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Bounds for low energy spectral properties of center-of-mass conserving positive two-body interactions

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We study the low energy spectral properties of positive center-of-mass conserving two-body Hamiltonians as they arise in models of fractional quantum Hall states. Starting from the observation that positive many-body Hamiltonians must have ground state energies that increase monotonously in particle number, we explore what general additional constraints can be obtained for two-body interactions with "center-of-mass conservation" symmetry, both in the presence and absence of particle-hole symmetry. We find general bounds that constrain the evolution of the ground state energy with particle number, and in particular constrain the chemical potential at $T = 0$. Special attention is given to Hamiltonians with zero modes, in which case similar bounds on the first excited state are also obtained, using a duality property. In this case, in particular an upper bound on the charge gap is also obtained. We further comment on center-of-mass and relative-decomposition in disk geometry within the framework of second quantization.

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I. INTRODUCTION

A cornerstone of the theory of fractional quantum Hall liquids is the construction and study of special parent Hamiltonians that stabilize prototypical wave functions such as the Laughlin state. The properties of such Hamiltonians have been well characterized analytically where their rich structure of so called "zero modes" is concerned, i.e., states at zero, or the lowest possible, energy. These states are of fundamental importance to the physics of a quantum Hall phase, since, in known examples, they fully describe in particular the low energy edge physics. In contrast, very little is known rigorously about the finite energy properties of such special Hamiltonians and their more generic deformations. This article reports an effort at improving this situation. Our starting point is a general monotony property, in particle number, of the ground state energy of positive many-body Hamiltonians. We observe that the strategy leading to this result gives rise to further interesting bounds when combined with other properties of general interest in fractional quantum Hall model Hamiltonians, most importantly center-of-mass conservation and the focus on two-body interaction. Our main result is a general bound on the step size of the ground state energy (and in some cases the first excitation energy) with particle number. In its simplest version, it is obtained in situations with particle-hole symmetry, but is subsequently improved and generalized to situations without particle-hole symmetry, including bosons. As a special application, an upper bound on the charge gap in special model Hamiltonians with zero modes is obtained. In the latter case, we also manage to give bounds on the evolution with particle number of the first excited state, by observing a certain invariance property of the zero mode subspace and then introducing a dual version thereof.

Technically, we work with second quantized forms of projection-operator type interactions. This is worth noting, since in this field, there is much history of deriving analytic results in a first quantized picture employing analytic wave functions[1–3] and correspondingly constructed first quantized parent Hamiltonians.[2,4] As far as wave functions are concerned, their spectral decomposition in a particle number basis has become of interest in recent years, through the study of the Jack-polynomial structure of special wave functions,[5,6] and through the more recently discovered matrix-product structure of these states.[7–9] In contrast, the use of second-quantized *Hamiltonians*, with some exceptions,[10] has long been reserved for numerical work, though their popularity has recently increased as well, in part due to interest in fractional Chern insulators,[11–16] purely technical reasons,[17,18] as well as more general ones.[19–22] The preference for first quantized descriptions of parent Hamiltonians can perhaps be attributed to the fact that these are, by construction, most suitable for studying the zero mode space, though it was recently shown (in some cases) that this is also possible in a purely second quantized framework.[19–21] Arguably, however, the advantage of working with first quantized Hamiltonians is lost when the focus is on finite energy spectral properties. There, and moreover when studying more generic Hamiltonians without any particularly interesting zero mode structure, arguments in favor of the greater efficiency of a "pure guiding center" description[23] are, in our opinion, particularly appealing. The second quantized presentation of Hamiltonians is one possible way to achieve such a pure guiding center description. Our study can thus also be viewed as adding further emphasis to the utility of such an approach.

II. MONOTONY OF GROUND STATE ENERGY

We begin by discussing the monotony in particle number and related general properties of the ground state energy of positive many-body Hamiltonians. To attain the desired level of generality, we will first consider a second-quantized k-body interaction of the form

$$H_k = \sum_{n_1,\ldots,n_{2k}} V_{n_1\ldots n_{2k}}\, c^\dagger_{n_1}\ldots c^\dagger_{n_k} c_{n_{k+1}}\ldots c_{n_{2k}}\,. \qquad (1)$$

The operators c_n may satisfy bosonic or fermionic commutation relations. We will later focus on the special case where a "center-of-mass" conservation law is explicit, as is appropriate for model Hamiltonians of fractional quantum Hall type systems in various geometries. For the moment, however, the only additional property we will require is *positivity*, i.e., $\langle\psi|H|\psi\rangle \geq 0$ for all k-particle (and hence N-particle) kets $|\psi\rangle$.

