Slow quenches in a quantum Ising chain: Dynamical phase transitions and topology

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Slow quenches in a quantum Ising chain; dynamical phase transitions and topology

Shraddha Sharma,[1] Uma Divakaran,[2] Anatoli Polkovnikov,[3] and Amit Dutta[1]

[1]*Department of Physics, Indian Institute of Technology, Kanpur 208 016, India*
[2]*UM-DAE Center for Excellence in Basic Sciences, Mumbai 400 098, India*
[3]*Department of Physics, Boston University, 590 Commonwealth Ave., Boston, MA 02215, USA*

We study the slow quenching dynamics (characterized by an inverse rate, τ^{-1}) of a one-dimensional transverse Ising chain with nearest neighbor ferromagentic interactions across the quantum critical point (QCP) and analyze the Loschmidt overlap measured using the subsequent temporal evolution of the final wave function (reached at the end of the quenching) with the final time-independent Hamiltonian. Studying the Fisher zeros of the corresponding generalized "partition function", we probe non-analyticities manifested in the rate function of the return probability known as dynamical phase transitions (DPTs). In contrast to the sudden quenching case, we show that DPTs survive in the subsequent temporal evolution following the quenching across two critical points of the model for a sufficiently slow rate; furthermore, an interesting "lobe" structure of Fisher zeros emerge. We have also made a connection to topological aspects studying the dynamical topological order parameter ($\nu_D(t)$), as a function of time (t) measured from the instant when the quenching is complete. Remarkably, the time evolution of $\nu_D(t)$ exhibits drastically different behavior following quenches across a single QCP and two QCPs. In the former case, $\nu_D(t)$ increases step-wise by unity at every DPT (i.e., $\Delta\nu_D = 1$). In the latter case, on the other hand, $\nu_D(t)$ essentially oscillates between 0 and 1 (i.e., successive DPTs occur with $\Delta\nu_D = 1$ and $\Delta\nu_D = -1$, respectively), except for instants where it shows a sudden jump by a factor of unity when two successive DPTs carry a topological charge of same sign.

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I. INTRODUCTION

The phase transition in a thermodynamic system is marked by non-analyticities in the free-energy density which can be detected by analyzing the zeroes of the partition function in a complex temperature plane as proposed by Fisher[1]; a similar proposal was also given earlier in the presence of a complex magnetic field[2] (See also [3]). When the line of Fisher zeros cross the real axis, there are non-analyticities in the free-energy density which signal the existence of a finite temperature phase transition in the thermodynamic limit.

In a recent work, Heyl *et al.*[4], introduced the notion of *dynamical phase transitions* (DPTs) exploiting the formal similarity between the canonical partition function $Z(\beta) = \mathrm{Tr}\ e^{-\beta H}$, of an equilibrium system described by a Hamiltonian H (where β is the inverse temperature) and that of the overlap amplitude or Loschmidt overlap (LO) defined for a quantum system which is suddenly quenched. Denoting the ground state of the initial Hamiltonian as $|\psi_0\rangle$ and the final Hamiltonian reached through the quenching process as H, the LO is defined as $G(t) = \langle\psi_0|e^{-iHt}|\psi_0\rangle$. Generalizing $G(t)$ to $G(z)$ defined in the complex time (z) plane, one can introduce the notion of a dynamical free energy density, $f(z) = -\lim_{L\to\infty} \ln G(z)/L^d$, where L is the linear dimension of a d-dimensional system.

In a spirit similar to the classical case, one then looks for the zeros of the $G(z)$ (or non-analyticities in $f(z)$) to define a dynamical phase transition. For a transverse Ising chain, it has been observed[4] that when the system is suddenly quenched across the quantum critical point (QCP)[5,6], the line of Fisher zeros crosses the imaginary time axis at instants t^*; at these instants the rate function of the return probability defined as $I(t) = -\ln|G(t)|^2/L$ shows sharp non-analyticities.

The initial observation by Heyl *et al.*[4] was verified in several subsequent studies[7–12] which established that similar DPTs are observed for sudden quenches across the QCP for both integrable and non-integrable models. Subsequent works however showed that DPTs can occur following a sudden quench even within the same phase (i.e., not crossing the QCP) for both integrable[13] as well as non-integrable models[14]. Furthermore, DPTs have been explored in two-dimensional systems[15,16] where topology of the equilibrium system plays a non-trivial role[15]; also, the notion of a local dynamical topological order parameter (DTOP) which assumes integer values and changes by unity at every DPT has been proposed[17]. We note in the passing that the rate function $I(t)$ is related to the Loschmidt echo which has been studied in the context of decoherence at zero[18–26] and also at finite temperatures[27] and has also been useful in studies of the work-statistics[28] and the entropy generation[29–31] in quenched quantum systems. In fact, the rate function (of the return probability) discussed above in the context of DPTs can be connected to the singularities in the work distribution function corresponding to the zero work following a double quenching experiment[4]; in this process, the initial Hamiltonian is suddenly quenched to the final Hamiltonian at $t = 0$ and then quenched back to the initial one at a time t.

It should be noted that the periodic occurrences of non-analyticities in the rate function for an integrable model was first reported in the reference [32], in the context of a slow quenching of the transverse Ising chain though the connection to DPTs and Fisher zeros remain unexplored. In the present work, we shall address that particular issue and probe the behavior of the line of Fisher zeros following slow quenches (dictated by a rate τ^{-1}) of integrable spin chains across their QCPs; we shall also connect this to point out the occurrences of such DPTs or non-analyticities manifested in the rate function of the return probability. Furthermore, we address the question concerning the fate of these DPTs when the transverse Ising chain is slowly driven across both the QCPs of the model; it is noteworthy that for the sudden quenching, the passage through two critical points wipes out the non-analyticities in the rate function[4]. We note at the outset that in the context of the slow quenching there are two times denoted by $\tilde{t}$ and t, respectively, appearing in the discussion here: $\tilde{t}$ is related to the ramping of a parameter of the Hamiltonian (namely, the transverse field h) using the quenching protocol $\tilde{t}/\tau$ to reach the final state $|\psi_f\rangle$ at a desired final value of the field h_f; the other one, denoted by t, that appears in $G(t)$ describes the subsequent temporal evolution of the state $|\psi_f\rangle$ with the time-independent final Hamiltonian H_f. Hence, once the linear-ramping (with time $\tilde{t}$) of the parameter is complete, we set the time $t = 0$, and probe the non-analyticities in $I(t)$ as a function of t (analytically continuing it to the complex time plane). In other words, the role of the slow quenching is to prepare the system in a desired state; we then probe the time evolution of this state with the final Hamiltonian as a function of t. Secondly, we study Fisher zeros numerically using a finite system, hence they do not really form a line, rather generate a set of closely spaced points.

