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Going beyond the BCS level in the superfluid path integral: A consistent treatment of electrodynamics and thermodynamics

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In this paper we derive the full gauge-invariant electromagnetic response beyond the BCS level using the fermionic superfluid path integral. In the process we identify and redress a failure to satisfy the compressibility sum rule; this shortcoming is associated with the conventional path-integral formulation of BCS-level electrodynamics. The approach in this paper builds on an alternative saddle point scheme. At the mean field level, this leads to the well known gauge-invariant electrodynamics of BCS theory and to the satisfaction of the compressibility sum rule. Moreover, this scheme can be readily extended to address arbitrary higher order fluctuation theories (for example, at the Gaussian level.) At any level this approach will lead to a gauge invariant and compressibility sum rule consistent treatment of electrodynamics and thermodynamics.

There is a great interest from diverse physics communities in understanding superfluids [1–4] and superconductors [5, 6] with stronger than BCS correlations. These strong correlations are present in both high temperature superconductors and in ultracold Fermi superfluids. At the heart of probes of superfluidity are electrodynamic and thermodynamic responses. It is, therefore, important to have a consistent theory for addressing both of these. One consistency requirement is that of gauge invariance. This affects only the electrodynamics, and importantly introduces collective modes of the order parameter. Another consistency requirement involves the inter-connection between electrodynamics and thermodynamics. This is encapsulated in the compressibility sum rule [7].

The path integral scheme is particularly well suited to consistency checks related to this inter-connection because it *simultaneously* derives electrodynamics and thermodynamics. However, this scheme, as it is applied in the literature, is not compatible with the compressibility sum rule [8]. This inconsistency shows up at the widely applied [9–12] saddle point plus Gaussian fluctuation level of approximation. This is the approximation level which is argued to be essential for obtaining the gauge invariant electrodynamics of BCS theory.

In this paper we present an alternative to this standard literature path integral approach [9–12]. The main goals are: (i) To obtain a fully gauge invariant theory of electrodynamics *beyond* BCS theory. (ii) To show how to make thermodynamics and electrodynamics consistent with the compressibility sum rule. (iii) To establish the physically observable consequence that if the electrodynamics are described by strict BCS theory, then the *thermodynamics should not include collective mode contributions*. There seems to be no consensus about whether these non-BCS terms should or should not be considered in thermodynamics [13, 14]. In our path integral re-formulation, for both the lowest order mean-field, and Gaussian fluctuation levels, we will derive theories fully consistent with gauge invariance and the compressibility sum rule. Indeed, this consistency can in principle be achieved at all orders of approximation.

We begin by addressing the compressibility sum rule. We define $\Omega = \Omega_{\rm mf} + \Omega_{\rm fl}$ as the thermodynamic potential resulting from a calculation that uses Gaussian fluctuations (fl) around mean field theory (mf) to establish a BCS-level gauge invariant electrodynamic response. We consider n particles having chemical potential μ. Within this formulation, which we call the gauge restoring Gaussian fluctuation (GRGF) theory, the number of particles $n = -\partial\Omega/\partial\mu$ has a leading order mean-field term $n_{\rm mf}$ and a fluctuation contribution $n_{\rm fl}$. Similarly the electrodynamic kernel which derives from Ω contains the counterpart mean-field and fluctuation terms, both of which combined lead to a proper gauge invariant BCS density-density correlation function $K^{00}(\omega, \mathbf{q})$. One can show that $n = n_{\rm mf} + n_{\rm fl}$ satisfies

$$K^{00}(\omega = 0, \mathbf{q} \to 0) = -\frac{\partial n_{\rm mf}}{\partial\mu} \neq -\frac{\partial n}{\partial\mu}. \qquad (1)$$

This demonstrates an explicit violation [8] of the compressibility sum rule, which should read $K^{00}(\omega = 0, \mathbf{q} \to 0) = -\partial n/\partial\mu$. It also demonstrates (at least at an empirically suggestive level) what assumptions need to be made to satisfy the compressibility sum rule within BCS theory.