We now consider an N-particle mixed state described by a density matrix ρ_N. From ρ_N, we may define various N'-particle reduced density matrices $\rho_{N'}$, $N' < N$, given recursively via

$$\rho_{N'-1} = \frac{1}{N'}\sum_n c_n \rho_{N'} c^\dagger_n\,. \qquad (2)$$

We note that $\hat{N}\rho_{N'} = \rho_{N'}\hat{N} = N'\rho_{N'}$, where $\hat{N} = \sum_n c^\dagger_n c_n$ is the particle number operator, and $\mathrm{Tr}\,\rho_{N'-1} = \mathrm{Tr}\,\rho_{N'} = 1$. For both fermions and bosons, one easily verifies the relation

$$\hat{N}H_k = kH_k + \sum_n c^\dagger_n H_k c_n\,, \qquad (3)$$

obtained by commuting c_n to the right. This gives

$$\begin{aligned}\mathrm{Tr}\,\rho_{N'}H_k &= \frac{1}{N'}\,\mathrm{Tr}\,\rho_{N'}\hat{N}H_k \\ &= \frac{k}{N'}\,\mathrm{Tr}\,\rho_{N'}H_k + \mathrm{Tr}\,\rho_{N'-1}H_k\,, \end{aligned} \qquad (4)$$

or

$$\mathrm{Tr}\,\rho_{N'-1}H_k = \frac{N'-k}{N'}\,\mathrm{Tr}\,\rho_{N'}H_k\,, \qquad (5)$$

and by induction:

$$\mathrm{Tr}\,\rho_{N'}H_k = \frac{(N-k)(N-1-k)\ldots(N'+1-k)}{N(N-1)\ldots(N'+1)}\,\mathrm{Tr}\,\rho_N H_k. \qquad (6)$$

We now denote the ground state energy of H_k in the N-particle sector as $E^k_0(N)$. Then choosing ρ_N such that $\mathrm{Tr}\,\rho_N H_k = E^k_0(N)$, and noting $\mathrm{Tr}\,\rho_{N'}H_k \geq E^k_0(N')$ by the variational principle, we have

$$E^k_0(N') \leq \frac{(N-k)(N-1-k)\ldots(N'+1-k)}{N(N-1)\ldots(N'+1)}E^k_0(N)\,. \qquad (7)$$

So far we have not used positivity yet. A result similar to Eq. (7) can also be obtained for general Hamiltonians of the form

$$H = \sum_{k=k_{\mathsf{min}}}^{k_{\mathsf{max}}} H_k\,, \qquad (8)$$

where each term represents a positive k-body interaction, with k_{min} (k_{max}) being the minimum (maximum) k. In this case we still have Eq. (5) for each H_k, which, for positive interaction, in particular implies

$$\mathrm{Tr}\,\rho_{N'-1}H_k \leq \frac{N'-k_{\mathsf{min}}}{N'}\,\mathrm{Tr}\,\rho_{N'}H_k\,, \qquad (9)$$

and thus we have the same relation for H in place of H_k. For the ground state energy within the N-particle sector $E_0(N)$, we thus obtain Eq. (7) with k_{min} in place of k:

$$E_0(N') \leq \frac{(N-k_{\mathsf{min}})(N-1-k_{\mathsf{min}})\ldots(N'+1-k_{\mathsf{min}})}{N(N-1)\ldots(N'+1)}E_0(N)\,. \qquad (10)$$

Clearly, this then implies in particular the monotony of the ground state energy with particle number,

$$E_0(N-1) \leq E_0(N)\,, \qquad (11)$$

with equality *only* for $E_0(N) = 0$. This result and a wealth of similar ones all flow from Eq. (10) and have no doubt appeared previously in the literature, though we are unable to determine original references. For example, as another special case of Eq. (10), one obtaines the *superadditivity*[24] of the ground state energy. For this consider $N' = N_1$ and $N' = N_2$ with $N_1 + N_2 = N$, and add the corresponding instances of Eq. (10):

$$E_0(N_1) + E_0(N_2) \leq ([N,N_1,k_{\mathsf{min}}] + [N,N_2,k_{\mathsf{min}}])E_0(N)\,, \qquad (12)$$

where we have denoted the numerical factor in Eq. (10) as $[N,N',k] = (N-k)!N'!/(N!(N'-k)!)$. It is easy to see that $[N,N_1,k_{\mathsf{min}}]+[N,N_2,k_{\mathsf{min}}] \leq 1$. To see this, one first observes that the left hand side is equal to one for $k_{\mathsf{min}} = 1$. Furthermore, $[N,N',k]$ monotonously decreases with increasing k. Hence we have the superadditivity

