Topological quantum field theory of three-dimensional bosonic Abelian-symmetry-protected topological phases

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Topological Quantum Field Theory approach to Three-Dimensional Bosonic Abelian-Symmetry-Protected Topological Phases

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Symmetry-protected topological phases (SPT) are short-range entangled gapped states protected by global symmetry. Nontrivial SPT phases cannot be adiabatically connected to the trivial disordered state(or atomic insulator) as long as certain global symmetry G is unbroken. At low energies, most of two-dimensional SPTs with Abelian symmetry can be described by topological quantum field theory (TQFT) of multi-component Chern-Simons type. However, in contrast to the fractional quantum Hall effect where TQFT can give rise to interesting bulk anyons, TQFT for SPTs only supports trivial bulk excitations. The essential question in TQFT descriptions for SPTs is to understand how the global symmetry is implemented in the partition function. In this paper, we systematically study TQFT of three-dimensional SPTs with unitary Abelian symmetry (e.g., $\mathbb{Z}_{N_1} \times \mathbb{Z}_{N_2} \times \cdots$). In addition to the usual multi-component BF topological term at level-1, we find that there are new topological terms with quantized coefficients (e.g., $a^1 \wedge a^2 \wedge da^2$ and $a^1 \wedge a^2 \wedge a^3 \wedge a^4$) in TQFT actions, where $a^1, a^2, \cdots$ are 1-form U(1) gauge fields. These additional topological terms cannot be adiabatically turned off as long as G is unbroken. By investigating symmetry transformations for the TQFT partition function, we end up with the classification of SPTs that is consistent with the well-known group cohomology approach. We also discuss how to gauge the global symmetry and possible TQFT descriptions of Dijkgraaf-Witten gauge theory.

I. INTRODUCTION

A. Background and motivation

At low energies, universal properties of quantum many-body systems may be governed by continuum quantum field theory. For symmetry-breaking phases, Ginzburg-Landau field theory with symmetry-breaking potentials is utilized, where the concept of local order parameters is introduced. For topological phases of quantum matter, such as fractional quantum Hall effects (FQHE), local order parameters vanish and thereby fail to characterize the ground state. However, such phases can be described by *topological* quantum field theory (TQFT)[1,2]. As a renormalization fixed point theory, TQFT partition functions are usually formulated in the Euclidean path-integral formalism where the classical action is purely imaginary and all correlation functions of local operators are invariant under diffeomorphism of the spacetime manifold. In this sense, TQFTs are expected to efficiently capture the global phenomena that are insensitive to local energetic details. For example, the (2+1)-dimensional Chern-Simons field theory was introduced to describe FQHE[1–12]. It reveals the key mechanism of flux-charge attachment for FQHE and successfully predicts the anyon statistics and quantized Hall conductance[2]. Very recently, there is intensive ongoing research focusing on the interplay between symmetry and topology in strongly correlated systems. For example, the so-called "symmetry-protected topological phases" (SPT) in interacting bosonic / spin systems[13–35,37–64] have been drawing much attention. In contrast to FQHE where the bulk is fractionalized and supports topologically anyonic

quasiparticles, the bulk of bosonic SPT phase is non-fractionalized and only supports trivial bosonic quasiparticles. Nevertheless, distinct SPT phases can still be identified as long as a symmetry group, say, G is unbroken. In other words, two distinct SPT phases with G cannot be adiabatically connected to each other unless G symmetry is broken[14].

Since TQFT approach has been successfully applied in topological phases of quantum matter (e.g., FQHE), a natural question would be: Can we also use TQFT to describe universal properties of SPT phases that, by definition, do not support fractionalized bulk excitations? If so, we expect that such TQFTs must have two key properties:

1. Bulk excitations are non-fractionalized. In other words, it has a unique ground state on any closed manifold and there do not exist nontrivial topological sectors.

2. TQFTs of distinct SPT phases are topologically distinguishable from each other if and only if certain unbroken symmetry G is considered. It indicates that nontrivial properties of TQFT are topologically robust only when G symmetry is unbroken.

In Ref. 18, it was proposed to use multicomponent Abelian Chern-Simons theory to describe 2D SPT phases with Abelian symmetry. The Chern-Simons theory applied in SPT phases must have a unique ground state on a torus. SPT phases are classified by distinct ways of anomalous symmetry transformations implemented on the 1D boundary. In Ref. 25, a complete TQFT description for 2D SPT phases with Abelian symmetry has been
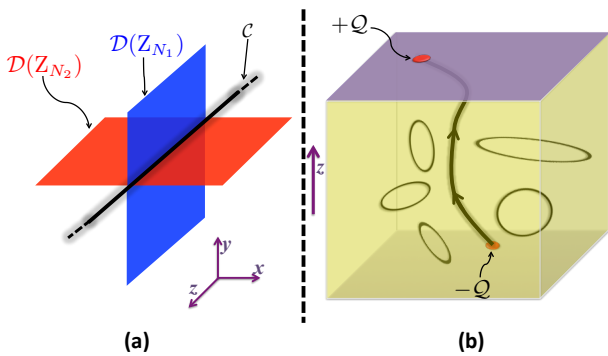
FIG. 1. (Color online) Bosonic wormhole effect as a physical mechanism of SPT phases with $\prod_I^2 \mathbb{Z}_{N_I}$ symmetry. **(a)** A $\mathbb{Z}_{N_1}$ symmetry domain wall "$\mathcal{D}(\mathbb{Z}_{N_1})$" and a $\mathbb{Z}_{N_2}$ symmetry domain wall "$\mathcal{D}(\mathbb{Z}_{N_2})$" intersect along a closed loop "$\mathcal{C}$" (only a segment is shown due to the limited space). **(b)** shows the bosonic wormhole effect induced by intersections of symmetry domain walls. Each intercepting line forms a closed loop in the bulk, but some of them may form open strings and end at two 2D boundaries between which the bulk is sandwiched. Along such an open string $\mathcal{C}$ (the line with arrow), $\mathbb{Z}_{N_2}$ symmetry charge $\mathcal{Q}$ is pumped from the endpoint on the lower boundary to the endpoint on the upper boundary. See main text for more detailed explanation.

achieved. Nevertheless, a complete TQFT description of 3D SPT phases is more challenging and still not conclusive even for Abelian symmetry groups. In Ref. 55, bosonic topological insulators (BTI) as specific cases of 3D SPT phase are studied and the corresponding TQFTs are derived via the vortex-line condensation scenario, where the cosmological constant-type term "$b \wedge b$" is used. Ref. 53 proposes to use topological BF field theory to describe some SPT phases in the presence of U(1) charge conservation symmetry. But except these special cases, we actually know very little about the TQFT description of 3D SPT, especially with unitary discrete symmetry. Indeed, there have already been some interesting proposals of 3D SPT materials, such as "topological paramagnetism" in 3D integer-spin Mott insulators[65]. We hope that a full establishment of TQFT in every spatial dimension will help us to systematically identify low-energy properties of bosonic SPT phases and realize them in a lab in future.

B. Main results and outline

In this paper, we attempt to establish a TQFT description of 3D SPT phases with unitary Abelian symmetry (see Table I). Instead of the abstract group cohomology approach[14,17], the strategy we adopt in this paper is based on Ginzburg-Landau (GL) type actions with global symmetry (Sec. II). Such an approach is more physical and accessible. The physical mechanism for realizing SPT phases is pictorially illustrated in Fig. 1, where the "bosonic wormhole effect" is introduced. Then, by du-

TABLE I. A brief summary of *irreducible* 3D SPT phases with unitary Abelian symmetry. By "irreducible", we means that all subgroups of symmetry group play nontrivial roles in protecting the nontrivial SPT phases. a^I and b^I are 1-form and 2-form U(1) gauge fields, respectively. "$(\mathbb{Z}_{N_{12}}) \cdots$" denote the corresponding classifications, where $N_{IJ\cdots}$ are greatest common divisors of $N_I, N_J, \cdots$. SPT phases with either $\mathbb{Z}_N$ or U(1)^k or $\mathbb{Z}_N \times$ U(1)^k are trivial and not included below. All other SPT's with unitary Abelian group symmetries can be obtained directly by using this table. For example, with arbitrary unitary finite Abelian group $G = \mathbb{Z}_{N_1} \times \mathbb{Z}_{N_2} \cdots$, it is sufficient to use this table to reproduce all SPT phases classified by the group cohomology, since $\mathcal{H}^4(\mathbb{Z}_{N_1} \times \mathbb{Z}_{N_2} \cdots, \mathrm{U}(1)) = \Pi_{I<J}(\mathbb{Z}_{N_{IJ}})^2 \times \Pi_{I<J<K}(\mathbb{Z}_{N_{IJK}})^2 \times \Pi_{I<J<K<L}\mathbb{Z}_{N_{IJKL}}$.