In the GRGF approach leading to Eq. (1) and presented in a fairly extensive literature [9–12], fluctuations of the mean-field phase ϕ were used to restore gauge invariance. These fluctuations enter as a "dressed" vector potential $\tilde{A}_\mu = A_\mu + \partial_\mu\phi$, which is then expanded to quadratic order. Integration of the fluctuations ϕ resulted in the standard electromagnetic response kernel of strict BCS theory. We emphasize here [9–12] that the focus was on electrodynamics while the thermodynamic implications were of no concern.

In contrast, understanding thermodynamics associated with Gaussian fluctuation theories (beyond the BCS level) was the focus of work by a different community that studied ultracold Fermi superfluids [15–20]. In these neutral superfluids, soft bosonic collective modes arising from fluctuations were shown to provide new thermodynamic contributions in addition to those of the fermionic quasi-particles of BCS theory.

Yet another series of studies incorporated these Gaussian-level (beyond BCS) fluctuations to revisit electrodynamics in a higher level theory. By introducing a small phase twist in the thermodynamic potential, it was argued that one could determine the superfluid density ρ_s [21, 22]. This leads to bosonic contributions, not present in BCS theory. These were somewhat similar (but not equivalent) to contributions found within a very different diagrammatic formalism [23]. In contrast to the present work, these electrodynamic calculations did not establish consistency with gauge invariance.

All this previous literature relating to Gaussian fluctuations can be summarized by noting that there have been separate path integral studies of superfluid electrodynamics and of thermodynamics. What is missing is an analysis of the constraints which relate the two. In this paper we address this shortcoming.

Path integral and mean field.– Here we consider a fermionic partition function for a neutral, attractive, Fermi gas with s-wave pairing. The techniques presented here can be readily extended to higher order pairing, and Coulomb interactions can be included at the RPA level [11]. The partition function is calculated using the Hubbard-Stratonovich (HS) path integral

$$\mathcal{Z}[A] = \int \mathcal{D}[\boldsymbol{\Delta}]\, e^{-S_{\text{HS}}[\boldsymbol{\Delta},A]}, \tag{2}$$

where the HS action takes the usual form $S_{\text{HS}}[\boldsymbol{\Delta},A] = \int dx \frac{|\Delta|^2}{g} - \text{Tr}\ln\left[-\mathcal{G}^{-1}[\boldsymbol{\Delta},A]\right]$ [9, 24], $g>0$ is an interaction constant, and $\text{Tr}[\cdot]$ includes a trace over both position and Nambu indices; throughout we set $\hbar = k_B = 1$. The inverse Nambu Green's function $\mathcal{G}^{-1}[\boldsymbol{\Delta},A] = \mathcal{G}_0^{-1}[A] - \Sigma[\boldsymbol{\Delta}]$ is constructed from a single particle Green's function $\mathcal{G}_0[A]$ and a self-energy $\Sigma = -\boldsymbol{\Delta}\cdot\boldsymbol{\tau}$, with $\boldsymbol{\tau} = (\tau_1,\tau_2)$ a vector of Nambu Pauli matrices. Throughout we use the notation $\boldsymbol{\Delta} = (\Delta_1,\Delta_2)$ to represent two real HS fields $\Delta_a(x)$, with $a = 1,2$, consistent with previous literature [25]. The single particle Green's function $\mathcal{G}_0[A]$ is kept general, but we note that an electromagnetic vector potential A_μ has been explicitly included.

We now calculate $\mathcal{Z}[A]$ at the mean-field level using the saddle point approximation $\delta S_{\text{HS}}[\boldsymbol{\Delta},A]/\delta\Delta_a = 0$ in the presence of $A_\mu \neq 0$. This is to be contrasted with previous work (belonging to the GRGF scheme) [9–12] where the saddle point condition assumed $A_\mu = 0$. Here, explicit calculation produces the standard BCS gap equation, $0 = 2\Delta_a[A]/g - \text{Tr}[\mathcal{G}[\boldsymbol{\Delta}[A],A]\tau_a]$, *in the presence of a non-zero vector potential* A_μ. We define the solution to this gap equation as $\boldsymbol{\Delta}^{\text{mf}}[A]$, which depends on A_μ. We note that other communities have also exploited the advantages of considering alternative saddle point schemes [26, 27].