$$E_0(N_1) + E_0(N_2) \leq E_0(N)\,,\ N_1 + N_2 = N\,. \qquad (13)$$

At the level of generality assumed thus far, Eq. (10) appears to be the strongest statement that can be made, containing a multitude of ground state monotony properties as special cases. In the following, we will be interested in a more restricted but physically relevant class of Hamiltonians that arises in particular when models of states in the fractional quantum Hall regime are considered. These Hamiltonians quite generally have an additional symmetry that in the second quantized form Eq. (1) manifests itself as "center-of-mass" conservation.[25] It turns out that in this case, further bounds on the evolution of the ground state energy with particle number can be given, and in some cases this is also true of the first excited state energy.

III. SPECIALIZATION TO CENTER-OF-MASS CONSERVING HAMILTONIANS

Many known parent Hamiltonians for various types of interesting fractional quantum Hall states have a peculiar way of satisfying Eq. (11): The ground state energy $E_0(N)$ is exactly zero until the particle number reaches some value $N = N_I$, where N_I/L approaches the incompressible filing factor and L is the number of Landau level orbitals available to the system due to finite size geometry (e.g. finite disk, sphere, or torus). We will comment on the situation in the infinite disk geometry in Sec. III C, where strictly speaking, absent any other constraints, $E_0(N) = 0$ for any finite N in the case of such special model Hamiltonians. It turns out that for quantum Hall type interaction Hamiltonians, additional constraints beyond Eq. (10) can be given. This is chiefly due to the general presence of another symmetry, that of the conservation of the center-of-mass. Related to that and in addition, some models of fermions possess a particle-hole symmetry. It is then natural to surmise that in cases where Eq. (11) is saturated for $N < N_I$, another inequality should be saturated in the particle-hole symmetric region $N > L - N_I$. This turns out to be an *upper* bound on the step size in ground state energy with particle number, $E_0(N) - E_0(N-1)$. This in particular provides an upper bound on the charge gap at the incompressible filling factor of special model Hamiltonians satisfying the "zero mode paradigm"

$$E_0(N) = 0 \;\text{ for }\; N \leq N_I\,, \tag{14}$$

but can be applied equally well to some more generic Hamiltonians. Before we derive these and other results, we will first write the Hamiltonian in a form in which all its pertinent properties are manifest.

In a constant magnetic background field, Landau level projection leads to a one-dimensional "lattice" Hilbert space of single particle orbitals labeled by an integer guiding center quantum number n, whose precise meaning depends on the geometry and choice of basis. Here, these orbitals are created by the operators $c_n^\dagger$. Certain rotational and or (magnetic) translational symmetries manifest themselves as "center-of-mass conservation", i.e., the matrix element in Eq. (1) is non-zero only when $n_1 + \ldots n_k = n_{k+1} + \ldots + n_{2k}$ is satisfied (on the torus modulo L). This can be made manifest by writing the Hamiltonian (8) in the form:

$$H = \sum_{m=1}^{M} \sum_{r=0}^{k_m-1} \sum_{R\in\mathbb{Z}+r/k_m} Q_R^{m\dagger} Q_R^m\,, \tag{15a}$$

where

$$Q_R^m = \sum_{n_1,\ldots,n_{k_m}} \eta^m_{R;n_1,\ldots,n_{k_m}}\, c_{n_1}\cdots c_{n_{k_m}} \tag{15b}$$

Eq. (15) can be obtained from Eqs. (8) and (1) by performing a spectral decomposition of the symbol $V_{n_1\ldots n_{2k}}$, viewed as a big matrix with multi-indices $(n_1,\ldots,n_k)$ and $(n_{k+1},\ldots,n_{2k})$. This matrix is block-diagonal in multi-indices of given $R = (n_1+\ldots+n_k)/k$, and so eigenvectors $\eta^m_{R;n_1\ldots n_k}$ can be labeled by R. These eigenvectors are normalized such that $\sum_{\{n_i\}} |\eta^m_{R;n_1\ldots n_k}|^2$ equals the corresponding eigenvalue, and the absence of negative coefficients signifies the positivity of the Hamiltonian. In Eq. (15), M different terms labeled by m are considered, each of which corresponds to an eigenvector of the aforementioned kind, obtained for a k_m-body operator in Eq. (8), with non-zero eigenvalue. To establish full equivalence between Eqs. (8) and (15), the case $M = \infty$ must be considered, whereas often M will be finite in quantum Hall model Hamiltonians. In the following, we will refer to the Hamiltonian either in the form (15) or in the less explicit but more condensed form Eqs. (1), (8), whichever is more convenient. We note that when working on the torus, center-of-mass conservation strictly holds only "modulo L". In this case we will still take the Hamiltonian to be of the form Eq. (15), where $c_n \equiv c_{n+L}$, and all symbols $\eta^m_{R;n_1\ldots n_k}$ are likewise invariant under the shift $n_i \to n_i + L$.

