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Michael Stone and Pedro L. e S. Lopes

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# Effective action and electromagnetic response of topological superconductors and Majorana-mass Weyl fermions

MICHAEL STONE

*University of Illinois,*

*Department of Physics*

*1110 W. Green St.*

*Urbana, IL 61801 USA*

*e-mail: m-stone5@illinois.edu*

PEDRO L. e S. LOPES

*Instituto de Física Gleb Wataghin,*

*Universidade Estadual de Campinas,*

*Campinas, SP 13083-970, Brazil*

*e-mail: pedrolslopes@gmail.com*

## Abstract

Motivated by an apparent paradox in [X-L. Qi, E. Witten, S-C. Zhang, Phys. Rev. B **87** 134519 (2013)] we use the method of gauged Wess-Zumino-Witten functionals to construct an effective action for a Weyl fermion with a Majorana mass that arises from coupling to a charged condensate. We obtain expressions for the current induced by an external gauge field and observe that the topological part of the current is only one-third of that that might have been expected from the gauge anomaly. The anomaly is not changed by the induced mass gap however. The topological current is supplemented by a conventional supercurrent that provides the remaining two-thirds of the anomaly once the equation of motion for the Goldstone mode is satisfied. We apply our formula for the current to resolve the apparent paradox, and also to the chiral magnetic effect (CME) where it predicts a reduction of the CME current to one third of its value for a free Weyl gas in thermal equilibrium. We attribute this reduction to a partial cancelation of the CME by a chiral vortical effect (CVE) current arising from the persistent rotation of the fluid induced by the external magnetic field.

## I. INTRODUCTION

In [1] Qi, Zhang and Witten (QZW) consider the electromagnetic response of a 3+1 dimensional topological superconductor in which two Fermi surfaces of opposite Chern number are each provided with their own independent superconducting order parameter. When the Fermi surfaces are realized as a pair of opposite-chirality Weyl fermions, the superconducting gap induced by the order parameter is an example of Majorana mass generation similar to that proposed for standard-model neutrinos. The analysis in [1] therefore has potential applications well beyond condensed matter physics.

One of the intriguing topological effects deduced by QZW applies when one (but not both) of the two condensate order parameters contains a vortex line about which the phase of the charged condensate winds through  $2\pi$ . If an electric field is directed along the vortex line they find an inflow of electric charge into the vortex core. This inflowing current is similar to that which appears for vortex strings in an uncharged Higgs field that induces a mass for conventional Dirac fermions. In the Dirac case the inflowing charge is soaked up by the  $U(1)$  anomaly of a 1+1 dimensional charged chiral fermion mode that is bound in the vortex core. Indeed the Dirac case is the simplest illustration of the Callan-Harvey anomaly-inflow mechanism [2].

For our Majorana-mass Weyl fermion, the charge inflow poses something of a paradox. The vortex core still confines a 1+1 chiral fermion — indeed a Weyl fermion gapped by a charged Higgs field is a system for which a low energy chiral vortex-core mode is guaranteed by the Erick Weinberg index theorem [3, 4] — however the chiral mode is a chiral-*Majorana* mode. It is electrically neutral (see appendix A) and hence possesses no anomaly that can absorb the inflowing current.

This paradox leads us to reconsider the derivation of the effective action in [1]. We follow the route pioneered in [5] and seek an effective action that contains as its degrees of freedom the ungapped phase degrees of freedom on the two Fermi surfaces. These Goldstone modes are then coupled to the external gauge field through the simplest set of interaction terms that are compatible with the anomalous realization of the gauge symmetry. The result is an action functional that is rather different from that obtained in [1] and enables us to resolve the inflow paradox.

In section II we describe how adding a Majorana mass to a charged Weyl fermion turns it

into a superconductor, and note a second potential paradox that this threatens. In section III we review the strategy for systematically constructing the Wess-Zumino-Witten (WZW) effective action for the chiral dynamics of anomalous system. We apply this strategy to two-dimensional systems of charge density waves and superconductors in section IV and demonstrate that it reproduces familiar physics. The more complicated case of four dimensions is addressed in section V where we find the topological currents and equation of motion for a superconducting Weyl fermion. Equations (86) and (87) of this section are the principal results of this paper. In section VI we show how our expression for the current resolves the inflow paradox, and also discuss the implications of these equations for the chiral magnetic effect. Finally in VII we summarize and contrast our results with those of [1].

## II. WEYL FERMIONS, SUPERCONDUCTIVITY AND MAJORANA MASS

The prototype of a system whose Fermi surface possesses a non-trivial Berry connection is a 3+1 dimensional Weyl fermion, where the first Chern numbers of the Berry curvature are  $C_1 = \pm 1$  for right- and left-handed particles respectively.

The second-quantized Hamiltonian for a right-handed Weyl particle with charge  $e$  and coupled to an external Maxwell field  $A^\mu = (\phi, \mathbf{A})$ ,  $A_\mu = (\phi, -\mathbf{A})$  is

$$\begin{aligned}\hat{H}_{\text{Weyl}}[A] &= \int \psi^\dagger \{ \boldsymbol{\sigma} \cdot (\mathbf{p} - e\mathbf{A}) + e\phi \} \psi d^3x \\ &= \int \psi^\dagger \{ -i\boldsymbol{\sigma} \cdot (\nabla - ie\mathbf{A}) + e\phi \} \psi d^3x.\end{aligned}\tag{1}$$

The anti-commuting two-component Fermi fields  $\psi$ ,  $\psi^*$  obey the canonical anti-commutation rules (CAR)

$$\{ \psi_\alpha(x), \psi_\beta^*(x') \} = \delta_{\alpha\beta} \delta^3(x - x'),\tag{2}$$

where by  $\psi^*$ , we mean the Hermitian conjugate of  $\psi$  as a Hilbert-space operator, but not a matrix transpose of the two-component column spinor into a two-component row spinor.

Thus

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \quad \psi^\dagger = ( \psi_1^* \quad \psi_2^* ).\tag{3}$$

The field  $\psi_c = i\sigma_2\psi^*$  possesses identical Lorentz transformation properties as  $\psi$ , and also obeys the same CAR:

$$\{ \psi_{c,\alpha}(x), \psi_{c,\beta}^*(x') \} = \delta_{\alpha\beta} \delta^3(x - x').\tag{4}$$

The Pauli-matrix identity

$$(i\sigma_2)^\dagger \boldsymbol{\sigma} (i\sigma_2) = -\boldsymbol{\sigma}^* \quad (5)$$

and an integration by parts permits us to rewrite  $\hat{H}_{\text{Weyl}}$  in terms of  $\psi_c$  as

$$\hat{H}_{\text{Weyl}}[A] = \int \psi_c^\dagger \{ -\boldsymbol{\sigma} \cdot (\mathbf{p} + e\mathbf{A}) - e\phi \} \psi_c d^3x. \quad (6)$$

The rewrite shows that  $\psi_c$  is the charge conjugate (antiparticle) field of  $\psi$ . The original field  $\psi$  annihilates positive energy states that have charge  $+e$  and are right handed in that the spin  $\boldsymbol{\sigma}$  is parallel to  $\mathbf{p}$ . The field  $\psi_c$  annihilates positive energy particles of charge  $-e$  that are left-handed in that their spin is anti-parallel to  $\mathbf{p}$ .

The skew symmetry of  $(i\sigma_2)_{\alpha\beta} = \epsilon_{\alpha\beta}$  allows us to add to  $\hat{H}_{\text{Weyl}}$  a term

$$\hat{H}_1 = \frac{1}{2}(\Phi\psi^\dagger\psi_c + \Phi^*\psi_c^\dagger\psi) = \frac{1}{2}(\Phi\epsilon_{\alpha\beta}\psi_\alpha^*\psi_\beta^* - \Phi^*\epsilon_{\alpha\beta}\psi_\alpha\psi_\beta) \quad (7)$$

that couples pairs of particles or antiparticles to a charged  $c$ -number Higgs field  $\Phi = |\Phi|e^{i\theta}$ .

We can write the resultant Hamiltonian in Bogoliubov-de-Gennes (BdG) form [6] as

$$\hat{H}_{\text{BdG}}[A] = \frac{1}{2} \int d^3x \begin{pmatrix} \psi^\dagger & \psi_c^\dagger \end{pmatrix} \begin{bmatrix} \boldsymbol{\sigma} \cdot (\mathbf{p} - e\mathbf{A}) + e\phi & \Phi \\ \Phi^* & -\boldsymbol{\sigma} \cdot (\mathbf{p} + e\mathbf{A}) - e\phi \end{bmatrix} \begin{pmatrix} \psi \\ \psi_c \end{pmatrix}. \quad (8)$$

The factor of 1/2 outside the integral ensures that  $\hat{H}_{\text{BdG}} \rightarrow \hat{H}_{\text{Weyl}}$  as  $\Phi \rightarrow 0$ .

