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Shifted-Action Expansion and Applicability of Dressed Diagrammatic Schemes

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While bare diagrammatic series are merely Taylor expansions in powers of interaction strength, dressed diagrammatic series, built on fully or partially dressed lines and vertices, are usually constructed by reordering the bare diagrams, which is an *a priori* unjustified manipulation, and can even lead to convergence to an unphysical result [Kozik, Ferrero, and Georges, PRL **114**, 156402 (2015)]. Here we show that for a broad class of partially dressed diagrammatic schemes, there exists an action $S^{(\xi)}$ depending analytically on an auxiliary complex parameter ξ, such that the Taylor expansion in ξ of correlation functions reproduces the original diagrammatic series. The resulting applicability conditions are similar to the bare case. For fully dressed skeleton diagrammatics, analyticity of $S^{(\xi)}$ is not granted, and we formulate a sufficient condition for converging to the correct result.

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Much of theoretical physics is formulated in the language of Feynman diagrams, in various fields such as condensed matter, nuclear physics, and QCD. A powerful feature of the diagrammatic technique, used in each of the above fields, is the possibility to build diagrams on partially or fully dressed propagators or vertices, see, *e.g.*, Refs.[1–5]. In quantum many-body physics, notable examples include dilute gases, whose description is radically improved if ladder diagrams are summed up so that the expansion is done in terms of the scattering amplitude instead of the bare interaction potential, and Coulomb interactions, which one has to screen to have a meaningful diagrammatic technique.

With the development of Diagrammatic Monte Carlo, it becomes possible to compute Feynman diagrammatic expansions to high order for fermionic strongly correlated quantum many-body problems[6–11]. The number of diagrams grows factorially with the order, even for a fully irreducible skeleton scheme[12]. Nevertheless, for fermionic systems on a lattice at finite temperature, diagrammatic series (of the form $\sum_n a_n$ with a_n the sum of all order-n diagrams) are typically convergent in a broad range of parameters, due to a nearly perfect cancellation of contributions of different sign within each order, as proven mathematically[13] and seen numerically[6,7,9–11].

One might think that partial or full renormalization of diagrammatic elements (propagators, interactions, vertices, etc.) always leads to more compact and better behaving diagrammatic expansion. However, such a dressed diagrammatic series cannot be used blindly: Even when it converges, the result is not guaranteed to be correct, since it is *a priori* not allowed to reorder the terms of a series that is not absolutely convergent (the sum of the absolute values of *individual diagrams* is typically infinite, due to factorial scaling of the number of diagrams with the order). And indeed, for a skeleton series, *i.e.*, a series built on the fully dressed propagator, convergence to a wrong result does occur in the case of the Hubbard model in the strongly correlated regime near half filling[14], and preliminary results suggest that the corresponding self-consistent skeleton scheme converges to a wrong result as a function of the maximal self-energy diagram order $\mathcal{N}$[15]. Both of these phenomena are clearly seen in the exactly solvable zero space-time dimensional case[16,17].

In this work, we establish a condition that is necessarily violated in the event of convergence to a wrong result of the self-consistent skeleton scheme. Furthermore, we show that this convergence issue is absent for a broad class of partially dressed schemes. In particular, we propose a simple scheme based on the truncated skeleton series. The underlying idea is to construct an action $S^{(\xi)}$ that depends on an auxiliary complex parameter ξ such that the Taylor series in ξ of correlation functions reproduces the dressed diagrammatic series built on a given partially or fully dressed propagator. This makes the dressed scheme as mathematically justified as a bare scheme, provided $S^{(\xi)}$ is analytic with respect to ξ and $S^{(\xi=1)}$ coincides with the physical action; these conditions hold automatically in the partially dressed case, while in the fully dressed case they hold under a simple sufficient condition which we provide. Our construction applies to a general class of diagrammatic schemes built on dressed lines and vertices, including two-particle ladders and screened long-ranged potentials.