The study of slow quenching dynamics has gained importance in recent years because of the possible Kibble-Zurek (KZ) scaling[33,34] of the defect density and the residual energy following a quench across (or to) a QCP by tuning a parameter of the Hamiltonian slowly[35,36]. This scaling has been verified and also modified in various situations[37–47]. (For reviews, see [48–50].)

The paper is organized in the following manner: in Sec II, we discuss a transverse Ising chain driven across its QCP by varying the transverse field following a linear protocol; studying the behavior of Fisher zeros we determine the location of non-analyticities in the rate function. We study both the situations when the transverse field is quenched across a single critical point and both the critical points of the model, as shown in Figs. 1 and 2, respectively, and compare the results with those of the sudden quenching. Finally, in Sec. III, we present the rich behavior that emerges from the study of the variation of the DTOP ($\nu_D(t)$) as a function of time (t) following a slow quench and explore the topology of DPTs which is dictated by the topological properties of the equilibrium system. We investigate both the situations, when starting from a non-topological phase, the system is quenched to a topological phase (crossing a single QCP) or to the other non-topological phase across both the QCPs separating the topological phase and the non-topological phases; the behavior of the DTOP is significantly different in these two situations. In the former case, $\nu_D(t)$ increases in a step-like fashion by unity at every DPT while in the latter it oscillates between zero and unity except at the special instants when there are two successive DPTs with the same sign of the topological charges; in that situation ν_D shows a discrete jump of unit magnitude.

II. TRANSVERSE ISING CHAIN AND QUENCHING OF THE TRANSVERSE FIELD

In this section, we shall study the slow dynamics of a transverse Ising chain and illustrate the flow of the line of the Fisher zeros studying the temporal evolution as a function of time t following such a quench. We shall show that the Fisher zeros cross the imaginary time axis in the complex time plane for a particular momentum leading to non-analyticites of the rate function at the corresponding real time. Let us first consider a ferromagnetic transverse Ising chain described by the Hamiltonian

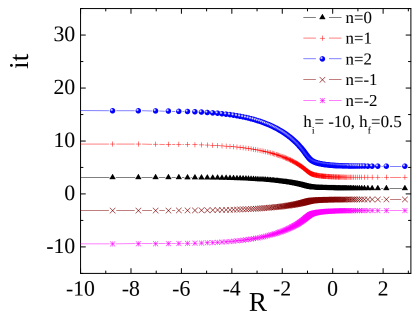
$$H = -J_x \sum_i \sigma_i^x \sigma_{i+1}^x - h \sum_i \sigma_i^z, \tag{1}$$

where σ_i's are the Pauli spin matrices satisfying the standard commutation relations, J_x is the ferromagnetic nearest neighbor interactions in the x directions (set equal to unity in the subsequent discussion) and h denotes the strength of the non-commuting transverse field. The model being translationally invariant (and hence momentum k being a good quantum number), we use Fourier transformation and employ Jordan-Wigner transformation to map spins to spin-less fermions. Furthermore noting that the 'parity' (of the number of Jordan-Wigner fermions) is conserved, one can arrive at decoupled 2×2 Hamiltonians for each mode k in the basis $|0\rangle$ (no fermion) and $|k, -k\rangle$ (with two fermions with quasi-momenta k and $-k$, respectively) given by[50]

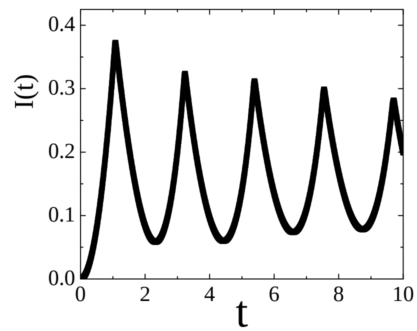
$$H_k = \begin{pmatrix} -h + \cos k & -i \sin k \\ i \sin k & h - \cos k \end{pmatrix}. \tag{2}$$

Analyzing the gap in the energy spectrum ($2\epsilon_k = 2\sqrt{(h - \cos k)^2 + \sin^2 k}$) of the Hamiltonian, it can be shown that the model has two QCPs at $h = \pm 1$ where the gap vanishes for the momentum modes $k = 0$ and $k = \pi$, respectively. These QCPs separate the ferromagnetic phase (for $|h| < 1$) from the paramagnetic phase.

For all the slow quenching schemes to be analyzed in the subsequent discussions, we shall assume that the system is initially prepared in the ground state of the initial Hamiltonian. We first consider a linear quenching of the transverse field $h = \tilde{t}/\tau$, with $\tilde{t}$ going from a large negative value (i.e., the initial field h_i is large and



(a)



(b)

FIG. 1: (Color online) (a) Fisher zeros as obtained from Eq. (4) are plotted in complex z plane for different n, where R denotes Re (z) and it is the Im (z). The line of Fisher zeros cross the imaginary axis for a momentum value k_*. (b) Sharp non-analyticities in the rate function are observed periodically at (real) times t_n^* (Eq. (6)) calculated at the momenta values k_* as shown in the panel (a). Here we have followed the quenching protocol $h(\tilde{t}) = \tilde{t}/\tau$ with $\tau = 1$, when the field is quenched from $h = -\infty$ to $h = 0.5$ crossing the QCP at $h = -1$. The time t is measured from the instant immediately after the quenching is complete, when we set $t = 0$.

negative) to a desired final value so that $h_f < 1$ (chosen to be equal to 0.5 here); hence, the system crosses the QCP at $h = -1$ where the relaxation time diverges leading to the breakdown of the adiabatic evolution and hence the final state ($|\psi_f\rangle$) reached is not the ground state of the final Hamiltonian (H_f). Considering the reduced Hamiltonian (2), one can define the dynamical free energy[4]: $f(z) = -\ln\langle\psi_f|\exp(-H_f z)|\psi_f\rangle/L$, where $\langle\psi_f|\exp(-H_f z)|\psi_f\rangle$ is the Loscmidt overlap while z is the complex time. It is to be noted that we have set $\hbar = 1$ throughout.