At the present mean-field (saddle point) level, we can write $\mathcal{Z}_{\text{mf}}\left[\boldsymbol{\Delta}^{\text{mf}}[A],A\right] = e^{-S_{\text{mf}}}$, where the mean-field action $S_{\text{mf}} = S_{\text{HS}}\left[\boldsymbol{\Delta}^{\text{mf}}[A],A\right]$ is the HS action evaluated at the solution to the saddle point equations. In general we cannot explicitly calculate the solution to the gap equation for $A_\mu \neq 0$. Instead, we will first use the self-consistent gap equation to find the variation of $\boldsymbol{\Delta}^{\text{mf}}[A]$ with respect to a variation in A_μ. We then take the $A_\mu \to 0$ limit, after which all quantities are calculated using $\boldsymbol{\Delta}^{\text{mf}} \equiv \boldsymbol{\Delta}^{\text{mf}}[0]$. Thus, no additional computational difficulties arise when using this self-consistency condition compared to the GRGF formalism.

Response functions at saddle point level.– Given an arbitrary "effective action" $S_{\text{eff}}[A] = -\ln\mathcal{Z}[A]$ in the presence of a weak perturbation A_μ, the response kernel comes from the second functional derivative of the action in the $A_\mu \to 0$ limit [24]. As such, we can expand $S_{\text{eff}}[A] \approx S_{\text{eff}}[0] + \frac{1}{2}\int dx \int dx' A_\mu(x)\, K^{\mu\nu}(x,x')\, A_\nu(x')$ to second order in the vector potential A_μ, where

$$K^{\mu\nu}(x,x') = \left.\frac{\delta^2 S_{\text{eff}}[A]}{\delta A_\mu(x)\,\delta A_\nu(x')}\right|_{A\to 0} \tag{3}$$

is the response kernel for an arbitrary action $S_{\text{eff}}[A]$.

We now calculate the mean-field response using the definition in Eq. (3) by including a nonzero vector potential in the saddle point condition, i.e., replace $S_{\text{eff}}[A]$ by $S_{\text{mf}} = S_{\text{mf}}\left[\boldsymbol{\Delta}^{\text{mf}}[A],A\right]$. When taking a functional derivative with respect to A_μ, new terms arise from a "functional chain rule" [9] applied to the self-consistent gap $\boldsymbol{\Delta}^{\text{mf}}[A]$. These terms, which do not not emerge for a gap calculated around $A_\mu = 0$ as in GRGF, are crucial for maintaining gauge invariance. The full response kernel then takes the form:

$$\begin{aligned} K^{\mu\nu}_{\text{mf}}(x,x') = {} & \left.\frac{\delta^2 S_{\text{mf}}}{\delta A^x_\mu \delta A^{x'}_\nu}\right|_{\boldsymbol{\Delta}^{\text{mf}}} + \frac{\delta\Delta^y_a}{\delta A^x_\mu}\left.\frac{\delta^2 S_{\text{mf}}}{\delta\Delta^y_a \delta\Delta^{y'}_b}\right|_{\boldsymbol{\Delta}^{\text{mf}}}\frac{\delta\Delta^{y'}_b}{\delta A^{x'}_\nu} \\ & + \frac{\delta\Delta^y_a}{\delta A^x_\mu}\left.\frac{\delta^2 S_{\text{mf}}}{\delta\Delta^y_a \delta A^{x'}_\nu}\right|_{\boldsymbol{\Delta}^{\text{mf}}} + \left.\frac{\delta^2 S_{\text{mf}}}{\delta A^x_\mu \delta\Delta^y_a}\right|_{\boldsymbol{\Delta}^{\text{mf}}}\frac{\delta\Delta^y_a}{\delta A^{x'}_\nu} \\ & + \left.\frac{\delta S_{\text{mf}}}{\delta\Delta^y_a}\right|_{\boldsymbol{\Delta}^{\text{mf}}}\frac{\delta^2\Delta^y_a}{\delta A^x_\mu \delta A^{x'}_\nu}, \end{aligned} \tag{4}$$

where the $A_\mu \to 0$ limit is applied after taking all derivatives. In this equation we have introduced the notation $\Delta^x_a \equiv \Delta_a(x)$ and $A^x_\mu \equiv A_\mu(x)$; repeated subscript (superscript) indices a,b (y,y') should be interpreted as an implied Einstein summation (integration.)