Partially dressed single-particle propagator. We consider a generic fermionic many-body problem described by an action

$$S[\psi,\bar\psi] = \langle\psi|G_0^{-1}|\psi\rangle + S_{\rm int}[\psi,\bar\psi] \tag{1}$$

where $\psi, \bar{\psi}$ are Grassmann fields[18], and we use bra-ket notations to suppress space, imaginary time, possible internal quantum numbers, and integrals/sums over them, *i.e.*, $\langle\psi|G_0^{-1}|\psi\rangle$ denotes the integral/sum over $\mathbf{r}$, τ and σ of $\bar{\psi}_\sigma(\mathbf{r},\tau)\,(G_{0,\sigma}^{-1}\,\psi_\sigma)(\mathbf{r},\tau)$. G_0^{-1} stands for the inverse, in the sense of operators, of the free propagator. The full propagator G and the self-energy Σ are related through the Dyson equation $G^{-1} = G_0^{-1} - \Sigma$. The bare Feynman diagrammatic expansion corresponds to perturbation theory in $S_{\rm int}$. In order to generate a diagrammatic expansion built on a partially dressed single-particle propagator $\tilde{G}_\mathcal{N}$, we introduce an auxiliary action of the form

$$S_\mathcal{N}^{(\xi)}[\psi,\bar{\psi}] = \langle\psi|G_{0,\mathcal{N}}^{-1}(\xi)|\psi\rangle + \xi S_{\rm int}[\psi,\bar{\psi}], \qquad (2)$$

where

$$G_{0,\mathcal{N}}^{-1}(\xi) = \tilde{G}_\mathcal{N}^{-1} + \xi\Lambda_1 + \ldots + \xi^\mathcal{N}\Lambda_\mathcal{N}, \qquad (3)$$

ξ is an auxiliary complex parameter, and $\Lambda_1, \ldots, \Lambda_\mathcal{N}$ are appropriate operators. $\tilde{G}_\mathcal{N}$ is the single particle propagator for $S_\mathcal{N}^{(\xi=0)}$. At $\xi \neq 0$, one can still view $\tilde{G}_\mathcal{N}$ as the free propagator, provided one includes in the interaction terms not only $\xi S_{\rm int}$, but also the quadratic terms $\langle\psi|\xi^n\Lambda_n|\psi\rangle$. Accordingly, ξ is interpreted as a coupling constant, and the $\xi^n\Lambda_n$ acquire the meaning of counter-terms. These counter-terms can be tuned to cancel out reducible diagrams, thereby enforcing the dressed character of the diagrammatic expansion. A natural requirement is that $S_\mathcal{N}^{(\xi=1)}$ coincides with the physical action S, *i.e.*, that

$$\tilde{G}_\mathcal{N}^{-1} + \sum_{n=1}^{\mathcal{N}} \Lambda_n = G_0^{-1}. \qquad (4)$$

For given G_0, this should be viewed as an equation to be solved for $\tilde{G}_\mathcal{N}$ (it is non-linear if the Λ_n's depend on $\tilde{G}_\mathcal{N}$). The unperturbed action for the dressed expansion, $\langle\psi|\tilde{G}_\mathcal{N}^{-1}|\psi\rangle$, is shifted by the Λ_n's with respect to the unperturbed action for the bare expansion, $\langle\psi|G_0^{-1}|\psi\rangle$.

We can then use any action of the generic class (2) for producing physical answers in the form of Taylor expansion in powers of ξ, provided the propagator $\tilde{G}_\mathcal{N}$ and the shifts Λ_n satisfy Eq. (4). More precisely, consider the full single-particle propagator $G_\mathcal{N}(\xi)$ of the action $S_\mathcal{N}^{(\xi)}$, and the corresponding self-energy

$$\Sigma_\mathcal{N}(\xi) \coloneqq G_{0,\mathcal{N}}^{-1}(\xi) - G_\mathcal{N}^{-1}(\xi). \qquad (5)$$