A. Particle-hole symmetry

We will now demonstrate that the results already established have further powerful implications on the evolution of the lowest eigenvalue with particle number in the presence of center-of-mass conservation as discussed above. A strikingly simple special instance of this is the case of $k = 2$-body interactions for fermions. In this case, the spatial symmetries of the problem often also imply a particle-hole symmetry, as we will now discuss.

We introduce the charge conjugation operator as a linear unitary operator C defined via $Cc_nC = c_n^\dagger$, where $C = C^\dagger$, or $C^2 = \mathbb{1}$. Consider now a two-body Hamiltonian H_2 as given in Eq. (1), and assuming that the interaction matrix element V_{n_1,n_2,n_3,n_4} is proportional to $\delta_{n_1+n_2,n_3+n_4}$, easy calculation gives

$$CH_2C = 2\sum_n \Delta_n - 4\sum_n \Delta_n c_n^\dagger c_n + H_2\,, \tag{16a}$$

where

$$\Delta_n = \sum_m V_{mnnm} \tag{16b}$$

We emphasize that even though there is no strict center-of-mass conservation on the torus, but only "modulo L", Eq. (16) is also obtained on the torus where the operators and η–form-factors have the aforementioned periodicity. Specifically, if we write a *translationally invariant* two-body interaction on the torus in the form (15) via

$$Q_R^m = \frac{1}{2}\sum_x \eta^m(x) c_{R-x} c_{R+x}\,, \tag{17}$$

where x runs over integer (half-odd-integer) values in the interval $[0, L)$ for integer (half-odd-integer) R, and $\eta^m(x)$ satisfies $\eta^m(x + L) = \eta^m(x)$ along with, for fermions, $\eta^m(x) = -\eta^m(-x)$, we find

$$\Delta_n = \frac{1}{4} \sum_m \sum_{x \in \frac{1}{2}\mathbf{Z}} |\eta^m(x)|^2 \equiv \Delta, \tag{18}$$

where the sum is over all integer *and* half-odd integer values in $[0, L)$. In particular, it is apparent that $\Delta_n \equiv \Delta$ does not depend on n at all. The same result can also be obtained in the presence of spherical symmetry, where again it can be shown that[26]

$$\Delta_n \equiv \Delta = \frac{1}{L} \sum_{m,n} V_{mnnm} \,. \tag{19}$$

We thus write Eq. (16) in its final form,

$$CH_2C = 2\Delta L - 4\Delta \hat{N} + H_2 \,, \tag{20}$$

which directly relates the spectrum at particle number N to that at $L - N$. We read off:

$$E_0(N) = E_0(L - N) + (4N - 2L)\Delta \,. \tag{21}$$

Eq. (21) applies not only to the ground state energy, but to the entire spectra at N and $L - N$, respectively, and is the manifestation of particle-hole symmetry of fermionic two-body interactions on the sphere or torus. It is straightforward to combine this last equation with the monotony result Eq. (11) into a new bound on the step size of energy with particle number:

$$E_0(N + 1) - E_0(N) \leq 4\Delta \,. \tag{22}$$

In the thermodynamic limit, this in particular constrains the chemical potential at zero temperature. We emphasize, however, that the validity of Eq. (22) is not limited to large system size. Also, we will in the following derive similar relations that can be applied to excitations. Hence we will refer to the quantity on the left hand side of this equation by the more generic term "energy step size" in the following.

Some remarks are in order to demonstrate that the above result is meaningful. We may, for example, consider the V_1 Haldane pseudo-potential on the torus, with coefficients normalized such that

$$\frac{1}{2} \sum_{x \in \mathbf{Z} \text{ or } x \in \mathbf{Z} + \frac{1}{2}} |\eta^1(x)|^2 \doteq 1 \,, \tag{23}$$

as befits a projection operator (note double counting due to the fact that x and $-x$ lead to identical terms in Eq. (17)). The $\doteq$ symbol signifies that on the torus, small deviations from the value of 1 appear due to the standard periodization of pseudo-potentials, which vanish in the thermodynamic limit and are not present on the sphere. In either case, in the thermodynamic limit, Eq. (18) gives $\Delta = 1$, owing to the fact that now x runs over both integer and half-odd integer values. In particular, the right hand side of Eq. (22) is of order unity.