To express Eq. (4) in a more suggestive form, we define the set of two-point response functions [8, 28–30]:

$$\mathcal{Q}^{\alpha\beta}_{\text{mf}}(x,x') \equiv \left.\frac{\delta^2 S_{\text{mf}}\left[\boldsymbol{\Delta}^{\text{mf}},A\right]}{\delta\mathcal{A}_\alpha(x)\,\delta\mathcal{A}_\beta(x')}\right|_{A\to 0}, \tag{5}$$

where $\mathcal{A}_\alpha = \left(\Delta^{\text{mf}}_1, \Delta^{\text{mf}}_2, A_\mu\right)$ parameterizes both gap and vector potential response. The kernel $K^{\mu\nu}_{0,\text{mf}} \equiv \mathcal{Q}^{\mu\nu}_{\text{mf}}$ is the standard (non-gauge invariant) response as calculated with a gap $\mathbf{\Delta}^{\mathrm{mf}}$; the functions $Q^{a\mu}_{\mathrm{mf}} = \mathcal{Q}^{a\mu}_{\mathrm{mf}}$ and $Q^{ab}_{\mathrm{mf}} = \mathcal{Q}^{ab}_{\mathrm{mf}}$ come from "partial" derivatives in the functional chain rule. We note that the propagator Q^{ab}_{mf} is equivalent to a "GG" t-matrix for a BCS self-energy, and therefore can be interpreted as an emergent bosonic propagator [6, 31]. Using these definitions, the mean-field level gauge invariant response is compactly written

$$K^{\mu\nu}_{\mathrm{mf}} = K^{\mu\nu}_{0,\mathrm{mf}} + \Pi^{\mu}_{a} Q^{a\nu}_{\mathrm{mf}} + Q^{\mu a}_{\mathrm{mf}} \Pi^{\nu}_{a} + \Pi^{\mu}_{a} Q^{ab}_{\mathrm{mf}} \Pi^{\nu}_{b}, \qquad (6)$$

where we henceforth include an implicit integration over y, y' for every Einstein summation over a, b. In Eq. (6) we have introduced the collective mode terms $\Pi^{\mu}_{a}(x, x') \equiv \delta\Delta^{\mathrm{mf}}_{a}[A](x') / \delta A_{\mu}(x)\big|_{A\to 0}$; these explicitly restore gauge invariance beyond the "bubble" response kernel $K^{\mu\nu}_{0,\mathrm{mf}}$ [8, 28–30]. In the saddle point response, the third line in Eq. (4) vanishes.

Using the revised saddle point condition, along with the above definitions, the collective modes are $\Pi^{\mu}_{a} = -\left[Q^{ab}_{\mathrm{mf}}\right]^{-1} Q^{b\mu}_{\mathrm{mf}}$ where the inverse $\left[Q^{ab}_{\mathrm{mf}}\right]^{-1}$ is taken over both position and Nambu indices (see Supplemental Material [32]). We emphasize that these collective modes are associated with the mean-field level of approximation. Finally, after taking the $A_{\mu} \to 0$ limit, the momentum space response is

$$K^{\mu\nu}_{\mathrm{mf}}(q) = K^{\mu\nu}_{0,\mathrm{mf}}(q) - Q^{\mu a}_{\mathrm{mf}}(-q) \left[Q^{ab}_{\mathrm{mf}}(q)\right]^{-1} Q^{b\nu}_{\mathrm{mf}}(q). \qquad (7)$$

This is the usual gauge invariant response kernel in BCS theory [28] which includes both amplitude and phase collective modes.

Importantly, the response kernel $K^{\mu\nu}_{\mathrm{mf}}$, which is explicitly gauge invariant, was obtained without including Gaussian fluctuations, which are usually invoked in the GRGF literature. In this way *the self-consistent treatment of the gap in the presence of a vector potential restores gauge invariance at the mean-field level.* Because there are no accompanying bosonic degrees of freedom in the thermodynamics, the compressibility sum rule will be shown to be exactly satisfied using this method, in contrast to the more conventional path integral methodology.

