Interaction-enabled topological crystalline phases

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Interaction Enabled Topological Crystalline Phases

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In this article we provide a general mechanism for generating interaction-enabled fermionic topological phases. We illustrate the mechanism with crystalline symmetry-protected topological phases in one, two and three spatial dimensions. These non-trivial phases require interactions for their existence and, in the cases we consider, the free-fermion classification yields only a trivial phase. For the 1D and 2D phases we consider, we provide explicit exactly solvable models which realize the interaction-enabled phases. Similar to the interpretation of the Kitaev Majorana wire as a mean-field p-wave superconductor Hamiltonian arising from an interacting model with quartic interactions, we show that our systems can be interpreted as "mean-field" charge-$4e$ superconductors arising, e.g., from an interacting model with eight-body interactions, or through another physical mechanism. The quartet superconducting nature allows for the teleportation of full Cooper pairs, and in 2D for interesting semiclassical crystalline defects with non-Abelian anyon boundstates.

I. INTRODUCTION

Symmetry protected topological phases (SPTs) have risen to the forefront of condensed matter physics. The impetus for such an explosion of interest began with the theoretical prediction and experimental discovery of 2D and 3D topological insulators protected by time-reversal symmetry[1], and has carried on through the classification of all weakly-interacting fermionic topological phases protected by discrete (anti-unitary) symmetries[2–4]. From here the field has now spread to encompass topological crystalline phases protected by spatial symmetries[5–19], and bosonic counterparts of the fermionic phases[20–24], which recently culminated into a classification of some interacting SPTs[25–27].

In this article we develop a mechanism for interaction-enabled fermionic crystalline SPTs, and provide explicit 1D and 2D models that realize the putative strongly-interacting topological phases[28]. We show that, without interactions, the symmetry classes we consider have no non-trivial topological phases, yet with interactions a non-trivial SPT exists. This is quite different from the case of, for example, 3D time-reversal invariant insulators, where it was shown in Ref. 27 that interactions extend the classification from $\mathbb{Z}_2$ to $\mathbb{Z}_2^3$. In that case the non-interacting classification still yields a non-trivial phase, whereas in our examples *only* the trivial phase is possible without interactions. It is also known that interactions can extend the classification of two-dimensional fermions in the unitary class A from $\mathbb{Z}$ to $\mathbb{Z}\times\mathbb{Z}$[29], but in this case as well, non-trivial phases still exist without interactions. The models we consider are essentially related to charge-$4e$ superconductors and have no mean-field (free-fermion) description. We also discuss the strongly interacting topological phase transition between the trivial and interaction-enabled topological phase, and the properties of topological bound states on defects.

This paper is organized in the following way. In Sec. II we give an example and introduce a general mechanism for the generation of interaction-enabled crystalline topological phases. In Sec. III we expand on the arguments from Sec. II and show the possibility of a topological phase in 1D fermionic systems (in symmetry class BDI with additional inversion symmetry) which *requires* interactions for its existence. In Sec. IV we construct an exactly solvable 1D fermionic model which realizes this interaction-enabled topological phase and we discuss its properties, paying particular attention to the projective action of symmetry operators at the boundary of the system, and to the stability of the model away from the exactly solvable point. We also discuss an interpretation of our 1D model as a "mean-field" description of a charge-$4e$ topological superconductor. In Sec. V we discuss 2D and 3D examples of interaction-enabled topological phases as well. Finally, in Sec. VI we summarize the results of the paper and propose an experimental signature which could in principle identify a charge-$4e$ topological superconductor of the type predicted here.