Moreover, we observe that the inequality Eq. (22) may be saturated. If we sum over the first M odd Haldane pseudo-potentials, it is well known[27] that the resulting Hamiltonian satisfies Eq. (14) with N_I approaching $L/(M + 1)$. From Eq. (21), it is then clear that in this case,

$$E_0(N) = (4N - 2L)\Delta \,, \quad \text{for } N \geq L - N_I \,. \tag{24}$$

Therefore, Eq. (22) is the best possible bound on the energy step size that is uniform in N. Below we will see that a slight improvement is possible at the expense of bringing in more complicated, N-dependent coefficients. The main benefit of the following considerations is, however, their greater generality. We finally remark that the single-particle charge gap may be defined as $\Delta_c = E_0(N + 1) + E_0(N - 1) - 2E_0(N)$. For the special (Laughlin state) parent Hamiltonians discussed above, the energy then jumps from $E(N \leq N_I) = 0$ to the charge gap $E(N_I + 1) \equiv \Delta_c$ at the incompressible filling factor, and we have in particular obtained an upper bound on this charge gap:

$$\Delta_c \leq 4\Delta \,. \tag{25}$$

We note that this last relation is, in its functional form, reminiscent of results obtained via "single mode approximation".[28,29] However, it essentially complements the latter, which provide a variational upper bound on the *neutral* gap. It is further worth pointing out that the above was obtained solely by appealing to the two principles of ground state monotony and particle-hole symmetry, bypassing the need for the construction of clever variational wave functions.

B. Bosons, excited states, and duality

The bound Eq. (22) has the advantage of simplicity. However, some limitations thus far apply. So far, we have only considered particle-hole symmetric two-body interactions of fermions. Interestingly, a road to generalization of this result manifests itself if we first limit our attention to special model Hamiltonians, and inquire about the evolution, in N, of the first excited state for such N where zero modes are present (and the ground state energy thus vanishes exactly). It turns out that this question can be investigated with methods similar to that of Sec. II, thanks to the following fortuitous circumstance. It has been pointed out that the zero mode space $\mathcal{H}_Z$ of the Hamiltonian is generally invariant under the action of destruction operators c_n (see, e.g., Refs. 20 and 21). Indeed, this follows from the zero mode condition $Q_R^m |\psi\rangle = 0$ for all n, R, and from the commutation relation $[Q_R^m, c_n] = 0$. Much less appreciated seems to be the fact that there is a dual version of this statement: Let $\mathcal{H} = \mathcal{H}_Z \oplus \mathcal{H}_{NZ}$ be the decomposition of the

Hilbert space into the zero mode subspace and its orthogonal complement, the latter being spanned by all finite energy eigenstates. Then in fact $\mathcal{H}_{NZ}$ is invariant under the action of all creation operators $c_n^\dagger$. For, if $|\psi\rangle \in \mathcal{H}_{NZ}$, and $|\phi\rangle$ is any zero mode, then $c_n|\phi\rangle$ is also a zero mode. Thus $\langle\psi|c_n|\phi\rangle = 0 = \langle\phi|c_n^\dagger|\psi\rangle$. So $c_n^\dagger|\psi\rangle$ is orthogonal to any zero mode. Thus $c_n^\dagger|\psi\rangle \in \mathcal{H}_{NZ}$ if $|\psi\rangle$ is.

We now consider $E_1(N,L)$, the lowest *non-zero* eigenenergy for given particle number N, where in the following, we will make both the N- and the L-dependence explicit. Note that for such N where there are no zero modes, $E_1(N,L) = E_0(N,L)$, and the following considerations equally apply to this situation. In general, $E_1(N,L)$ is the ground state energy of $\mathcal{H}_{NZ}$ for fixed N, and the invariance of $\mathcal{H}_{NZ}$ under the creation operators $c_n^\dagger$ allows us to proceed in a manner that parallels the considerations of Sec. II, except stepping up in particle number instead of stepping down.

To this end, we restrict ourselves for now to two body-interactions of fermions, which we simply denote by $H_2 \equiv H$ below. For such interactions, we note the identity

$$(L-\hat{N})H = \sum_n c_n c_n^\dagger H = 2H - 4\Delta\hat{N} + \sum_n c_n H c_n^\dagger\,, \quad (26)$$

where again center-of-mass conservation and symmetries have been used to extract the term Δ, Eq. (19). It is then straightforward to proceed along the lines of Eqs. (3)-(10), where only the concept of a reduced density matrix Eq. (2) must be replaced by a dual counterpart of an "enlarged" (in particle number) density matrix,

$$\rho_{N'+1} = \frac{1}{L-N'}\sum_n c_n^\dagger \rho_{N'} c_n\,. \quad (27)$$

This then leads to the relation

$$E_1(N+1,L) - \frac{L-N-2}{L-N}E_1(N,L) \le \frac{4N}{L-N}\Delta\,. \quad (28)$$