Were our Weyl field electrically neutral,  $\hat{H}_{\text{BdG}}$  would contain the one-particle four-component Dirac hamiltonian

$$H_{\text{Dirac}} = \begin{bmatrix} \boldsymbol{\sigma} \cdot \mathbf{p} & \Phi \\ \Phi^* & -\boldsymbol{\sigma} \cdot \mathbf{p} \end{bmatrix}, \quad (9)$$

and the added term would be a *Majorana mass*. A Majorana mass term opens a gap at the  $\mathbf{p} = 0$  Dirac point, but couples the right-handed particle to its own left-handed antiparticle rather than to an independent left-handed Weyl fermion. It is as yet uncertain whether the masses of standard-model neutrinos arise from Dirac or Majorana terms [7]. The presence of the gauge field  $A^\mu$ , however, reveals a key difference between the matrix appearing in (8) and the conventional one-particle Hamiltonian for charged Dirac particles

$$H_{\text{Dirac}}(A) = \begin{bmatrix} \boldsymbol{\sigma} \cdot (\mathbf{p} - e\mathbf{A}) + e\phi & \Phi \\ \Phi^* & -\boldsymbol{\sigma} \cdot (\mathbf{p} - e\mathbf{A}) + e\phi \end{bmatrix}. \quad (10)$$

The sign-difference in the  $(\phi, \mathbf{A})$  coupling to the left- and right-handed fields in (8) means that the BdG fermions are coupled to an *axial-vector* gauge field. The four-dimensional Lagrange density is therefore of the form

$$\mathcal{L}_{\text{BdG}} = \bar{\Psi} \left( i\gamma^\mu (\partial_\mu + ie\gamma_5 A_\mu) - |\Phi| e^{i\gamma_5 \theta} \right) \Psi. \quad (11)$$

One consequence of the axial character of the gauge field appears when we set  $\mathbf{A} = 0$  and let  $-e\phi = -eA_0$  be regarded as a chemical potential  $\mu$  that fixes the Fermi-surface of the Weyl fermions to be at  $|\mathbf{p}| = \mu$ . We find that the energy gap in the spectrum of  $\hat{H}_{\text{BdG}}[A]$  appears at this Fermi surface rather than at the Dirac point  $\mathbf{p} = 0$ . Consequently a degenerate gas of gauge-coupled neutrinos with a Majorana mass is really a topological superconductor [8].

A second and potentially paradoxical issue is the question of the gauge anomaly. The original massless Weyl fermion possesses an anomaly in the conservation law for the particle number current  $j^\mu = (\psi^\dagger \psi, \psi^\dagger \boldsymbol{\sigma} \psi)$  that modifies it to read

$$\partial_\mu j^\mu = \frac{e^2}{32\pi^2} \epsilon^{\mu\nu\sigma\tau} F_{\mu\nu} F_{\sigma\tau}. \quad (12)$$

The non-conservation of charge arises from a flux of particles from the negative-energy Dirac sea (regarded as a charge-neutral vacuum) into the positive-energy Fermi sea *via* the Dirac point at a rate  $\dot{N} = e^2 \mathbf{E} \cdot \mathbf{B} / 4\pi^2$  per unit volume. This anomaly is no trouble for the gauge invariance of our topological superconductor as each Fermi surface with a positive Chern number is paired with one with a negative Chern number and hence a canceling anomaly. The entire theory is therefore anomaly-free. What *is* a potential problem is that the axial-current anomaly for Dirac particle coupled to a *axial-vector* gauge field is only 1/3 that of the axial current anomaly for Dirac particle coupled to a conventional vector gauge field [9, 10]. The 1/3 is a puzzle because a mere cosmetic rewrite combined with the introduction of a mass term (a “soft” low-energy perturbation) should not be able to alter an anomaly that arises from high-energy effects. Note also that compared to the usual axial anomaly, there is an additional factor of 1/2 to be taken into account in the current because the fermi field in (11) obeys a Majorana condition  $\Psi = C\Psi^*$ , but this is already included in (12) because we have only right-handed particles. The usual Dirac-particle axial anomaly counts the difference between the number of right handed and left handed particles and so has a RHS that is twice that of (12).

In the next to section we will set out to resolve the two potential paradoxes by investi-

gating the form of the effective action

$$S[\theta, A] = -\frac{i}{2} \ln \text{Det} \left( i\gamma^\mu (\partial_\mu + ie\gamma_5 A_\mu) - |\Phi| e^{i\gamma_5 \theta} \right) \quad (13)$$

that arises from integrating out the right-handed Weyl fermion. In (13) the factor of  $1/2$  comes from the Majorana/BdG condition.

### III. WESS-ZUMINO EFFECTIVE ACTIONS

We begin by reviewing Witten's Wess-Zumino strategy [5] that enables us to deduce, with minimal labour, the topological part of the effective action for a massive fermion coupled to a gauge field. To appreciate the underlying structure of the method we first consider the general case of  $N$  flavours of Weyl fermions  $\psi_L$ ,  $\psi_R$  that are coupled to non-Abelian  $U(N)_L \times U(N)_R$  gauge fields. Only then will we restrict ourselves to fermions obeying  $\psi_L = (\psi_R)_c$  and Abelian axial-vector gauge fields.

The action for the fields  $\psi_R$ ,  $\psi_L$  is built from the gauge covariant derivatives

$$\begin{aligned} \nabla_\mu \psi_R &= (\partial_\mu + R_\mu) \psi_R, \\ \nabla_\mu \psi_L &= (\partial_\mu + L_\mu) \psi_L. \end{aligned} \quad (14)$$

We have here absorbed the customary factors of  $i$  and  $e$  into the definition of the gauge potentials  $R$  and  $L$  so as to improve the readability of our formulæ. These factors will be restored when we consider physical effects. We also sometimes write  $\Psi = (\psi_R, \psi_L)^T$  and

$$\nabla \Psi = (\partial + \mathcal{V} + \gamma_5 \mathcal{A}) \Psi, \quad (15)$$

where the vector and axial-vector gauge fields  $\mathcal{V}$  and  $\mathcal{A}$  are related to the right and left gauge fields by  $\mathcal{V} + \mathcal{A} = R$  and  $\mathcal{V} - \mathcal{A} = L$ .

Our fermions are gapped by coupling to a nonlinear  $\sigma$ -model field  $U \in U(N)$  through a term

$$H_{\text{mass}} = \Delta (\psi_{L,i}^* U_{ij} \psi_{R,j} + \psi_{R,i}^* U_{ij}^\dagger \psi_{L,j}). \quad (16)$$

This form for the gap-inducing interaction makes the  $\Phi$  appearing in (7) and (8) correspond with  $U^\dagger$  rather than  $U$ , but we have adopted it so as to facilitate comparison with the notation in [5].

The resulting classical action is invariant under the transformation  $(h_L, h_R) \in U(N)_L \times U(N)_R$  that acts to take  $\psi_R \rightarrow h_R^{-1}\psi_R$ ,  $\psi_L \rightarrow h_L^{-1}\psi_L$ ,  $U \rightarrow h_L^{-1}U h_R$ , while the gauge-potential 1-forms  $L$ ,  $R$  and their associated curvature 2-forms  $F_L = dL + L^2$  and  $F_R = dR + R^2$  transform as

$$\begin{aligned} L &\rightarrow L^{h_L} = h_L^{-1}L h_L + h_L^{-1}dh_L, \\ R &\rightarrow R^{h_R} = h_R^{-1}R h_R + h_R^{-1}dh_R, \\ F_L &\rightarrow h_L^{-1}F_L h_L, \\ F_R &\rightarrow h_R^{-1}F_R h_R. \end{aligned} \tag{17}$$

This gauge invariance will be partially violated in the quantum theory by anomalies.

The transformation rules (17) show that the appropriate covariant derivative for the non-linear  $\sigma$ -model field  $U$  is

$$\nabla_\mu U = \partial_\mu U + L_\mu U - U R_\mu. \tag{18}$$

This derivative transforms in the same manner as  $U$  itself

$$\nabla_\mu U \rightarrow h_L^{-1}(\nabla_\mu U)h_R. \tag{19}$$

Because the fermions are fully gapped, they are slaved to the external gauge and mass-generating fields. Their response to adiabatic changes in those fields is therefore governed by an effective action  $S[R, L, U]$  which contains a topological part, the gauged Wess-Zumino-Witten (WZW) action  $W[R, L, U]$ , that is entirely determined by anomalies [5, 12]. The functional  $W[R, L, U]$  can be systematically constructed by imagining that our four-dimensional theory lives on the boundary of a five manifold (the “bulk”) from which a current inflow is the source of the anomalous conservation laws [2]. The action in the bulk will involve a five-dimensional Chern-Simons form constructed so that the complete topological action functional (bulk-plus-boundary) is fully gauge invariant.

