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Missing Fractional Quantum Hall States in ZnO

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We have analyzed the crucial role a proper form of the Coulomb interaction plays on the even and odd denominator fractional quantum Hall effects in a two-dimensional electron gas (2DEG) in the ZnO heterointerface. In this system, the Landau level gaps are much smaller than those in conventional GaAs systems. The Coulomb interaction is also very large compared to the Landau level gap even in very high magnetic fields. We therefore consider the influence of higher Landau levels by considering the screened Coulomb potential in the random phase approximation. Interestingly, our exact diagonalization studies of the collective modes with this screened potential successfully explain recent experiments of even and odd denominator fractional quantum Hall effects, in particular, the unexpected absence of the $5/2$ state and the presence of $9/2$ state in ZnO. Additionally, our study also reveals a strong presence of spin-reversed excitations in the $7/2$ state in accordance with the experimental observation.

Discovery of the odd-denominator fractional quantum Hall effects (FQHE) in GaAs heterojunctions in 1982 [1] and its subsequent explanation by Laughlin [2, 3], has remained the ‘gold standard’ for novel quantum states of correlated electrons in a strong magnetic field. These effects also have been observed in ‘Dirac materials’ such as graphene [4, 5, 9]. and are expected to be present in other graphene-like materials [6–8] with novel attributes. The FQHE states in monolayer and bilayer graphene were investigated theoretically [9–12] and experimentally [13, 14]. For example, in bilayer graphene the application of a bias voltage results in a phase transition between incompressible FQHE and compressible phases [11, 12] some Landau levels (LLs). The FQHE in silicene and germanene indicated that because of the strong spin-orbit interaction present in these materials as compared to graphene, the electron-electron interaction and the FQHE gap are significantly modified [15]. The puckered structure of phosphorene exhibits a lower symmetry than graphene. This results in anisotropic energy spectra and other physical characteristics of phosphorene, both in momentum and real space in the two-dimensional (2D) plane [16, 17]. The anisotropic band structure of phosphorene causes splitting of the magnetoroton mode into two branches with two minima. For long wavelengths, we also found a second mode with upward dispersion that is clearly separated from the magnetoroton mode and is entirely due to the anisotropic bands [18].

In 1987, a discovery of the quantum Hall state at the LL filling factor $\nu = \frac{5}{2}$, the first even-denominator state observed in a single-layer system [19] added to the mystery of the FQHE. It soon became clear that this state must be different from the FQHE in predominantly odd-denominator filling fractions [1]. Understanding this enigmatic state has remained a major challenge in all

these years [20, 21]. At this half-filled first excited LL, a novel state described by a pair wave function involving a Pfaffian [12, 22], where the low-energy excitations obey non-Abelian exchange statistics, has been the strongest candidate.

The field of FQHE has now witnessed a very exciting development with the the observation of the effect in high-mobility MgZnO/ZnO heterointerfaces [23, 24]. The odd-denominator fractional states such as $\nu = \frac{4}{3}, \frac{5}{3}, \frac{8}{3}$ were observed here with indications of the $\nu = \frac{5}{2}$ state in the extreme quantum limit. Soon after, the even-denominator states, such as $\nu = \frac{3}{2}$, and $\frac{7}{2}$ were also observed [25], but surprisingly, the most prominent even-denominator state of the GaAs systems, the $\nu = \frac{5}{2}$ was found to be conspicuously absent in the ZnO system. The system of 2DEG in ZnO is unique as compared to that in GaAs. In the case of GaAs-based 2DEG, the LL gap is large compared to that for the Coulomb interaction ($e^2/\epsilon\ell$, where ϵ is the dielectric constant and $\ell = \sqrt{\hbar/eB}$ is the magnetic length with a magnetic field B). However, in a ZnO heterosturcutre [23–25] the LL gap is very small. The ratio κ between the Coulomb interaction and the LL gap is the relevant parameter in this context. In GaAs, $\kappa = 2.5/\sqrt{B}$, which would be very small in a strong magnetic field. In the ZnO heterointerface, where the dielectric constant is 8.5, that ratio is $\kappa = 25.1/\sqrt{B}$, i.e., about an order of magnitude larger than that of GaAs (as observed in Ref. [25], $\kappa = 9.7, 14.5, 16.5$ for $\nu = 3/2, 7/2, 9/2$, respectively). Therefore, considering the electron system in a single LL may not be appropriate. On the other hand, in graphene the ratio depends only on the dielectric constant of the substrate [26]. In the case of boron nitride as the substrate, $\kappa = 0.5 \sim 0.8$, which is smaller than one. Hence, a perturbative scheme of the effective Coulomb potential [27], in which higher LLs are projected onto the lowest Landau level by expanding the Coulomb potential in order of κ can be useful. However, those theories are only useful when κ is comparable to or smaller than unity. In ZnO, this ratio