II. A GENERAL MECHANISM FOR INTERACTION-ENABLED TOPOLOGICAL PHASES

In this section we describe a general mechanism for generating interaction-enabled topological phases of fermions. We start by introducing the mechanism via a simple example: the Kitaev p-wave wire (also known as a Majorana chain), with an additional time-reversal symmetry T $(T^2 = +1)$[30]. This model belongs to the BDI Altland-Zirnbauer class[3,31], and is classified, in the non-interacting limit, by an integer winding number ν. A model of spinless fermions ψ_n in this class realizing the $\nu = 1$ phase is given by the following Hamiltonian (and illustrated in Fig. 1a)

$$H_{Kitaev} = \sum_{n=1}^{N-1} ib_n a_{n+1} \ , \tag{2.1}$$

where a_n and b_n are Majorana fermions, related to the original spinless fermions by $\psi_n = \frac{1}{2}(a_n + ib_n)$. The

Majorana fermions a_n and b_n are Hermitian, they anticommute with each other, and square to 1. The spinless fermions ψ_n annihilate the Fock vacuum state $|0\rangle$. The time-reversal operator preserves the Fock vacuum, $T|0\rangle = |0\rangle$, and acts as $T\psi_n T^{-1} = \psi_n$ on each fermion operator. This means that the time-reversal operator acts on the Majorana fermions as $Ta_nT^{-1} = a_n$ and $Tb_nT^{-1} = -b_n$. This model is time-reversal invariant and gapped in the bulk, but possesses low-energy unpaired Majorana zero modes, e.g., in this limit the modes are just a_1 on the left end and b_N on the right end. The bulk-boundary correspondence dictates that the number of boundary zero modes of a-type minus the number of b-type on a single end is $|\nu|$, so the system in Eq. (2.1) realizes the $\nu = 1$ phase. Larger values of ν are topologically equivalent to multiple copies of the $\nu = 1$ case and exhibit $|\nu|$ stable Majorana modes of an identical type on a single end. Negative values of ν correspond to chains with unpaired a-modes (b-modes) on the right (left).

Next we wish to consider the possibility of a topological crystalline superconductor (TCS) in this class by requiring inversion symmetry R with $R^2 = 1$ and $[R, T] = 0$, which is natural for spinless (or spin-polarized) fermions. Unfortunately the classification is not very interesting. In class BDI, the fact that R and T commute means that R does not act within a unit cell to interchange a and b type modes. We can see from Fig. 1a that acting with R just flips the chain from left to right, which converts an a-type end to a b-type end, and so we find that $R\colon \nu \to -\nu$. Thus, when the symmetry is enforced we must have $\nu = -\nu$, but the only solution is $\nu = 0$ since ν is an integer. Hence, there are no free-fermion SPTs for the BDI class with the inversion symmetry of this type, nor any weakly-interacting SPTs in this symmetry class that can be adiabatically connected to the non-interacting limit. In fact, in Sec. III we give a more formal proof that the winding number for free fermions in class BDI vanishes when the additional inversion symmetry R, with $[R, T] = 0$, is imposed.

Recently, however, it has been explicitly shown that the classification of the vanilla BDI class with interactions is deformed from its non-interacting limit[21,32–34]. In a seminal paper, Fidkowski and Kitaev showed that eight copies of the $\nu = 1$ chain (i.e., $\nu = 8$) can be adiabatically deformed to $\nu = 0$ by passing through a gapped, interacting phase that preserves all of the required symmetries[32]. Hence, the classification is reduced from $\mathbb{Z}$ to $\mathbb{Z}_8$. Now if we add inversion symmetry, we find that the constraint $\nu = -\nu$ has a non-trivial solution! Since $\nu \in \mathbb{Z}_8$, the solution $\nu = 4 \equiv -4 \mod 8$ indicates the existence of a non-trivial crystalline SPT that *requires* strong interactions for its existence. We provide further evidence for the existence of a non-trivial phase with $\nu = 4$ in Sec. III where we construct a general argument which shows that for a 1D system in class BDI, either zero *or* four Majorana end modes are consistent with the additional inversion symmetry R. Additionally, we then provide an explicit, exactly-solvable model which realizes the non-trivial $\nu = 4$ phase in Sec. IV, thereby acting as a proof of principle for the existence of this phase as the ground state of a local Hamiltonian.