When the bulk is  $2n - 1$  dimensional, its Chern-Simons action density  $\omega_{2n-1}(L, R)$  is to be a solution to

$$d\omega_{2n-1}(R, L) = \Omega_{2n}(F_R) - \Omega_{2n}(F_L), \tag{20}$$

where  $\Omega_{2n}(F) = \text{tr} \{F^n\}$  is the (unnormalized) Chern-character anomaly polynomial. The minus sign between the two terms reflects the fact that left and right handed fermions have



opposite anomalies. An obvious way to satisfy (20) would be to set

$$\omega_{2n-1}(R, L) = \omega_{2n-1}(R) - \omega_{2n-1}(L), \quad (21)$$

where  $\omega_{2n-1}(R)$ ,  $\omega_{2n-1}(L)$  are the standard Chern-Simons  $(2n-1)$ -forms for a single gauge field. For example

$$\begin{aligned} \omega_3(A) &= \text{tr} \left\{ AF - \frac{1}{3} A^3 \right\}, \\ &= \text{tr} \left\{ AdA + \frac{2}{3} A^3 \right\}; \end{aligned} \quad (22)$$

$$\begin{aligned} \omega_5(A) &= \text{tr} \left\{ A(dA)^2 + \frac{3}{2} A^3 dA + \frac{3}{5} A^5 \right\}, \\ &= \text{tr} \left\{ AF^2 - \frac{1}{2} F A^3 + \frac{1}{10} A^5 \right\}. \end{aligned} \quad (23)$$

For constructing a Wess-Zumino-Witten action with a *single* non-linear  $\sigma$ -model field  $U$  — as opposed to one with separate fields  $g_L \in \text{U}(N)_L$  and  $g_R \in \text{U}(N)_R$  — it is necessary to have a solution  $\tilde{\omega}_{2n-1}(L, R)$  to (20) that is invariant under a diagonal ( $h_R = h_L$ ) gauge transformation, *i.e.*

$$\tilde{\omega}_{2n-1}(R^h, L^h) = \tilde{\omega}_{2n-1}(R, L). \quad (24)$$

How to arrange for this is shown by Mañes [11]. His idea is to consider

$$\Omega_{2n}(t) \stackrel{\text{def}}{=} \Omega_{2n}(F_{+,t}) - \Omega_{2n}(F_{-,t}) \quad (25)$$

where

$$A_{+,t} = tR + (1-t)L, \quad A_{-,t} = tL + (1-t)R. \quad (26)$$

Then  $\Omega_{2n}(1) = \Omega_{2n}(F_R) - \Omega_{2n}(F_L) = -\Omega_{2n}(0)$ , so

$$2\Omega_{2n}(1) = \Omega_{2n}(1) - \Omega_{2n}(0) = \int_0^1 \partial_t \Omega_{2n}(t) dt. \quad (27)$$

The transgression formula for the variation of the Chern character gives

$$\partial_t \Omega_{2n}(t) = n d \left( \text{tr} \{ (R - L) F_{+,t}^{n-1} \} - \text{tr} \{ (L - R) F_{-,t}^{n-1} \} \right) \quad (28)$$

and we can take

$$\tilde{\omega}_{2n-1}(R, L) = \frac{n}{2} \int_0^1 \left( \text{tr} \{ (R - L) F_{+,t}^{n-1} \} - \text{tr} \{ (L - R) F_{-,t}^{n-1} \} \right) dt. \quad (29)$$

Under a diagonal gauge transformation the inhomogeneous terms cancel so that  $(R - L) \rightarrow h^{-1}(R - L)h$ , and  $F_{\pm,t} \rightarrow h^{-1}F_{\pm,t}h$ . Consequently the integrand in (29) is manifestly invariant under this transformation and the invariance is inherited by  $\tilde{\omega}_{2m-1}(R, L)$ . Solutions to (20) can differ only by the  $d$  of something, and so

$$\tilde{\omega}_{2m-1}(R, L) = \omega_{2n-1}(R) - \omega_{2n-1}(L) + dS_{2n-2}(R, L) \quad (30)$$

for some  $S_{2n-2}(L, R)$ .

We now define

$$\tilde{C}[R, L] = \frac{i^n}{(2\pi)^{n-1}n!} \int_{M_{2n-1}} \tilde{\omega}_{2n-1}[R, L], \quad (31)$$

where the normalization has been chosen so as to reproduce the perturbation-theory anomaly. We have

$$\tilde{C}[R^{g_R}, L^{g_L}] = \tilde{C}[R, L^{g_L g_R^{-1}}] = \tilde{C}[R, L^U], \quad (32)$$

where  $U = g_L g_R^{-1}$ . This allows us to define the gauged Wess-Zumino-Witten functional  $W[R, L, U]$  by setting

$$\tilde{C}[R, L^U] = C[R, L] + W[R, L, U], \quad (33)$$

where

$$C[R, L] = \frac{i^n}{(2\pi)^{n-1}n!} \int_{M_{2n-1}} \omega_{2n-1}[R, L], \quad (34)$$

as  $S_{2n-2}[R, L^U]$  has no  $U$ -independent part. Variations of the functional  $W[R, L, U]$  depend only on the values of the fields  $L$ ,  $R$  and  $U$  on the boundary of  $M_{2n-1}$ , and so  $W$  can serve as an action on the  $2n - 2$  dimensional space-time  $M_{2n-1} = \partial M_{2n-1}$ . The functional  $\tilde{C}[R^{g_R}, L^{g_L}]$  has been constructed to be invariant under

$$\begin{aligned} R &\rightarrow R^{h_R} = h_R^{-1} R h_R + h_R^{-1} d h_R, \\ L &\rightarrow L^{h_L} = h_L^{-1} L h_L + h_L^{-1} d h_L, \\ U &\rightarrow h_L^{-1} U h_R. \end{aligned} \quad (35)$$

which coincides with (17). The equivalent functional  $\tilde{C}[R, L^U]$  is therefore also gauge invariant. Its bulk Chern-Simons and boundary Wess-Zumino functionals,  $C[R, L]$  and  $W[R, L, U]$  respectively, are not separately gauge invariant. The gauge dependence of  $W[R, L, U]$  is the source of the anomaly.

A key ingredient in  $W[R, L, U]$  is the  $(2n - 2)$ -form  $S_{2n-2}$ . This form is the ‘‘Bardeen counterterm’’ that was originally introduced by W. Bardeen to ensure that the ‘‘consistent

anomaly” vanished for vector currents. Here it must be included in the action for a rather different reason: the left and right Dirac seas are being glued together by the single mass-generating field multiplet  $U$  [11].

#### IV. TWO-DIMENSIONS: APPLICATION TO SUPERCONDUCTORS AND CHARGE-DENSITY WAVES

##### A. Currents and anomalies

As an illustration of the WZW strategy consider a theory on a two space-time dimensional surface  $M_2$  that is the boundary of a three-dimensional bulk  $M_3$ . In this case  $n = 2$  and Mañes’ construction gives

$$\tilde{\omega}_3(R, L) = \omega_3(R) - \omega_3(L) + d \operatorname{tr} \{LR\}. \quad (36)$$

We can verify the diagonal invariance by using

$$\omega_3(A^g) = \omega_3(A) - \frac{1}{3} \operatorname{tr} \{(g^{-1}dg)^3\} - d \operatorname{tr} \{dgg^{-1}A\} \quad (37)$$

to find that

$$\begin{aligned} \tilde{\omega}_3(R^g, L^g) - \tilde{\omega}_3(R, L) &= -\operatorname{tr} \{dgg^{-1}R\} + \operatorname{tr} \{dgg^{-1}L\} \\ &\quad + \operatorname{tr} \{Ldgg^{-1}\} + \operatorname{tr} \{dgg^{-1}R\} + \operatorname{tr} \{(g^{-1}dg)^2\} \\ &= 0 \end{aligned} \quad (38)$$

We have taken note that the last term in the penultimate line in (38) is zero. The modified Chern-Simons action is therefore invariant under *vector* gauge transformations even if  $F_R \neq F_L$ .

The bulk-plus-boundary topological and gauge-invariant action is now

$$\begin{aligned} \tilde{C}[R, L^U] &= \frac{1}{4\pi} \int_{M_3} \operatorname{tr} \{\omega_3(R) - \omega_3(L^U)\} - \frac{1}{4\pi} \int_{M_2} \operatorname{tr} \{RL^U\} \\ &= \frac{1}{4\pi} \int_{M_3} \operatorname{tr} \{\omega_3(R) - \omega_3(L) + \frac{1}{3}(U^{-1}dU)^3\} \\ &\quad + \frac{1}{4\pi} \int_{M_2} \operatorname{tr} \{dUU^{-1}L - RU^{-1}dU - RU^{-1}LU\}. \end{aligned}$$

We compute the currents  $J_R^\mu$  and  $J_L^\mu$  that flow in  $M_2$  from the boundary part of the variations of  $\tilde{C}$  with respect to  $R$  and  $L$ . These variations are

$$\begin{aligned}\delta_R \tilde{C} &= \frac{1}{4\pi} \int_{M_2} \text{tr} \{ \delta R (R - U^{-1} L U - U^{-1} dU) \} \\ &= \frac{1}{4\pi} \int_{M_2} \text{tr} \{ \delta R (-U^{-1} \nabla U) \} \\ &\equiv i \int_{M_2} d^2 x \sqrt{g} \text{tr} \{ \delta R_\mu J_R^\mu \},\end{aligned}\tag{39}$$

and

$$\begin{aligned}\delta_L \tilde{C} &= \frac{1}{4\pi} \int_{M_2} \text{tr} \{ \delta L (-L - U R U^{-1} - dU U^{-1}) \} \\ &= \frac{1}{4\pi} \int_{M_2} \text{tr} \{ \delta L (-\nabla U U^{-1}) \}. \\ &\equiv i \int_{M_2} d^2 x \sqrt{g} \text{tr} \{ \delta L_\mu J_L^\mu \}.\end{aligned}\tag{40}$$