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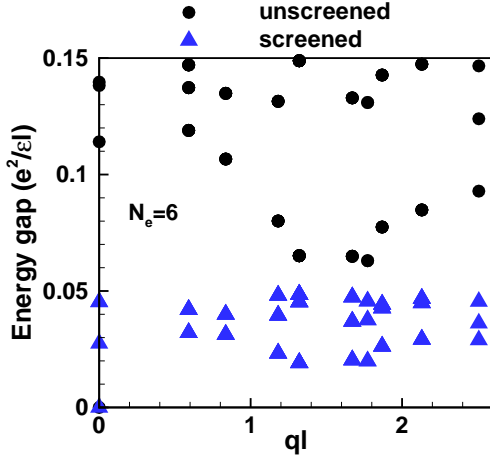


FIG. 1: (Color Online) The collective mode of $\nu = 1/3$ for six electrons, (a) without and (b) with screening.

is experimentally found to be much larger than 1, even an order of magnitude higher than unity.

In the experiment of Ref. [25] the FQHE was found to be missing at $5/2$ but survives at $7/2$, which suggests that the electron-hole symmetry must be broken in the $N = 1$ LL. Hence, the Coulomb interaction in the two cases has to be different to make the two different spins distinguishable. From the arguments above it is amply clear that we need to introduce an appropriate method to project the higher empty LLs onto the relevant LL by the virtual process between the empty LLs and full (or partly occupied) LLs. The Coulomb potential is screened by all the electrons below the Fermi level, and consequently depends on the filling factor. The dielectric constant is then replaced by the dielectric function of the momentum. The screened Coulomb potential is calculated in the random phase approximation (RPA) [28], and is useful for any κ . When the LL gap is infinitely large the screened Coulomb potential returns to the original (unscreened) value. This form of screened Coulomb interaction was used earlier in higher LLs [29] and in the case of skyrmions [30] in the Hartree-Fock approximation. Here we use this screened Coulomb interaction to study the collective modes of the FQHE states in the ZnO system using the exact diagonalization scheme. Interestingly, in our present scheme, we are able to satisfactorily explain the unique experimental observations by Falson et al. [25], such as the missing $5/2$ state but the presence of the $9/2$ state and spin-reversed excitations [31–33] of the $7/2$ state.

A screened Coulomb interaction but with a simpler form was also used in bilayer graphene [34] in which the screening was strong but is still much weaker than that in ZnO. In our scheme for the screened Coulomb potential [30], the interaction between electrons in the relevant LL is renormalized by the polarizability of all the other Landau levels. We consider here only the static screening so

that only the zero-frequency response function is taken into consideration. The Coulomb potential in the momentum space is $V(\mathbf{q}) = \frac{2\pi e^2}{\epsilon q}$. The screened Coulomb potential is then written

$$V_s(\mathbf{q}) = \frac{2\pi e^2}{\epsilon \epsilon_s(\mathbf{q}) q},$$

where $\epsilon_s(\mathbf{q})$ is the screened dielectric function [35],

$$\epsilon_s(\mathbf{q}) = 1 - V(\mathbf{q}) \chi_{nn}^R(\mathbf{q}, \omega \rightarrow 0^+),$$