Note that since $S_\mathcal{N}^{(\xi=1)} = S$, we have $G_\mathcal{N}(\xi{=}1) = G$ and hence also $\Sigma_\mathcal{N}(\xi{=}1) = \Sigma$. We assume for simplicity that $\Sigma_\mathcal{N}(\xi)$ is analytic at $\xi = 0$, and that its Taylor series $\sum_{n=1}^\infty \Sigma_\mathcal{N}^{(n)}[\tilde{G}_\mathcal{N}]\,\xi^n$, converges at $\xi = 1$. We expect these assumptions to hold for fermionic lattice models at finite temperature in a broad parameter regime, given that the action $S_\mathcal{N}^{(\xi)}$ is analytic in ξ[6,7,9–11,13,19]. Then, since $S_\mathcal{N}^{(\xi)}$ is an entire function of ξ, we can conclude that

$$\Sigma = \sum_{n=1}^\infty \Sigma_\mathcal{N}^{(n)}[\tilde{G}_\mathcal{N}], \qquad (6)$$

i.e., the physical self-energy is equal to the dressed diagrammatic series.

This last step of the reasoning can be justified using the following presumption: *Let $\mathcal{D}$ be a connected open region of the complex plane containing* 0. *Assume that $S^{(\xi)}$ is analytic in $\mathcal{D}$, that the corresponding self-energy $\Sigma(\xi)$ is analytic at $\xi = 0$, and that $\Sigma(\xi)$ admits an analytic continuation $\tilde{\Sigma}(\xi)$ in $\mathcal{D}$. Then, Σ and $\tilde{\Sigma}$ coincide on $\mathcal{D}$.* This presumption is based on the following argument: Since $S^{(\xi)}$ is analytical, if no phase transition occurs when varying ξ in $\mathcal{D}$, then $\Sigma(\xi)$ is analytical on $\mathcal{D}$, and by the identity theorem for analytic functions, Σ and $\tilde{\Sigma}$ coincide on $\mathcal{D}$. If a phase transition would be crossed as a function of ξ in $\mathcal{D}$, analytic continuation through the phase transition would not be possible[20], contradicting the above assumption on the existence of $\tilde{\Sigma}$. Applying this presumption to $\tilde{\Sigma}(\xi) := \sum_{n=1}^\infty \Sigma_\mathcal{N}^{(n)}[\tilde{G}_\mathcal{N}]\,\xi^n$, which has a radius of congergence $R \geq 1$ (from the Cauchy-Hadamard theorem), and taking for $\mathcal{D}$ the open disc of radius R, we directly obtain Eq. (6) provided $R > 1$. If $R{=}1$, we can still derive Eq. (6), using Abel's theorem and assuming that $\Sigma_\mathcal{N}(\xi)$ is continuous at $\xi{=}1$, which, given that the action in entire in ξ, is generically expected (except for physical parameters fined-tuned precisely to a first-order phase transition, where Σ is not uniquely defined).

Semi-bold scheme. We first focus on the choice

$$\Lambda_n = \Sigma_{\rm bold}^{(n)}[\tilde{G}_\mathcal{N}] \qquad (1 \leq n \leq \mathcal{N}), \qquad (7)$$

where $\Sigma_{\rm bold}^{(n)}[\mathcal{G}]$ is the sum of all skeleton diagrams of order n, built with the propagator $\mathcal{G}$ and the bare interaction vertex corresponding to $S_{\rm int}$, that remain connected when cutting two $\mathcal{G}$ lines. This means that $\tilde{G}_\mathcal{N}$ is the solution of the bold scheme for maximal order $\mathcal{N}$, cf. Eq. (4). For a given $\mathcal{N}$, higher-order dressed graphs can then be built on $\tilde{G}_\mathcal{N}$. The numerical protocol corresponding to this 'semi-bold' scheme consists of two independent parts: Part I is the Bold Diagrammatic Monte Carlo simulation of the truncated order-$\mathcal{N}$ skeleton sum employed to solve iteratively for $\tilde{G}_\mathcal{N}$ satisfying Eqs. (4,7); Part II is the diagrammatic Monte Carlo simulation of higher-order terms, $\Sigma_\mathcal{N}^{(n)}[\tilde{G}_\mathcal{N}], n > \mathcal{N}$, that uses $\tilde{G}_\mathcal{N}$ as the bare propagator. Note that here $\mathcal{N}$ is fixed (contrarily to the conventional skeleton scheme discussed below), and the infinite-order extrapolation is done only in Part II.