We read off that

$$\begin{aligned}J_R^\mu &= -\frac{\epsilon^{\mu\nu}}{4\pi i} U^{-1} \nabla_\nu U \\ J_L^\mu &= -\frac{\epsilon^{\mu\nu}}{4\pi i} (\nabla_\nu U) U^{-1}.\end{aligned}\tag{41}$$

The currents (41) automatically include the Bardeen polynomial terms that convert “consistent” currents to “covariant” currents [13]. The Bardeen polynomials are here simply the integrated-out boundary parts of the variation of the bulk Chern-Simons action (see, for example, [14]). In the absence of the  $M_3$  bulk these polynomials have to be motivated and added by hand as was done in [13]. The currents being covariant means that that under a gauge transformation  $(h_R, h_L)$  each current transforms in the adjoint representation of its appropriate group

$$\begin{aligned}J_R^\mu &\rightarrow h_R^{-1} J_R^\mu h_R, \\ J_L^\mu &\rightarrow h_L^{-1} J_L^\mu h_L.\end{aligned}$$

This property is easily verified. As a consequence of the transformation properties of the currents their appropriate covariant derivatives are

$$\begin{aligned}\nabla_\mu J_L^\nu &= \partial_\mu J_L^\nu + [L_\mu, J_L^\nu], \\ \nabla_\mu J_R^\nu &= \partial_\mu J_R^\nu + [R_\mu, J_R^\nu].\end{aligned}$$

While we are envisaging the gauge fields as being externally imposed, the non-linear  $\sigma$ -model field  $U$  is autonomous. In order for our currents to satisfy their conservation laws we need  $U$  to obey its equation of motion. This we obtain by setting to zero the variation of  $\tilde{C}[R, L^U]$  due to an arbitrary change in  $U$ . The variation is the integral of

$$\frac{1}{4\pi} \text{tr} \left\{ U^{-1} \delta U \left( (U^{-1} dU)^2 + [U^{-1} dU, U^{-1} L U]_+ - U^{-1} dL U - dR - [R, U^{-1} dU + U^{-1} L U]_+ \right) \right\}, \quad (42)$$

and gives the matrix-valued equation,

$$0 = \frac{1}{4\pi} \epsilon^{\mu\nu} \left( U^{-1} \partial_\mu U U^{-1} \partial_\nu U + [U^{-1} \partial_\mu U, U^{-1} L_\nu U] - U^{-1} (\partial_\mu L_\nu) U - \partial_\mu R_\nu - [R_\mu, U^{-1} \partial_\nu U + U^{-1} L_\nu U] \right). \quad (43)$$

When we substitute

$$J_R^\mu = -\frac{\epsilon^{\mu\nu}}{4\pi i} U^{-1} (\partial_\nu U + L_\nu U - U R_\nu) \quad (44)$$

into

$$\nabla_\mu J_R^\mu = \partial_\mu J_R^\mu + [R_\mu, J_R^\mu] \quad (45)$$

and make use of the equation of motion, we verify the covariant anomalous conservation law

$$\nabla_\mu J_R^\mu = \frac{1}{4\pi i} \epsilon^{\mu\nu} F_{\mu\nu}^R. \quad (46)$$

A similar equation

$$\nabla_\mu J_L^\mu = -\frac{1}{4\pi i} \epsilon^{\mu\nu} F_{\mu\nu}^L \quad (47)$$

holds for  $J_L^\mu$ .

When we restrict ourselves to an Abelian gauge group, equation (43) reduces to  $dL + dR = 0$ , which is a constraint on the external gauge fields rather than an equation of motion. This awkwardness is resolved by remembering that in addition to the topological terms the complete action will contain non-topological but manifestly gauge invariant terms such as an ordinary non-linear  $\sigma$ -model action

$$\begin{aligned} S_{\text{conventional}}[U, R, L] &= \frac{f^2}{2} \int d^2x g^{\mu\nu} \text{tr} \{ \nabla_\mu U \nabla_\nu U^\dagger \} \\ &= -\frac{f^2}{2} \int d^2x g^{\mu\nu} \text{tr} \{ (U^{-1} \nabla_\mu U) (U^{-1} \nabla_\nu U) \}. \end{aligned} \quad (48)$$

After including the contribution from this action we recover a proper equation of motion.

## B. Abelian applications

In the Abelian case we can set  $U = e^{-i\theta}$  (recall that our  $\Phi$  corresponds to  $U^\dagger$ , so  $\Phi = |\Phi|e^{i\theta}$ ) and restore the factors of  $i$  so that  $R = iR_\mu dx^\mu$ ,  $L = iL_\mu dx^\mu$  then (48) becomes

$$S_{\text{conventional}}[\theta, R, L] = \frac{f^2}{2} \int d^2x \{(\partial_\mu \theta - L_\mu + R_\mu)(\partial^\mu \theta - L^\mu + R^\mu)\}. \quad (49)$$

The currents become

$$\begin{aligned} J_R^\mu &= -f^2(\partial^\mu \theta - L^\mu + R^\mu) + \frac{\epsilon^{\mu\nu}}{4\pi}(\partial_\nu \theta - L_\nu + R_\nu), \\ J_L^\mu &= +f^2(\partial^\mu \theta - L^\mu + R^\mu) + \frac{\epsilon^{\mu\nu}}{4\pi}(\partial_\nu \theta - L_\nu + R_\nu), \end{aligned} \quad (50)$$

and the equation of motion for  $\theta$  is modified to

$$-f^2 \partial_\mu (\partial^\mu \theta - L^\mu + R^\mu) - \frac{\epsilon^{\mu\nu}}{8\pi} (F_{\mu\nu}^L + F_{\mu\nu}^R) = 0. \quad (51)$$

The anomalous conservation laws (46) and (47) remain unchanged.

If we restrict ourselves to the case of vector gauge fields only ( $L = R$ ,  $\mathcal{A} = 0$ ) the vector current  $J_V^\mu \equiv J_R^\mu + J_L^\mu$  reduces to

$$J_V^\mu = \frac{\epsilon^{\mu\nu}}{2\pi} \partial_\nu \theta, \quad (52)$$

which is the automatically-conserved current found by Goldstone and Wilczek [15]. The axial current

$$J_A^\mu \equiv J_R^\mu - J_L^\mu = -2f_\pi^2 \partial^\mu \theta \quad (53)$$

obeys

$$\partial_\mu J_A^\mu = \frac{1}{2\pi} \epsilon^{\mu\nu} F_{\mu\nu}^V, \quad (54)$$

(where  $F^V \equiv F^L = F^R$ ) by virtue of the equation of motion (51) for  $\theta$ .

This vector gauge-field case provides a model for the conductivity of a sliding charge-density wave (CDW). In a one-dimensional CDW the Fermi surface is gapped by a potential

$$V(x, t) = \Delta \cos(2k_f x - \theta_{\text{CDW}}(x, t)) \quad (55)$$

which arises from a Peierls distortion of the lattice and couples the two Fermi points at  $k = \pm k_f$ . The electronic states near the Fermi energy are described by a Hamiltonian

$$\hat{H}_{\text{CDW}} = \int dx \begin{pmatrix} \psi_R^* & \psi_L^* \end{pmatrix} \begin{bmatrix} -iv_f(\partial_x - ieA_x) + e\phi & \Delta e^{-i\theta_{\text{CDW}}} \\ \Delta e^{+i\theta_{\text{CDW}}} & iv_f(\partial_x - ieA_x) + e\phi \end{bmatrix} \begin{pmatrix} \psi_R \\ \psi_L \end{pmatrix} \quad (56)$$

that is of Dirac form (and with a Dirac mass), but with the speed of light replaced by the Fermi velocity  $v_f$  so that the metric is  $g_{\mu\nu} = \text{diag}(v_f^2, -1)$ . The resulting number density and current are [17]

$$\begin{aligned} J_V^0 &= \langle \psi_R^* \psi_R + \psi_L^* \psi_L \rangle = -\frac{1}{2\pi} \partial_x \theta_{\text{CDW}}, \\ J_V^1 &= v_f \langle \psi_R^* \psi_R - \psi_L^* \psi_L \rangle = +\frac{1}{2\pi} \partial_t \theta_{\text{CDW}}, \end{aligned} \quad (57)$$

and coincide with (52) once we notice that our convention for the CDW phase  $\theta_{\text{CDW}}$  gives it the opposite sign to our previous  $\theta$ . That the equation of motion for  $\theta_{\text{CDW}}$  provides a condensed-matter analogue of the axial anomaly was pointed out in [18, 19].

