Left) The normalized plasmon propagation length ($q_r/q_i$) as a function of frequency for graphene with a carrier density of $n = 3 \times 10^{12}$ cm$^{-2}$ at three different temperatures (200, 300, and 400 K) and on three different substrates: DLC, SiO$_2$, and hBN. The impurity densities are chosen to yield the room-temperature electron mobility of 3000 cm$^2$ V$^{-1}$ s$^{-1}$ (DLC), 2500 cm$^2$ V$^{-1}$ s$^{-1}$ (SiO$_2$), and 9900 cm$^2$ V$^{-1}$ s$^{-1}$ (hBN). Inset: Plasmon dispersions. (Right) Plasmon propagation length as a function of frequency.

IV. CONCLUSION

In summary, we presented a method to calculate the linear-response dielectric function of a dissipative electronic system with an arbitrary band structure and Bloch wave functions. We calculated the induced charge density as a function of the self-consistent field by solving a generalized Markovian master equation of motion for coherences. The SCF-MMEF preserves the positivity of the density matrix and, consequently, conserves the number of electrons. The technique can be readily generalized to low-dimensional structures on other materials.

We employed the SCF-MMEF to study the electrical and optical properties of graphene. In our calculation, we considered intrinsic phonon scattering, ionized-impurity scattering, and SO-phonon scattering. We calculated the complex conductivity of graphene and showed that ionized impurities improve screening when the carrier density is comparable to or lower than the impurity density, a phenomenon that the Mermin-Lindhard approach cannot capture.

We calculated the dielectric function and loss function for graphene on three substrates (the nonpolar DLC, and the polar SiO$_2$ and hBN) and for different values of the impurity density, carrier density, and temperature. From the loss-function maximum, the plasmon dispersions and propagation length were computed. On polar substrates (SiO$_2$ and hBN), plasmons are strongly suppressed and have short propagation lengths at frequencies below the highest SO-phonon mode. DLC, being nonpolar, provides a broader spectrum for graphene plasmons.

We also investigated the effect of impurity density, carrier density, and temperature on plasmon dispersion and propagation length. Plasmon dispersion is fairly insensitive to varying impurity density or temperature, but is pushed towards shorter wave vectors with increasing carrier density. Plasmon damping – inversely proportional to the number of wave lengths that a plasmon propagates before dying out – worsens with more pronounced scattering, such as when the impurity density or temperature is increased. However, damping drops with increasing carrier density, benefiting plasmon propagation. Overall, the plasmon propagation length in absolute units is better on substrates with fewer impurities, and improves at lower temperatures and at higher carrier densities. We note that the calculated propagation lengths, comparable on polar and nonpolar substrates and roughly tens of nanometers, are an order of magnitude shorter than previously reported based on the Mermin-Lindhard approach.

This work underscores the importance of treating the dissipative mechanisms accurately. The SCF-MMEF may lead to improved understanding of the dielectric function and collective electronic excitations in graphene and related nanomaterials, such as nanoribbons or van der Waals structures.

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Appendix A: The SCF-MMEF for a quasi-one-dimensional material

For quasi-one-dimensional materials such as graphene nanoribbons (GNRs), Eqs. (4) and (6) are modified. Assuming a GNR is extended in the $x$-direction and has a width of $W$ in the $y$-direction, the induced charge density can be written as

$$n(x, y, z, t) = n_l(x, t) \left[ \frac{1}{W} \Pi \left( \frac{y}{W} \right) \right] \delta(z). \quad (A1)$$

$\delta(\cdot)$ denotes the Dirac delta function and $\Pi(\cdot)$ denotes the rectangular function [1 for its argument being within $(0,1), 1/2$ for argument equal to 0 or 1, and zero elsewhere]. With an approach similar to the derivation of Eq. (3), the induced potential reads

$$V_{\text{ind}}(q, y = 0, z = 0, \omega) = \frac{-ie^2}{4\epsilon_r \epsilon_0} n_l(q, \omega) \times \int \frac{1}{\omega^2} d\eta H_0^{(1)}(i Q W |\eta|), \quad (A2)$$

where $H_0^{(1)}(\cdot)$ is the zeroth-order Hankel function of the first kind. It has been assumed that the potential does not vary significantly across the GNR. To simplify the notation, henceforth we drop the $y$ and $z$ arguments. The dielectric function for a quasi-one-dimensional system may be written as

$$\varepsilon(q, \omega) = 1 + \frac{ie}{4\epsilon_r \epsilon_0} P_l(q, \omega) \int \frac{1}{\omega^2} d\eta H_0^{(1)}(i Q W |\eta|), \quad (A3)$$

where here $P_l(q, \omega) = \frac{e n_l(q, \omega)}{\sqrt{V}} \cdot s_j$ is the linear polarization. Analogously, the conductivity of GNRs is

$$\sigma_l(q, \omega) = -\frac{ie \omega}{q^2} P_l(q, \omega). \quad (A4)$$

In solving the SCF-MMEF for a quasi-one-dimensional material, Eqs. (28) and (29) remain unchanged, but the Brillouin zone is one-dimensional. Also, Eq. (30) should be modified to

$$P_l(q, \omega) = -\frac{e}{L} \mathcal{X}^T \mathcal{X}. \quad (A5)$$

Appendix B: The complex dielectric function of SiO$_2$ and hBN

We use the Lorentz oscillator model for the frequency-dependent dielectric function of SiO$_2$ and hBN.