Let us first illustrate the slow quenching scheme following the protocol $h = \tilde{t}/\tau$, with the initial state is $|k, -k\rangle$ for very large negative h. (In this limit, the reduced 2×2 Hamiltonian in Eq. (2) effectively becomes diagonal and hence the ground state, for the mode k, becomes $|k, -k\rangle$; on the other hand, for large positive h, the ground state is $|0\rangle$. Other than these extreme cases, the ground state is in general a linear superposition of these two basis states.) The final state reached after the ramping (for the k-th mode) can then be written as $|\psi_{f_k}\rangle = v_k|1_k^f\rangle + u_k|2_k^f\rangle$, with $|u_k|^2 + |v_k|^2 = 1$: here, $|1_k^f\rangle$ and $|2_k^f\rangle$ are the ground state and the excited states of the Hamiltonian $H_k(h_f)$ with energy eigenvalues $-\epsilon_k^f$ and ϵ_k^f, respectively. Clearly $|u_k|^2 = |\langle 2_k^f|k, -k\rangle|^2 = p_k$ denote the non-adiabatic transition probability that the system ends up at the excited state after the quench.

Once the quenching is complete, we bring in the second time t, set equal to zero, and study the evolution of the state $|\psi_{f_k}\rangle$ with the final Hamiltonian $H_k(h_f)$; generalizing to the complex time plane z, the LO for the k-th mode ($L_k = \langle\psi_{f_k}|\exp(-H_k(h_f)z)|\psi_{f_k}\rangle$) is then given by $(|v_k|^2 + |u_k|^2\exp(-2\epsilon_k^f z))$. Here, we have rescaled the ground state energy of the final Hamiltonian $H_k(h_f)$ to 0, so that the excited state energy $\epsilon_f^{\text{excited}} = 2\epsilon_k^f$. Summing over the contributions from all the momenta modes and converting summation to the integral, in the thermodynamic limit we obtain:

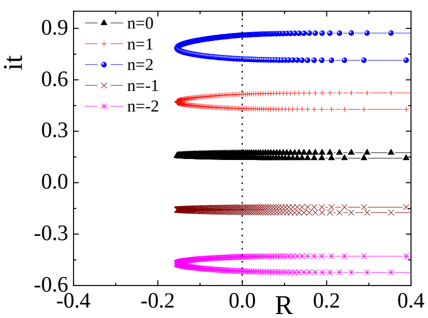
$$
\begin{aligned}
f(z) &= -\int_0^\pi \frac{dk}{2\pi}\ln\left(|v_k|^2 + |u_k|^2\exp(-2\epsilon_k^f z)\right) \\
&= -\int_0^\pi \frac{dk}{2\pi}\ln\left((1-p_k) + p_k\exp(-2\epsilon_k^f z)\right). \quad (3)
\end{aligned}
$$

We then immediately find the zeros (i.e., the Fisher zeros) of the "effective" partition function given by:

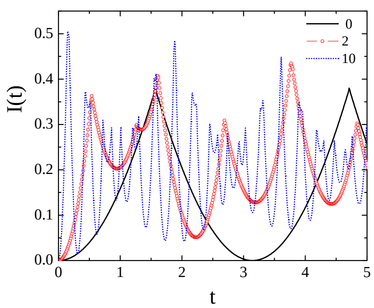
$$
z_n(k) = \frac{1}{2\epsilon_k^f}\left(\ln(\frac{p_k}{1-p_k}) + i\pi(2n+1)\right), \quad (4)
$$

where $n = 0, \pm1, \pm2, \cdots$.

The Fisher zeros following the slow quenching across one critical point and terminating the driving at $h_f = 0.5$ are shown in Fig. 1(a). Let us now probe the question whether the line of Fisher zeros cross the imaginary axis for a particular momenta mode k_*. Clearly, this happens when $p_{k_*} = 1/2$ (when the Re$(z_n(k))$ vanishes); this in fact corresponds to "infinite temperature" state when both the levels of the two-level system are equally populated. Furthermore, for the critical mode $k = \pi$ (which does not evolve in the process of quenching) $p_k = 1$ while in the case of high energy modes (i.e., modes close to $k \sim 0$), $p_k \to 0$ (see discussion around Eq. (8) and Fig. 3(a)), since according to KZ theory, modes $k > \tilde{k} \sim \tau^{-1/2}$ move adiabatically, i.e., do not sense the passage through the QCP. Using Eq. (4), we therefore conclude that $z_n(k \to 0) \to -\infty$ while for $z_n(k \to \pi) \to +\infty$. This immediately implies that $z_n(k)$ indeed goes from ∞ to $-\infty$, crossing the imaginary time axis for a particular k_* as shown in Fig. 1(a). (It should be noted that numerically we are dealing with a finite system and hence Fisher zeros do not really go from $-\infty$ to $+\infty$ but they indeed cross the imaginary axis for k_*.) An exact expression for the rate function $I(t)$, (where, we reiterate, the time t is measured after the quenching is complete) can be exactly derived in the present case,



(a)



(b)

FIG. 2: (Color online) (a) Lines of Fisher zeros for each integer n following quenching across two QCPs when h is quenched from $h_i = -10$ to $h_f = 10$ with $h(\tilde{t}) = \tilde{t}/\tau$ and $\tau = 1$. Corresponding to each lobe denoted by n, we find two k_*s close to two different critical modes ($k = 0$ and π); consequently, there exist two t_n^*s which we shall denote by t_n^+ and t_n^-, with $t_n^+ < t_n^-$. (b) The figure shows the corresponding non-analyticities in $I(t)$ for different final values of h ($h_f = 0, 2, 10$) at different instants of times which match perfectly with those predicted by Eq. (6) for $h_f = 0$ and Eq. (7) for $h_f = 2, 10$.

$$I(t) = -\int_0^{\pi} \frac{dk}{2\pi} \ln\left(1 + 4p_k(p_k - 1)\sin^2 \epsilon_k^f t\right). \qquad (5)$$

As presented in Fig. 1(b), the non-analyticities in $I(t)$ appear at the values of the real time t_n^*s

$$t_n^* = \frac{\pi}{\epsilon_{k_*}^f}\left(n + \frac{1}{2}\right), \qquad (6)$$

derived by setting $\mathrm{Re}(z_n(k_*)) = 0$ in Eq. (4) because the argument of log in Eq. (5) vanishes when $k = k_*$ and $t = t_n^*$.