For us, a more interesting case occurs when we specialize to axial gauge fields  $A_\mu \equiv R_\mu = -L_\mu$ . Then the axial gauge current

$$J_A^\mu = -2f^2(\partial^\mu \theta + 2A^\mu) \quad (58)$$

is conserved by the equation of motion

$$\partial_\mu J_A^\mu = \partial_\mu (-f^2(\partial^\mu \theta + 2A^\mu)) = 0, \quad (59)$$

while the vector current

$$J_V^\mu = \frac{\epsilon^{\mu\nu}}{2\pi} (\partial_\nu \theta + 2A_\nu) \quad (60)$$

is anomalous

$$\partial_\mu J_V^\mu = \frac{\epsilon^{\mu\nu}}{2\pi} \partial_\mu (\partial_\nu \theta + 2A_\nu) = \frac{\epsilon^{\mu\nu}}{2\pi} F_{\mu\nu}^A. \quad (61)$$

Here  $F^A \equiv F^R = -F^L$ .

These equations are applicable to a non-relativistic 1+1 dimensional BCS superconductor. If we linearize near the Fermi surface, the BdG Hamiltonian becomes

$$\hat{H}_{\text{SC}} = \int dx \begin{pmatrix} \psi_{\uparrow,R}^* & \psi_{\downarrow,L} \end{pmatrix} \begin{bmatrix} -iv_f(\partial_x - ieA_x) + e\phi & \Delta e^{i\theta} \\ \Delta e^{-i\theta} & iv_f(\partial_x + ieA_x) - e\phi \end{bmatrix} \begin{pmatrix} \psi_{\uparrow,R} \\ \psi_{\downarrow,L}^* \end{pmatrix}, \quad (62)$$

together with another term for the opposite spin components.

This Hamiltonian is again of Dirac form (again with a Dirac mass), but involves a two-component Nambu spinor

$$\Psi = \begin{pmatrix} \psi_{\uparrow,R} \\ \psi_{\downarrow,L}^* \end{pmatrix} \quad (63)$$

The physical number current is therefore the axial current

$$\begin{aligned}
J_{\text{Num}}^0 &= : \Psi^\dagger \sigma_3 \Psi : \\
&= : (\psi_{\uparrow,R}^* \psi_{\uparrow,R} - \psi_{\downarrow,L} \psi_{\downarrow,L}^*) : \\
&= \psi_{\uparrow,R}^* \psi_{\uparrow,R} + \psi_{\downarrow,L}^* \psi_{\downarrow,L}
\end{aligned} \tag{64}$$

$$\begin{aligned}
J_{\text{Num}}^1 &= v_f : \Psi^\dagger \Psi : \\
&= v_f : (\psi_{\uparrow,R}^* \psi_{\uparrow,R} + \psi_{\downarrow,L} \psi_{\downarrow,L}^*) : \\
&= v_f (\psi_{\uparrow,R}^* \psi_{\uparrow,R} - \psi_{\downarrow,L}^* \psi_{\downarrow,L}).
\end{aligned} \tag{65}$$

In a superconductor the U(1) particle-number symmetry is spontaneously broken by the condensate. However conservation of the number current (here the axial current) is recovered once we impose the equation of motion for the condensate order parameter — just as happens in (59).

What about the anomalous conservation law (61) for the vector current? After multiplication by the Fermi momentum  $k_f$  we can identify the normal-ordered vector-current density with the electron momentum density  $T^0_1$ , and its space component with the momentum flux  $T^1_1$ . Thus

$$\begin{aligned}
T^0_1 &= k_f (: \Psi^\dagger \Psi :) = k_f (\psi_{\uparrow,R}^* \psi_{\uparrow,R} - \psi_{\downarrow,L}^* \psi_{\downarrow,L}), \\
T^1_1 &= k_f v_f (: \Psi^\dagger \sigma_3 \Psi :) = k_f v_f (\psi_{\uparrow,R}^* \psi_{\uparrow,R} + \psi_{\downarrow,L}^* \psi_{\downarrow,L}).
\end{aligned} \tag{66}$$

By using (60) for the vector current we find

$$\begin{aligned}
T^0_1 &= \frac{k_f}{2\pi} (\partial_1 \theta + 2eA_1) \\
T^1_1 &= -\frac{k_f}{2\pi} (\partial_0 \theta + 2eA_0),
\end{aligned} \tag{67}$$

so the anomaly in the vector current therefore reads

$$\begin{aligned}
\partial_0 T^0_1 + \partial_1 T^1_1 &= \frac{k_f}{\pi} e (\partial_0 A_1 - \partial_1 A_0) \\
&= \rho e E_1.
\end{aligned} \tag{68}$$

Here  $\rho = k_f/\pi$  is the particle-number density, and we have used that in our  $(+, -, \dots)$  metric convention  $A_1$  is *minus* the  $x$  component  $A_x$  of the physical vector potential. The vector-current anomaly therefore describes the force exerted on the superfluid by the electric field  $E_1$ .



In two dimensions, and in the absence of a gauge field, there is relation between the vector and axial vector Dirac currents in these models:

$$\begin{aligned} J_A^0 &= \frac{1}{v_f} J_V^1, \\ J_A^1 &= v_f J_V^1. \end{aligned} \tag{69}$$

In light of  $\sqrt{g} = v_f$ , this relation can be written in Lorentz covariant form

$$J_A^\mu = -\frac{\epsilon^{\mu\nu}}{\sqrt{g}} J_{\nu,V} \tag{70}$$

and tells us that  $f^2 = 1/4\pi$  and is independent of  $v_f$ . In higher dimensions  $f^2$  will be non-universal.

## V. FOUR DIMENSIONS

Having seen that our strategy for obtaining an effective action for Weyl and Dirac fermions coupled to left and right gauge fields gives physically correct results, we apply it to the 3+1 dimensional case.

For four dimensions the Bardeen counterterm is

$$S_4(R, L) = \frac{1}{2} \text{tr} \{ (LR - RL)(F_R + F_L) + R^3 L - L^3 R + \frac{1}{2} LRLR \}, \tag{71}$$

and the WZW functional becomes

$$W[R, L, U] = -\frac{i}{240\pi^2} \int_M \text{tr} \{ (U^{-1} dU)^5 \} - \frac{i}{48\pi^2} \int_{\partial M} Z(L, R, U), \tag{72}$$

where [5, 11, 20]

$$\begin{aligned} Z(R, L, U) &= -\text{tr} \{ U_L (LdL + dLL + L^3) - U_L^3 L \} - \text{tr} \{ R \leftrightarrow L \} \\ &\quad + \frac{1}{2} \text{tr} \{ U_L L U_L L \} - \frac{1}{2} \{ R \leftrightarrow L \} \\ &\quad - \text{tr} \{ U^{-1} L U R^3 \} + \text{tr} \{ U R U^{-1} L^3 \} \\ &\quad - \text{tr} \{ U^{-1} L U (RdR + dRR) \} + \text{tr} \{ U R U^{-1} (LdL + dLL) \} \\ &\quad - \text{tr} \{ U R U^{-1} L U_L L \} - \text{tr} \{ U^{-1} L U R U_R R \} \\ &\quad + \text{tr} \{ L d U U_R R U^{-1} \} + \text{tr} \{ R d U^{-1} U_L L U \} \\ &\quad - \text{tr} \{ d L d U R U^{-1} \} + \text{tr} \{ d R d U^{-1} L U \} \\ &\quad + \frac{1}{2} \text{tr} \{ R U^{-1} L U R U^{-1} L U \}. \end{aligned} \tag{73}$$

We are using the notation  $U_L = dUU^{-1}$ , and  $U_R = U^{-1}dU$  from [5].

The rather long and complicated expression (73) simplifies greatly in the Abelian case where  $U = e^{-i\theta}$  because all terms with more than one  $d\theta$  go to zero. If we then set  $L = R = A$ , we find

$$Z \rightarrow 6i d\theta \operatorname{tr} \left\{ AdA + \frac{2}{3}A^3 \right\}, \quad (74)$$

making

$$\begin{aligned} W[A, \phi] &= \frac{1}{8\pi^2} \int_{\partial M} d\theta \operatorname{tr} \left\{ AdA + \frac{2}{3}A^3 \right\} \\ &= -\frac{1}{8\pi^2} \int_{\partial M} \theta \operatorname{tr} \{F^2\}. \end{aligned} \quad (75)$$

This is the usual “ $\theta$ ” term that appears in topological insulators.