Beyond saddle point.– Often it is desirable to calculate the path integral beyond the saddle point approximation. In order to do this, one changes variables from the HS field $\mathbf{\Delta}$ to a fluctuation $\boldsymbol{\eta} = (\eta_1, \eta_2)$ around the saddle point solution defined through $\mathbf{\Delta} = \mathbf{\Delta}^{\mathrm{mf}}[A] + \boldsymbol{\eta}$. We note that since $\boldsymbol{\eta}$ is a dynamical variable it does not have any dependence on A_{μ}. The full action is then expressed exactly as $S_{\mathrm{HS}}[\mathbf{\Delta}, A] = S_{\mathrm{mf}} + S_{\eta}$, where the action $S_{\eta} \equiv S_{\eta}\left[\mathbf{\Delta}^{\mathrm{mf}}[A], A, \boldsymbol{\eta}\right] = S_{\mathrm{HS}}\left[\mathbf{\Delta}^{\mathrm{mf}}[A] + \boldsymbol{\eta}, A\right] - S_{\mathrm{HS}}\left[\mathbf{\Delta}^{\mathrm{mf}}[A], A\right]$ is $\mathcal{O}\left(\eta^2\right)$ or higher, since any term linear in $\boldsymbol{\eta}$ vanishes by the saddle point condition. This definition allows for the exact factorization of the partition function $\mathcal{Z}[A] = \mathcal{Z}_{\mathrm{mf}}\left[\mathbf{\Delta}^{\mathrm{mf}}[A], A\right] \mathcal{Z}_{\mathrm{fl}}\left[\mathbf{\Delta}^{\mathrm{mf}}[A], A\right]$, where

$$\mathcal{Z}_{\mathrm{fl}}\left[\mathbf{\Delta}^{\mathrm{mf}}[A], A\right] = \int \mathcal{D}[\boldsymbol{\eta}]\, e^{-S_{\eta}\left[\mathbf{\Delta}^{\mathrm{mf}}[A], A, \boldsymbol{\eta}\right]} \qquad (8)$$

is the contribution due to fluctuations beyond mean field.

In calculations of response beyond saddle point, one uses Eq. (3) with an effective action $S_{\mathrm{eff}}[A] = -\ln \mathcal{Z}[A] = S_{\mathrm{mf}} + S_{\mathrm{fl}}$, and the fluctuation action $S_{\mathrm{fl}} = -\ln \mathcal{Z}_{\mathrm{fl}}\left[\mathbf{\Delta}^{\mathrm{mf}}[A], A\right]$ also depends on the self-consistent gap $\mathbf{\Delta}^{\mathrm{mf}}[A]$. The response kernel is linear in the action, so that $K^{\mu\nu} = K^{\mu\nu}_{\mathrm{mf}} + K^{\mu\nu}_{\mathrm{fl}}$, where the mean-field response is given in Eq. (7). The new contribution to the response, $K^{\mu\nu}_{\mathrm{fl}}$, has a form identical to Eq. (4), only with S_{mf} replaced by S_{fl}. Note, however, that the collective mode terms Π^{μ}_{a} still arise from the mean field self-consistent gap condition; these collective modes are always constructed from the Q_{mf} propagators, and not from an analogous Q_{fl}.

This higher order fluctuation response again contains a "bubble" term $K^{\mu\nu}_{0,\mathrm{fl}}$ that arises from bosonic fluctuations. On its own, $K^{\mu\nu}_{0,\mathrm{fl}}$ is not gauge invariant. Analogous to the saddle-point response, the collective modes Π^{μ}_{a}, along with the corresponding Q_{fl} response functions, are necessary to restore gauge invariance. To show that this arbitrary fluctuation theory is fully gauge invariant, one can verify that $\partial_{\mu} K^{\mu\nu}_{\mathrm{fl}} = 0$ is satisfied (see the Supplemental Material [32].) In this way, gauge invariance holds term by term in the expansion of the action beyond mean-field. This calculation scheme for gauge invariant response beyond-BCS is a completely general sum rule consistent scheme and a central result of this manuscript.

Compressibility sum rule.– Thermodynamic quantities can be calculated from derivatives of the thermodynamic potential, $\Omega = -T \ln \mathcal{Z} = T S_{\mathrm{eff}}$, which is the effective action up to the prefactor T. Since electromagnetic response functions also come from derivatives of the effective action, it is clear that there should be an intimate connection between the two. An important requirement for consistency between electrodynamics and thermodynamics is contained in the compressibility sum rule: $\partial n / \partial \mu = -K^{00}(0, \mathbf{q} \to 0)$.