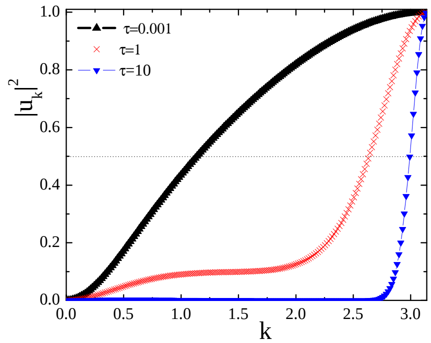
We then consider the case when the transverse field is quenched from a large negative value to a large positive value with a rate τ^{-1}, crossing both the QCPs in the process; the results are presented in Fig. 2. The field is varied as $h(\tilde{t}) = \tilde{t}/\tau$ so that the system crosses the critical point at $h = -1$ at time $\tilde{t} = -\tau$ and that at $h = 1$ at time $\tilde{t} = \tau$. At the end of the ramp, the spin chain is in the state $|\psi_{f_k}\rangle = v_k|1_k^f\rangle + u_k|2_k^f\rangle$, (with $|u_k|^2 + |v_k|^2 = 1$); the variation of $|u_k|^2$ as a function of k is shown in the Fig. 3(b). As $h(\tilde{t})$ is varied the system detects both these critical points which are gapless at different k values (0 and π). For $\tau \gg 1$, the transition probability p_k is close to unity for both $k \to 0, \pi$ and falls off exponentially for $k \gg 0$ or $k \ll \pi$.

We now proceed to analyze the Fisher zeros and probe the non-analyticties in the rate function $I(t)$, again setting $t = 0$ when the ramping is complete; the LO derived using the time evolution of $|\psi_{f_k}\rangle$ with $H_k(h_f)$ generalizing the real t to the complex z plane. As shown in Fig. 3(b), the profile of p_k clearly marks the fact that one is expected to obtain two separate k_*'s owing to these two individual critical points which can be treated independent of each other. Hence, although $z_n(k) \to \infty$ for both $k = 0$ and $k = \pi$, there exist two intermediate values k_* (one close to $k \to 0$ and other close to $k \to \pi$) for which $p_k = 1/2$. For these two k_* values, the line of Fisher zeros will indeed cross the imaginary axis marking DPTs at two different instants of time for each value of the integer n (Eq. (6)). This is numerically verified and shown in Figs. (2(a)) and (2(b)). Interestingly, we note that corresponding to each lobe (denoted by n), there are two distinct time instants $t_n^* = t_n^{\pm}$ (where Fisher zeros cross the imaginary axis) emerging due to the passage through two QCPs with

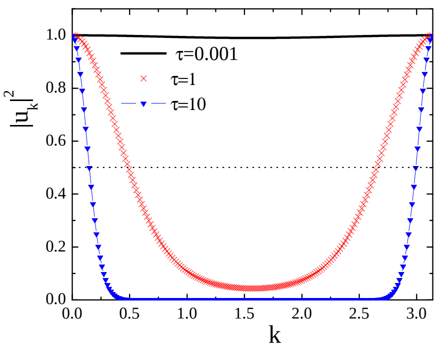
$$t_n^+ = \frac{(2n+1)\pi}{2\epsilon_{(\pi - k_*)}^f}$$
$$t_n^- = \frac{(2n+1)\pi}{\epsilon_{k_*}^f} \quad . \qquad (7)$$

The DPT corresponding to the quench across the QCP at $h = -1$ occurs at t_n^+ while the instant t_n^- is associated with the QCP at $h = 1$ and $t_n^- > t_n^+$. (The physical interpretation of this notation will be clear following the discussion of the next section where a positive (negative) topological charge will be attributed to a DPT occurring at $t_n^{(+)}(t_n^{(-)})$.) In the case of quenching across a single QCP in Fig. 1(a), we only get zeros corresponding to $t_n^* = t_n^+$. We can contrast this result with the case where h is changed abruptly from a large positive to a negative value across both the critical points, one can straightway verify that the lines of Fisher zeros never cross the imaginary axis and hence no DPT is observed as predicted in the earlier study by Heyl et al.[4].

The slow quenching scheme analyzed here brings in another time scale in the problem, namely, the inverse quenching rate τ. A pertinent question at this point would be how does the scenario depicted in Figs. 1 and 2 gets altered when τ is varied. Let us first consider the case of crossing a single QCP at $h = -1$ (Fig. 1), where gap vanishes for the critical mode $k = \pi$; expanding around this mode, one can rewrite the reduced 2×2



(a)



(b)

FIG. 3: (Color online) Numerically obtained values of the adiabatic transition probability (or the probability of the excited state) $p_k = |u_k|^2 = |\langle\psi_{f_k}|2_f^k\rangle|^2$, for quenching across a single QCP (Fig. (a)) and two QCPs (Fig. (b)) plotted as a function of k for several values of τ. The horizontal line denotes $p_k = 1/2$. Fig. (a) confirms that the value of k_* depend on τ and there exists a value of k_* even in the limit $\tau \to 0$. On the other hand, as shown in Fig. (b), there are two k_* for each τ leading to two instants of time t_n^+ and t_n^- where DPTs occur for each lobe denoted by n as shown in Fig. 2. The value k_* depend on τ and in the limit $\tau \to 0$, the situation becomes similar to the sudden quenching case as $|u_k|^2 > 1/2$ for all values of k and hence no k_* exists.

Hamiltonian

$$H_k = \begin{pmatrix} -h - 1 + \frac{(\pi-k)^2}{2} & -i(\pi - k) \\ i(\pi - k) & h + 1 - \frac{(\pi-k)^2}{2} \end{pmatrix}. \quad (8)$$

Analyzing the Hamiltonian (8), we shall now probe the non-adiabatic transition probability (i.e., the probability of the excited state $|2_k^f\rangle$) given by $p_k = |u_k|^2$, at the end of the quenching process; it is obvious that the mode $k = \pi$ is temporally frozen as the off-diagonal term vanishes, and hence $|u_{k=\pi}|^2 = 1$. On the other hand, modes away from $k = \pi$ evolve adiabatically and hence $|u_{k\ll\pi}|^2 = 0$. From the continuity argument, one then immediately expects a value of $k = k_*$ for which $p_{k=k_*} = |u_{k=k_*}|^2 = 1/2$ which is essential for observing a DPT. In Fig. 3(a), we plot numerically obtained $|u_k|^2$ as a function of k; indeed an analytical expression for the same can be derived using the Landau-Zener (LZ) transition formula for non-adiabatic transition probability[51–54] which can be approximated as $|u_{k\sim\pi}|^2 = \exp(-\pi(\pi - k)^2\tau)$. From these observations, it is straightforward to conclude that it is the value of k_* and hence, those of t_n^*, related through Eq. (6), which get modified when τ is changed. On the other hand, we would like to emphasize that the behavior of Fisher zeros and non-analyticities in $I(t)$ do not change qualitatively; the non-analyticities only occur at different instants of time depending upon the value of τ. This claim will be further illustrated in Fig. 5(b). The existence of DPTs for $\tau \to 0$ is expected since for a large amplitude sudden quench across a single QCP a k_* will always be present as shown in Fig. 3(a)[4].