$$\varepsilon = \varepsilon_\infty + \sum_j \frac{s_j^2}{\omega_j^2 - \omega^2 - i\Gamma_j \omega} \quad (B1)$$

In Fig. 7 the real part and imaginary parts of the complex dielectric function of a roughly 300-nm-thick slab of SiO$_2$ adopted from measurement[94,95] and the Lorentz oscillator fit are shown. In the Lorentz oscillator model, $\varepsilon_\infty = 2.3$, and the other parameters are

$$\begin{array}{ccc}
\omega(\text{meV}) & s_j^2(\text{meV}^2) & \Gamma(\text{meV}) \\
142 & 812 & 7.4 \\
133 & 7832 & 5.4 \\
100 & 537 & 4 \\
57 & 3226 & 6.2 \\
47 & 1069 & 24.5 \\
\end{array} \quad (B2)$$

The above choice of parameters results in $\varepsilon_0 = 4.4$. The surface phonon modes can be obtained by solving $\varepsilon + 1 = (\mathcal{X})^2$, which for SiO$_2$ results in three dominant modes: 65 meV, 102 meV, and 147 meV.

The Lorentz oscillator fit of the complex dielectric function of hBN is adopted from Ref[96]. For hBN, $\varepsilon_\infty = 4.95$, and the other parameters are

$$\begin{array}{ccc}
\omega(\text{meV}) & s_j^2(\text{meV}^2) & \Gamma(\text{meV}) \\
169 & 5364 & 3.6 \\
95 & 1895 & 4.34 \\
\end{array} \quad (B3)$$
The above choice of parameters results in $\varepsilon_0 = 7.03$ for hBN. Similarly, we obtain the surface phonon modes by solving $\varepsilon + 1 = 0$, which results in two dominant modes for hBN: 98 meV and 195 meV.

**Appendix C: Scattering Mechanisms**

1. **Phonon scattering**

The interaction Hamiltonian of electrons and phonons reads

$$\hat{H}_{e-ph} = \sum_{kq,\ell l} \mathcal{M}_{ph}(q)(k + q\ell|k|)c_{k+q\ell}^\dagger c_{k\ell}(b_q + b_{-q})^\dagger,$$

(C1)

where $b_{-q}$ and $b_q$ are the phonon creation and destruction operators, respectively. The scattering weights are defined as

$$\begin{align*}
\mathcal{M}_{k-k',ph} &= N_{k-k',ph}|\mathcal{M}_{ph}(k - k')|^2, \\
\mathcal{M}_{k-k',ph} &= (N_{k-k',ph} + 1)|\mathcal{M}_{ph}(k - k')|^2.
\end{align*}$$

(C2)

For longitudinal acoustic (LA) phonons

$$\mathcal{M}_{LA}(q) = D_{ac} \left( \frac{1}{2m\omega_{q,LA}} \right)^{\frac{3}{2}} (i\mathbf{q} \cdot \mathbf{e}_q),$$

(C3)

$m$ is the mass of the graphene sheet (with mass density of $7.6 \times 10^{-7}$ kg/m$^2$), $\omega_{q,LA}$ is the frequency of phonons in the acoustic branch, $D_{ac} = 12$ eV is the deformation potential for acoustic phonons, and $\mathbf{e}_q$ is the unit vector along the displacement direction. The acoustic phonon scattering may be approximated as an elastic scattering mechanism. We assume linear dispersion for acoustic phonons ($\omega_q = v_q|\mathbf{q}|$), where $v_q = 2 \times 10^4 \frac{\text{m}}{\text{s}}$ is the sound velocity in graphene. By employing the equipartition approximation at room temperature, it can be shown that

$$\mathcal{M}^{\pm}_{k,k',LA} \approx \mathcal{M}^{\pm}_{k,k',LA} \approx \frac{D^2_{ac} k_{BT}}{2mu_{q}^2}.$$  

(C4)

For longitudinal optical (LO) phonons in a nonpolar material

$$\mathcal{M}_{LO}(q) = D_{op} \left( \frac{1}{2m\omega_{q,LO}} \right)^{\frac{3}{2}},$$