The Feynman rules for this scheme are as follows:

$$\Sigma_\mathcal{N}^{(n)}[\tilde{G}_\mathcal{N}] = \Sigma_{\rm bold}^{(n)}[\tilde{G}_\mathcal{N}] \quad \text{for } n \leq \mathcal{N}; \qquad (8)$$

while for $n \geq \mathcal{N}+1$, $\Sigma_\mathcal{N}^{(n)}[\tilde{G}_\mathcal{N}]$ is the sum of all bare diagrams, built with $\tilde{G}_\mathcal{N}$ as free propagator and the bare interaction vertex corresponding to $S_{\rm int}$, which do not contain any insertion of a subdiagram contributing to $\Sigma^{(n)}_{\rm bold}[\tilde{G}_{\mathcal{N}}]$ with $n \le \mathcal{N}$. Indeed, each such insertion is exactly compensated by the corresponding counterterm. To derive Eq. (8), we will use the relation

$$\Sigma_{\mathcal{N}}(\xi) \mathrel{\hat{=}} \sum_{n=1}^{\infty} \Sigma^{(n)}_{\rm bold}[G_{\mathcal{N}}(\xi)]\, \xi^n \tag{9}$$

where $\hat{=}$ stands for equality in the sense of formal power series in ξ, and we will show the proposition

$$\Sigma_{\mathcal{N}}(\xi) \mathrel{\hat{=}} \sum_{n=1}^{k} \Sigma^{(n)}_{\rm bold}[\tilde{G}_{\mathcal{N}}]\, \xi^n + O(\xi^{k+1}) \tag{$\mathcal{P}_k$}$$

for any $k \in \{0,\dots,\mathcal{N}+1\}$, by recursion over k. $(\mathcal{P}_{k=0})$ clearly holds. If $(\mathcal{P}_k)$ holds for some $k \le \mathcal{N}$, then we have $G_{\mathcal{N}}(\xi) \mathrel{\hat{=}} \tilde{G}_{\mathcal{N}} + O(\xi^{k+1})$, as follows from Eqs. (5), (3) and (7). Substitution into Eq. (9) then yields $(\mathcal{P}_{k+1})$.

Alternatively to the semi-bold scheme Eq. (7), other choices are possible for the shifts $\Lambda_1,\dots,\Lambda_{\mathcal{N}}$ and the dressed propagator $\tilde{G}_{\mathcal{N}}$. For example, the shifts can be based on diagrams containing the original bare propagator G_0 instead of $\tilde{G}_{\mathcal{N}}$. In the absence of exact cancellation, all diagrams should be simulated in Part II of the numerical protocol, and Λ_n will enter the theory explicitly. This flexibility of choosing the form of Λ_n's, along with the obvious option of exploring different $\mathcal{N}$'s, provides a tool for controlling systematic errors coming from truncation of the ξ-series.