A formal derivation of this sum rule, for the exact action, arises from twice invoking the identity $\int dx\, \delta \mathcal{G}^{-1}_{0} / \delta A_0(x) = -\partial \mathcal{G}^{-1}_{0} / \partial \mu$ on the partition function in Eq. (2). A more intuitive derivation of this sum rule follows from the fermionic path integral, before applying the HS transformation. The atom number is $n \equiv \left\langle \int dx\, \hat{n}(x) \right\rangle = -\partial \Omega / \partial \mu$, where $\hat{n}(x) = \sum_{s=\uparrow,\downarrow} \psi^{\dagger}_{s}(x) \psi_{s}(x)$ is the local fermion density operator. A second derivative gives $\partial n / \partial \mu = -\partial^2 \Omega / \partial \mu^2 = -\left\langle \left(\int dx\, \hat{n}(x)\right)^2 \right\rangle$. On the other hand, the small momentum limit of the density-density correlation function is $K^{00}(0, \mathbf{q} \to 0) = \int dx \int dx' K^{00}(x, x')$, where $K^{00}(x, x') = \langle \hat{n}(x) \hat{n}(x') \rangle$ follows from Eq. (3). It is straightforward to see this response function is just

$K^{00}(0, \mathbf{q} \to 0) = -\partial n/\partial\mu$ as defined above. Therefore, the compressibility sum rule is an exact consequence of a path integral approach *provided no approximations are made*.

When considering *only* thermodynamics, it is not necessary to keep track of the vector potential in the self-consistent solution, and S_{eff} can be calculated for $A_\mu = 0$ and $\mathbf{\Delta}^{\text{mf}}[0]$. However, when simultaneously considering electrodynamics and thermodynamics it is important to calculate $S_{\text{eff}}[A]$ to the same level of approximation for both quantities. Due to the linear dependence of both electrodynamic and thermodynamic quantities on the effective action, any theory studying both quantities, which considers a *consistent* approximation scheme, will also satisfy the compressibility sum rule.

Gaussian fluctuations.– An exact calculation of $\mathcal{Z}_{\text{fl}}$ is in general difficult and is frequently treated at the Gaussian level in the literature. We similarly consider response at this level: fluctuations $\boldsymbol{\eta}$ about the saddle point solution are assumed small and the fluctuation action is expanded to quadratic order: $S_\eta\left[\mathbf{\Delta}^{\text{mf}}[A], A\right] \approx \frac{1}{2}\eta_a \widetilde{Q}^{ab}_{\text{mf}}\eta_b$. The path integral can then be solved exactly; integration of the fluctuation field $\boldsymbol{\eta}$ gives an effective action $S^{(2)}_{\text{fl}} = \frac{1}{2}\text{Tr}\ln\left[\widetilde{Q}^{ab}_{\text{mf}}\right]$ at the Gaussian level. We emphasize that in the calculation of the fluctuation response kernel, $K^{\mu\nu}_{\text{fl}}$, the propagator $\widetilde{Q}^{ab}_{\text{mf}} = \widetilde{Q}^{ab}_{\text{mf}}\left[\mathbf{\Delta}^{\text{mf}}[A], A\right]$ includes dependence on A_μ both explicitly, and through the mean-field solution. This is in contrast to previous literature which used the fluctuation propagator Q^{ab}_{mf} in Eq. (5).

It is clear that setting $A_\mu = 0$ will reproduce beyond-BCS thermodynamics found in the literature [15–20]. Our formalism can also recover the bosonic contributions to the superfluid density $\rho_s \sim K^{ii}(0, \mathbf{q} \to 0)$ found in Refs. [21, 22]. This calculation introduced a phase twist $\mathbf{Q} \to 0$ that is equivalent to our counterpart calculation after the replacement $A_\mu \to \mathbf{Q}$. Since ρ_s is a purely transverse quantity, this prior work did not need to include collective mode contributions emphasized in the present formalism. In this way our results reproduce and extend previous explorations of Gaussian fluctuations, now establishing consistency with the compressibility sum rule.