Symmetry Group	Topological Quantum Field Theory and Classification
$\mathbb{Z}_{N_1} \times \mathbb{Z}_{N_2}$	$\frac{i}{2\pi}\int \sum_I^2 b^I \wedge da^I + ip_1 \int a^1 \wedge a^2 \wedge da^2$ $(\mathbb{Z}_{N_{12}})$; $\frac{i}{2\pi}\int \sum_I^2 b^I \wedge da^I + ip_2 \int a^2 \wedge a^1 \wedge da^1$ $(\mathbb{Z}_{N_{12}})$
$\mathbb{Z}_{N_1} \times$ $\mathbb{Z}_{N_2} \times \mathbb{Z}_{N_3}$	$\frac{i}{2\pi}\int \sum_I^3 b^I \wedge da^I + ip_1 \int a^1 \wedge a^2 \wedge da^3$ $(\mathbb{Z}_{N_{123}})$; $\frac{i}{2\pi}\int \sum_I^3 b^I \wedge da^I + ip_2 \int a^2 \wedge a^3 \wedge da^1$ $(\mathbb{Z}_{N_{123}})$
$\prod_I^4 \mathbb{Z}_{N_I}$	$\frac{i}{2\pi}\int \sum_I^4 b^I \wedge da^I + ip \int a^1 \wedge a^2 \wedge a^3 \wedge a^4$ $(\mathbb{Z}_{N_{1234}})$
$\mathbb{Z}_{N_1} \times$ $\mathbb{Z}_{N_2} \times$U(1)	$\frac{i}{2\pi}\int \sum_I^3 b^I \wedge da^I + ip \int a^1 \wedge a^2 \wedge da^3$ $(\mathbb{Z}_{N_{12}})$

alizing GL actions, we obtain TQFT partition functions (Sec. III). We quantitatively study how symmetry transformations are realized in TQFTs, which gives rise to the same SPT classification results that were previously obtained from group cohomology approach[14]. While SPT phases with $\prod_I^2 \mathbb{Z}_{N_I}$ are studied in details, other SPT phases can be understood in a similar manner (Sec. IV), which are summarized in Table I. As such, TQFTs in this paper may be regarded as continuum field theory of group cohomological lattice models. We also connect our TQFT results to the "decorated domain wall picture"[40] as well as the interesting '3-loop statistics' phenomena[62] (Sec. IV). A brief summary of the paper including some future directions is given in Sec. V. Several technical details are collected in Appendices.

II. SPT AS A QUANTUM DISORDERED PHASE: GINZBURG-LANDAU ACTION APPROACH

A. A review of quantum disordered phase

We start with a brief review of the quantum field theory description for quantum disordered phase. Let us consider the following partition function of four-dimensional classical XY model in Villain form:

$$\mathcal{Z} = \int D\theta \sum_{\{N_{ij}\}} e^{-\frac{\chi}{2}\sum_{\langle ij \rangle}(\theta_i - \theta_j - 2\pi N_{ij})^2}, \quad (1)$$

where $N_{ij} \in \mathbb{Z}$ is an integer link variable defined on the NNN link $\langle ij \rangle$ of 4D cubic lattice. χ is inverse temperature. $\theta_i \in (0, 2\pi]$ is a U(1) phase angle defined on lattice

site i. This classical model corresponds to a quantum XY model in $3+1$D spacetime. There is a phase transition which separates two quantum phases at $\chi = \chi_c$. At small χ region near χ_c where the Villain approximation works, we have a "quantum disordered" phase where the ground state respects U(1) global symmetry and the lowest excitations are gapped. Such a quantum disordered phase serves as our starting point for understanding SPT phases.

To proceed further with field theory language, we rewrite the above lattice theory in terms of continuous spacetime variables with the following Lagrangian:

$$\mathcal{L}_0 = \frac{\chi}{2}(\partial_\mu \theta)^2 = \frac{\chi}{2}(\partial_\mu \theta_s + \partial_\mu \theta_v)^2 = \frac{\chi}{2}(\partial_\mu \theta_s - a_\mu)^2, \quad (2)$$

where we decompose θ as $\theta = \theta_s + \theta_v$. The θ_s variable in Eq. (2) is a smooth function. The singular part(or vortex part) θ_v of original lattice model can be incorporated in the newly-introduced vector variable a_μ. It can be viewed as a U(1) gauge field since the action is manifestly invariant under $\theta \to \theta + \chi, a_\mu \to a_\mu + \partial_\mu \chi$ where χ is a scalar gauge parameter. Apparently, such a gauge degree of freedom origins from the ambiguity(up to the scalar field χ) of the decomposition $\theta = (\theta_s + \chi) + (\theta_v - \chi)$.

We note that the action Eq. (2) actually only describes one of local minima of the Villain form Eq. (1). Since θ is by definition smooth, the large gauge transformation of a with singular χ and $\int_{S^1} d\chi = 2\pi n$ (with n an arbitrary integer) will transform the action Eq. (2) to another local minimum. Therefore, in order to maintain large gauge invariance, it is required to sum over all local minima, which essentially reproduce the Villain form Eq. (1). Thus, in the path integral formulation $\mathcal{Z} = \int D\theta Da e^{-\int d^4 x \mathcal{L}_0}$, all the large gauge transformations of a will be naturally included.

B. Ginzburg-Landau action description for a $3+1$D bosonic SPT phase with $G = \mathbb{Z}_{N_1} \times \mathbb{Z}_{N_2}$ symmetry

Since there are no interesting SPT phases with $\mathbb{Z}_N$ or $\mathrm{U}(1)^k$ symmetry in $3+1$D, here we consider the simplest $3+1$D bosonic SPT phase with Abelian symmetry $G = \mathbb{Z}_{N_1} \times \mathbb{Z}_{N_2}$. We begin with the following Ginzburg-Landau action with U(1) × U(1) symmetry.

$$S_0 = \sum_I^2 \int d^4 x \left[\frac{\chi}{2}(\partial_\mu \theta_s^I - a_\mu^I)^2 + \frac{1}{4g^2}(f_{\mu\nu}^I)^2 \right] + \cdots. \quad (3)$$

At the quadratic level of the low energy effective theory, we may also add a "Maxwell" term $\frac{1}{4g^2}(f_{\mu\nu}^I)^2$ where $f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu$ and g is gauge coupling constant.

In addition to S_0, we may add the following topological term:

$$S_{\mathrm{Top}} = -ip \int d^4 x \epsilon^{\mu\nu\lambda\rho}(\partial_\mu \theta_s^1 - a_\mu^1)(\partial_\nu \theta_s^2 - a_\nu^2)\partial_\lambda a_\rho^2. \quad (4)$$

Mathematically, S_{Top} is a wedge product of differential forms that is invariant under diffeomorphism. S_{Top} is the only term in (3+1)D spacetime that is topological and gauge-invariant with two independent 1-form gauge fields. Although S_{Top} formally has a U(1) × U(1) symmetry, below we will show that such a symmetry is anomalous and can not be realized in a strictly 3D system. However, when p takes certain quantized value, Eq.(4) has an anomaly-free $\mathbb{Z}_{N_1} \times \mathbb{Z}_{N_2}$ symmetry. In Sec. III, we will rigorously show that p is not only quantized but also takes values in a compact space. Distinct SPTs may be labeled by distinct p's, which may unveil the classifications of SPT phases. As a matter of fact, a usual dimension reduction scheme may already provide us a simple picture about the quantization condition on p.

For this purpose, let us consider a special 4D manifold $\mathcal{M}^4 = T^2 \times T^2$, where $\int_{T^2} da^2 = 2\pi$ on one of the torus T^2. After compactifying over this T^2, we end up with a $1+1$D SPT phases described by the following topological term:

$$S'_{\mathrm{Top}} = -2\pi ip \int d^2 x \epsilon^{\mu\nu}(\partial_\mu \theta_s^1 - a_\mu^1)(\partial_\nu \theta_s^2 - a_\nu^2). \quad (5)$$

In Ref. 25, it has been shown that if $4\pi^2 p = 0$ mod 1, the above topological term has an anomalous U(1) × U(1) symmetry and can not be realized as a pure $1+1$D system. However, if $4\pi^2 p = 0$ mod $\frac{N_1 N_2}{N_{12}}$ with N_{12} the greatest common divisor of N_1 and N_2, the above topological term has an anomaly-free $\mathbb{Z}_{N_1} \times \mathbb{Z}_{N_2}$ symmetry. Therefore, if $p = \frac{N_1 N_2 k}{4\pi^2 N_{12}}$ where $k = 0, 1, \cdots N_{12}$, the original topological term Eq. (4) will describe N_{12} different $3+1$D SPT phases with $\mathbb{Z}_{N_1} \times \mathbb{Z}_{N_2}$ symmetry. In Sec. III, the classification will be systematically derived from a TQFT which is a dual theory of the above GL action.