Skeleton scheme. We turn to the conventional scheme in which diagrams are built on the *fully* dressed single-particle propagator. The corresponding numerical protocol is identical to Part I of the above one, with the additional step of extrapolating $\mathcal{N}$ to infinity, as done in[8–11,21]. Accordingly, we assume that the 'skeleton sequence' $\tilde{G}_{\mathcal{N}}$ converges to a limit $\tilde{G}$ when $\mathcal{N} \to \infty$. The crucial question is under what conditions one can be confident that $\tilde{G}$ is the genuine propagator G of the original model. The answer comes from the properties of the sequence of functions

$$L^{(\xi)}_{\mathcal{N}} := \sum_{n=1}^{\mathcal{N}} \Sigma^{(n)}_{\rm bold}[\tilde{G}_{\mathcal{N}}]\, \xi^n. \tag{10}$$

Let us show that $\tilde{G} = G$ holds under the following sufficient condition:

(i) for any ξ in a disc $\mathcal{D} = \{|\xi| < R\}$ of radius $R > 1$, and for all $(\mathbf{p},\tau)$, $L^{(\xi)}_{\mathcal{N}}(\mathbf{p},\tau)$ converges for $\mathcal{N}\to\infty$; moreover this sequence is uniformly bounded, *i.e.*, there exists a function $C_1(\mathbf{p},\tau)$ such that $\forall\xi \in \mathcal{D}, \forall(\mathcal{N},\mathbf{p},\tau)$, $|L^{(\xi)}_{\mathcal{N}}(\mathbf{p},\tau)| \le C_1(\mathbf{p},\tau)$; and

(ii) $\tilde{G}_{\mathcal{N}}(\mathbf{p},\tau)$ is uniformly bounded, *i.e.*, there exists a constant C_2 such that for all $(\mathcal{N},\mathbf{p},\tau)$, $|\tilde{G}_{\mathcal{N}}(\mathbf{p},\tau)| \le C_2$.

Our derivation is based on the action

$$S^{(\xi)}_{\infty} := \lim_{\mathcal{N}\to\infty} S^{(\xi)}_{\mathcal{N}}. \tag{11}$$

Clearly,

$$S^{(\xi)}_{\infty} = \langle\psi|\, \tilde{G}^{-1} + L^{(\xi)}\, |\psi\rangle + \xi S_{\rm int} \tag{12}$$

with

$$L^{(\xi)}(\mathbf{p},\tau) := \lim_{\mathcal{N}\to\infty} L^{(\xi)}_{\mathcal{N}}(\mathbf{p},\tau). \tag{13}$$

Since $S^{(\xi=1)}_{\mathcal{N}} = S$, we have $S^{(\xi=1)}_{\infty} = S$, and thus $G_\infty(\xi{=}1) = G$ where $G_\infty(\xi)$ is the full propagator of the action $S^{(\xi)}_{\infty}$.

We first observe that $L^{(\xi)}(\mathbf{p},\tau)$ is an analytic function of $\xi \in \mathcal{D}$ for all $(\mathbf{p},\tau)$, and that

$$\left.\frac{1}{n!}\,\frac{\partial^n}{\partial\xi^n} L^{(\xi)}(\mathbf{p},\tau)\right|_{\xi=0} = \Sigma^{(n)}_{\rm bold}[\tilde{G}](\mathbf{p},\tau). \tag{14}$$

This follows from conditions (i,ii), given that momenta are bounded for lattice models. Indeed, for any triangle $\mathcal{T}$ included in $\mathcal{D}$, $\oint_{\mathcal{T}} d\xi\, L^{(\xi)}_{\mathcal{N}}(\mathbf{p},\tau) = 0$. Thanks to condition (i), the dominated convergence theorem is applicable, yielding $\oint_{\mathcal{T}} d\xi\, L^{(\xi)}(\mathbf{p},\tau) = 0$. The analyticity of $\xi \mapsto L^{(\xi)}(\mathbf{p},\tau)$ follows by Morera's theorem. To derive Eq. (14) we start from

$$\left.\frac{1}{n!}\,\frac{\partial^n}{\partial\xi^n} L^{(\xi)}_{\mathcal{N}}(\mathbf{p},\tau)\right|_{\xi=0} = \Sigma^{(n)}_{\rm bold}[\tilde{G}_{\mathcal{N}}](\mathbf{p},\tau). \tag{15}$$