Now keep  $L$  and  $R$  distinct, but make them Abelian. Then

$$\begin{aligned} \tilde{C}[R, L, U] &= \frac{1}{24\pi^2} \int_{M_5} (RF_R^2 - LF_L^2) \\ &\quad - \frac{1}{48\pi^2} \int_{M_4} \{id\theta(2L dL + 2R dR + RdL + LdR) + 2LR dR - 2RL dL\} \end{aligned} \quad (76)$$

The variation of the Chern-Simons terms requires knowing

$$\delta \int_{M_5} AF^2 = 3 \int_{M_5} \delta A F^2 + 2 \int_{M_4} \delta A AF. \quad (77)$$

Varying  $R$  gives a surface contribution to the current from

$$\delta_R \tilde{C}[R, L, U] = \frac{1}{12\pi^2} \int_{M_4} \delta R \left\{ (id\theta - L + R)dR + \frac{1}{2}(id\theta - L + R)dL \right\}. \quad (78)$$

Varying  $L$  gives

$$\delta_L \tilde{C}[R, L, U] = \frac{1}{12\pi^2} \int_{M_4} \delta L \left\{ (id\theta - L + R)dL + \frac{1}{2}(id\theta - L + R)dR \right\}. \quad (79)$$

Both currents make use of the appropriate covariant derivative. Some integration by parts is necessary to get these, so there may be extra terms in the presence of boundaries or singularities in the  $\theta$  field.

There will also be a non-topological part of the action such as

$$S[\phi] = \int d^4x \left\{ \frac{f^2}{2} (\partial_\mu \theta - L_\mu + R_\mu) (\partial^\mu \theta - L^\mu + R^\mu) \right\}, \quad (80)$$

where  $f^2$  might be a superfluid density. Here we have again set

$$R = iR_\mu dx^\mu \quad (81)$$

and similarly for  $L$ .

The non-topological action contributes currents

$$\begin{aligned} j_R^\mu &= -f^2(\partial^\mu\theta - L^\mu + R^\mu), \\ j_L^\mu &= +f^2(\partial^\mu\theta - L^\mu + R^\mu) \end{aligned} \quad (82)$$

that are to be added to the topological currents found above to make

$$\begin{aligned} J_R^\mu &= -f^2(\partial^\mu\theta - L^\mu + R^\mu) + \frac{1}{24\pi^2}\epsilon^{\mu\nu\sigma\tau}(\partial_\nu\theta - L_\nu + R_\nu)\left(F_{\sigma\tau}^R + \frac{1}{2}F_{\sigma\tau}^L\right), \\ J_L^\mu &= +f^2(\partial^\mu\theta - L^\mu + R^\mu) + \frac{1}{24\pi^2}\epsilon^{\mu\nu\sigma\tau}(\partial_\nu\theta - L_\nu + R_\nu)\left(F_{\sigma\tau}^L + \frac{1}{2}F_{\sigma\tau}^R\right). \end{aligned} \quad (83)$$

The equation of motion for the  $\theta$  field is

$$-f^2\partial_\mu(\partial^\mu\theta - L^\mu + R^\mu) = \frac{1}{96\pi^2}\epsilon^{\mu\nu\sigma\tau}(F_{\mu\nu}^L F_{\sigma\tau}^L + F_{\sigma\tau}^R F_{\mu\nu}^R + F_{\mu\nu}^R F_{\sigma\tau}^L), \quad (84)$$

the RHS coming from the four-dimensional  $\theta$  term.

Using the equation of motion gives

$$\begin{aligned} \partial_\mu J_R^\mu &= \frac{1}{32\pi^2}\epsilon^{\mu\nu\sigma\tau}F_{\mu\nu}^R F_{\sigma\tau}^R, \\ \partial_\mu J_L^\mu &= -\frac{1}{32\pi^2}\epsilon^{\mu\nu\sigma\tau}F_{\mu\nu}^L F_{\sigma\tau}^L. \end{aligned} \quad (85)$$

We see that, as expected, the coupling to the mass-generating  $\theta$  field does not affect the anomaly.

For our chiral superfluid we must set  $eA_\mu \equiv R_\mu = -L_\mu$  so that  $F_{\mu\nu}^R = -F_{\mu\nu}^L = eF_{\mu\nu}$ . We must also divide by two because of the BdG/Majorana over-counting. The physical particle-number current for our right-handed Weyl superfluid is therefore

$$J_{\text{Num}}^\mu = -f^2(\partial^\mu\theta + 2eA^\mu) + \frac{e}{48\pi^2}\epsilon^{\mu\nu\sigma\tau}(\partial_\mu\theta + 2eA_\mu)F_{\sigma\tau}. \quad (86)$$

This current has the same anomaly as a massless right-handed chiral fermion:

$$\partial_\mu J_{\text{Num}}^\mu = \frac{e^2}{32\pi^2}\epsilon^{\mu\nu\sigma\tau}F_{\mu\nu}F_{\sigma\tau} = \frac{e^2}{4\pi^2}\mathbf{E} \cdot \mathbf{B}, \quad (87)$$

so, again as is to be expected, a Majorana mass does not affect the anomaly. Equations (86) and (87) are the principal results of this paper.

## VI. DISCUSSION AND APPLICATION TO THE CHIRAL MAGNETIC EFFECT

By the end of section V we have seen that the action

$$S[\theta, A] = \frac{1}{2} \left[ \int_{M_4=\partial M_5} d^4x \left\{ \frac{f^2}{2} (\partial_\mu \theta + 2eA_\mu) (\partial^\mu \theta + 2eA^\mu) - \frac{\theta}{96\pi^2} \epsilon^{\mu\nu\sigma\tau} F_{\mu\nu} F_{\sigma\tau} \right\} \right] - \frac{1}{96\pi^2} \int_{M_5} d^5x \epsilon^{\mu\nu\rho\sigma\tau} A_\mu F_{\nu\rho} F_{\sigma\tau}, \quad (88)$$

is invariant under the gauge transformation

$$\begin{aligned} \theta &\rightarrow \theta - 2\alpha e, \\ A_\mu &\rightarrow A_\mu + \partial_\mu \alpha, \end{aligned} \quad (89)$$

and gives a current on the  $M_4$  space-time boundary of

$$J_{\text{Num}}^\mu = -f^2 (\partial^\mu \theta + 2eA^\mu) + \frac{e}{48\pi^2} \epsilon^{\mu\nu\sigma\tau} (\partial_\mu \theta + 2eA_\mu) F_{\sigma\tau}. \quad (90)$$

This current has the same chiral anomaly

$$\partial_\mu J_{\text{Num}}^\mu = \frac{e^2}{32\pi^2} \epsilon^{\mu\nu\sigma\tau} F_{\mu\nu} F_{\sigma\tau} \quad (91)$$

as the original ungapped Weyl fermion. One third of the anomaly comes from the topological second term in (90) and two-thirds from the non-universal first term and its associated equation of motion

$$-f^2 \partial_\mu (\partial^\mu \theta + 2eA^\mu) = \frac{1}{96\pi^2} \epsilon^{\mu\nu\sigma\tau} F_{\mu\nu} F_{\sigma\tau}, \quad (92)$$

where the RHS arises from the  $\theta$ -term in the  $M_4$  part of (88). The space-time current non-conservation is accounted for by the inflow from the  $M_5$  bulk current described by the last (Chern-Simons) term in (88). In all these equations  $A_\mu$  and  $F_{\mu\nu}$  are the physical Maxwell fields.

For a topological superconductor, the space-time part of the action — the first line in (88) — describes only one of the two opposite-Chern-number Fermi surfaces. The second surface will have a similar action, but with the signs of the space-time  $\theta$  term and the bulk Chern-Simons term reversed. The resultant cancellation of the  $M_5$  terms ensures that the complete current that couples to the external electromagnetic field is free of anomalies.

One may worry that the division between the topological term and the non-universal term in (90) is an artifact of the ambiguity of the definition of the currents in an anomalous

theory — particularly as a key ingredient in our derivation of the effective action involved the Bardeen counter-term, which was originally introduced as an ad-hoc modification of the “consistent” currents so as to conserve the vector current even in the presence of an axial gauge field. Bardeen was allowed to add such terms because the AVV and AAA triangle Feynman diagrams are only *conditionally* convergent, and therefore both they and the currents whose response they capture are intrinsically ambiguous. We claim however that the statement that only one third of the anomaly comes from the topological term is unambiguous. This is because the coefficient  $1/48\pi^2$  of  $\partial_\nu\theta$  in the topological part of the current comes from the *absolutely convergent*  $\gamma_5 AA$  triangle diagram. It is shown in appendix B that the  $\gamma_5 AA$  diagram evaluates to  $1/3$  of the absolutely convergent  $\gamma_5 VV$  triangle diagram that gives the corresponding topological part

$$J_{\text{top}}^\mu = \frac{e}{8\pi^2} \epsilon^{\mu\nu\sigma\tau} \partial_\nu \theta F_{\sigma\tau}^V \quad (93)$$

of the current for a four-component Dirac particle given mass by a neutral Higgs field [2]. (The extra factor of  $1/2$  in (90) is from the Majorana condition.)

We now have enough information to resolve the paradox described in the introduction. Recall how it comes about: The total current from the two Fermi surfaces includes a topological term

$$J_{\text{top}}^\mu = \frac{e}{48\pi^2} \epsilon^{\mu\nu\sigma\tau} (\partial_\nu \theta_+ - \partial_\nu \theta_-) F_{\sigma\tau}, \quad (94)$$

where  $\theta_\pm$  are the order-parameter phases on the  $C_1 = \pm 1$  Fermi surfaces. Both Fermi surfaces see the same gauge field, so the connection terms in the covariant derivatives have canceled. This current can, however, be non zero when the two order parameters are free to wind independently. In particular, in the presence of a  $2\pi$  winding in one of the Fermi surfaces and an electric field  $\mathbf{E}$  directed parallel to the vortex, we have an inflowing current of

$$\dot{N} = \frac{e|\mathbf{E}|}{12\pi} \quad (95)$$

particles per unit length. This inflow is only one-third that found in [1], but is still an embarrassment as the topologically bound mode in the vortex core is uncharged and has no anomaly that can absorb it.