The physical meaning of S_{Top} can be understood via bosonic wormhole effect induced by intersections of symmetry domain walls. On one hand, the wormhole effect of topological insulators was discussed in Ref. 69, where the bulk is a fermionic system. On the other hand, by definition, a $\mathbb{Z}_{N_I}$ symmetry domain wall appears as a 2D closed manifold that separates two 3D regions with two distinct $\theta^I = \frac{2\pi}{N_I} k_I$ with $k_I \in \mathbb{Z}_{N_I}$. We consider $a_\mu^1 = a_\mu^2 = 0$ configurations such that $\mathbb{Z}_{N_I}$ symmetry domain walls $(I = 1, 2)$ do not proliferate. As a result, there are many closed loops that are intersections between $\mathbb{Z}_{N_1}$ and $\mathbb{Z}_{N_2}$ symmetry domain walls as shown in Fig.1-(a). Such closed loops do not contribute a $\mathbb{Z}_{N_2}$ symmetry charge $\mathcal{Q}$ from S_{Top} since $\mathcal{Q} = \partial_i (\partial_j \theta^1 \partial_k \theta^2)\epsilon^{ijk} = 0$. However, if open boundaries are considered, intersections may form open strings that end at two 2D boundaries as shown in Fig.1-(b). At these endpoints, nonzero $\mathcal{Q}$ is realized as a pumped charge due to wormhole effect. Each symmetry domain wall is labeled by $\frac{2\pi K_I}{N_I}$, where $K_I \in \mathbb{Z}_{N_I}$ $(I = 1, 2)$. As such, $\mathcal{Q} = p \frac{2\pi K_1}{N_1} \frac{2\pi K_2}{N_2} = \frac{k K_1 K_2}{N_{12}}$ $(k \in \mathbb{Z}_{N_{12}})$ where the quantization condition Eq. (14) is applied. N_{12} is the greatest common divisor of N_1 and N_2. Once a_μ^1 and a_μ^2 are turned on, the energetic cost of

symmetry domain walls is compensated by gauge configurations. Thus, the above novel domain wall configurations proliferate leading to a symmetric ground state.

III. TOPOLOGICAL QUANTUM FIELD THEORY OF SYMMETRY-PROTECTED TOPOLOGICAL PHASES

A. Duality between GL theory and TQFT

In the following, we may apply duality transformations to obtain a TQFT description of GL action. We start with the continuum action $S_0 + S_{\text{Top}}$ and derive TQFT. Next step is to apply Hubbard-Stratonovich transformation:

$$\mathcal{L} = i\sum_{I}^{2} n_{\mu}^{I}(a_{\mu}^{I} - \partial_{\mu}\theta_s^{I}) + \sum_{I}^{2} \frac{(f_{\mu\nu}^{I})^2}{4g^2}$$
$$- ip\epsilon^{\mu\nu\lambda\rho}(a_{\mu}^1 - \partial_{\mu}\theta_s^1)(a_{\nu}^2 - \partial_{\nu}\theta_s^2)\partial_{\lambda}a_{\rho}^2 + \frac{1}{2\chi}\sum_{I}^{2}(n_{\mu}^I)^2,$$

where the vector fields n_{μ}^I are auxiliary fields in order to linearize the quadratic terms. After total derivative terms are dropped off, integrating out θ_s^1 leads to: $\partial_{\mu}(n_{\mu}^1 - pa_{\nu}^2\partial_{\lambda}a_{\rho}^2\epsilon^{\mu\nu\lambda\rho}) = 0$ which can be resolved by introducing a 2-form gauge field $b_{\mu\nu}^1$: ($b_{\mu\nu}^1 = -b_{\nu\mu}^1$)

$$n_{\mu}^1 = pa_{\nu}^2\partial_{\lambda}a_{\rho}^2\epsilon^{\mu\nu\lambda\rho} + \frac{1}{4\pi}\partial_{\nu}b_{\lambda\rho}^1\epsilon^{\mu\nu\lambda\rho}.$$

After replacing n_{μ}^1 by the above identity, we end up with:

$$\mathcal{L} = in_{\mu}^2(a_{\mu}^2 - \partial_{\mu}\theta_s^2) + i\frac{1}{4\pi}a_{\mu}^1\partial_{\nu}b_{\lambda\rho}^1\epsilon^{\mu\nu\lambda\rho}$$
$$+ ip\epsilon^{\mu\nu\lambda\rho}a_{\mu}^1\partial_{\nu}\theta_s^2\partial_{\lambda}a_{\rho}^2 + \sum_{I}^{2}\frac{(f_{\mu\nu}^I)^2}{4g^2} + \sum_{I}^{2}\frac{1}{2\chi}(n_{\mu}^I)^2,$$

where the last term will be replaced at the final step.

Now, further integrating out θ_s^2 leads to: $\partial_{\mu}(n_{\mu}^2 + pa_{\nu}^1\partial_{\lambda}a_{\rho}^2\epsilon^{\mu\nu\lambda\rho}) = 0$ which can be resolved by introducing a 2-form gauge field $b_{\mu\nu}^2$: ($b_{\mu\nu}^2 = -b_{\nu\mu}^2$)

$$n_{\mu}^2 = -pa_{\nu}^1\partial_{\lambda}a_{\rho}^2\epsilon^{\mu\nu\lambda\rho} + \frac{1}{4\pi}\partial_{\nu}b_{\lambda\rho}^2\epsilon^{\mu\nu\lambda\rho}.$$

After replacing n_{μ}^2 by the above identity, we end up with the partition function with the TQFT action:

$$S_{\text{TQFT}} = \frac{i}{2\pi}\sum_{I}^{2}\int b^I \wedge da^I + ip\int a^1 \wedge a^2 \wedge da^2 + S_M, \quad (6)$$

where $b_{\mu\nu}^I$ are three 2-form U(1) gauge fields. As a side note, the differential form notation 'b^I' is replaced by '$\frac{b_{\mu\nu}^I}{2!}$' once spacetime indices are written explicitly.

To the best of our knowledge, the term $\int a^1 \wedge a^2 \wedge da^2$ as a classical action was introduced in Ref. [73] where two additional constraints $da^1 \wedge da^2 = 0$ and $da^2 \wedge da^2 = 0$ are imposed to recover gauge invariance[73]. And S_M, as an ultraviolet regulator, is given by:

$$S_M = \int d^4x \frac{1}{2\chi}(pa_{\nu}^2\partial_{\lambda}a_{\rho}^2\epsilon^{\mu\nu\lambda\rho} + \frac{1}{4\pi}\partial_{\nu}b_{\lambda\rho}^1\epsilon^{\mu\nu\lambda\rho})^2$$
$$+ \int d^4x \frac{1}{2\chi}(-pa_{\nu}^1\partial_{\lambda}a_{\rho}^2\epsilon^{\mu\nu\lambda\rho} + \frac{1}{4\pi}\partial_{\nu}b_{\lambda\rho}^2\epsilon^{\mu\nu\lambda\rho})^2$$
$$+ \sum_{I}^{2}\int d^4x \frac{(f_{\mu\nu}^I)^2}{4g^2}. \quad (7)$$

Interestingly, from S_M, one can explicitly read the gauge transformations (9) that play a very important role in the remaining discussions.

B. Exotic gauge transformations

The first term in Eq. (6) is two copies of topological BF term[70–72] at level-1. As usual, all gauge fields are subject to the following usual Dirac quantization conditions:

$$\frac{1}{2\pi}\int_{\mathcal{M}^2} da^I \in \mathbb{Z}, \quad \frac{1}{2\pi}\int_{\mathcal{M}^3} db^I \in \mathbb{Z}, \quad (8)$$

where $\mathcal{M}^2$ is any closed two-dimensional surface. As an Abelian gauge theory, the gauge transformations are given by:

$$a^I \to a^I + d\chi^I, \, b^I \to b^I + dV^I - 2\pi p\,\epsilon^{IJ3}\chi^J\,da^2, \quad (9)$$

where $\chi^I \in \mathbb{R}$ and $V_{\mu}^I \in \mathbb{R}$ are independent scalars and vectors, respectively. Here, both 1-form $d\chi^I$ and 2-form dV^I are closed but allowed to be non-exact if the following integers are nonzero on non-contractable manifolds.

$$\int_{\mathcal{M}^1} d\chi^I = 2\pi n^I, \quad \int_{\mathcal{M}^2} dV^I = 2\pi k^I, \quad (10)$$

where n^I and k^I are six independent integers. For nonzero n^I and k^I, the corresponding gauge transformations are said to be "large gauge transformations" labeled by the winding numbers n^I and k^I.