χ_{nn}^R is the retarded density-density response function and the associated response function χ_{nn} is defined as

$$\chi_{nn}(\mathbf{q}, \tau) = -\frac{1}{\hbar S} \langle T_\tau \delta n(\mathbf{q}, \tau) \delta n(-\mathbf{q}, 0) \rangle,$$

with time ordering operator T_τ , system area S and the density operator $n(\mathbf{q})$. If we consider only the non-interacting response function χ_{nn}^0 without LL mixing in the Matsubara frequency Ω_n , then

$$\chi_{nn}^0(\mathbf{q}, i\Omega_n) = \frac{N_s}{\hbar S} \sum_{\sigma, n, n'} |F_{n', n}(\mathbf{q})|^2 \frac{\nu_{\sigma, n} - \nu_{\sigma, n'}}{i\Omega_n + (E_n - E_{n'})/\hbar},$$

where N_s is the LL degeneracy, σ is the spin index, n, n' are the LL indices, E_n is the kinetic energy of the LL n , and the form factor is defined by

$$F_{n, n'}(\mathbf{q}) = \frac{\sqrt{\min(n, n')!}}{\sqrt{\max(n, n')!}} e^{-q^2 \ell^2 / 4} L_{\min(n, n')}^{|n-n'|} \left(\frac{q^2 \ell^2}{2} \right) \times \left[\frac{(\text{sign}(n - n') q_y + i q_x) \ell}{\sqrt{2}} \right]^{|n-n'|}$$

with a Laguerre function $L(x)$. The parameter $\nu_{\sigma, n}$ is the filling factor of the level with spin σ in the LL n . In our exact diagonalization scheme $\nu = N_e/N_\phi$, where N_e is the electron number of the finite-size system.

In order to study the collective modes for odd- and even-denominator FQHE states, we follow the standard procedure of finite-size systems in a periodic rectangular geometry [3, 36]. The Hamiltonian for the Coulomb interaction is

$$H_C = \frac{1}{2} \sum_{\alpha, \beta} \sum_{n_1, n_2, n_3, n_4} \sum_{i_1, i_2, i_3, i_4} V_{i_1, i_2, i_3, i_4}^{n_1, n_2, n_3, n_4} \times c_{\alpha, n_1, i_1}^\dagger c_{\beta, n_2, i_2}^\dagger c_{\beta, n_3, i_3} c_{\alpha, n_4, i_4},$$

where n_i is the LL index, i_j is the guiding center index, α, β are spin indices, and c is the electron operator. The Coulomb interaction elements are given by [37]

$$V_{i_1, i_2, i_3, i_4}^{n_1, n_2, n_3, n_4} = \frac{1}{N_s} \frac{e^2}{\epsilon \ell} \sum_{\mathbf{q}} \frac{1}{(\epsilon_s) q \ell} \delta'_{i_1, i_4 + q_y \ell^2} \delta'_{i_2, i_3 - q_y \ell^2} \times e^{i q_x (i_3 - i_1)} F_{n_1, n_4}(\mathbf{q}) F_{n_2, n_3}(-\mathbf{q}),$$

where $\overline{\sum}$ excludes the term for $\mathbf{q} = \mathbf{0}$, δ' includes the periodic boundary condition, and the momentum is discrete $\mathbf{q} = \left(\frac{2\pi i}{L_x}, \frac{2\pi j}{L_y}\right)$ with the sample length L_x and width L_y . If a screened Coulomb interaction is taken into consideration, we just need to add the dielectric function ϵ_s in the denominator. The classical interaction term in the Hamiltonian which is induced by the periodic geometry is neglected even in the screened case, since the term is always a constant.

In the present case of ZnO the Zeeman energy ($0.2489B$ meV) is very close to the LL gap ($0.26311B$ meV). For example, the level $|1, \uparrow\rangle$ is only a little higher than $|0, \downarrow\rangle$. For odd denominator FQHE, for simplicity and without loss of generality, we consider only one LL and compare the collective modes with and without screening for filling factors $\nu = k/3$, since the spin is polarized. Our present work focuses solely on the even denominator FQHE [25]. In a perpendicular magnetic field, $\nu = 3/2$ state is not observed as is the case in GaAs system. Electrons in the half filled level $|0, \downarrow\rangle$ is compressible. In a tilted field there is a crossover of kinetic energies between LL 1 and LL 0 with different spins. The exact diagonalization method in a tilted magnetic field is quite involved [38] and will be reported in a future publication.