Amplitude and Phase fluctuations.– While not explicitly discussed, amplitude fluctuations of the gap were implicitly included in the compressibility sum rule arguments presented in this paper. These are often ignored, although they have been introduced in the literature via an alternative parameterization of the gap, by writing $\Delta = \rho e^{i2\phi}$, where $\rho = |\Delta|$ and $2\phi = \arg\Delta$ are respectively the amplitude and phase of the order parameter. Including amplitude fluctuations by setting $\rho = \rho_0 + \delta\rho$ and integrating out both $\partial_\mu\phi$ and $\delta\rho$ fluctuations results in a different gauge invariant formulation but one which is equivalent to the $\boldsymbol{\eta}$ fluctuation used above. It should be noted that while amplitude fluctuations result in a contribution to electrodynamic (and thermodynamic) response, phase fluctuations alone are sufficient to restore gauge invariance at both the mean-field and fluctuation levels. We note, however, that by neglecting amplitude fluctuations, the compressibility sum rule will be violated and this violation is apparent even at the mean field level of strict BCS theory.

Discussion.– In this paper we have presented a path integral formulation for superfluids and superconductors which: (1) allows for a consistent calculation of (gauge invariant) electrodynamic and thermodynamic response at any desired level of approximation, and (2) gives the full gauge invariant response kernel for beyond mean-field physics. The consistency of our formulation is apparent in the compressibility sum rule which related electrodynamics and thermodynamics. This sum rule is not satisfied at the BCS level in the path integral formalism if Gaussian fluctuations are invoked as in GRGF; instead a consistent treatment involves finding the saddle point solution in the presence of a vector potential. Our way of introducing collective mode effects is closer in spirit to earlier self consistency schemes [33–35] derived using strict BCS theory.

We stress an important physical implication of the current scheme. Within the conventional path integral approach, Gaussian fluctuations are needed to arrive at gauge invariant electrodynamics. One might posit that there ought to be fluctuation contributions to thermodynamics. Specifically, in a neutral superfluid these collective modes would seem to require power law contributions, say in the specific heat. We argue here, despite some controversy in the literature [13], including these correction terms in strict BCS theory is unphysical, as they are inconsistent with the compressibility sum rule.

Within the present formalism, the next level approximation, involving Gaussian fluctuations then emerges as a true beyond-BCS theory in which there are interrelated (by the compressibility sum rule) contributions to both thermodynamics and the electromagnetic response. This beyond-BCS level of approximation provides a starting point for studying strongly correlated superfluids. It should be viewed as an alternative to schemes which build on a correlation self energy and the Ward-Takahashi identity [8].

This approach provides a promising new route to bench marking beyond-BCS calculations derived from path integral approaches. There are indications from the superfluid density at the Gaussian level that possibly unphysical non-monotonicities appear [22]. These may also be present when comparing with density correlation functions which are measured in Bragg scattering experiments. Nevertheless it will be interesting to look at these higher level (Gaussian) corrections in a variety of physical contexts, including, for example, their role in topological [10–12] or disordered [26] superfluids. Quite generally, this work should be viewed as providing a new paradigm for exploring beyond-BCS physics using path integral techniques.

Note Added.– After the submission of our manuscript, a preprint appeared [36] that considered electrodynamic response at the Gaussian level. Our more general framework includes their results at the Gaussian fluctuation level, where $S_{\text{fl}}^{(2)} = \frac{1}{2}\text{Tr}\ln\left[\widetilde{Q}_{\text{mf}}^{ab}\right]$ presented above.