This mechanism for, what we call, an interaction-enabled SPT is quite general. Given any topological integer property ν, and a symmetry R under which ν transforms non-trivially, then the constraint $\nu = R\nu$ has no non-trivial solutions, i.e., $\nu \equiv 0$ is the only solution. The property ν could be a scalar, vector, tensor etc., but for now let us focus on the scalar case where ν can only transform non-trivially to $-\nu$. By including interactions, the integer classification of ν could be reduced to a cyclic group $\mathbb{Z}_n$. If n is even, then $\nu = n/2$ is a non-trivial solution, and the classification of the interacting system with R symmetry is $\mathbb{Z}_2$ valued where $\nu = 0 \mod n$ and $\nu = n/2 \mod n$ are the trivial and non-trivial values respectively. In the remainder of the article we will construct and discuss the properties of 1D and 2D models which have interaction-enabled topological phases.

III. INVERSION SYMMETRY AND 1D FERMIONS IN CLASS BDI

In this section we reinforce the results argued for in the previous section by providing: (i) an explicit proof that the winding number ν for free 1D fermions in class BDI vanishes when the additional inversion symmetry R is required, and (ii) a general argument, closely paralleling the arguments in Refs. 33 and 34, which shows that *interacting* fermions in class BDI with the extra symmetry R can have zero *or* four unpaired Majorana fermions at the end of an open chain. Our argument (ii) represents an extension of the classification of 1D fermionic systems in class BDI, worked out in Refs. 33 and 34 (see also Ref. 21), to the case with additional inversion symmetry R. Since in Sec. IV we construct a concrete model realizing the interaction-enabled $\nu = 4$ phase, the results of these two sections imply that the interacting classification for 1D fermions in class BDI with the extra inversion symmetry R is indeed $\mathbb{Z}_2$, i.e., it has a non-trivial topological phase, contrary to the non-interacting limit.

A. Vanishing of the BDI winding number when inversion is added and $[R, T] = 0$

We now prove that the winding number $\nu \in \mathbb{Z}$ of class BDI vanishes when the extra inversion symmetry R, which satisfies $R^2 = 1$ and $[R, T] = 0$, is enforced. We first briefly review the construction of the winding number (for translationally invariant systems) from the Bogoliubov-de-Gennes (BdG) Hamiltonian following Ref. 35, and then show that incorporating the inversion symmetry R forces the winding number to be zero. In this subsection we use $\mathbb{I}_m$ to denote the $m \times m$ identity matrix for any positive integer m, and σ^a, $a = x, y, z$, are the usual Pauli matrices.

Consider a system of M flavors of spinless fermions on a one-dimensional lattice, and write the M annihilation operators as an M-component vector $\vec{\psi}_n$, where n is the position on the lattice. The anti-unitary time-reversal operator is defined by the relations $T\vec{\psi}_n T^{-1} = \vec{\psi}_n$ and $T|0\rangle = |0\rangle$, where $|0\rangle$ is the Fock vacuum annihilated by the fermions $\vec{\psi}_n$, and we have $T^2 = 1$. We Fourier transform to momentum space by defining $\vec{\psi}_k = \frac{1}{\sqrt{N}}\sum_n \vec{\psi}_n e^{-ikn}$, where N is the number of unit cells. Time-reversal now acts as $T\vec{\psi}_k T^{-1} = \vec{\psi}_{-k}$. The Hamiltonian operator for a generic system of free fermions then takes the form

$$H = \frac{1}{2}\sum_k \vec{\Psi}_k^\dagger \mathcal{H}(k)\vec{\Psi}_k \ , \tag{3.1}$$

where $\vec{\Psi}_k = (\vec{\psi}_k, \vec{\psi}^\dagger_{-k})^T$ and $\mathcal{H}(k)$ is the $2M \times 2M$ BdG Hamiltonian.