As mentioned above, in the experiment of [25] there is no indication of the $5/2$ state, which has a strong presence in the GaAs system. There could be several possible reasons for this: (i) the LL mixing may decrease or even close the gap of the incompressible ground state; (ii) a spin-mixed charge density wave state may exist between $|0, \downarrow\rangle$ and $|1, \uparrow\rangle$, since the gap Δ between the two levels is very small (for $B = 3.75\text{T}$, the gap is only $\Delta = 0.05329$ meV $= 0.004167e^2/\epsilon\ell$ [25]); or (iii) the screened Coulomb potential which integrates out all other LLs, changes the ground state. To test the first possibility we performed an exact diagonalization study including the LL mixing which includes LL $|1, \uparrow\rangle$ and $|2, \uparrow\rangle$. The results indicate that the collective modes are just slightly changed and the ground state is still an incompressible liquid. The spin remains fully polarized in our numerical calculations that includes $|1, \uparrow\rangle$ and $|1, \downarrow\rangle$, as in previous theoretical works [21] and in some of the experimental works [39]. On the other hand, if the LL mixing or spin mixing changes the ground state at $5/2$, then the incompressible ground state at $7/2$ would also be changed. But the FQHE experiment shows a robust $\nu = 7/2$. To test the second possibility, we also perform an exact diagonalization calculation where we class the Hamiltonian by the spin polarization [32, 33]. The ground state always has all electrons occupied in $|0, \downarrow\rangle$ when the gap $\Delta \geq 0$. Even for a negative gap $\Delta_C < \Delta < 0$, i.e., $|1, \uparrow\rangle$ is a little lower than $|0, \downarrow\rangle$, the electrons of the ground state are still in $|0, \downarrow\rangle$. Note that Δ_C can not be too negative: if $\Delta_C \rightarrow -\infty$, then all electrons would be flipped to $|1, \uparrow\rangle$.

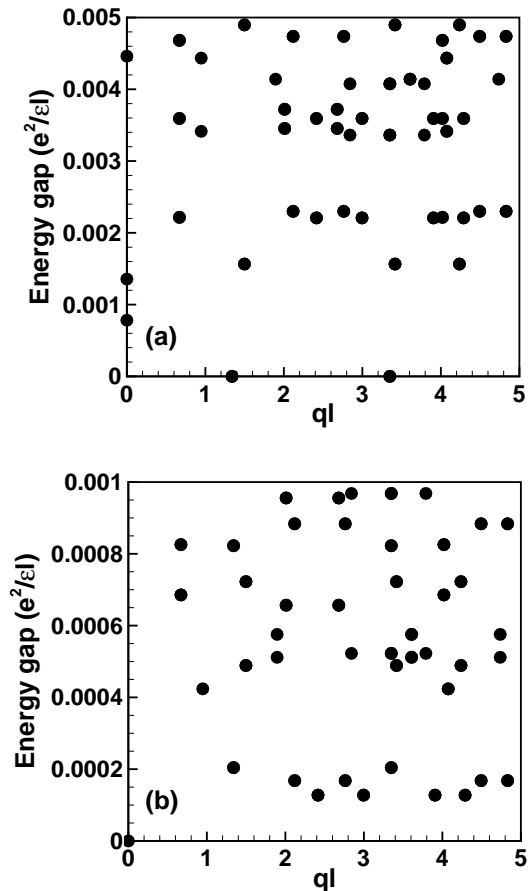


FIG. 2: The low-lying excitations in a $N_e = 7$ electrons system. (a) The collective mode for $5/2$: the ground state is degenerate and compressible. (b) The collective mode for $7/2$ indicates an incompressible ground state.

Only the third possibility seems to explain the experiment, i.e., the absence of the $5/2$ state, but the appearance of $\nu = 7/2$. For simplicity, we consider only a single LL with the screened potential. In our work that follows, the aspect ratio is $L_x/L_y = 1$. Figure 1 shows the comparison of the unscreened and screened collective modes at $\nu = 1/3$ for six electrons. The shape of the characteristic FQHE collective mode does not change, only the gap is reduced by the screening. For other odd denominator filling factors, $\nu = k/3$, ($k = 2, 4, 5, 7, 8, 10, 11$), we are also able to observe the characteristic FQHE collective modes, and the ground states indicate the incompressible liquid phase. Without screening, the collective modes in the exact diagonalization are calculated in GaAs in Ref. [21], where the screening effect is much weaker than for the ZnO heterojunction. First, we use the system parameters of GaAs to perform the exact diagonalization with screened Coulomb potential, and it shows that the FQHE survives for both $5/2$ and $7/2$. It clearly shows that our screen-