The resolution of the problem resides in the remark made after equation (79) that we might have missed boundary terms arising from integrations by parts in our four-dimensional

space-time. Furthermore there will be boundaries whenever we have a vortex: We must exclude from our manifold  $M_4$  any line about which  $\theta$  winds by  $2\pi$  as at such places  $d^2\theta \neq 0$ . Finding such boundary terms by inspection of the algebra is tedious, but we can locate one of them by observing that there can be no physical effect from a singular gauge transformation that is implemented by inserting a half-unit of magnetic along a line, and simultaneously making  $\theta$  wind about the line by  $2\pi$  so that no covariant derivatives are changed. Because our currents are built from covariant derivatives most of our expressions are unchanged by this process. An exception is the source term  $e^2 \mathbf{E} \cdot \mathbf{B}/12\pi^2$  on the RHS of the equation of motion for  $\theta$ . Because a singular gauge transformation located on the  $z$  axis inserts a flux tube of strength  $(\pi/e)\delta^2(x, y)$ , this source term is modified so that it would appear that charge is being absorbed by the singular line at rate proportional to the component of  $\mathbf{E}$  tangential to the flux line. This cannot be so. We must therefore have missed a compensating source term proportional to  $d^2\theta$  that will remain present when  $\theta$  winds but there is no inserted flux. This extra term on the RHS of the equation of motion for  $\theta$  solves our problem. It provides a source that under the condition of the paradoxical topological inflow supplies an equal and opposite outflow in the *non-topological* part of the current. The net result is that there is no net inflow, and no paradox.

To further illustrate the consequences of (90) we consider the chiral magnetic effect (CME) [21–24] in which a static magnetic field  $\mathbf{B}$  induces a current parallel to the field. (In our discussion of the CME we consider only the effect of the *external* field on the superfluid. We are not accounting for any field generated by the currents induced in the condensate. Such additional geometry-dependent fields would lead to a Meissner effect and tend to screen the fluid from the external field. Ignoring these screening fields is standard in the usual derivations of the CME.)

Suppose we have a field  $\mathbf{B} = (0, 0, B_3 = F_{12})$  that arises from static and  $x^3$  independent  $A_1, A_2$ . If we allow  $A_0$  to depend on  $x^3$  or  $A_3$  to depend on  $x^0 = t$ , then the anomaly equation becomes

$$\partial_0 J_{\text{Num}}^0 + \partial_3 J_{\text{Num}}^3 = \frac{1}{4\pi^2} (\partial_0 A_3 - \partial_3 A_0) B_3. \quad (96)$$

This is satisfied by

$$\begin{aligned} J_{\text{Num}}^0 &= \rho_0 + \frac{e^2}{4\pi^2} A_3 B_3, \\ J_{\text{Num}}^1 &= J_{\text{Num}}^2 = 0, \end{aligned}$$

$$J_{\text{Num}}^3 = -\frac{e^2}{4\pi^2} A_0 B_3. \quad (97)$$

The last line of (97) leads to the usual static CME current [21–24]

$$\mathbf{J}_{\text{CME}} = \frac{e\mu_5}{2\pi^2} \mathbf{B}. \quad (98)$$

Here we have replaced  $-eA_0$  by separate chemical potentials  $\mu_R, \mu_L$  for a pair of right- and left-handed Weyl fermions, and then defined the axial chemical potential  $\mu_5$  by setting  $\mu_R = \mu + \mu_5, \mu_L = \mu - \mu_5$ .

For our superfluid, and when  $B_3 = \partial_1 A^2 - \partial_2 A^1 \neq 0$ , we cannot find a  $\theta$  such that  $(\partial^1 \theta + 2eA^1) = (\partial^2 \theta + 2eA^2) = 0$ . Consequently our formula (90) for the current cannot be coerced to give (98). The simplest solution to the equation of motion for the condensate in the presence of the magnetic field is to take  $\theta$  constant, and this gives us a London-equation-like current

$$\begin{aligned} J_{\text{Num}}^0 &= -2ef^2 A^0, \\ J_{\text{Num}}^1 &= -2ef^2 A^1, \\ J_{\text{Num}}^2 &= -2ef^2 A^2, \\ J_{\text{Num}}^3 &= -\frac{e^2}{12\pi^2} A^0 B_3 = \frac{\mu_R e}{12\pi^2} B_3. \end{aligned} \quad (99)$$

Taking  $(A^1, A^2) = B_3/2(-y, x)$  and comparing the  $x, y$  current components with the number density  $\rho = J_{\text{Num}}^0 = 2f^2 \mu_R$ , we see that this solution corresponds to the fluid rotating rigidly with angular velocity

$$\boldsymbol{\Omega} = -\left(\frac{e}{2\mu_R}\right) \mathbf{B}, \quad (100)$$

and possessing a CME current

$$J_{\text{Num}}^3 = \frac{\mu_R e}{12\pi^2} B_3, \quad (101)$$

that is only 1/3 of the usual equilibrium value.

We might expect some reduction in the strength of the CME because a degenerate gas of non-interacting Weyl fermions that is rigidly rotating with angular velocity  $\boldsymbol{\Omega}$  possesses an equilibrium *chiral vortical* effect (CVE) current [25, 26] of

$$\mathbf{J}_{\text{CVE}} = \frac{\mu^2}{4\pi^2} \boldsymbol{\Omega}. \quad (102)$$

Given (100), this CVE current would cancel 1/2 of the usual CME. We find only 1/3 rather than 1/2 of the noninteracting CME current remaining, but it is not unreasonable that the CVE of a superconductor should differ from that of the free gas.

Is our rotating solution physically relevant? Imagine starting with our chiral superfluid at rest and in the absence of any external field. Now slowly switch on the magnetic field. The circulating electric field from  $\text{curl } \mathbf{E} = -\dot{\mathbf{B}}$  will spin-up the fluid (this is the origin of the London moment of a rotating superconductor) to give the  $J_{\text{Num}}^1, J_{\text{Num}}^2$  of (99). Consequently our  $\theta = \text{const.}$  solution corresponds to this low-frequency response. Finite-frequency computations of the CVE (see for example [26] eqs (47,48), or [27, 28]) show that the CME current drops from its  $\omega = 0$  value (98) to exactly one-third of this value once the frequency  $\omega$  becomes non-zero (*i.e.* larger than the equilibrium relaxation time). It remains at this reduced value as long as  $\omega$  is small compared to the temperature or the chemical potential. The physical difference between  $\omega = 0$  and  $\omega > 0$  in these calculations is that in the former case the fluid has had time to relax to an equilibrium state in which all rotational momentum has been shed. Being a superfluid, our system will have persistent rotational currents and the relaxation time is infinite. Our result (99) applies at both  $\omega = 0$  and  $\omega > 0$  and is nicely consistent with the results of [26–28].

It should be appreciated that there are a number of subtleties associated with the presence of a persistent chiral magnetic current. As pointed out by Bloch and reported by Bohm [29], no system can support a persistent transport current in its ground state. If it did, we could extract energy from it in contradiction to the assumption that we are in a minimum-energy state. This issue has been discussed in the CME literature [30] and is connected with the proper method of relating gauge potential  $A_0$  to the right and left chemical potentials  $\mu_R, \mu_L$ . When this identification is correctly made it becomes clear that a system with non-zero  $\mu_R - \mu_L$  is not in equilibrium [31, 32].

## VII. CONCLUSIONS

We have found an action functional, gauge current, and equation of motion for the low energy degrees of freedom of a Weyl fermion whose superconducting gap (or Majorana mass) is induced by coupling to a charged condensate. Our expressions for all these quantities differ from those in [1]. In particular our expression for the charge current involves a topological



term that is smaller by a factor of a one-third than that in [1]. It also contains a covariant derivative of the charged order parameter rather than a plain derivative. Despite the coefficient of the topological term being reduced by a factor of one-third, we find that the chiral anomaly is unchanged. The deficit is made up of a contribution from a non-universal and non-topological part of the action that nonetheless makes a universal contribution to the anomaly through the influence of the topological term on the equation of motion obeyed by the Goldstone mode. This additional contribution resolves the threatened paradox mentioned in section II, where a simple cosmetic rewrite of the Hamiltonian appears to have reduced the gauge anomaly by a factor of one-third.

The contribution of the non-universal terms to the anomaly is independent of their detailed form so long as they are conventionally gauge invariant — *i.e.* are constructed from gauge-covariant derivatives.