If $p = 0$, b^I are transformed in the usual definitions of 2-form gauge fields. Remarkably, the presence of p term in Eq. (6) induces a p-dependent term in the gauge transformations. Due to such unusual gauge transformations, we may construct the following two gauge-invariant operators:

Wilson loops: $e^{i\int_{\mathcal{M}^1} a^I}$

Modified Wilson surfaces: $e^{i\int_{\mathcal{M}^2} b^I - i2\pi p\int_{\mathcal{V}^3}\epsilon^{IJ3}a^J \wedge da^2}$

where $\mathcal{V}^3$ is an open volume enclosed by $\mathcal{M}^2$, i.e., $\partial\mathcal{V}^3 = \mathcal{M}^2$. It means that the Wilson surface operators of b^1 and b^2 are not gauge invariant alone, in contrast to the usual definitions. In Fig. 2, these gauge invariant operators are shown pictorially.
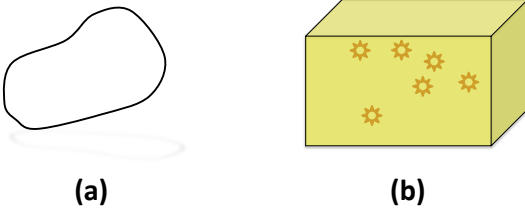
FIG. 2. (Color online) Illustration of gauge-invariant operators introduced in Sec. III B. (a) The Wilson loop operator $e^{i \int_{\mathcal{M}^1} a^I}$. (b) The operator formed by $\exp\{i \int_{\mathcal{M}^2} b^I - i2\pi p \int_{\mathcal{V}^3} \epsilon^{IJ3} a^J \wedge da^2\}$ where $\mathcal{M}^2 = \partial\mathcal{V}^3$. In (b), the cube represents $\mathcal{V}^3$ and its surface represents $\mathcal{M}^2$. The star symbols in (b) represent nonzero contributions of the "Chern-Simons density" $\int_{\mathcal{V}^3} \epsilon^{IJ3} a^J \wedge da^2$ in $\mathcal{V}^3$.

C. Global symmetry $G = \mathbb{Z}_{N_1} \times \mathbb{Z}_{N_2}$ in TQFT

In TQFT action (6), the physical meanings of the two form gauge field b can be understood by identifying its curvature as four-current of point-particles: $J_\mu^I = \frac{1}{4\pi}\epsilon^{\mu\nu\lambda\rho}\partial_\nu b^I_{\lambda\rho}$. In terms of differential forms, we have:

$$\star J^I = \frac{1}{2\pi} db^I, \qquad (11)$$

where $\star$ denotes Hodge dual operation. If the symmetry is U(1), then, we may add a U(1) external background gauge potential A^I to impose the U(1) symmetry:

$$S_{\text{TQFT}} \to S = S_{\text{TQFT}} + \sum_I^2 \frac{i}{2\pi} \int A^I \wedge db^I, \qquad (12)$$

where constant terms are neglected unless otherwise specified. In our case of $G = \prod_I^3 \mathbb{Z}_{N_I}$, the particle current is conserved mod N_I. To impose this condition, the closed loop integral of the 1-form A^I must be quantized at $\frac{2\pi}{N_I}$:

$$\int_{\mathcal{S}^1} A^I = \frac{2\pi}{N_I} \times 0, \pm 1, \cdots \qquad (13)$$

D. Symmetry-protected quantization

The implementation of the discrete symmetry leads to the following quantization and periodicity of p:

$$p = \frac{k}{4\pi^2} \frac{N_1 N_2}{N_{12}}, \quad k \in \mathbb{Z}_{N_{12}}, \qquad (14)$$

where N_{12} denotes the greatest common divisor (GCD) of N_1 and N_2. Eq. (14) indicates that there are totally N_{12} topologically distinct SPT phases that are described by TQFT Eq. (6). For example, the SPT phase labeled by k is equivalent to that labeled by $k + N_{12}$. In addition to Eq. (6), we may also consider a TQFT (as well as its

related Ginzburg-Landau action) with $\frac{i}{2\pi}\sum_I^2 \int b^I \wedge da^I + ip\int a^2 \wedge a^1 \wedge da^1$, where $p = \frac{k}{4\pi^2}\frac{N_1 N_2}{N_{12}}$, $k \in \mathbb{Z}_{N_{12}}$. As a result, the total classification of irreducible SPT phases with $\prod_I^2 \mathbb{Z}_{N_I}$ symmetry is given by $(\mathbb{Z}_{N_{12}})^2$.

Physically, the quantization in Eq. (14) can be understood in the following way. By performing the gauge transformations (9), the total $\mathbb{Z}_{N_I}$ symmetry charges $(I = 1, 2; \bar{I} = 3 - I)$ are not gauge invariant:

$$\frac{1}{2\pi} \int_{\mathcal{M}^3} db^I \to \frac{1}{2\pi} \int_{\mathcal{M}^3} db^I - p \int_{\mathcal{M}^3} d\chi^{\bar{I}} \wedge da^2. \qquad (15)$$

Since the 1-form external gauge potential A^I is quantized at $\frac{2\pi}{N_I}$, we may enforce the additional terms are divisible by N_I, i.e. $p\int_{\mathcal{M}^3} d\chi^{\bar{I}} \wedge da^2/N_I \in \mathbb{Z}$ such that the partition function is still invariant ($e^{iS} \to e^{iS+2\pi i} = e^{iS}$ with the action S defined in Eq. (12)). Concretely speaking, p should be quantized properly in order to ensure that there exists an integer pair (m^1, m^2) such that the two equations below always hold: ($I = 1, 2$; $\bar{I} = 3 - I$.) $p\int_{\mathcal{M}^3} d\chi^{\bar{I}} \wedge da^2 = N_I m_I$. To proceed further, let us consider $\mathcal{M}^3 = S^1 \times T^2$ such that for $I = 1$: $2\pi p\int_{\mathcal{M}^3} d\chi^2 \wedge da^2 = 2\pi p\int_{S^1} d\chi^2 \int_{T^2} da^2 = 2\pi p(2\pi)^2 \times$ integers where Eqs. (8,10) are applied. Therefore, the existence of integer m_1 requires that "$4\pi^2 p$ is divisable by N_1". Likewise, we can obtain a similar condition: "$4\pi^2 p$ is divisable by N_2". In summary, p is quantized as: $p = \frac{k}{4\pi^2}\frac{N_1 N_2}{N_{12}}$, $k \in \mathbb{Z}$, where N_{12} is the greatest common divisor (GCD) of N_1 and N_2.

However, $k \in \mathbb{Z}$ doesn't mean that the classification of bosonic SPT is necessarily $\mathbb{Z}$. Let us consider the following *shift operation*:

$$db^1 \to db^1 - \frac{K^1 N_1 N_2}{2\pi N_{12}} a^2 \wedge da^2, \qquad (16)$$

$$db^2 \to db^2 + \frac{K^2 N_1 N_2}{2\pi N_{12}} a^1 \wedge da^2, \qquad (17)$$

$$k \to k + K^1 + K^2, \qquad (18)$$

where K^1 and K^2 are two integers parametrizing the above shift operation. Likewise, one can check that TQFT partition function is invariant under the above shift operation if and only if

$$\frac{1}{2\pi} \int_{\mathcal{M}^3} db^I \bigg|_{\text{after shift}} - \frac{1}{2\pi} \int_{\mathcal{M}^3} db^I \bigg|_{\text{before shift}}$$
is divisible by N_I (for $I = 1, 2$). $\qquad (19)$

As a result, $e^{iS} \to e^{iS+2\pi i} = e^{iS}$ with the action S defined in Eq. (12). By using the shift operations Eqs. (16,17), the two equations in Eq. (19) reduce to:

$$\frac{K^1 N_1 N_2}{4\pi^2 N_{12}} \int a^2 \wedge da^2 \text{ is divisible by } N_1,$$

$$\frac{K^2 N_1 N_2}{4\pi^2 N_{12}} \int a^1 \wedge da^2 \text{ is divisible by } N_2.$$

Integrating out b^I leads to: $\int_{S^1} a^I = -\int_{S^1} A^I = \frac{2\pi}{N_I} \times 0, \pm 1, \cdots$. By further using the Dirac conditions (8), we end up with two conditions: $\frac{K^1 N_1 N_2}{4\pi^2 N_{12}} \frac{4\pi^2}{N_2}$ is divisible by N_1 and $\frac{K^2 N_1 N_2}{4\pi^2 N_{12}} \frac{4\pi^2}{N_1}$ is divisible by N_2. The general solution is $K^1 = l N_{12}, K^2 = l' N_{12}$ with $l, l' \in \mathbb{Z}$. Therefore, by using Bezout's lemma, $K^1 + K^2$ is quantized at GCD of N_{12} and N_{12}, i.e., N_{12}. In other words, the minimal shift of k [see Eq. (18)] is N_{12}. For our bosonic SPT phases, let us start with $k = 0$ which describes the trivial phase. Then, by increasing k, distinct bosonic SPT phases labeled by $k = 1, 2, \cdots$ are obtained in succession. But once $k = N_{12}$, the state returns to the trivial state immediately since it can be connected to $k = 0$ trivial state just by the above shift operation. One may also consider TQFT with action:

$$S_{\text{TQFT}} = \frac{i}{2\pi} \sum_I^2 \int b^I \wedge da^I + ip \int a^2 \wedge a^1 \wedge da^1 + S_M \quad (20)$$

which leads to another set of SPT states with $p = \frac{k N_1 N_2}{4\pi^2 N_{12}}$ and classified by $k \in \mathbb{Z}_{N_{12}}$. As a result, the total number of distinct 3D SPT states with $\prod_I^2 \mathbb{Z}_{N_I}$ symmetry is given by the cyclic group $(\mathbb{Z}_{N_{12}})^2$, which is also consistent to group cohomology: $\mathcal{H}^4(\prod_I^2 \mathbb{Z}_{N_I}, \text{U}(1)) = (\mathbb{Z}_{N_{12}})^2$.