We emphasize once more that Eq. (28) describes both the first excited state energy in the presence of zero modes, as well as, in the absence of the latter, the ground state energy. As far as this second application is concerned, it is quite similar to Eq. (22), as the coefficient on the left hand side is close to unity for large $L-N$, and the bound on the left hand side even represents an improvement over Eq. (22) for $N < L/2$. One may again check that Eq. (28) is saturated in the regime discussed in and around Eq. (24).

Eq. (28) has been derived by taking the expectation value of Eq. (26) in the ground state of $\mathcal{H}_{NZ}$. If instead we again consider $N = N_I$, the largest N for which $E_0(N,L) = 0$, and take the expectation value of Eq. (26) for the corresponding zero mode ground state, we obtain the following upper bound on the charge gap:

$$\Delta_c \le \frac{4N_I}{L-N_I}\Delta\,, \quad (29)$$

which is usually (for $N_I/L < 2$) an improvement over Eq. (25).

We emphasize that while in deriving Eq. (26), we used the same symmetries that lead to particle-hole symmetry for fermions, particle-hole symmetry does itself not seem to play any essential role here. To make this point, we now derive analogous results for two-body interactions of bosons. In this case, the analog of Eq. (26) is given by

$$(L+\hat{N})H = -2H - 4\Delta\hat{N} + \sum_n c_n H c_n^\dagger\,. \quad (30)$$

This then leads in an analogous manner to

$$E_1(N+1,L) - \frac{L+N+2}{L+N}E_1(N,L) \le \frac{4N}{L+N}\Delta\,. \quad (31)$$

This is again similar in spirit to the "step size" equation (22), and, in addition to generalizing the latter to bosons, has the same benefit as Eq. (28), applying also to the first excited state in the presence of zero modes. We may now further generalize Eq. (29) to bosons via

$$\Delta_c \le \frac{4N_I}{L+N_I}\Delta\,. \quad (32)$$

We note one more subtle difference between Eqs. (28) and (31). In Eq. (28), the positive term $2E_1/(L-N)$ may always be dropped if desired, in order to bound the step size more directly. This is not immediately possible in Eq. (31), where a similar term appears with opposite sign. While this term appears innocent at first, it gets somewhat out of control when E_1 approaches order L. A priori, we do not know when that happens. However, it is easy enough to use Eq. (31) in order to bound $E_1(N,L)$ directly. To demonstrate this, let us focus on the region $N > N_I$. In this case, we prove from Eq. (31) by easy induction that

$$E_1(N,L) \le \frac{2N(N-1)}{L+1}\Delta \quad (N > N_I)\,, \quad (33)$$

using Eq. (32) with $N = N_I + 1$ as the starting point of the induction. It is clear that Eq. (33) may be much improved if N_I is appreciably larger than 1. Proceeding with the general Eq. (33), however, we in particular see that

$$E_0(N,L) \le 2L\Delta \quad (N < L)\,, \quad (34)$$

where we note again that E_1 and E_0 are defined to be the same for $N > N_I$. This in Eq. (31) actually reproduces the original step size equation (22), now for bosons, with the additional restriction of $N \le L$, i.e., filling factor no greater than 1.

In closing this section, we evaluate the bound (29) for the important special case of the V_1 Haldane-pseudopotential on a sphere threaded by $N_\Phi = L-1$ flux quanta, which stabilizes a $\nu = 1/3$ Laughlin state of N_I particles, where $N_\Phi = 3(N_I-1)$. Table I summarizes our results. Note that as defined above Eq. (25), the charge

N	L	Δ	$(\Delta_c)_{ub}$	$(\Delta_c)_{ED}$	$\frac{(\Delta_c)_{ub}}{(\Delta_c)_{ED}}$
5	10	0.8500	2.2667	1.6406	1.38
6	13	0.8846	2.2115	1.4600	1.52
7	16	0.9063	2.1750	1.4208	1.53
8	29	0.9211	2.1491	1.3692	1.57
9	22	0.9318	2.1299	1.3367	1.60
10	25	0.9400	2.1150	1.3100	1.62

TABLE I. Charge gap for a sphere threaded by $N_\Phi = L - 1$ flux quanta, at the incompressible filling factor of the V_1 Haldane-pseudopotential (see text). N represents particle number, the parameter Δ defined by Eq. (19) equals $1 - 3/(2L)$ for the V_1 pseudopotential, $(\Delta_c)_{ub}$ is the upper bound on the charge gap as given by Eq. (29), and $(\Delta_c)_{ED}$ is the actual charge gap as determined by exact diagonalization.