In [1] the non-universal current is set equal to the topological current for reasons that we do not understand, but it is possible that our decomposition of the current into topological and non-topological parts is somehow equivalent to their expression. For example our distinction between the gauge-covariant derivative and the plain partial derivative in the topological current is insignificant because the connection part cancels when we add the contributions from the two Fermi surfaces. Full equivalence seems unlikely, however, because we have no inflow into vortex lines, and because our factor of one-third in the topological current has a real physical effect of reducing the CME to one third of its free equilibrium value.

After this paper was written we came across a work [33] in which the part of the effective action in (88) that arises from the second Fermi surface (including the  $1/3$  coefficient before the  $\theta$ -term) is used to cancel the anomaly of a massless chiral fermion. In [33] the anomaly-cancelling term is interpreted as a four-dimensional analogue of the Green-Schwarz mechanism [34] rather than as a physical effect of a second gapped chiral superconductor.

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## Appendix A: Vortex core states

Consider a vortex lying along the  $z$  axis. The Erick Weinberg index theorem [3, 4] guarantees that when  $\Phi$  has winding number  $\pm 1$  there will be a zero energy solution to the Weyl equation

$$H\Psi \equiv \begin{bmatrix} \boldsymbol{\sigma} \cdot (\mathbf{p} - e\mathbf{A}) & \Phi \\ \Phi^* & -\boldsymbol{\sigma} \cdot (\mathbf{p} + e\mathbf{A}) \end{bmatrix} \Psi = E\Psi, \quad (\text{A1})$$

when we restrict  $\mathbf{p}$  and  $\mathbf{A}$  to the  $x$ - $y$  plane. If we consider the rotationally symmetric field  $A_\mu dx^\mu = A_\theta d\theta$  and take unit positive winding  $\Phi = e^{i\theta} \Delta(r)$  (where  $\theta$  is the polar angle, and not the dynamical Goldstone field) this solution is

$$\Psi = \begin{bmatrix} e^{i\pi/4} \\ 0 \\ 0 \\ e^{-i\pi/4} \end{bmatrix} \exp \left\{ - \int_0^r \left( \Delta(\rho) + \frac{A_\theta(\rho)}{\rho} \right) d\rho \right\}. \quad (\text{A2})$$

If the winding goes the other way  $\Phi = e^{-i\theta} \Delta(r)$  it will be

$$\Psi = \begin{bmatrix} 0 \\ e^{i\pi/4} \\ e^{-i\pi/4} \\ 0 \end{bmatrix} \exp \left\{ - \int_0^r \left( \Delta(\rho) - \frac{A_\theta(\rho)}{\rho} \right) d\rho \right\}. \quad (\text{A3})$$

Now we allow motion in the  $z$  direction by letting  $\Psi \rightarrow e^{ip_3 z} \Psi$  and including a gauge field component  $A_3$ . This leads to an additional term in the Hamiltonian

$$H(p_3, A_3) = \begin{bmatrix} \sigma_3(p_3 - eA_3) & 0 \\ 0 & -\sigma_3(p_3 + eA_3) \end{bmatrix}. \quad (\text{A4})$$

When  $A_3 = 0$  this new operator is diagonal in the zero-mode basis with eigenvalue  $E(p_3) = +p_3$  in the winding number  $+1$  case and  $E(p_3) = -p_3$  in the winding number  $-1$  case. The zero mode therefore metamorphoses into a family of chiral modes running up (down) the positive (negative) unit-winding vortex.

If the coupling to the gauge field were vector-like, the sign before  $eA_3$  in the diagonal terms in (A4) would be the same and the zero-mode wavefunction would remain an eigenstate but

with the energies shifted by  $eA_3$ . Then, when  $A_3 = -A_z = E_3 t$  we would have spectral flow, and hence a 1+1 dimensional anomaly

$$\partial_t \rho + \partial_z j_z = \pm \frac{eE_3}{2\pi}. \quad (\text{A5})$$

This does not work in the axial case (A4) as the added term is no longer diagonal in the zero-mode basis. Indeed when we restrict to the zero mode subspace (which is separated by an energy gap from the rest of the two-dimensional spectrum) the matrix elements of the perturbation are zero. Consequently, provided the  $\mathbf{E}$  field is not sufficiently strong as to disrupt the zero mode, the vortex core modes behave as if they were electrically neutral. The vanishing of the matrix elements actually holds at second order, and this remains true even if we include a non-zero chemical potential  $\mu = -eA_0$ , although the eigenfunctions are more complicated [35].

## Appendix B: Feynman diagrams

Here we evaluate the triangle diagrams that determine the coefficients  $C_{A,V}$  in the parity violating part of the gradient expansion of vector and axial currents  $j_V^\mu = C_V \epsilon^{\mu\nu\sigma\tau} \partial_\mu \theta F_{\nu\sigma}^V$  and  $j_A^\mu = C_A \epsilon^{\mu\nu\sigma\tau} \partial_\mu \theta F_{\nu\sigma}^A$  induced by a spatially varying Goldstone field and vector and axial-vector gauge fields respectively. We wish to show that  $C_A = C_V/3$ .

We work in the Euclidean region where the Feynman integrals are

$$I_{\gamma_5 VV}^{\mu\nu}(q_1, q_2) = \int \frac{d^4 k}{(2\pi)^4} \frac{\text{tr} \{ \gamma^5 (\not{k} + \not{q}_1 + m) \gamma^\mu (\not{k} + m) \gamma^\nu (\not{k} - \not{q}_2 + m) \}}{((k + q_1)^2 + m^2)(k^2 + m^2)((k - q_2)^2 + m^2)}, \quad (\text{B1})$$

and

$$I_{\gamma_5 AA}^{\mu\nu}(q_1, q_2) = \int \frac{d^4 k}{(2\pi)^4} \frac{\text{tr} \{ \gamma^5 (\not{k} + \not{q}_1 + m) \gamma^5 \gamma^\mu (\not{k} + m) \gamma^5 \gamma^\nu (\not{k} - \not{q}_2 + m) \}}{((k + q_1)^2 + m^2)(k^2 + m^2)((k - q_2)^2 + m^2)}. \quad (\text{B2})$$

Both integrals are convergent. We only need to evaluate them for small  $q_1, q_2$ .

We use

$$\text{tr} \{ \gamma^5 \gamma^\mu \gamma^\nu \gamma^\sigma \gamma^\tau \} = 4\epsilon^{\mu\nu\sigma\tau} \quad (\text{B3})$$

to evaluate the traces in the numerators. We find

$$\begin{aligned} & \text{tr} \{ \gamma^5 (\not{k} + \not{q}_1 + m) \gamma^\mu (\not{k} + m) \gamma^\nu (\not{k} - \not{q}_2 + m) \} \\ &= -4m \epsilon^{\mu\nu\alpha\beta} q_1^\alpha q_2^\beta, \end{aligned} \quad (\text{B4})$$

and

$$\begin{aligned}
& \text{tr} \{ \gamma^5 (\not{k} + \not{q}_1 + m) \gamma^5 \gamma^\mu (\not{k} + m) \gamma^5 \gamma^\nu (\not{k} - \not{q}_2 + m) \} \\
&= -\text{tr} \{ \gamma^5 (\not{k} + \not{q}_1 + m) \gamma^\mu (-\not{k} + m) \gamma^\nu (\not{k} - \not{q}_2 + m) \} \\
&= +4m \epsilon^{\mu\nu\alpha\beta} (q_1^\alpha q_2^\beta + 2k^\alpha q_2^\beta - 2q_1^\alpha k^\beta).
\end{aligned} \tag{B5}$$

We need the standard integrals

$$\int \frac{d^4 k}{(2\pi)^4} \frac{1}{(k^2 + m^2)^3} = \frac{1}{32\pi^2} \frac{1}{m^2}, \tag{B6}$$

$$\int \frac{d^4 k}{(2\pi)^4} \frac{k^\mu k^\nu}{(k^2 + m^2)^4} = \frac{1}{32\pi^2} g^{\mu\nu} \frac{1}{m^2} \frac{1}{6}, \tag{B7}$$

from which we obtain

$$\int \frac{d^4 k}{(2\pi)^4} \frac{1}{((k + q_1)^2 + m^2)(k^2 + m^2)((k - q_2)^2 + m^2)} = \frac{1}{32\pi^2} \frac{1}{m^2} + O\left(\frac{|q|^2}{m^2}\right), \tag{B8}$$

and

$$\int \frac{d^4 k}{(2\pi)^4} \frac{k^\alpha}{((k + q_1)^2 + m^2)(k^2 + m^2)((k - q_2)^2 + m^2)} = -\frac{1}{3}(q_1^\alpha - q_2^\alpha) \frac{1}{32\pi^2} \frac{1}{m^2} + O\left(\frac{|q|^2}{m^2}\right). \tag{B9}$$

The last integral leads to the substitution

$$k^\alpha \rightarrow -\frac{1}{3}(q_1^\alpha - q_2^\alpha) \tag{B10}$$

in the second trace, and gives

$$I_{\gamma_5 AA}^{\mu\nu}(q_1, q_2) = \frac{1}{3} I_{\gamma_5 VV}^{\mu\nu}(q_1, q_2) \tag{B11}$$

for small  $q$ .

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