For a system in class BDI, the BdG Hamiltonian takes the special form

$$\mathcal{H}(k) = \begin{pmatrix} h_0(k) & i\Delta(k) \\ -i\Delta(k) & -h_0(k) \end{pmatrix} \tag{3.2}$$

$$= \sigma^z \otimes h_0(k) - \sigma^y \otimes \Delta(k) \ , \tag{3.3}$$

where $h_0(k)$ and $\Delta(k)$ are $M\times M$ matrices. The fact that the Hamiltonian should be Hermitian and time-reversal invariant implies that these matrices satisfy the relations

$$\Delta^T(k) = \Delta^*(k) \tag{3.4}$$

$$\Delta^*(k) = -\Delta(-k) \tag{3.5}$$

and

$$h_0^T(k) = h_0^*(k) \tag{3.6}$$

$$h_0^*(k) = h_0(-k) \ . \tag{3.7}$$

In this case, applying the unitary transformation $S = e^{-i\frac{\pi}{4}\sigma^y\otimes\mathbb{I}_M}$ gives us

$$S\mathcal{H}(k)S^\dagger = \begin{pmatrix} 0 & A(k) \\ A^T(-k) & 0 \end{pmatrix} \equiv \tilde{\mathcal{H}}(k) \ , \tag{3.8}$$

where $A(k) = h_0(k) + i\Delta(k)$ and satisfies

$$A^*(k) = A(-k) \ . \tag{3.9}$$

For later use we note that this relation implies that

$$\det(A(k))^* = \det(A(-k)) \ . \tag{3.10}$$

A winding number for the BdG Hamiltonians of class BDI can be defined from the phase of the determinant of $A(k)$. Write

$$\det(A(k)) = |\det(A(k))|e^{i\theta(k)} \ . \tag{3.11}$$

In terms of $z(k) = e^{i\theta(k)}$, the expression for the winding number given in Ref. 35 is

$$\nu = \frac{-i}{\pi}\int_{k=0}^{k=\pi}\frac{dz(k)}{z(k)} = \frac{1}{\pi}(\theta(\pi) - \theta(0)) \ . \tag{3.12}$$

Next we consider adding (unitary) inversion symmetry R to the system. We define R to act on $\vec{\psi}_n$ as $R\vec{\psi}_n R^{-1} = U_R\vec{\psi}_{-n}$, where U_R is an $M\times M$ matrix. The relations $R^2 = 1$ and $[R, T] = 0$ imply that $U_R^2 = \mathbb{I}_M$ and that U_R is a real matrix. These properties imply that R acts on $\vec{\Psi}_k$ as $R\vec{\Psi}_k R^{-1} = (\mathbb{I}_2\otimes U_R)\vec{\Psi}_k$. For inversion to be a symmetry of the system we require $RHR^{-1} = H$. This implies that

$$(\mathbb{I}_2\otimes U_R)\mathcal{H}(k)(\mathbb{I}_2\otimes U_R) = \mathcal{H}(-k) \ . \tag{3.13}$$

Multiplying both sides of this equation by S on the left and $S^\dagger$ on the right gives

$$(\mathbb{I}_2\otimes U_R)\tilde{\mathcal{H}}(k)(\mathbb{I}_2\otimes U_R) = \tilde{\mathcal{H}}(-k) \ , \tag{3.14}$$

which reduces to

$$U_R^T A(k) U_R = A(-k) \ . \tag{3.15}$$

Now since $U_R^2 = \mathbb{I}_M$, this relation implies that

$$\det(A(k)) = \det(A(-k)) \ . \tag{3.16}$$

Combined with Eq. (3.10), we see that the extra inversion symmetry R forces $\det(A(k))$ to be real. This means that the phase of $\det(A(k))$ cannot vary smoothly in the Brillouin zone, and so the BDI winding number ν must vanish!