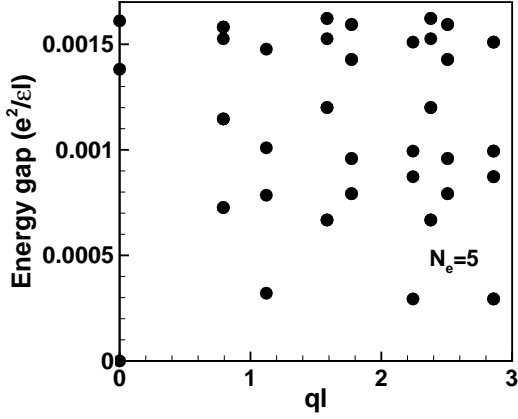


FIG. 3: The low-lying excitation spectrum with screening for a $N_e = 5$ system at $\nu = 9/2$.

ing calculations are compatible with the GaAs systems. For the ZnO system, we adopt the experimental parameters of Ref. [25]. The $7/2$ and $5/2$ are equivalent without screening due to the electron-hole symmetry in the $n = 1$ LL. The Coulomb interactions are distinguishable with screening included: the screening at $7/2$ is stronger than that at $5/2$, and there is a step in the dielectric function ϵ_s versus ql in the $7/2$ filling factor. Hence the ground state and collective modes can be different in the two cases.

We have tested different system sizes: $N_e = 4 \dots 11$. For simplicity, only the case of $N_e = 7$ is shown in Fig. 2. Clearly, the FQHE state is absent for $5/2$, but survives at $7/2$, even though the screening of the latter is stronger. The ground state of $5/2$ is a *degenerate compressible* state, while the ground state of $7/2$ is always an incompressible state. Note that for odd electrons, the ground states of $7/2$ are at $\mathbf{q} = \mathbf{0}$, but for even electrons, the ground states are always located at $\mathbf{q} = \sqrt{2\pi/N_s}(N/2, N/2)$. So the ground state could become an incompressible liquid state by a global translation, which was already pointed out in Ref. [21]. The collective modes at $7/2$ seem to have two minimum that are located at about $ql = 2.5$ and 3.8 . The energy gap, however, is very small compared to other systems. It is because the screened Coulomb interaction reduces the gap. Interestingly, the screening of $7/2$ is stronger, but the FQHE is still not destroyed. The energy gap for a larger system (more electrons) is larger than that of a smaller system (for example, when $N_e = 11$, the lowest gap is $0.0004e^2/\epsilon l$). So we expect that for a real system, the energy gap is large enough to be observable.

For higher LLs such as at $\nu = 9/2$, κ is even larger than that in LL 1 and the screening is stronger. The Coulomb potential thus be changed more by the screening induced by other LLs. Our exact diagonalization re-

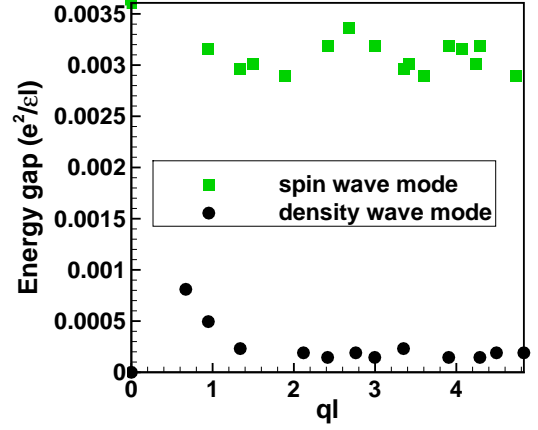


FIG. 4: (Color Online) Spin and charge density modes in the collective excitations of $\nu = 7/2$. Only the lowest lying excitations are shown.

sults are presented for 5 electrons for the experimental value of $B = 2.1$ T. The collective modes clearly show an incompressible state (Fig. 3). However, the gap is very small. Incidentally, the experimental signal is also very weak. Finally, we have studied the spin-reversed excitations [31–33] in the $7/2$ state [Fig. 4]. The spin wave mode is well separated from the density wave mode, and the spin wave mode is gapped. The spin wave mode is quite different from that of the FQHE state at $\nu = 1/3$. The latter can be described by the Laughlin's wave function [31]. Here the Goldstone spin mode disappears. It is not surprising since the nature of the ground state of an even denominator FQHE is quite different from those of the odd denominator FQHE. Interestingly, a signature of the spin state was observed at $\nu = 7/2$ in ZnO by Falson et al. [25].