IV. OTHER SYMMETRIES AND APPLICATIONS

A. Other Abelian symmetries

In Appendix A, the *irreducible* (see Table I) SPT phases with $\mathbb{Z}_{N_1} \times \mathbb{Z}_{N_2} \times \mathbb{Z}_{N_3}$ are derived, where the actions of TQFT are given by:

$$S_{\text{TQFT}} = \frac{i}{2\pi} \sum_I^3 \int b^I \wedge da^I + ip \int a^1 \wedge a^2 \wedge da^3 + S_M, \quad (21)$$

and,

$$S_{\text{TQFT}} = \frac{i}{2\pi} \sum_I^3 \int b^I \wedge da^I + ip \int a^2 \wedge a^3 \wedge da^1 + S_M. \quad (22)$$

(21) and (22) produce $(\mathbb{Z}_{N_{23}})^2$ classification of irreducible SPT phases. Since SPT phases with only two $\mathbb{Z}_N$ symmetries can be viewed as *reducible* SPT phases with three $\mathbb{Z}_N$ symmetries, we conclude that from TQFT approach the full classification of SPT phases with G= $\prod_I^3 \mathbb{Z}_{N_I}$ symmetry is given by: $\mathcal{H}^4(\prod_I^3 \mathbb{Z}_{N_I}, \text{U}(1)) = (\mathbb{Z}_{N_{12}})^2 \times (\mathbb{Z}_{N_{23}})^2 \times (\mathbb{Z}_{N_{13}})^2 \times (\mathbb{Z}_{N_{123}})^2$. Therefore, we have achieved all SPT phases with either $\prod_I^3 \mathbb{Z}_{N_I}$ or $\prod_I^2 \mathbb{Z}_{N_I}$ symmetries that were predicted in group cohomology.

The irreducible SPT phases with $\prod_I^4 \mathbb{Z}_{N_I}$ symmetry have been studied in Ref. 25 where TQFT is given by:

$$S_{\text{TQFT}} = \frac{i}{2\pi} \sum_I^4 \int b^I \wedge da^I + ip \int a^1 \wedge a^2 \wedge a^3 \wedge a^4 \quad (23)$$

with conditions Eq. (8) with $I = 1, \cdots, 4$. $p = \frac{k N_1 N_2 N_3 N_4}{(2\pi)^3 N_{1234}}$ with $k \in \mathbb{Z}_{N_{1234}}$ where N_{1234} is GCD of all N_I's. Thus, for arbitrary finite unitary Abelian group $G = \mathbb{Z}_{N_1} \times \mathbb{Z}_{N_2} \cdots$, we can derive the TQFT descriptions for all SPT phases classified by the group cohomology:

$$\mathcal{H}^4(\mathbb{Z}_{N_1} \times \mathbb{Z}_{N_2} \cdots, \text{U}(1))$$
$$= \Pi_{I<J}(\mathbb{Z}_{N_{IJ}})^2 \times \Pi_{I<J<K}(\mathbb{Z}_{N_{IJK}})^2 \times \Pi_{I<J<K<L}\mathbb{Z}_{N_{IJKL}}.$$

In fact, all the results for finite unitary Abelian group can be generalized into cases that involve U(1) symmetry as well. In Appendices, we show that there is no nontrivial SPT with $\text{U}(1)^k$ or $\mathbb{Z}_{N_1} \times \text{U}(1)^k$ symmetry, and the irreducible SPT phase with $\prod_I^2 \mathbb{Z}_{N_I} \times \text{U}(1)$ symmetry can be described by $\frac{i}{2\pi} \int \sum_I^3 b^I \wedge da^I + ip \int a^1 \wedge a^2 \wedge da^3$, where $p = \frac{k}{4\pi^2} \frac{N_1 N_2}{N_{12}}$, $k \in \mathbb{Z}_{N_{12}}$ gives rise to a $\mathbb{Z}_{N_{12}}$ classification. Together with the $(\mathbb{Z}_{N_{12}})^2$ reducible SPT phases contributed from SPT phases protected merely by the $\prod_I^2 \mathbb{Z}_{N_I}$ part, we derive the TQFT descriptions for all the $(\mathbb{Z}_{N_{12}})^3$ SPT phases predicted by group cohomology classifications. The above scheme can be easily generalized to arbitrary Abelian group symmetry with the form $\tilde{G} = \mathbb{Z}_{N_1} \times \mathbb{Z}_{N_2} \cdots \times \text{U}(1)^k \equiv G \times \text{U}(1)^k$, that is, in addition to the TQFT descriptions for finite unitary Abelian group piece G, one can always pick up a subgroup $\mathbb{Z}_{N_I} \times \mathbb{Z}_{N_J} \times \text{U}(1)$ to construct $\mathbb{Z}_{N_{IJ}}$ additional SPT phases.

B. Connection to "decorated domain walls" picture

Ref. 40 proposed a way called "decorated domain walls" to construct the ground state wave function of SPT states. Take $\prod_I^2 \mathbb{Z}_{N_I}$ symmetry as an example, the SPT ground state can be formed as an equal weight superposition of 2D domain walls of $\mathbb{Z}_{N_1}$ on which a nontrivial 2D $\mathbb{Z}_{N_2}$ SPT is placed. Here we use our TQFT approach to understand the "decorated domain walls" picture. Let us start with the action S in Eq. (12) and set $A^1 = 0, A^2 = A$. We also add a dynamical Higgs term:

$$S_{\text{Higgs}} = \int \frac{i N_I}{2\pi} B \wedge dA, \quad (24)$$

where A and B are 1-form and 2-form dynamical gauge fields respectively. S_{Higgs} breaks U(1) gauge symmetry of A^2 gauge field down to $\mathbb{Z}_{N_2}$. Integrating out B and b^2 leads to $\int_{\mathcal{M}^1} a^2 = \int_{\mathcal{M}^1} A = -\frac{2\pi}{N_2} \times$ integer. $a^1 \wedge a^2 \wedge da^2$ becomes: $ip \int a^1 \wedge A \wedge dA$. a^1 can be viewed

as domain wall of $\mathbb{Z}_{N_1}$ symmetry such that it may be replaced by $\frac{2\pi k_1}{N_1}$ where $k_1 \in \mathbb{Z}_{N_1}$. In other words, given a $\mathbb{Z}_{N_1}$ symmetry domain wall "$\mathcal{D}(\mathbb{Z}_{N_1})$" labeled by k_1, there is a topological gauge theory on this 2D manifold:

$$S[A] = \frac{i}{4\pi} \int_{\mathcal{D}(\mathbb{Z}_{N_1})} \frac{2N_2 k_1 k}{N_{12}} A \wedge dA\,, \qquad (25)$$

where we have used $p = \frac{k}{4\pi^2} \frac{N_1 N_2}{N_{12}}$ with $k \in \mathbb{Z}_{N_{12}}$. By noting that $\frac{2N_2 k_1 k}{N_{12}}$ is even[18–24], Eq. (25) suggests that the 2D state decorated on the domain wall is indeed a nontrivial 2D SPT with $\mathbb{Z}_{N_2}$ symmetry. Such a way of thinking can be naturally extended to SPT phases with $\prod_I^3 \mathbb{Z}_{N_I}$ described by TQFT Eq. (6). For this purpose, we may introduce A and $\tilde{A}$ that respectively gauge $\mathbb{Z}_{N_2}$ and $\mathbb{Z}_{N_3}$ symmetry groups. As a result, on the $\mathbb{Z}_{N_1}$ symmetry domain wall, we have the topological gauge action for $\mathbb{Z}_{N_2} \times \mathbb{Z}_{N_3}$ SPT phases:

$$S[A, \tilde{A}] = \frac{i}{4\pi} \int_{\mathcal{D}(\mathbb{Z}_{N_1})} \frac{2N_2 k_1 k}{N_{12}} A \wedge d\tilde{A}\,, \qquad (26)$$

where Eq. (14) is applied and $k \in \mathbb{Z}_{N_{123}}$.