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[1] S. Giorgini, L. P. Pitaevskii, and S. Stringari, Rev. Mod. Phys. **80**, 1215 (2008).
[2] P. D. Drummond, H. Hu, and X. J. Liu, J. Mod. Opt. **56**, 2076 (2009).
[3] H. Hu, Front. Phys. **7**, 98 (2012).
[4] W. Zwerger, *The BCS-BEC Crossover and the Unitary Fermi Gas*, Lecture Notes in Physics (Springer Berlin Heidelberg, 2011).
[5] K. L. Hur and T. Maurice Rice, Ann. Phys. **324**, 1452 (2009).
[6] Q. Chen, J. Stajic, S. Tan, and K. Levin, Phys. Rep. **412**, 1 (2005).
[7] D. Pines and P. Nozieres, *The theory of quantum liquids: normal Fermi liquids*, Vol. 1 (WA Benjamin, 1966).
[8] R. Boyack, B. M. Anderson, C.-T. Wu, and K. Levin, ArXiv e-prints (2016), arXiv:1602.02156 [cond-mat.supr-con].
[9] A. Altland and B. Simons, *Condensed Matter Field Theory* (Cambridge University Press, 2006) cambridge Books Online.
[10] J. Goryo and K. Ishikawa, Phys. Lett. A **260**, 294 (1999).
[11] R. M. Lutchyn, P. Nagornykh, and V. M. Yakovenko, Phys. Rev. B **77**, 144516 (2008).
[12] T. Ojanen and T. Kitagawa, Phys. Rev. B **87**, 014512 (2013).
[13] Z. Yu, G. M. Bruun, and G. Baym, Phys. Rev. A **80**, 023615 (2009).
[14] D. Vollhardt and P. Wolfle, *The superfluid phases of helium 3* (Courier Corporation, 2013).
[15] J. R. Engelbrecht, M. Randeria, and C. A. R. Sáde Melo, Phys. Rev. B **55**, 15153 (1997).
[16] V. M. Loktev, R. M. Quick, and S. G. Sharapov, Phys. Rep. **349**, 1 (2001).
[17] H. Hu, X.-J. Liu, and P. D. Drummond, Europhys. Lett. **74**, 574 (2006).
[18] E. Taylor, A. Griffin, and Y. Ohashi, Phys. Rev. A **76**, 023614 (2007).
[19] R. B. Diener, R. Sensarma, and M. Randeria, Phys. Rev. A **77**, 023626 (2008).
[20] L. He, H. Lü, G. Cao, H. Hu, and X.-J. Liu, Phys. Rev. A **92**, 023620 (2015).
[21] E. Taylor, A. Griffin, N. Fukushima, and Y. Ohashi, Phys. Rev. A **74**, 063626 (2006).
[22] N. Fukushima, Y. Ohashi, E. Taylor, and A. Griffin, Phys. Rev. A **75**, 033609 (2007).
[23] N. Andrenacci, P. Pieri, and G. C. Strinati, Phys. Rev. B **68**, 144507 (2003).
[24] E. Fradkin, *Field theories of condensed matter systems*, 82 (Addison Wesley Publishing Company, 1991).
[25] This notation is equivalent to that in Refs. [8, 28–30]. The reality condition is expressed in position space; a momentum-space parametrization has a different condition [30]. The conventional BCS self energy $\Sigma\left[\mathbf{\Delta}\right] = -\left(\Delta\tau_{+} + \Delta^{*}\tau_{-}\right)$ suggests an equivalent complex parametrization $\Delta_{\pm} = \Delta_1 \pm i\Delta_2$ where the BCS gap is identified through $\Delta \equiv \Delta_{-}$. See the Supplemental Material [32] for details.
[26] A. Kamenev and A. Andreev, Phys. Rev. B **60**, 2218 (1999).
[27] E. Ercolessi, G. Morandi, P. Pieri, and M. Roncaglia, Phys. Rev. B **62**, 14860 (2000).
[28] I. O. Kulik, O. Entin-Wohlman, and R. Orbach, Journal of Low Temperature Physics **43**, 591 (1981).
[29] I. Kosztin, Q. Chen, Y.-J. Kao, and K. Levin, Phys. Rev. B **61**, 11662 (2000).
[30] H. Guo, C.-C. Chien, and Y. He, J. Low Temp. Phys. **172**, 5 (2013).
[31] P. Pieri and G. C. Strinati, Phys. Rev. B **71**, 094520 (2005).
[32] See Supplemental Material for more details.
[33] V. Ambegaokar and L. P. Kadanoff, Nuovo Cimento **22**, 914 (1955).
[34] R. Parks, *Superconductivity: Part 1 (In Two Parts)*, Superconductivity (Taylor & Francis, 1969).
[35] G. Rickayzen, *Theory of superconductivity*, Interscience monographs and texts in physics and astronomy No. v. 14 (Interscience Publishers, 1965).
[36] L. He, ArXiv e-prints (2016), arXiv:1602.06502 [cond–mat.quant-gas].