B. Class BDI with extra inversion symmetry allows 0 or 4 Majorana end modes

In this subsection we give a general argument which shows that for 1D systems in class BDI with the extra inversion symmetry R, satisfying $R^2 = 1$ and $[R, T] = 0$, it is possible to have zero *or* four Majorana modes at the ends of an open chain, the latter representing the non-trivial topological phase. Our argument is an extension of the classification in Refs. 33 and 34 of 1D fermions in class BDI to the case with additional inversion symmetry R. Briefly, the classification in Refs. 33 and 34 was determined by studying the algebra obeyed by the symmetry operators relevant for class BDI (the operators T, P and TP discussed below), and then considering all possible projective representations of this algebra when the operators act on the degrees of freedom localized at a single end of an open chain. In our case there is an additional complication because the inversion operator R maps degrees of freedom at one end of the chain to those at the other end of the chain. We show that this extra complication places consistency conditions on how all the symmetry operators act in the *total* ground state subspace of both ends of an open chain, which allows us to determine which numbers of Majorana end modes are consistent with the additional inversion symmetry.

To derive this result we consider a generic 1D chain in class BDI with open boundary conditions on both ends of the chain. We assume that the degrees of freedom in the bulk of the chain are gapped, but that there are ν unpaired Majorana fermions at each end of the chain. Based on the arguments of Ref. 32, we only need to consider $\nu = 1, \ldots, 8$. We will deduce the allowed numbers of Majorana end modes consistent with all symmetries of the system by carefully considering the action of the symmetry operators within the ground state subspace of an open chain.

Before we describe our general argument, we first review the symmetries of a 1D system in class BDI. For a system in class BDI the Hamiltonian H commutes with the time-reversal operator T, the fermion parity operator P, and the composite operator TP. These operators satisfy $P^2 = T^2 = 1$ and $[P, T] = 0$ as operator relations in the Hilbert space of a chain which is open at both ends. We now add the inversion symmetry R to this setup, and require $R^2 = 1$ and $[R, T] = [R, P] = 0$ (R should commute with P and T since the latter are local operators whereas R acts globally).

Since R, P, and T all commute with each other *and* with the Hamiltonian H, it follows that R will commute with T and P in each constant energy subspace of the Hilbert space. In particular, R should commute with T and P in the *ground state* subspace of the open chain. Now comes the crucial observation, which is based on the results of Refs. 33 and 34. When the Hilbert space is restricted to the ground state subspace of the open chain, the operators T and P may be written in terms of the Majorana zero modes located at the two ends of the chain. We will determine the number of allowed Majorana zero modes at the ends of an open chain by requiring that $[R, P] = [R, T] = 0$ in the ground state subspace of an open chain.

We consider the case of $\nu = 1, \ldots, 8$ Majorana modes at each end of the chain (since $\nu = 8$ can be adiabatically connected to $\nu = 0$ via the FK interaction). A basic point is that the relation $[R, T] = 0$ requires the Majorana modes on the two ends of the chain to be of the same type (a-type or b-type) so that they transform in the same way under T. We may therefore suppose that there are ν Majorana modes a_l^J at the left end of the chain and another ν Majorana modes a_r^J at the right end of the chain, where $J = 1, \ldots, \nu$ (a similar argument will hold if we instead take the end modes to be of b-type). In this section we assume that the inversion operator R acts by simply switching a right end-mode with a left one, i.e. $R a_l^J R^{-1} = a_r^J$.

In the ground state subspace, and up to an overall minus sign, the fermion parity operator P can be written as

$$P = \prod_{J=1}^{\nu} (i a_l^J a_r^J) \ . \tag{3.17}$$

We find that $RPR^{-1} = (-1)^\nu P$, which means that $[R, P] = 0$ only if ν is even. We conclude that in the presence of inversion symmetry R, only an even number of Majorana zero modes are allowed at each end of the chain.