gap Δ_c corresponds to a state of $N = N_I + 1$ particles, or the insertion of three quasi-particles into the Laughlin state, which must be well separated before the thermodynamic limit is reached. While this is not quite the case for the system sizes shown in Table I yet, we emphasize that the bounds derived here apply equally well to finite particle number. Moreover, one may be confident from the data given that even in the thermodynamic limit, our upper bound Eq. (29) overestimates the charge gap by less than a factor of 2. This seems quite reasonable, given the great generality of the bounds derived here. As we stressed above, these bounds are saturated in certain cases, hence there is not much room for improvement at this level of generality. In this light, the fact that Eq. (29) is within less than a factor of 2 of the actual gap seems quite satisfactory.

C. Considerations for the disk

Most of the above results, except for the general monotony Eq. (11), are not in any obvious way applicable or sensible in the infinite disk geometry, where $L = \infty$. In this case, the Hilbert space of any rotationally invariant Hamiltonian nonetheless decomposes into finite dimensional subspaces of given particle number N *and* given angular momentum $\mathcal{L}_z$. The lowest energy $E_0(N, \mathcal{L}_z)$ then satisfies a fairly obvious monotony relation in the angular momentum variable $\mathcal{L}_z$, which we wish to mention here for completeness.

In the disk geometry, a decomposition into center-of-mass and relative degrees of freedom is possible, and any Hamiltonian with translational and rotational invariance will decouple from the center-of-mass degrees of freedom. In this context, it is useful to introduce ladder operators a_i, $a_i^\dagger$, $[a_i, a_j^\dagger] = \delta_{ij}$ such that $a_i^\dagger a_i$ is the angular momentum of the i-th particle. We stress that these operators are very different from the "second-quantized" operators c_n, which carry orbital indices and preserve the symmetry of the wave function. In contrast, the a_i carry particle indices like any first quantized single particle operators, thus not by themselves preserving the symmetry of the wave-function, which is also not in any way encoded in the commutation relations of the a_i. As a result, the following is independent of particle statistics.

In this description, the relative degrees of freedom are (over-)completely described by the operators $a_i - a_j$ and their Hermitian adjoints. The total angular momentum operator may be decomposed into a center-of-mass part and a relative part, respectively, via

$$\mathcal{L}_z = \mathcal{L}_z^{\mathsf{C}} + \mathcal{L}_z^{\mathsf{rel}} \,. \tag{35}$$

The operator $b = \frac{1}{\sqrt{N}} \sum_i a_i$ and its adjoint $b^\dagger$ commute with all $a_i - a_j$, $a_i^\dagger - a_j^\dagger$, and thus with the Hamiltonian and with $\mathcal{L}_z^{\mathsf{rel}}$. Clearly, also, $b^\dagger$ raises $\mathcal{L}_z = \sum_i a_i^\dagger a_i$, and thus $\mathcal{L}_z^C$, by 1. $b^\dagger$ and b are thus ladder operators for the center-of-mass part of the angular momentum. From the commutation relation

$$[b, b^\dagger] = 1 \tag{36}$$

it follows that $b^\dagger$ cannot annihilate any non-zero ket of the Hilbert space. From the above it thus follows that the entire spectrum $\Sigma_{\mathcal{L}_z}$ for given value of $\mathcal{L}_z$ is contained in that for $\mathcal{L}_z + 1$, $\Sigma_{\mathcal{L}_z+1}$: $b^\dagger$ always raises the value of $\mathcal{L}_z$ while keeping the energy the same. This in particular implies the monotony

$$E_0(N, \mathcal{L}_z) \geq E_0(N, \mathcal{L}_z + 1) \,. \tag{37}$$

The above is simply a manifestation of center-of-mass degeneracy in the infinite plane, and is not too surprising. Note that unlike when using conjugate magnetic translations to establish a similar degeneracy (cf., e.g., Ref. 25), equality of the spectra at $\mathcal{L}_z$ and $\mathcal{L}_z + 1$ does not follow, since b can, in general, annihilate non-zero kets. However, $\Sigma_{\mathcal{L}_z}$ is identical to the spectrum associated to the subspace having angular momentum $\mathcal{L}_z + 1$ and $\mathcal{L}_z^C > 0$. In determining the full spectrum, it is thus sufficient to focus on the subspaces characterized by $\mathcal{L}_z^C = 0$ and all possible values for $\mathcal{L}_z = \mathcal{L}_z^{\mathsf{rel}}$. A more interesting question is thus whether the ground state energy $E_0(N, \mathcal{L}_z, \mathcal{L}_z^C = 0)$ as a function of given N, $\mathcal{L}_z$ and subject to the constraint $\mathcal{L}_z^C = 0$ satisfies a monotony similar to Eq. (37). We leave detailed analysis as an interesting problem for the future.