C. Connection to 3-loop statistics

It was first shown in Ref. 62 that, after the global symmetry $\mathbb{Z}_{N_1} \times \mathbb{Z}_{N_2} \times \cdots$ of SPT phases is gauged at weak gauge coupling limit, the loop excitations of resultant gapped state exhibit the so-called 3-loop statistics. Based on our TQFT approach to SPT with $G = \mathbb{Z}_{N_1} \times \mathbb{Z}_{N_2}$, we may formally gauge the symmetry by adding the coupling term:

$$S_{\text{coupling}} = \int \sum_I^2 \frac{i}{2\pi} A^I \wedge db^I + \int \sum_I^2 \frac{iN_I}{2\pi} B^I \wedge dA^I\,, \quad (27)$$

where A^I and B^I are 1-form and 2-form dynamical gauge fields respectively. The first term is the minimal coupling term between A^I and matter field current $\star \frac{1}{2\pi} db^I$; the second term is a Higgs term that breaks U(1) gauge symmetry of A^I gauge field down to $\mathbb{Z}_{N_I}$. After integrating out a^I and b^I, it turns out that the resultant new TQFT is given by (e.g. symmetry G$= \prod_I^3 \mathbb{Z}_{N_I}$):

$$S = \sum_I^2 \int \frac{iN_I}{2\pi} B^I \wedge dA^I + ip \int A^1 \wedge A^2 \wedge dA^2\,. \quad (28)$$

By integrating out B^I, we will end up with an action for Dijkgraaf-Witten[74] gauge theory proposed in Ref. 50. We conjecture that Eq. (28) gives rise to the 3-loop statistics. Some progress has been made along this line[66–68].

V. CONCLUSIONS AND FUTURE DIRECTIONS

In this paper, we attempt to provide a complete TQFT description for all 3D SPT phases with unitary Abelian group symmetry. Taking $\prod_I^2 \mathbb{Z}_{N_I}$ as an example, we start with a Ginzburg-Landau type action and illustrate the key mechanism via proliferating "nontrivial" symmetry domain walls for SPT phases with finite unitary Abelian group symmetry. We then rigorously derive the corresponding TQFT and compute the level quantization. Finally, we consider generic 3D SPT phases with arbitrary unitary Abelian group symmetry, including U(1). All irreducible TQFT results for unitary Abelian group symmetry are collected in Table I. In Appendices, we also discuss possible TQFT descriptions with anti-unitary time reversal symmetry. Together with the previous work of bosonic topological insulators[55], we provide a route towards a complete TQFT description for 3D SPT phases and reveal the key mechanism of these exotic quantum phases of matter.

There are several interesting directions for future studies. One of them is to construct the boundary of 3D SPT phases by using the present TQFT framework. There are two steps. First, we should consider TQFT in an open manifold and try to derive the boundary effective field theory. It is possible that the gauge invariant argument in boundary CFT derivation of Chern-Simons theory[2] is still applicable in the present TQFT. However, the exotic gauge transformations, e.g. Eq. (9), may lead to technical challenge. Second, the symmetry transformations on the boundary effective field theory should reveal some nontrivial properties that forbid the realization of the boundary theory in a 2D plane alone (i.e. no 3D bulk). In addition, it is also interesting to study the symmetry-enriched topological phases (SET) by using TQFT. SET has topological excitations in the bulk and symmetry acts on excitations in a fractionalized manner (e.g. projective representation). For this purpose, one may replace $\frac{1}{2\pi} \sum_I b^I \wedge da^I$ by $\sum_I \frac{k_I}{2\pi} b^I \wedge da^I$ with $k_I = 2, 3, \cdots$ and then study how to impose symmetry G. Along this line, one may study 3D SET phases with gauge group $\mathbb{Z}_{k_1} \times \mathbb{Z}_{k_2} \times \cdots$ and symmetry group G. Finally, it is interesting to extend all of the above studies to fermionic SPT and fermionic SET where the transparent quasiparticles are fermionic (e.g. electrons in FQHE).

Appendix A: TQFT of irreducible bosonic SPT phases with global symmetry $\mathbb{Z}_{N_1} \times \mathbb{Z}_{N_2} \times \mathbb{Z}_{N_3}$

In this Appendix, we derive TQFT and its classification when the symmetry is $\mathbb{Z}_{N_1} \times \mathbb{Z}_{N_2} \times \mathbb{Z}_{N_3}$. The action is given by:

$$S_{\text{TQFT}} = \frac{i}{2\pi}\sum_I^3 \int b^I \wedge da^I + ip\int a^1 \wedge a^2 \wedge da^3 + S_M \tag{A1}$$

The implementation of the discrete symmetry leads to the following quantization and periodicity of p:

$$p = \frac{k}{4\pi^2}\frac{N_1 N_2}{N_{12}}, \quad k \in \mathbb{Z}_{N_{123}}, \tag{A2}$$

where N_{12} (N_{123}) denotes the greatest common divisor (GCD) of N_1 and N_2 (N_1, N_2 and N_3). Eq. (A2) indicates that there are totally N_{123} topologically distinct SPT phases that are described by TQFT Eq. (A1). For example, the SPT phase labeled by k is equivalent to that labeled by $k + N_{123}$. In addition to Eq. (A1), we may also consider a TQFT (as well as its related Ginzburg-Landau action) with $\frac{i}{2\pi}\sum_I^3 \int b^I \wedge da^I + ip\int a^2 \wedge a^3 \wedge da^1$, where $p = \frac{k}{4\pi^2}\frac{N_3 N_2}{N_{23}}$, $k \in \mathbb{Z}_{N_{123}}$. As a result, the total classification of irreducible SPT phases with $\prod_I^3 \mathbb{Z}_{N_I}$ symmetry is given by $(\mathbb{Z}_{N_{123}})^2$.

Physically, the quantization in Eq. (A2) can be understood in the following way. By performing the gauge transformations (9), the total $\mathbb{Z}_{N_I}$ symmetry charges ($I = 1, 2; \bar{I} = 3 - I$) are not gauge invariant:

$$\frac{1}{2\pi}\int_{\mathcal{M}^3} db^I \rightarrow \frac{1}{2\pi}\int_{\mathcal{M}^3} db^I - p\int_{\mathcal{M}^3} d\chi^{\bar{I}} \wedge da^3. \tag{A3}$$

Since the 1-form external gauge potential A^I is quantized at $\frac{2\pi}{N_I}$, we may enforce the additional terms are $N_I \times$integer: $p\int_{\mathcal{M}^3} d\chi^{\bar{I}} \wedge da^3 = N_I \times \mathbb{Z}$ such that the partition function is still invariant. Concretely speaking, p should be quantized properly in order to ensure that there exists an integer pair (m^1, m^2) such that the two equations below always hold: ($I = 1, 2; \bar{I} = 3 - I$.) $p\int_{\mathcal{M}^3} d\chi^{\bar{I}} \wedge da^3 = N_I m_I$. To proceed further, let us consider $\mathcal{M}^3 = S^1 \times T^2$ such that for $I = 1$:

$$2\pi p\int_{\mathcal{M}^3} d\chi^2 \wedge da^3 = 2\pi p\int_{S^1} d\chi^2\int_{T^2} da^3$$
$$= 2\pi p(2\pi)^2 \times \text{integers},$$

where Eqs. (8,10) are applied. Therefore, the existence of integer m_1 requires that "$4\pi^2 p$ is divisable by N_1" such that $m_1 = \frac{4\pi^2 p}{N_1} \times \text{integer} \in \mathbb{Z}$. Likewise, we can obtain a similar condition: "$4\pi^2 p$ is divisable by N_2". In summary, p is quantized as: $p = \frac{k}{4\pi^2}\frac{N_1 N_2}{N_{12}}$, $k \in \mathbb{Z}$, where N_{12} is the greatest common divisor (GCD) of N_1 and N_2.