Next we consider the action of time-reversal symmetry in the ground state subspace. We now have to consider only the case where ν is even, so we write $\nu = 2m$ for $m = 1, 2, 3, 4$. Following Ref. 34, T can be written in the form $T = \mathcal{T}_l \mathcal{T}_r K^*$, where $\mathcal{T}_l$ ($\mathcal{T}_r$) is an operator localized at the left (right) end of the chain, and in this case K^* is complex conjugation in the basis defined by locally pairing up Majorana fermions at each end of the chain. The precise definition of K^* is a subtle point, so we describe it here following the definition given in Ref. 33. Pair up the Majorana zero modes a_l^J at the left end of the chain into new complex fermions ζ_l^I as

$$\zeta_l^I = \frac{1}{2}\left(a_l^{2I} + i a_l^{2I-1}\right) \ , \ I = 1, \ldots, m \ , \tag{3.18}$$

and likewise for ζ_r^I. Let $|\omega\rangle$ be the state annihilated by all of the ζ_l^I and ζ_r^I. Then we can write a state in this basis as

$$|\psi\rangle = \sum_{\alpha_i=0,1} \sum_{\beta_i=0,1} C_{\alpha_1\cdots\alpha_m\beta_1\cdots\beta_m} (\zeta_l^{1,\dagger})^{\alpha_1} \cdots (\zeta_l^{m,\dagger})^{\alpha_m} (\zeta_r^{1,\dagger})^{\beta_1} \cdots (\zeta_r^{m,\dagger})^{\beta_m} |\omega\rangle \ . \tag{3.19}$$

The operator K^* is defined as complex conjugation in this basis, i.e., acting with K^* on $|\psi\rangle$ sends $C_{\alpha_1\cdots\alpha_m\beta_1\cdots\beta_m} \to C^*_{\alpha_1\cdots\alpha_m\beta_1\cdots\beta_m}$. Therefore K^* commutes with Majorana modes $a_{r/l}^J$ for even J, and anti-commutes with $a_{r/l}^J$ for odd J. With this definition of K^*, the explicit form of the operators $\mathcal{T}_l$ and $\mathcal{T}_r$ was derived in Ref. 33. They found that

$$\mathcal{T}_l = \begin{cases} \prod_{J=1}^{m} a_l^{2J-1} \ , & m = \text{even} \\ \prod_{J=1}^{m} a_l^{2J} \ , & m = \text{odd} \end{cases} \tag{3.20}$$

with a similar form for $\mathcal{T}_r$. For $\nu = 2, 6$, $\mathcal{T}_l$ and $\mathcal{T}_r$ are products of an odd number of Majorana fermions, while for $\nu = 4, 8$, $\mathcal{T}_l$ and $\mathcal{T}_r$ are products of an even number of Majorana fermions. Since inversion acts as $R\mathcal{T}_l\mathcal{T}_r R^{-1} = \mathcal{T}_r\mathcal{T}_l$, we can only get $[R, T] = 0$ in the ground state subspace if $\mathcal{T}_l\mathcal{T}_r = \mathcal{T}_r\mathcal{T}_l$. This only happens in the cases $\nu = 4$ and $\nu = 8$.

Therefore we find that requiring R to commute with P and T in the ground state subspace of an open chain only allows for 4 or $8 \equiv 0$ Majorana zero modes at each end of an open chain. In particular, a non-trivial phase with $\nu = 4$ unpaired Majorana end modes is allowed according to this argument. We have seen that this phase cannot be constructed with free fermions, but in the next section we show that the non-trivial $\nu = 4$ phase with inversion symmetry can be realized in a concrete model of interacting fermions.

IV. THE FIDKOWSKI-KITAEV CHAIN MODEL

In this section we construct a one-dimensional wire model in class BDI with extra inversion symmetry which realizes the $\nu = 4$ interaction-enabled phase. Let us briefly illustrate the complication with generating the non-trivial $\nu = 4$ state from a free-fermion (quadratic) Hamiltonian. To preserve T the only allowed quadratic terms must have the form $ia_n b_m$. To preserve inversion symmetry, and satisfy $[R, T] = 0$, each end of the chain must have the same number, and type, of low-energy Majorana modes. Thus, beginning with the ends of a topological chain and working backward to form the gapped bulk, one always reaches a point where Majorana fermions of the same type must be coupled to open a gap, but this is forbidden by T. An example of this is shown in Fig. 1b. We can essentially think of this as trying to find a way to adiabatically connect a $\nu = 4$ chain to a $\nu = -4$ chain as shown in Fig. 1b. This cannot be done in the free-fermion limit, and thus interactions are required.