To summarize, we have studied the FQHE states in the ZnO system with screened Coulomb interaction that incorporates the influence of other Landau levels. For the odd-denominator filling factors, our work agrees with the present system of ZnO and with earlier GaAs systems as well. However, for the even-denominator filling factors, we are able to explain the absence of $3/2, 5/2$ FQHE states, while the presence of $7/2, 9/2$ FQHE states, by introducing screened interaction which integrates out all the other LLs.

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- [1] D. C. Tsui, H. L. Störmer, and A. C. Gossard, *Phys. Rev. Lett.* **48**, 1559 (1982).
- [2] R. B. Laughlin, *Phys. Rev. Lett.* **50**, 1395 (1983).
- [3] T. Chakraborty and P. Pietiläinen, *The Quantum Hall Effects* (Springer, New York, 1995); *The Fractional Quantum Hall Effect* (Springer, New York, 1988).
- [4] D.S.L. Abergel, V. Apalkov, J. Berashevich, K. Ziegler, and T. Chakraborty, *Adv. Phys.* **59**, 261 (2010).
- [5] H. Aoki and M.S. Dresselhaus (Eds.), *Physics of Graphene* (Springer, New York 2014).
- [6] P. De Padova, C. Ottaviani, C. Quaresima, B. Olivieri, P. Imperatori, E. Salomon, T. Angot, L. Quagliano, C. Romano, A. Vona, M. Muniz-Miranda, A. Generosi, B. Paci and G. Le Lay, *2D Materials* **1**, 021003 (2014); J. Sone, T. Yamagami, Y. Aoki, K. Nakatsuji and H. Hirayama, *New J. Phys.* **16**, 095004 (2014).
- [7] M.E. Dávila, L. Xian, S. Cahangirov, A. Rubio and G. Le Lay, *New J. Phys.* **16**, 095002 (2014).
- [8] Xi Ling, H. Wang, S. Huang, F. Xia, and M.S. Dresselhaus, *PNAS* **112**, 4523 (2015).
- [9] T. Chakraborty and V. Apalkov, in [5] Ch. 8; T. Chakraborty and V.M. Apalkov, *Solid State Commun.* **175**, 123 (2013).
- [10] V.M. Apalkov and T. Chakraborty, *Phys. Rev. Lett.* **97**, 126801 (2006).
- [11] V.M. Apalkov and T. Chakraborty, *Phys. Rev. Lett.* **105**, 036801 (2010).
- [12] V.M. Apalkov and T. Chakraborty, *Phys. Rev. Lett.* **107**, 186803 (2011).
- [13] X. Du, I. Skachko, F. Duerr, A. Luican and E.Y. Andrei, *Nature* **462**, 192 (2009); K.I. Bolotin, F. Ghahari, M.D. Shulman, H.L. Stormer and P. Kim, *Nature* **462**, 196 (2009).
- [14] P. Maher, L. Wang, Y. Gao, C. Forsythe, T. Taniguchi, K. Watanabe, D. Abanin, Z. Papić, P. Cadden-Zimansky, J. Hone, P. Kim and C.R. Dean, *Science* **345**, 61 (2014).
- [15] V.M. Apalkov and T. Chakraborty, *Phys. Rev. B* **90**, 245108 (2014).
- [16] L. Kou, C. Chen and S.C. Smith, *J. Phys. Chem. Lett.* **6**, 2794 (2015).
- [17] H. Liu, Y. Du, Y. Deng and P.D. Ye, *Chem. Soc. Rev.* **44**, 2732 (2015).
- [18] A. Ghazaryan and T. Chakraborty, *Phys. Rev. B* **92**, 165409 (2015).
- [19] R. Willett, J.P. Eisenstein, H.L. Störmer, D.C. Tsui, A.C. Gossard, and J.E. English, *Phys. Rev. Lett.* **59**, 1776 (1987).