However, $k \in \mathbb{Z}$ doesn't mean that the classification of bosonic SPT is necessarily $\mathbb{Z}$. Let us consider the following *shift operation*:

$$b^3 \rightarrow b^3 - \frac{K^3 N_1 N_2}{2\pi N_{12}}a^1 \wedge a^2, \tag{A4}$$

$$db^1 \rightarrow db^1 - \frac{K^1 N_1 N_2}{2\pi N_{12}}a^2 \wedge da^3, \tag{A5}$$

$$db^2 \rightarrow db^2 - \frac{K^2 N_1 N_2}{2\pi N_{12}}da^3 \wedge a^1, \tag{A6}$$

$$k \rightarrow k + K^1 + K^2 + K^3, \tag{A7}$$

where K^1, K^2, and K^3 are three integers parametrizing the above shift operation. One can check that TQFT is formally invariant under the above shift operation. Furthermore, the shifts in b^I fields should be consistent to the $\mathbb{Z}_{N_I}$ symmetry via the following relations:

$$\frac{1}{2\pi}\int_{\mathcal{M}^3} db^I\Big|_{\text{after shift}} - \frac{1}{2\pi}\int_{\mathcal{M}^3} db^I\Big|_{\text{before shift}}$$
$$= N_I \times \text{integer} \ (\text{for } I = 1, 2, 3) \tag{A8}$$

which means that the change amount should be divisible by N_I for index I. By using the shift operations Eqs. (A4,A5,A6), these relations reduce to:

$$\frac{K^3 N_1 N_2}{4\pi^2 N_{12}}\left(\int a^1 \wedge da^2 + \int da^1 \wedge a^2\right) \text{ is divisible by } N_3,$$

$$\frac{K^1 N_1 N_2}{4\pi^2 N_{12}}\int a^2 \wedge da^3 \text{ is divisible by } N_1,$$

$$\frac{K^2 N_1 N_2}{4\pi^2 N_{12}}\int da^3 \wedge a^1 \text{ is divisible by } N_2.$$

It means the integers (K^1, K^2, K^3) should be properly selected such that the three "integers" on the right hand sides always exist. Integrating out b^I leads to:

$$\int_{S^1} a^I = -\int_{S^1} A^I = \frac{2\pi}{N_I} \times 0, \pm 1, \cdots. \tag{A9}$$

By using Eqs. (8,A9), the above equations are reexpressed as:

$$\frac{K^3 N_1 N_2}{4\pi^2 N_{12}}\frac{4\pi^2 (N_1 \times \text{integer} + N_2 \times \text{integer})}{N_1 N_2}$$
$$\text{is divisible by } N_3, \tag{A10}$$

$$\frac{K^1 N_1 N_2}{4\pi^2 N_{12}}\frac{4\pi^2}{N_2} \text{ is divisible by } N_1, \tag{A11}$$

$$\frac{K^1 N_1 N_2}{4\pi^2 N_{12}}\frac{4\pi^2}{N_1} \text{ is divisible by } N_2. \tag{A12}$$

By noting that ($N_1 \times \text{integer} + N_2 \times \text{integer}$) is always divisible by N_{12} due to Bézout's lemma, we obtain the following solution: $K^3/N_3 \in \mathbb{Z}$, $K^1/N_{12} \in \mathbb{Z}$, and $K^2/N_{12} \in \mathbb{Z}$. By using Bézout's lemma again, we obtain the minimal shift of k is $GCD(N_{12}, N_3) =$

$GCD(N_1, N_2, N_3) \equiv N_{123}$. Thus, for our bosonic SPT phases, let us start with $k = 0$ which is trivial. Then, by increasing k, distinct bosonic SPT phases labeled by $k = 1, 2, \cdots$ are obtained in succession. But once $k = N_{123}$, the state returns the trivial state immediately since it can be connected to $k = 0$ trivial state just by the above shift operation.

Appendix B: TQFT of irreducible bosonic SPT phases with global symmetry $(\mathbb{Z}_{N_1} \times \cdots) \times \mathrm{U}(1)$

1. $\mathbb{Z}_N \times \mathrm{U}(1)$

In the following, we consider bosonic SPT phases with direct product of U(1) and several $\mathbb{Z}_N$'s. First, let us consider G= $\mathbb{Z}_N \times \mathrm{U}(1)$. In this case, we may consider TQFT with the conditions:

Dirac conditions: $\dfrac{1}{2\pi} \displaystyle\int_{\mathcal{M}^2} d\,a^I \in \mathbb{Z}$, $\dfrac{1}{2\pi} \displaystyle\int_{\mathcal{M}^3} db^I \in \mathbb{Z}$ (B1)

Symmetry: $\dfrac{1}{2\pi} \displaystyle\int_{\mathcal{M}^3} db^1 \to \dfrac{1}{2\pi} \displaystyle\int_{\mathcal{M}^3} db^1 + N$ (B2)

which means that all bosons of $I = 1$ carry $\mathbb{Z}_N$ symmetry while all bosons of $I = 2$ carry U(1) symmetry. The gauge transformations for both b^I are given by: $b^I \to b^I - 2\pi p\, \epsilon^{IJ} \chi^J\, da^2$. Again, we require that the additional terms in b^I fields after gauge transformations can be removed by the $\mathbb{Z}_N$ symmetry transformations Eq. (B2). However, this can only be done for $I = 1$. The absence of $I = 2$ in Eq. (B2) leads to $p = 0$.

In a similar way, we may show that $bda + a^2a^1da^1$ is also trivial. Note that bosonic SPT phases with either U(1) or $\mathbb{Z}_N$ symmetry are always trivial, we conclude that all bosonic SPT phases (both irreducible and reducible) with G= $\mathbb{Z}_N \times \mathrm{U}(1)$ symmetry are trivial.

2. $\prod_I^2 \mathbb{Z}_{N_I} \times \mathrm{U}(1)$

Next, we consider G= $\prod_I^2 \mathbb{Z}_{N_I} \times \mathrm{U}(1)$. Since there are three independent symmetry groups, we need to consider TQFT in Eq. (6). If we assume that the bosons of either $I = 1$ or $I = 2$ carry U(1) symmetry, then, it is still concluded that no irreducible bosonic SPT phases exist. Therefore, it is sufficient to only explore the possibility of nontrivial irreducible bosonic SPT phases where the bosons of $I = 3$ carry U(1) symmetry. Then, the following conditions should be imposed:

Dirac conditions: $\dfrac{1}{2\pi} \displaystyle\int_{\mathcal{M}^2} da^I \in \mathbb{Z}$, $\dfrac{1}{2\pi} \displaystyle\int_{\mathcal{M}^3} db^I \in \mathbb{Z}$; (B3)

Symmetry: $\dfrac{1}{2\pi} \displaystyle\int_{\mathcal{M}^3} db^I \to \dfrac{1}{2\pi} \displaystyle\int_{\mathcal{M}^3} db^I + N_I$

$(I = 1, 2)$. (B4)

Likewise, we can obtain the quantization of p: $p = \dfrac{k}{4\pi^2} \dfrac{N_1 N_2}{N_{12}}$, $k \in \mathbb{Z}$. We must also proceed further with the shift operation defined in Eqs. (A4,A5,A6,A7). By taking account of the symmetry transformations Eq. (B4), Eq. (A8) is changed to:

$$\dfrac{1}{2\pi} \int_{\mathcal{M}^3} db^I \bigg|_{\text{after shift}} - \dfrac{1}{2\pi} \int_{\mathcal{M}^3} db^I \bigg|_{\text{before shift}}$$
$$= N_I \times \text{integer}(\text{ for I=1,2}) \qquad \text{(B5)}$$

$$\dfrac{1}{2\pi} \int_{\mathcal{M}^3} db^3 \bigg|_{\text{after shift}} - \dfrac{1}{2\pi} \int_{\mathcal{M}^3} db^3 \bigg|_{\text{before shift}} = 0$$
$$\text{(B6)}$$

Eq. (B6) leads to:

$$K^3 = 0. \qquad \text{(B7)}$$

By using the shift operations Eqs. (A5,A6), Eq. (B5) leads to

$$\dfrac{K^1 N_1 N_2}{4\pi^2 N_{12}} \int a^2 \wedge da^3 \text{ is divisible by } N_1,$$

$$\dfrac{K^2 N_1 N_2}{4\pi^2 N_{12}} \int da^3 \wedge a^1 \text{ is divisible by } N_2.$$

By further using Eq. (A9) $(I = 1, 2)$, we end up with:

$$\dfrac{K^1 N_1 N_2}{4\pi^2 N_{12}} \dfrac{4\pi^2}{N_2} \text{ is divisible by } N_1,$$

$$\dfrac{K^1 N_1 N_2}{4\pi^2 N_{12}} \dfrac{4\pi^2}{N_1} \text{ is divisible by } N_2,$$

which means that: $K^1/N_{12} \in \mathbb{Z}, K^2/N_{12} \in \mathbb{Z}$. Thus the minimal shift of k is N_{12}. Therefore, the irreducible bosonic SPT phases are classified by cyclic group $\mathbb{Z}_{N_{12}}$.

In summary, the complete classification of bosonic SPT phases with $\prod_I^2 \mathbb{Z}_{N_I} \times \mathrm{U}(1)$ symmetry is given by $\mathbb{Z}_{N_{12}} \times (\mathbb{Z}_{N_{12}})^2 = (\mathbb{Z}_{N_{12}})^3$ where the additional two $\mathbb{Z}_{N_{12}}$ parts arise from bosonic SPT phases with $\prod_I^2 \mathbb{Z}_{N_I}$ symmetry only.