This failure, however, immediately gives the key to the correct construction. We see that what is needed is a perturbation that can open a gap by coupling eight Majorana fermions of the same type in an inversion and time-reversal symmetric way. Fortunately, the Fidkowski-Kitaev (FK) interaction is exactly what is needed. In this section we first discuss the FK interaction and its relation to the Heisenberg interaction for spin-$\frac{1}{2}$ systems. We then go on to discuss our FK chain *model*, which realizes the interaction-enabled $\nu = 4$ phase of class BDI with inversion symmetry R discussed in the previous sections of this paper. We then give a detailed discussion of the physics at the boundary of the FK chain, and we show that the degrees of freedom at the boundary form a projective representation of the group $\mathbb{Z}_2$ generated by the time-reversal symmetry operator T which satisfies $T^2 = 1$ when acting on the degrees of freedom in the bulk of the system.

A. Fidkowski-Kitaev interaction and relation to Heisenberg interaction for spins

Let $c^J, J = 1, \ldots, 8$, be any eight Majorana fermions which all transform in the same way under time-reversal. Then the Fidkowksi-Kitaev (FK) interaction for these eight Majorana fermions has the form

$$H_{FK} = u\left[c^{1234} + c^{5678}\right] + v\left[\sum_{\sigma\in A_4} c^{\sigma(1)\sigma(2)[\sigma(1)+4][\sigma(2)+4]}\left(\frac{1+c^{1234}}{2}\right)\right] , \quad (4.1)$$

where $c^{IJKL} \equiv c^I c^J c^K c^L$, σ runs over the even permutations of (1234), and $u, v > 0$ (Fidkowski and Kitaev take $u = v$, but this is not necessary for our purposes). This Hamiltonian is time-reversal invariant and has a unique ground state. The ground state may be constructed in the following way. First pair the Majorana fermions c^J into four new complex fermions χ_K, $K = 1, \ldots, 4$, where $\chi_K = \frac{1}{2}(c^{2K-1} + ic^{2K})$. For the fermions χ_K, we define the number operators $n_K = \chi_K^\dagger \chi_K$ and the modified number operators $N_K = 2n_K - 1$. In terms of these, the FK interaction takes the form

$$H_{FK} = -u(N_1N_2 + N_3N_4) - v(N_1 + N_2)(N_3 + N_4) + 8v(\chi_1\chi_2\chi_3\chi_4 + \text{h.c.}) . \quad (4.2)$$

Let $|\tilde{0}\rangle$ be the state annihilated by all of the χ_K. Then the Hilbert space of these eight Majorana fermions is generated by acting on $|\tilde{0}\rangle$ with the $\chi_K^\dagger$, and the ground state of Eq. (4.1) (for $u, v > 0$) is easily shown to be

$$|\omega\rangle = \frac{1}{\sqrt{2}}\left(|\tilde{0}\rangle - \chi_1^\dagger\chi_2^\dagger\chi_3^\dagger\chi_4^\dagger|\tilde{0}\rangle\right) . \quad (4.3)$$

We now show that when the Hilbert space of the fermions is restricted to the sector of even local fermion parity, the Fidkowski-Kitaev interaction can be reduced to to the Heisenberg exchange interaction for a spin-$\frac{1}{2}$ system.