By Cauchy's integral formula, the l.h.s. of Eq. (15) equals $1/(2i\pi)\oint_{\mathcal{C}} d\xi\, L^{(\xi)}_{\mathcal{N}}(\mathbf{p},\tau)/\xi^{n+1}$ where $\mathcal{C}$ is the unit circle. Using again condition (i) and the dominated convergence theorem, when $\mathcal{N}\to\infty$, this tends to $1/(2i\pi)\oint_{\mathcal{C}} d\xi\, L^{(\xi)}(\mathbf{p},\tau)/\xi^{n+1}$, which equals the l.h.s. of Eq. (14). To show that $\Sigma^{(n)}_{\rm bold}[\tilde{G}_{\mathcal{N}}](\mathbf{p},\tau)$ tends to $\Sigma^{(n)}_{\rm bold}[\tilde{G}](\mathbf{p},\tau)$, we consider each Feynman diagram separately; the dominated convergence theorem is applicable thanks to condition (ii), the boundedness of the integration domain for internal momenta and imaginary times, and assuming that interactions decay sufficiently quickly at large distances for the bare interaction vertex to be bounded in momentum representation.

Hence

$$L^{(\xi)} = \sum_{n=1}^{\infty} \Sigma^{(n)}_{\rm bold}[\tilde{G}]\ \xi^n. \tag{16}$$

As a consequence, the action $S^{(\xi)}_{\infty}$ generates the fully dressed skeleton series built on $\tilde{G}$, *i.e.*, its self-energy $\Sigma_\infty(\xi)$ has the Taylor expansion $\sum_{n=1}^{\infty} \Sigma^{(n)}_{\rm bold}[\tilde{G}]\, \xi^n$, and the Taylor series of $G_\infty(\xi)$ reduces to the ξ-independent term $\tilde{G}$. This can be derived in the same way as Eq. (8), by showing by recursion over k that for any $k \ge 0$, $\Sigma_\infty(\xi) = \sum_{n=1}^{k} \Sigma^{(n)}_{\rm bold}[\tilde{G}]\, \xi^n + O(\xi^{k+1})$. Furthermore, having shown above the analiticity of $L^{(\xi)}$, *i.e.*, of $S^{(\xi)}_{\infty}$, we again expect that $G_\infty(\xi)$ is analytic at $\xi = 0$ (for fermions on a lattice at finite temperature), and we can use again the above presumption to conclude that $G_\infty(\xi=1)=G=\tilde{G}$.

Dressed pair propagator. So far we have discussed dressing of the single-particle propagator while keeping the bare interaction vertices. We turn to diagrammatic schemes built on dressed pair propagators. We restrict to spin-1/2 fermions with on-site interaction:

$$S_{\rm int}[\psi,\bar\psi] \;=\; U\sum_{\mathbf{r}}\int_0^\beta d\tau\;\;(\bar\psi_\uparrow\bar\psi_\downarrow\psi_\downarrow\psi_\uparrow)(\mathbf{r},\tau), \quad (17)$$

where U is the bare interaction strength. For simplicity we discuss dressing of the pair propagator while keeping the bare G_0. It is necessary to perform a Hubbard-Stratonovich transformation in order to construct the appropriate auxiliary action. Introducing a complex scalar Hubbard-Stratonovich field η leads to the action

$$\mathcal{S}[\psi,\bar\psi,\eta,\bar\eta] = \langle\psi|G_0^{-1}|\psi\rangle - \langle\eta|\Gamma_0^{-1}|\eta\rangle - \langle\eta|\Pi_0|\eta\rangle + \langle\eta|\psi_\downarrow\psi_\uparrow\rangle + \langle\psi_\downarrow\psi_\uparrow|\eta\rangle, \quad (18)$$

where $\Pi_0(\mathbf{r},\tau)=-(G_{0,\uparrow}G_{0,\downarrow})(\mathbf{r},\tau)$ and Γ_0 is the sum of the ladder diagrams, $\Gamma_0^{-1}(\mathbf{p},\Omega_n)=U^{-1}-\Pi_0(\mathbf{p},\Omega_n)$ with Ω_n the bosonic Matsubara frequencies.