- [20] J.P. Eisenstein, R. Willett, H.L. Störmer, D.C. Tsui, A.C. Gossard, and J.E. English, *Phys. Rev. Lett.* **61**, 997 (1988).
- [21] T. Chakraborty and P. Pietiläinen, *Phys. Rev. B Phys. Rev. B* **38**, 10 097(R) (1988).
- [22] N. Read, *Physica B* **298**, 121 (2001); G. Moore and N. Read, *Nucl. Phys. B* **360**, 362 (1991).
- [23] A. Tsukazaki, A. Ohtomo, T. Kita, Y. Ohno, H. Ohno, and M. Kawasaki, *Science* **315**, 1388 (2007).
- [24] A. Tsukazaki, S. Akasaka, K. Nakahara, Y. Ohno, H. Ohno, D. Maryenko, A. Ohtomo and M. Kawasaki, *Nature Materials* **9**, 889 (2010).
- [25] J. Falson, D. Maryenko, B. Friess, D. Zhang, Y. Kozuka, A. Tsukazaki, J.H. Smet and M. Kawasaki, *Nature Physics* **11**, 347 (2015).
- [26] M.R. Peterson and C. Nayak, *Phys. Rev. B* **87**, 245129 (2013).
- [27] W. Bishara and C. Nayak, *Phys. Rev. B* **80**, 121302(R) (2009); E.H. Rezayi and F.D.M. Haldane, *Phys. Rev. B* **42**, 4532 (1990); E.H. Rezayi and S.H. Simon, *Phys. Rev. Lett.* **106**, 116801 (2011); I. Sodemann and A.H. MacDonald, *Phys. Rev. B* **87**, 245425 (2013); R.E. Wooten, J.H. Macek, and J.J. Quinn, *Phys. Rev. B* **88**, 155421 (2013); M.R. Peterson and C. Nayak, *Phys. Rev. Lett.* **113**, 086401 (2014).
- [28] I.L. Aleiner and L.I. Glazman, *Phys. Rev. B* **52**, 11296 (1995).
- [29] M.M. Fogler, in *High Magnetic Fields: Applications in Condensed Matter Physics and Spectroscopy*, edited by C. Berthier, L.-P. Levy, and G. Martinez (Springer-Verlag, Berlin 2002).
- [30] W. Luo and R. Côté, *Phys. Rev. B* **88**, 115417 (2013).
- [31] T. Chakraborty, P. Pietiläinen, and F.C. Zhang, *Phys. Rev. Lett.* **57**, 130 (1986); J.P. Eisenstein, H.L. Stormer, L. Pfeiffer, and K.W. West, *Phys. Rev. Lett.* **62**, 1540 (1989).
- [32] T. Chakraborty and F.C. Zhang *Phys. Rev. B* **29**, 7032(R) (1984); F.C. Zhang and T. Chakraborty, *Phys. Rev. B* **30**, 7320(R) (1984).
- [33] D. Yoshioka, *J. Phys. Soc. Jpn.* **55**, 3960-3968 (1986); M. Rasolt, F. Perrot, and A.H. MacDonald, *Phys. Rev. Lett.* **55**, 433 (1985).
- [34] Z. Papić and D. A. Abanin, *Phys. Rev. Lett.* **112**, 046602 (2014).
- [35] K. Shizuya, *Phys. Rev. B* **75**, 245417 (2007); R. Roldan, M.O. Goerbig, and J.-N. Fuchs, *Semicond. Sci. Technol.* **25**, 034005 (2010).
- [36] F.D.M. Haldane, *Phys. Rev. Lett.* **55**, 2095 (1985).
- [37] D. Yoshioka, B.I. Halperin, and P.A. Lee, *Phys. Rev. Lett.* **50**, 1219 (1983); D. Yoshioka, *Phys. Rev. B* **29**, 6833 (1984).
- [38] T. Chakraborty and P. Pietiläinen, *Phys. Rev. B* **39**, 7971 (1989); V. Halonen, P. Pietiläinen and T. Chakraborty, *Phys. Rev. B* **41**, 10202 (1990); Z. Papić, *Phys. Rev. B* **87**, 245315 (2013).
- [39] L. Tiemann, G. Gamez, N. Kumada, and M. Muraki, *Science* **335**, 828 (2012); M. Stern, B. A. Piot, Y. Vardi, V. Umansky, P. Plochocka, D. K. Maude, and I. Bar-Joseph, *Phys. Rev. Lett.* **108**, 066810 (2012).