Appendix C: TQFT of irreducible bosonic SPT phases with time-reversal symmetry $(\mathbb{Z}_2^T)$

1. $\mathbb{Z}_2^T$ and $\mathrm{U}(1) \rtimes \mathbb{Z}_2^T$

The complete classification of bosonic SPT phases with $\mathbb{Z}_2^T$ (time-reversal symmetry with $\mathcal{T}^2 = 1$) is given by $(\mathbb{Z}_2)^2$[17]. One $\mathbb{Z}_2$ index corresponds to bosonic SPT phases with surface "all-fermion topological order"[30]. Its TQFT description is proposed to be the form of "$b \wedge da + b \wedge b$" in Ref. 55. The other $\mathbb{Z}_2$ index corresponds to bosonic SPT phases with surface $\mathbb{Z}_2$ topological order where both e and m carry Kramers' doublet. Its TQFT description is just a multi-component $b \wedge da$ theory of level-1 but with unusual definition of bulk time-reversal symmetry transformation, as shown in Section

VII of Ref. 55. These bosonic SPT can be viewed as either bosonic topological insulators where U(1) symmetry doesn't play role of symmetry protection or "bosonic topological superconductors" where U(1) symmetry is completely broken.

Formally, $U(1) \rtimes \mathbb{Z}_2^T$ is a *non-Abelian* symmetry group due to the semi-product operation "$\rtimes$" but we still summarize known results below. The complete classification of bosonic SPT phases with $U(1) \rtimes \mathbb{Z}_2^T$ symmetry is given by $(\mathbb{Z}_2)^3$ [17], among which there are two $\mathbb{Z}_2$ indices are given by bosonic SPT phases with merely time-reversal symmetry. The remaining one $\mathbb{Z}_2$ index labels irreducible bosonic SPT phases with $U(1) \times \mathbb{Z}_2^T$ symmetry so that both U(1) and $\mathbb{Z}_2^T$ play nontrivial role of symmetry protection. In other words, both U(1) and $\mathbb{Z}_2^T$ symmetry play nontrivial role. It can be understood through Witten effect discussed in Refs. 55 and 56. The TQFT description of this $\mathbb{Z}_2$ index is given by a multi-component $b \wedge da$ in Sec. VI of Ref. 55.

2. $U(1) \times \mathbb{Z}_2^T$

The complete classification of bosonic SPT phases with $U(1) \times \mathbb{Z}_2^T$ symmetry is given by $(\mathbb{Z}_2)^4$ [17], among which there are two $\mathbb{Z}_2$ indices are given by bosonic SPT phases in C 1. The remaining two $\mathbb{Z}_2$ indices label irreducible bosonic SPT phases with $U(1) \times \mathbb{Z}_2^T$ symmetry.

One $\mathbb{Z}_2$ index can be understood in the similar way to the Witten effect in bosonic topological insulators discussed in Refs. 55 and 56. The response theory is still given by $\Theta = 2\pi F \wedge F$ response action where the external gauge field A here is "spin gauge field" that is pseudo-vector. Under time-reversal transformation "electric field" $\mathbf{E}$ changes sign while "magnetic field" $\mathbf{B}$ not[30]. The TQFT description of this $\mathbb{Z}_2$ index is the same as the multi-component $b \wedge da$ in Ref. 55 by just changing the time-reversal transformation of external A from polar-like to pseudo-like transformations.

The other $\mathbb{Z}_2$ index is signalled by surface $\mathbb{Z}_2$ topological order where e carries half charge (compared to the fundamental U(1) charge unit of bulk bosons) and m carries Kramers' doublet[54]. For the purpose of TQFT description, we still start with the following two-component $b \wedge da$ theory with a coupling to external "spin gauge field" A:

$$S = \int i \frac{1}{2\pi} b^1 \wedge \mathrm{d}a^2 + i \int \frac{1}{2\pi} b^2 \wedge \mathrm{d}a^1$$
$$+ i \int \frac{2}{2\pi} b^2 \wedge \mathrm{d}a^2 + S_{\text{coupling}}. \qquad \text{(C1)}$$

It corresponds to $\frac{i}{2\pi} K^{IJ} \int b \wedge da$ with $K = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 2 \end{smallmatrix} \right)$.

S_{coupling} is given by: $(q_1, q_2 \in \mathbb{Z}, F = dA)$

$$S_{\text{coupling}} = \int \frac{iq_1}{2\pi} F \wedge da^2 + \frac{iq_2}{2\pi} b^2 \wedge dA. \qquad \text{(C2)}$$

Time-reversal transformation is defined as: $(I = 1, 2; i, j, \dots = \hat{x}, \hat{y}, \hat{z})$

$$\mathcal{T} a_0^I \mathcal{T}^{-1} = a_0^I, \mathcal{T} a_i^I \mathcal{T}^{-1} = -a_i^I, \qquad \text{(C3)}$$
$$\mathcal{T} b_{0i}^I \mathcal{T}^{-1} = -b_{0i}^I, \mathcal{T} b_{ij}^I \mathcal{T}^{-1} = b_{ij}^I. \qquad \text{(C4)}$$

Integrating out a^1 leads to local flatness of b^2, which can be resolved by introducing a new 1-forma gauge field: $b^2 = d\tilde{a}^2$. Thus, the time-reversal transformation is given by:

$$\mathcal{T} \tilde{a}_0^2 \mathcal{T}^{-1} = -\tilde{a}_0^2, \mathcal{T} \tilde{a}_i^I \mathcal{T}^{-1} = \tilde{a}_i^2. \qquad \text{(C5)}$$

The term $\sim b^2 \wedge \mathrm{d}a^2$ in Eq. (C1) provides a surface Chern-Simons term:

$$\mathcal{L}_\partial = i \frac{1}{\pi} \epsilon^{\mu\nu\lambda} \tilde{a}_\mu^2 \partial_\nu a_\lambda^2 \qquad \text{(C6)}$$

which can also be reformulated by introducing a matrix $K_\partial \stackrel{\text{def.}}{=\!=\!=} \left(\begin{smallmatrix} 0 & 2 \\ 2 & 0 \end{smallmatrix} \right)$ in the standard convention of K-matrix Chern-Simons theory.[2] a_μ^2 and $\tilde{a}_\mu^2$ form a 2-dimensional vector $(a_\mu^2, \tilde{a}_\mu^2)^T$. The ground state of Eq. (C6) supports a $\mathbb{Z}_2$ topological order associated with four gapped quasiparticle excitations $(1, e, m, \varepsilon)$. By using $b^2 = d\tilde{a}^2$, Eq. (C2) reduces to its surface counterpart: $\frac{q_1}{2\pi} \epsilon^{\mu\nu\lambda} A_\mu \partial_\nu a_\lambda^2 + \frac{q_2}{2\pi} \epsilon^{\mu\nu\lambda} A_\mu \partial_\nu \tilde{a}_\lambda^2$.

Based on the Chern-Simons term in Eq. (C6), one may calculate the electric charge carried by each quasiparticle: $Q_e = (q_1, q_2)(K_\partial)^{-1}(1, 0)^T = \frac{q_2}{2}$, $Q_m = (q_1, q_2)(K_\partial)^{-1}(0, 1)^T = \frac{q_1}{2}$. Physically, both e and m quasiparticles can always attach trivial identity particles to change their charges by arbitrary integer so that q_1 and q_2 are integers mod 2, namely, $q_1 \sim q_1 + 2, q_2 \sim q_2 + 2$. If we consider $(q_1, q_2) = (2, 1)$, then $Q_e = \frac{1}{2}, Q_m = 1$ indicating that e carries half-charge of U(1) symmetry. Since $\tilde{a}_\mu^2$ transforms as a pseudo-vector under time-reversal transformation defined in Eq. (C5), its gauge charge, m particle, is able to carry Kramers' doublet. This surface is what we need: e carries half-charge while m carries Kramers' doublet. We like to point out that the surface state cannot be realized on 2D plane alone unless time-reversal symmetry is broken. More concretely, let us investigate the coupling term "$\frac{q_2}{2\pi} \epsilon^{\mu\nu\lambda} A_\mu \partial_\nu \tilde{a}_\lambda^2$" that changes sign under time-reversal transformation since A is a pseudo-like vector: $\mathcal{T} A_0 \mathcal{T}^{-1} = -A_0, \mathcal{T} A_i \mathcal{T}^{-1} = A_i$. Therefore, this state necessarily break time-reversal on 2D plane. Since the sign change can be remedied by shifting q_2 through the identification "$q_1 \sim q_1 + 2, q_2 \sim q_2 + 2$", the surface state is time-reversal invariant. Note that is the external gauge field is the usual electromagnetic field that is a polar-like vector, then, the symmetry group is $U(1) \rtimes \mathbb{Z}_2^T$ and there are no obstruction for realization of the above surface state on 2D plane.

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