We now proceed to analyze the situation presented in Fig. 2; in this case the system crosses both the critical points at $h = -1$ (with critical mode $k = \pi$) and $h = 1$ (with critical mode $k = 0$). As a results both the modes $k = 0$ and $k = \pi$ are temporally frozen leading to $|u_{k=0}|^2 = |u_{k=\pi}|^2 = 1$ (see Fig. 3(b)); the corresponding LZ formula is $p_k = |u_k|^2 = e^{-\pi\tau\sin^2 k}$. Therefore, for each value of τ, we find two values of k_* (with $p_{k=k_*} = 1/2$) yielding two real instants given by t_n^+ and t_n^- as defined before. We therefore again conclude that only the instants of real time at which DPTs occur will depend on the inverse rate τ. Referring to Eq. (6), we note that t_n^+ or t_n^- depends on $\epsilon_{k_*}^f = 2\sqrt{(h_f - \cos k_*)^2 + \sin^2 k_*}$, where k_* assumes two values (close to $k = 0$ and $k = \pi$). However, in the limit of small $\tau(\to 0)$, (i.e., in the sudden limit) as presented in Fig. 3(b), one does not find a value of k_* and no DPT is expected. Furthermore, it can be shown in the following straightforward manner that if quenching is fast (i.e., $\tau < \ln 2/\pi$), the DPTs disappear for passage across two critical points: analyzing the profile of $p_k = |u_k|^2 = e^{-\pi\tau\sin^2 k}$, we note that the minimum value of p_k (that occurs at $k = \pi/2$) should at least be less than $1/2$, for DPTs to be present. This implies that $p_{k\to\pi/2} = p_k^{\min} = e^{-\pi\tau_c} = 1/2$; here τ_c refers to a critical τ and DPTs exist for $\tau > \tau_c$; otherwise, the situation is identical to the sudden quenching across two QCPs where DPTs get wiped out.

III. TOPOLOGICAL ASPECTS:

Topological properties of DPTs are related to topological structure of equilibrium phase diagram, which for the present model in turn, is determined by the sign of the quantity $(h + \cos k)_{k=0}/(h + \cos k)_{k=\pi} = (h + 1)/(h - 1)$; clearly it is negative in the topological (ferromagnetic phase with $|h| < 1$) while positive in the non-topological (paramagnetic) phase, $|h| > 1$. We now proceed to investigate the behavior of the DTOP; to define it in the present context, we refer to the LO:

$$L_k = \left(|v_k|^2 + |u_k|^2 \exp(-2i\epsilon_k^f t)\right). \tag{9}$$

Let us first consider the situation of quenching from the non-topological (paramagnetic) to the topological (ferromagnetic) phase across a single QCP (take for example, the situation illustrated in the Fig. 1(a)). In a spirit similar to the Ref. [17], we can also express the LO as $L_k = |r_k| \exp(i\phi_k)$. In the present case of slow quenching,

$$\phi_k = \tan^{-1}\left(\frac{-|u_k|^2 \sin(2\epsilon_k^f t)}{|v_k|^2 + |u_k|^2 \cos(2\epsilon_k^f t)}\right), \tag{10}$$

and the corresponding dynamical phase $\phi_k^{\rm dyn} = -\int_0^t ds \langle\psi_{f_k}(s)|H_f|\psi_{f_k}(s)\rangle = -2|u_k|^2\epsilon_k^f t$ with $|\psi_{f_k}\rangle = v_k|1_k^f\rangle + u_k e^{-2i\epsilon_k^f s}|2_k^f\rangle$. Let us now define the geometric phase as $\phi_k^G = \phi_k - \phi_k^{\rm dyn}$, so that

$$\phi_k^G = \tan^{-1}\left(\frac{-|u_k|^2 \sin(2\epsilon_k^f t)}{|v_k|^2 + |u_k|^2 \cos(2\epsilon_k^f t)}\right) + 2|u_k|^2\epsilon_k^f t. \tag{11}$$

Note that for the mode $k = 0$, we have $|v_{k=0}|^2 = 1$, and the excitation probability $|u_{k=0}|^2 = 0$; on the other hand, for the critical mode $k = \pi$, $|v_{k=\pi}|^2 = 0$ and $|u_{k=\pi}|^2 = 1$. The geometric phase therefore satisfies the periodicity relation:

$$\phi_\pi^G - \phi_0^G = 0 \mod 2\pi \tag{12}$$

As the lattice momentum goes from 0 to π, ϕ_k^G goes from $-\pi$ to π thus completing a full circle (See Fig. 4). Focusing on the mode k_*, we find that

$$\phi_k^G|_{k_*} = \tan^{-1}\left(-\tan(\epsilon_{k_*}^f t)\right) + 2|u_{k_*}|^2\epsilon_{k_*}^f t, \tag{13}$$

with $|u_{k_*}|^2 = 1/2$; this shows that $\phi_k^G|_{k_*}$ is fixed to zero or π for all values of t except at DPTs given by the condition $\epsilon_{k_*}^f t_n^* = (n + 1/2)\pi$ where $\phi_k(t)$ is ill-defined; ϕ_k^G alternates between 0 and π between two DPTs at $k = k_*$.

We study the behavior of DTOP or the winding number

$$\nu_D = \frac{1}{2\pi}\oint_0^\pi \frac{\partial\phi_k^G}{\partial k}, \tag{14}$$

close to DPTs (discussed in the previous section) focusing on the quenching across a single QCP first. In the context of sudden quenches, the DTOP has been found to appropriately characterize a DPT[17]. We address the question whether the same is true in the case of slow quenches and show that it is indeed the case.