We first consider the diagrammatic scheme built on G_0 and Γ_0. We denote by $\Sigma^{(n)}_{\rm lad}[G_0,\Gamma_0]$ the sum of all self-energy diagrams of order n, *i.e.* containing n Γ_0-lines. This diagrammatic series is generated by the shifted action

$$\mathcal{S}^{(\xi)}_{\rm lad}[\psi,\bar\psi,\eta,\bar\eta] = \langle\psi|G_0^{-1}|\psi\rangle - \langle\eta|\Gamma_0^{-1}|\eta\rangle - \xi^2\langle\eta|\Pi_0|\eta\rangle + \xi\left(\langle\eta|\psi_\downarrow\psi_\uparrow\rangle + \langle\psi_\downarrow\psi_\uparrow|\eta\rangle\right), \quad (19)$$

in the sense that self-energy $\Sigma_{\rm lad}(\xi)$ corresponding to this action has the Taylor series $\sum_{n=1}^\infty \Sigma^{(n)}_{\rm lad}[G_0,\Gamma_0]\,\xi^{2n}$. Indeed, the counter-term $\xi^2\Pi_0$ cancels out the reducible diagrams contatining G_0G_0 bubbles. Therefore, if this diagrammatic series converges, then it yields the physical self-energy. This follows from the same reasoning as below Eq. (5). The same applies to the series for the pair self-energy Π in terms of $[G_0,\Gamma_0]$. Here Π is defined by $\Gamma^{-1}=\Gamma_0^{-1}-\Pi$, where Γ denotes the fully dressed pair propagator, used in[8,11].

More complex schemes, built on other dressed pair propagators than Γ_0, can be generated by the shifted action

$$\mathcal{S}^{(\xi)}_{\mathcal{N}}[\psi,\bar\psi,\eta,\bar\eta] = \langle\psi|G_0^{-1}|\psi\rangle - \langle\eta|\Gamma^{-1}_{0,\mathcal{N}}(\xi)|\eta\rangle - \xi^2\langle\eta|\Pi_0|\eta\rangle + \xi\left(\langle\eta|\psi_\downarrow\psi_\uparrow\rangle + \langle\psi_\downarrow\psi_\uparrow|\eta\rangle\right), \quad (20)$$

where

$$\Gamma^{-1}_{0,\mathcal{N}}(\xi) = \tilde\Gamma^{-1}_{\mathcal{N}} + \xi^2\,\Omega_1 + \ldots + \xi^{2\mathcal{N}}\,\Omega_{\mathcal{N}} \quad (21)$$

and one imposes $\Gamma_{0,\mathcal{N}}(\xi=1)=\Gamma_0$. In particular, the semi-bold scheme is defined by

$$\Omega_n = \Pi^{(n)}_{\rm bold}[\tilde\Gamma_{\mathcal{N}}], \quad (22)$$

where $\Pi^{(n)}_{\rm bold}[\gamma]$ is the sum of all skeleton diagrams of order n built with the pair-propagator γ that remain connected when cutting two γ-lines. As usual, $\Pi^{(1)}_{\rm bold} = -GG+G_0G_0$. This scheme was introduced previously for $\mathcal{N}$=1[22].

Finally, we consider the skeleton scheme built on G_0 and Γ. Assuming that the skeleton sequence $\tilde\Gamma_{\mathcal{N}}$ converges to some $\tilde\Gamma$, one can show analogously to the above reasoning that $\tilde\Gamma$ is equal to the exact Γ under the following sufficient condition:
(i) for any ξ in a disc $\mathcal{D}=\{|\xi|<R\}$ of radius $R>1$, and for all $(\mathbf{p},\Omega_n)$, $M^{(\xi)}_{\mathcal{N}}(\mathbf{p},\Omega_n) \coloneqq \sum_{n=1}^{\mathcal{N}} \Pi^{(n)}_{\rm bold}[\tilde\Gamma_{\mathcal{N}}](\mathbf{p},\Omega_n)\,\xi^n$ converges for $\mathcal{N}\to\infty$; moreover this sequence is uniformly bounded, *i.e.*, there exists $C(\mathbf{p},\Omega_n)$ such that $\forall\xi\in\mathcal{D}, \forall(\mathcal{N},\mathbf{p},\Omega_n), |M^{(\xi)}_{\mathcal{N}}(\mathbf{p},\Omega_n)|\le C(\mathbf{p},\Omega_n)$; and
(ii) $\tilde\Gamma_{\mathcal{N}}(\mathbf{p},\Omega_n)$ is uniformly bounded.