We emphasize that the above is true exactly only when no cutoff in orbital space is imposed, other than what naturally follows from fixing total angular momentum (i.e. for bosons, no orbitals with $n > \mathcal{L}_z$ allowed, and a correspondingly lower cutoff for fermions). Since, for example, in numerical calculations the second quantized framework used throughout most of this paper may be deemed preferable, we will give second quantized expressions for the operators b, $b^\dagger$, and the various components of angular momentum appearing in Eq. (35).

Clearly, the operators b, $b^\dagger$ are single body operators changing the total angular momentum by ± 1 while preserving energy. In particular, they preserve zero modes

of special Hamiltonians. This last circumstance allows us to make contact with our recent work,[19–21] where a class of second quantized single body operators was discussed, for various geometries, that preserve zero modes. These operators $\mathcal{O}_d$ are labeled by an integer d, and in disk geometry, raise $\mathcal{L}_z$ by d. Up to some arbitrary normalization which we will fix here for our purposes, the operator $\mathcal{O}_1$ for the disk is given by[19]

$$\mathcal{O}_1 = \sum_{n=0}^{\infty} \sqrt{n+1}\, c_{n+1}^\dagger c_n \,. \tag{38}$$

It is thus natural to assume that $b^\dagger$ is proportional to this operator. Indeed, the action of $\mathcal{O}_1$ can be seen,[19,21] at first quantized level, to correspond to multiplication of an analytic wave function with a factor proportional to $\sum_{i=1}^N z_i$. Here, $z_i = x_i + iy_i$ as usual. The same can easily be established for the operator $b^\dagger$. Hence up to normalization, these operators are the same. One easily verifies $[\mathcal{O}_1^\dagger, \mathcal{O}_1] = \hat{N}$, such that comparison with Eq. (36) gives

$$b^\dagger = \frac{1}{\sqrt{\hat{N}}} \mathcal{O}_1 \,. \tag{39}$$

Alternatively, one may verify directly, if desired, that Eq. (38) commutes with all Haldane pseudo-potentials, and thus only acts on center-of-mass degrees of freedom. The operators for various aspects of angular momentum are then given by

$$\mathcal{L}_z = \sum_{n=0}^{\infty} n\, c_n^\dagger c_n \,, \tag{40}$$

which is obvious, furthermore

$$\mathcal{L}_z^C = b^\dagger b \,, \tag{41}$$

which is, up to terms proportional to particle number, a two-body operator, and $\mathcal{L}_z^{\mathsf{rel}}$ is then obtained from Eq. (35).

IV. DISCUSSION AND CONCLUSION

In this paper, we have been interested in bounds describing the evolution in particle number of low energy spectral properties for both general and "special" positive two-body interaction Hamiltonians. Even in the latter, "special" case, we have accessed properties which are beyond the zero mode subspace that renders these Hamitonian special. We have thus naturally employed a second quantized framework, which we find superior for addressing any properties not directly related to zero modes, whether or not the latter are present.

Our starting point has been the general monotony of the ground state energy in particle number, for any positive interaction Hamiltonian. We then asked what additional information can be obtained using strategies related to those used in the proof of this monotony property when additional assumptions applying to a wide class of fractional quantum Hall model Hamiltonians are made. Specifically, we have focused on two-body interactions with center-of-mass conservation. For fermions, in many situations of interest one also has a particle-hole symmetry, and we observed that this alone can be used to immediately translate the monotony property into a general bound on the energy step size, giving in particular an upper bound on the charge gap at the "incompressible filling factor" of special model Hamiltonians. This bound was subsequently improved and generalized to bosons. We also used a dual argument to obtain similar bounds for the first excited state in the presence of zero modes, thus showing that for special model Hamiltonians possessing the latter, non-trivial statements are possible even for excited states at non-zero energy.

While we have been mostly concerned with compact geometries, in particular the torus and sphere, we have also commented on the situation in the infinite disk geometry, where center-of mass degeneracy leads to an obvious monotony property as function of angular momentum. In this context, we also commented on aspects of center-of-mass and relative-coordinate decomposition in the framework of second quantization.

We are hopeful that these results will spur further development regarding exact properties beyond zero modes in model Hamiltonians of fractional quantum Hall states and related systems.

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