As we are integrating in Eq. 14 the full derivative of a periodic function (modulo 2π), the integral remains constant unless there is some discontinuity in the phase, which should be manifested in the δ-function type contribution to the derivative of the geometric phase. We can anticipate that such discontinuities develop only at DPT therefore it suffices to explore the quantity $\partial\phi_k^G/\partial k$ for the momentum k_*:

$$\frac{\partial\phi_k^G}{\partial k}\Big|_{k_*} = 2\tan(\epsilon_{k_*}^f t)\frac{\partial|v_k|^2}{\partial k}\Big|_{k_*} + 2\frac{\partial|u_k|^2}{\partial k}\Big|_{k_*}(\epsilon_{k_*}^f t). \tag{15}$$

While the first term diverges at every DPT, the second term provides a linearly varying contribution with a slope determined by the sign of its coefficient $(\partial_k|u_k|^2)|_{k_*}$. (In fact, it can be shown that the term $\oint\partial_k\phi_k$ varies symmetrically between positive and negative values in the intermediate time between two DPTs while $\oint\partial_k\phi_k^{\rm dyn}$ increases or decreases monotonically with time depending on the sign of $(\partial_k|u_k|^2)|_{k_*}$). It is therefore clear that the sign of the change in ν_D at a DPT will be determined by the sign of the coefficient of the linearly increasing term $(\partial_k|u_k|^2)|_{k_*}(= -(\partial_k|v_k|^2)|_{k_*})$. (This has been loosely connected to an index theorem in ref. [17].) Positive sign of $(\partial_k|u_k|^2)|_{k_*}$ corresponds to a jump $\Delta\nu_D = +1$ while negative sign yields $\Delta\nu_D = -1$ at every DPT.

Fig. 5 shows the variation of ν_D as a function of time following a quench from large negative h to the topological phase (e.g., $h_f = 0.5$); $\nu_D(t)$ increases in a step-like fashion at every DPT with $\Delta\nu_D = 1$. Let us compare with Fig. 2 of Ref. [17], where a sudden quench from the topological phase to the trivial phase of a p-wave superconducting chain was studied. As discussed above, the sign of $\Delta\nu_D$ at a DPT is given in terms of the sign of $(\partial_k|u_k|^2|)|_{k_*}$; this is negative in the situation studied in Ref. [17] where $|u_{k=0}|^2 = 1$ and $|u_{k=\pi}|^2 = 0$. On the contrary, for a slow quenching of the transverse Ising chain starting from large negative h across the QCP at $h = -1$ (with the critical mode $k = \pi$), we find the value of ν_D increases at every DPT. Here, $|u_{k=\pi}|^2 = 1$ and $|u_{k=0}|^2 = 0$; from continuity one expects a k_* for which $|u_{k=k_*}|^2 = |v_{k=k_*}|^2 = 1/2$ and hence,

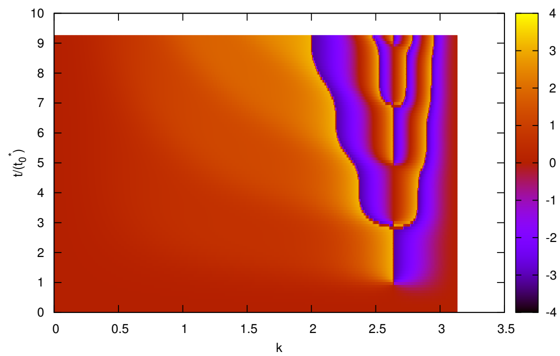
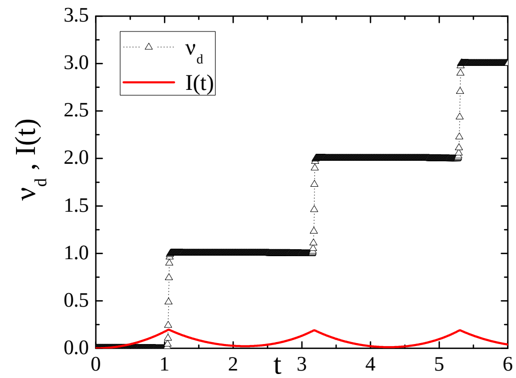
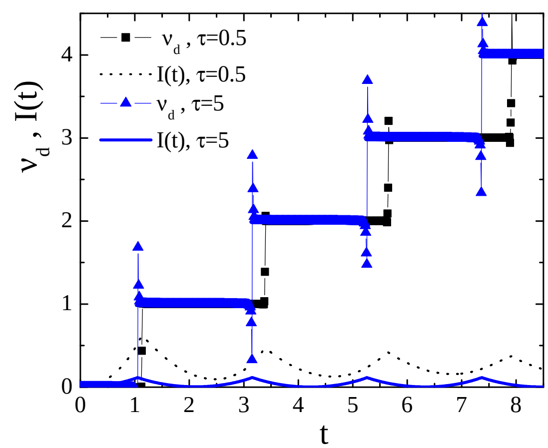


FIG. 4: (Color online) Variation of ϕ_k^G as a function of k and t/t_0^* (where $t_0^* = \pi/2\epsilon_{k_*}^f$) following a slow quench $h(\tilde{t}) = \tilde{t}/\tau$ (with $\tau = 1$) from the initial value of the field $h_i = -10$ to the final value $h_f = 0.5$ across the QCP at $h = -1$. As the lattice momentum k goes from 0 to π, ϕ_k^G varies from $-\pi$ to $+\pi$. It is to be noted that between t_0^* and $3t_0^*$ there is a single winding resulting in $\nu_D = 1$ while between $3t_0^*$ and $5t_0^*$ two full winding and hence $\nu_D = 2$.



(a)



(b)

FIG. 5: (Color online) (a) ν_D along with the rate function $I(t)$ are plotted as a function of time t following a slow quench $h(\tilde{t}) = \tilde{t}/\tau$ (with $\tau = 1$) from $h_i = -10$ to $h_f = 0.5$ so that the system crosses the QCP at $h = -1$. We find an increases in ν_D by a factor of unity whenever there is a DPT. (b) Using two different values of τ (0.5 and 5), we show that the time instants at which DPTs occur depend on τ. However, $\nu_D(t)$ indeed shows a jump of unit magnitude at every DPT for all τ as explained in the text.