Screened interaction potential. Finally, we briefly address the procedure of dressing the interaction line, which is particularly important for long-range interaction potentials. Restricting for simplicity to a spin-independent interaction potential $V(\mathbf{r})$, the interaction part of the action writes

$$\frac{1}{2}\sum_{\sigma,\sigma'}\sum_{\mathbf{r},\mathbf{r}'}\int_0^\beta d\tau\;(\bar\psi_\sigma\psi_\sigma)(\mathbf{r},\tau)\,V(\mathbf{r}-\mathbf{r}')\,(\bar\psi_{\sigma'}\psi_{\sigma'})(\mathbf{r}',\tau). \quad (23)$$

We again keep the bare G_0 for simplicity and consider dressing of V only. Introducing a real scalar Hubbard-Stratonovich field χ leads to the action

$$\mathcal{S}[\psi,\bar\psi,\chi] \;=\; \langle\psi|G_0^{-1}|\psi\rangle + \frac{1}{2}\langle\chi|\,V^{-1}\,|\chi\rangle + i\sum_\sigma\langle\chi|\bar\psi_\sigma\psi_\sigma\rangle. \quad (24)$$

Here we assume that the Fourier transform $V(\mathbf{q})$ of the interaction potential is positive, so that the quadratic form $\langle\chi|\,V^{-1}\,|\chi\rangle = (2\pi)^{-d}\int_0^\beta d\tau\int d^dq\,|\chi(\mathbf{q},\tau)|^2/V(\mathbf{q})$ is positive definite. The auxiliary action takes the form

$$\mathcal{S}^{(\xi)}_{\mathcal{N}}[\psi,\bar\psi,\chi] = \langle\psi|G_0^{-1}|\psi\rangle + \frac{1}{2}\langle\chi|\tilde V^{-1}_{\mathcal{N}} + \xi^2\Omega_1 + \ldots + \xi^{2\mathcal{N}}\Omega_{\mathcal{N}}|\chi\rangle + i\xi\sum_\sigma\langle\chi|\bar\psi_\sigma\psi_\sigma\rangle. \quad (25)$$

The semi-bold scheme corresponds to $\Omega_n = \Pi^{(n)}_{\rm bold}[\tilde V_{\mathcal{N}}]$ where Π now stands for the polarization. In particular, $\tilde V_1$ is the RPA screened interaction.

Summarizing, we have revealed an analytic structure behind dressed-line diagrammatics. More precisely, we have exhibited the function which analytically continues a dressed diagrammatic series. This function originates from an action that depends on an auxiliary parameter ξ. When the action is a polynomial in ξ, the situation reduces to the one of a bare expansion. Within this category, a particular case well suited for numerical implementation is the semi-bold scheme for which the bare propagator is taken from the truncated bold self-consistent equation. For the fully bold scheme, we construct an appropriate auxiliary action, but only under a certain condition. If this condition is verified numerically, it is safe to use the fully bold scheme. If not, the semi-bold scheme remains applicable.

Furthermore we have demonstrated the generality of the shifted-action construction by treating the case of a dressed pair propagator and of a screened long-range interaction. Further extensions left for future work are dressing of three-point vertices, as well as justifying resummation of divergent diagrammatic series by considering non disc-shaped analyticity domains $\mathcal{D}$.

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