(C5)

where $\omega_{q,LO}$ represents the frequency of LO phonons, and $D_{op} = 10^{11} \text{eV m}^{-1}$ is the deformation potential for nonpolar electron-LO phonon scattering. LO phonons may be assumed dispersionless, $\omega_{q,LO} = \omega_{LO} = 195$ meV. It can be shown that

$$\begin{align*}
\mathcal{M}^{\pm}_{k,k',LO} &= N_{LO} \frac{D^2_{op}}{2mu_0^2}, \\
\mathcal{M}^{\pm}_{k,k',LO} &= (N_{LO} + 1) \frac{D^2_{op}}{2mu_0^2}.
\end{align*}$$

(C6)

For the electron-SO phonon interaction we have

$$\mathcal{M}_{SO}(q) = \left[ \frac{e^2}{2 \Delta v_0} \frac{\tilde{\varepsilon}}{\varepsilon_s^\infty} \left( \frac{e^{-2qd}}{q^2 e^2(q, \omega = 0)} \right)^{\frac{3}{2}} \right],$$

(C7)

where $\tilde{\varepsilon} = \varepsilon_s^\infty \left( \frac{1}{\varepsilon_{s1}^\infty} - \frac{1}{\varepsilon_{s2}^\infty} \right)$, $\varepsilon_s^\infty$ and $\varepsilon_s^0$ are the high-frequency and low-frequency relative permittivities of the substrate, respectively. $\varepsilon(q, \omega = 0)$, calculated self-consistently according to Eq. (11), denotes the static dielectric function of the graphene sheet. The SO phonons may also be assumed dispersionless.

2. **Ionized-impurity scattering**

In graphene, the electron-ion interaction is related to the Coulomb potential in a two-dimensional electron gas. The screened potential of a buried ionized impurity at $z = -d$ in the substrate observed at the graphene sheet is

$$V_{eff}(q) = \frac{e^2}{2 A_{eff}^2 \tilde{\varepsilon}(q, \omega = 0)} \frac{e^{-qd}}{q},$$

(C8)

where $\tilde{\varepsilon} \equiv \frac{1 + \varepsilon(q, \omega = 0)}{\varepsilon_\infty}$. Thus, the interaction Hamiltonian of electrons and ionized impurities may be written as

$$\hat{H}_{e-n} = \sum_{kq,\ell l} \frac{e^2}{A_{eff}^2 \tilde{\varepsilon}(q, \omega = 0)} \frac{e^{-qd}}{2q} (k + q\ell|k|)c_{k+q\ell}^\dagger c_{k\ell}.$$ 

(C9)

The corresponding scattering weight for a sheet of impurities distributed uniformly on a plane located at a
distance \( d \) beneath the graphene sheet with the density of \( N_i \) reads

\[
\mathcal{M}_{k,k',ii} = \frac{N_i}{A} \left( \frac{e^2}{\epsilon_b \epsilon_0} \frac{e^{-|k'-k|d}}{2|k'-k|} \right)^2
\]  

(C10)

To have a unified notation, we define \( \mathcal{M}^{+}_{k,k',ii} = 2\mathcal{M}^{-}_{k,k',ii} = \frac{1}{2} \mathcal{M}_{k,k',ii} \).

3. **Comparison between SO-phonon scattering and ionized-impurity scattering**

Surface plasmons and SO phonons couple with each other and form interfacial plasmon-phonon (IPP) modes. The IPP modes are different from both pure plasmon and pure SO-phonon modes. Therefore, the electron–IPP scattering rates are different from the electron–SO phonon scattering rates. However, even in high-quality samples of supported graphene, ionized-impurity scattering is the dominant mechanism. In Fig. 9, we compare the momentum relaxation rates for electron scattering with ionized impurities at three different impurity sheet densities against the rates for scattering with IPPs (data from 48) for graphene on SiO\(_2\). Given that the ionized-impurity scattering dominates by over an order of magnitude, we consider the approximation of uncoupled SO phonons and plasmons (instead of IPPs) and the static screening of SO phonons 44,57,97 to be acceptable approximations, balancing simplicity with accuracy. The loss function does not get considerably altered in shape by the inclusion of IPP versus pure SO-phonon and plasmon modes.

**FIG. 9.** The momentum relaxation rate, \( \Gamma_m \), of electrons due to ionized-impurity scattering at the impurity sheet densities of \( 4 \times 10^{11} \text{ cm}^{-2} \) (yellow), \( 8 \times 10^{11} \text{ cm}^{-2} \) (orange), and \( 1.2 \times 10^{12} \text{ cm}^{-2} \) (red) and due to scattering from three IPP branches (blue; data from 48). Carrier density is \( n = 10^{12} \text{ cm}^{-2} \).