$\Delta\nu_D(t_c) = \text{sgn}(\partial_k |u_k|^2|)|_{k_*}$ is positive. To establish that this is indeed the case, we reverse the quenching scheme to $h(\tilde{t}) = \tilde{t}/\tau$ with $\tilde{t}$ starting from a large positive value and ending in the topological phase. The spin chain then crosses the QCP at $h = 1$ where the gap vanishes for the mode $k = 0$ and hence $|u_{k=\pi}|^2 = 0$ and $|u_{k=0}|^2 = 1$, leading to a negative $\Delta\nu_D(t_n^*) = \text{sgn}(\partial_k |u_k|^2|)|_{k_*}$; one can straightway verify numerically that $\Delta\nu_D$ is indeed negative in this case (see Fig. 6). Furthermore, one can investigate the variation $\partial\nu_D/\partial t$ as function of time: ν_D is independent of time always except at DPTs (t_n^*) when there is a diverging (δ-function) peak.

Equipped with above observations, we can now interpret Eq. (14) as a generalized Gauss's law; as the system evolves in time following a quench across a single QCP, one can assume that a positive topological charge enters the system at every t_n^*. On the other hand, if the system is initially chosen to be in the non-topological phase with a large positive h (i.e., the situation depicted in Fig. 6), we can infer that a negative change enters the system at every DPT.

Let us now address the question how does the above situation gets altered when the field is quenched from a large negative to a large positive value so that the system is swept across both the QCPs at $h = \pm 1$, from one topologically trivial phase to the other. In this case as well ν_D exhibits a discrete change at every DPT, however, there exist further intricacies as presented in Fig. 7. To analyze this remarkable behavior, we plot the flow of Fisher zeros following a double quench as shown in the Fig. 8. We note that corresponding to each lobe denoted by n, there are two DPTs occurring at two instants of times $t_n^{(+)}$ and $t_n^{(-)}$ with $t_n^{(-)} > t_n^{(+)}$. From Fig. (7), it is evident that at $t_n^{(+)}$, $\Delta\nu_D = 1$ (in other words, a positive charge enters the system as shown in Fig. 8); on the other hand, the DPT at $t_n^{(-)}$ is associated with $\Delta\nu_D = -1$ or a negative topological charge. This makes ν_D oscillate between 0 and 1. This pattern follows up to 5th lobe and there is a discontinuity at the 6th lobe: from Fig. 8, we find that $t_6^- > t_7^+$. Therefore, we have two successive DPTs both with $\Delta\nu_D = +1$ (i.e., with identical topological charges) and consequently, ν_D jumps from 1 to 2. In general, this type of change would first happen when

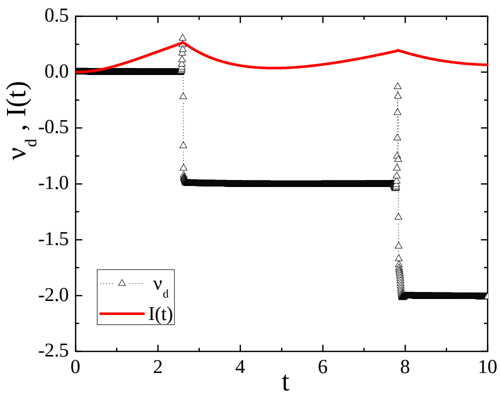


FIG. 6: (Color online) This shows that when the system quenched across the QCP at $h = 1$ to the topological phase ($h_f = 0.5$) starting from a large positive h ($h_i = 10$), $\text{sgn}(\partial_k |u_k|^2|_{k_*})$, and hence $\Delta\nu_D$ is indeed negative.

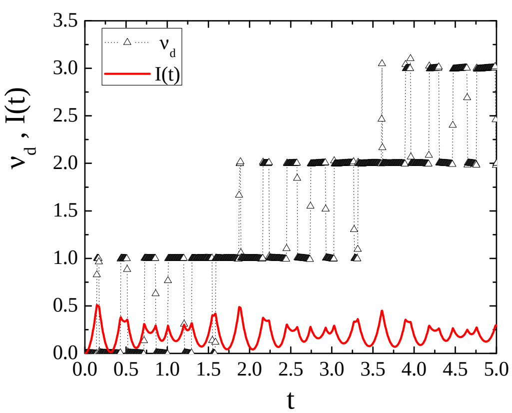


FIG. 7: (Color online) ν_D along with the rate function $I(t)$ are plotted as a function of time t following a slow quench $h(\tilde{t}) = \tilde{t}/\tau$ from $h_i = -10$ to $h_f = 10$, so that the system crosses both the QCPs at $h = -1$ and $h = +1$. We find that ν_D oscillates between 0 and 1, i.e., $\nu_D = \pm 1$ at successive DPTs. However there is a jump in ν_D by unity for some values of t when there are two successive DPTs with $\Delta\nu_D = 1$ (or when a positive charge crosses a negative charge as shown in Fig. 8); between two jumps the alternating pattern persists.

$t_n^{(-)} > t_{n+1}^{(+)}$, i.e., second DPT (with a negative topological charge) corresponding to the lobe n happens at a later (real) time compared to the first DPT (with a positive charge) associated with $(n+1)$-th lobe.

Finally, let us discuss the issue concerning the analytic continuation of the boundary partition function to the real time axis. Comparing Figs. 1(a) and 2(a) we see a very different topology pattern of the Fisher zeros for the quenches across one and two critical points. In the former case Fisher zeros slice the complex time space into disconnected regions. As discussed in Ref. [4] this prevents analytic continuation of a boundary partition function defined as $Z_\psi(z) = \langle\psi|\exp[-zH]|\psi\rangle$ into the real time-axis beyond the critical time t^*. On the other hand, for the quench across two critical points the time domain is split into alternating intervals. The analytic continuation of $Z(z)$ is only possible to the intervals reachable without crossing Fisher zeros, i.e. $0 < t < t_0^+$, $t_0^- < t < t_1^+$ and so on. As we discussed above such intervals where analytic continuation is possible correspond to the sectors with zero topological charge. As it is clear from Figs. 7 and 8 such piecewise analytic continuation is possible for finite time until the zero with positive topological change from the $n+1$-st branch of Fisher zeros crosses the zero with negative topological charge from the n-th branch, i.e. $t_{n+1}^+ < t_n^-$. Beyond this point no analytic continuation of the boundary partition function is possible.

Finally, we address the question how does this topological structure depend on τ; as argued in the previous section, τ determines the time instants at which DPTs occur. Hence there will not be any qualitative change in the topological pattern, i.e., $\Delta\nu_D = 1$ (or -1) at every DPT as illustrated in Fig. 5(b). However, in the case of crossing across two QCPs the values of n for which $t_{n+1}^+ < t_n^-$ and hence we have two successive DPTs with $\Delta\nu_D = 1$ may depend on τ.

IV. CONCLUDING COMMENTS

We have studied the slow quenching dynamics of a transverse Ising chain across its QCPs and analyzed the Fisher zeros and hence DPTs which are reflected in the non-analyticities of the rate function. We emphasize that although the non-analyticities in the dynamical free energy leading to periodic occurrences of DPTs have been reported in the context of slow quenching of the Hamiltonian (1)[32], they have not been connected to the Fisher zeros. Furthermore, we show that these DPTs survive even when the system is quenched across both the QCPs of the model when the line of Fisher zeros crosses the imaginary axis for two characteristic momenta values. This is in sharp contrast with the sudden quenching case when an abrupt passage across two QCPs wipes out the non-analyticities. In fact, for a sufficiently slow quenching, two QCPs of the model appear to be independent of each other. Secondly, in this case, it is essential to cross the QCP to observe the DPTs, otherwise the transition probability p_k is always less than 1/2 for all values of k and hence a DPT can not occur.

We have also analyzed the behavior of the DTOP (i.e., $\nu_D(t)$). The slow quenching scheme discussed in this paper, allows to study both cases: the quenching across a single QCP (i.e., from a non-topological phase to the topological phase) or two QCPs (i.e., from one non-topological phase to the other) depending upon the final value of the transverse field. In the former case, the DTOP increases (or decreases depending upon the direction of the quenching) step-wise by unity at those instants of time where DPTs occur. This can be interpreted as charges of the same sign entering the system at every DPT. On the contrary, in the latter case the DTOP oscillates between 1 and 0; this implies that a DPT with a positive topological charge $\Delta\nu_D = 1$ at real time t_n^+ is followed by a DPT with a negative topological charge $\Delta\nu_D = -1$ at t_n^-. Furthermore there are sharp increases by a factor of unity whenever there is a crossing of Fisher zeros belonging to neighboring lobes i.e., $t_n^- > t_{n+1}^+$, there are two successive DPTs with topological charges of same signs; this is indeed remarkable.

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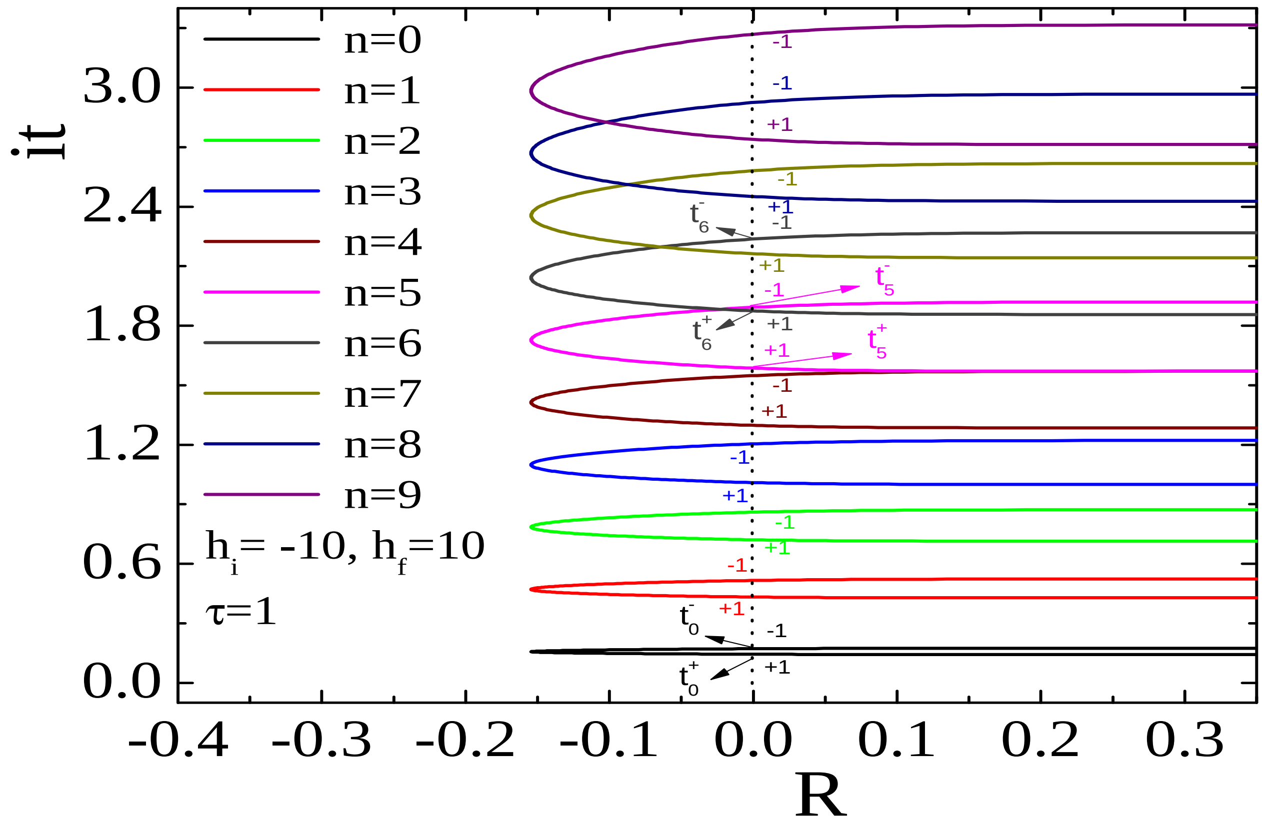

FIG. 8: (Color online) The flow of Fisher zeros following quenching from $h_i = -10$ to $h_f = 10$ across two QCPs with the protocol $h(\tilde{t}) = \tilde{t}/\tau$ with $\tau = 1$. Each lobe is characterized by the integer n which increases from bottom to top with the lowest lobe corresponding to $n = 0$. There exist two DPTs associated with each n; the DPT at t_n^+ is associated with a positive topological charge $(+1)$ while t_n^- denotes a DPT with a negative topological charge (-1). This explains the oscillation of ν_D between 0 and 1. However, we note that $t_6^+ < t_5^-$ and hence there are two successive DPTs both with positive topological charge occurring at t_5^+ and t_6^+, respectively; this leads to a jump in $\nu_D(t)$ by a factor of unity (as shown in Fig. 7) following which the oscillating pattern persists.

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