For the system consisting of eight Majorana fermions c^J considered here, the subspace with even local fermion parity is defined by the relations $N_1N_2 = N_3N_4 = 1$. There are four states in this subspace, which we label as

$$|+,+\rangle = \chi_1^\dagger\chi_2^\dagger\chi_3^\dagger\chi_4^\dagger|\tilde{0}\rangle \quad (4.4a)$$

$$|+,-\rangle = \chi_1^\dagger\chi_2^\dagger|\tilde{0}\rangle \quad (4.4b)$$

$$|-,+\rangle = \chi_3^\dagger\chi_4^\dagger|\tilde{0}\rangle \quad (4.4c)$$

$$|-,-\rangle = |\tilde{0}\rangle , \quad (4.4d)$$

where "+" and "−" are meant to represent up and down states of a spin 1/2 object. In this subspace one can check that

$$(N_1+N_2)(N_3+N_4) = \begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & -4 & 0 & 0 \\ 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix} = 4\sigma^z\otimes\sigma^z \quad (4.5)$$

and

$$\chi_1\chi_2\chi_3\chi_4 + \text{h.c.} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} = \frac{1}{2}(\sigma^x\otimes\sigma^x - \sigma^y\otimes\sigma^y) , \quad (4.6)$$

so that the FK interaction takes the form (we drop the term $N_1N_2+N_3N_4$ since it is a constant in this subspace)

$$H_{FK} = 4v\left[\sigma^x \otimes \sigma^x - \sigma^y \otimes \sigma^y - \sigma^z \otimes \sigma^z\right] \ . \tag{4.7}$$

Next, we can conjugate H_{FK} in this subspace by the unitary operator $U = \sigma^x \otimes \mathbb{I}$ to get

$$H'_{FK} = UH_{FK}U^\dagger = 4v\left[\sigma^x \otimes \sigma^x + \sigma^y \otimes \sigma^y + \sigma^z \otimes \sigma^z\right] \ , \tag{4.8}$$

so we see that in the subspace of even local fermion parity the FK interaction is equivalent to an antiferromagnetic ($v > 0$) Heisenberg interaction.

B. The Fidkowksi-Kitaev chain

We now use the FK interaction to create a translationally invariant wire model which we call the Fidkowski-Kitaev (FK)-chain. Instead of coupling Majorana fermions with quadratic tunneling terms, we couple the fermions with the quartic FK interaction in a "dimerized" pattern which we now discuss.

The FK-chain model consists of eight complex fermions ψ_n^J per unit cell, where $J = 1, \ldots, 8$ is a flavor index. Each complex fermion can be split into two Majorana modes $\psi_n^J = \frac{1}{2}(a_n^J + ib_n^J)$ as before. In the FK-chain we couple the eight b-type Majorana modes b_n^J *within* each unit cell using the FK-interaction Eq. (4.1), and we also couple the four a-type Majorana modes a_n^J, $J = 5, \ldots, 8$ in unit cell n with the four Majorana modes a_{n+1}^J, $J = 1, \ldots, 4$ in unit cell $n+1$ using Eq. (4.1), as shown in Fig. 1c. The resulting Hamiltonian has the following very important properties: (1) it is time-reversal and inversion symmetric, (2) it has a unique gapped ground state on a periodic chain, and (3) each end of an open chain harbors two effective spin $\frac{1}{2}$ degrees of freedom (4 Majorana modes), and the time-reversal operator acts projectively as $T^2 = -1$ on each of these spin $\frac{1}{2}$ degrees of freedom. This is the non-trivial 1D crystalline topological phase with $\nu = 4$.

The local fermion parity is conserved by the FK-chain Hamiltonian, even with arbitrary intra- and inter-unit cell FK interactions. This means that as long as u is sufficiently larger than v, the ground state of the chain with arbitrary FK interactions lies in the sector of even local fermion parity, and all low energy phenomena take place within that sector. Since the FK interaction reduces to a Heisenberg exchange interaction in the sector of even local fermion parity, it follows that with u large, the low-energy physics of the FK chain is *exactly* equivalent to the physics of the dimerized spin-$\frac{1}{2}$ chain (more precisely, we have two separate dimerized spin-$\frac{1}{2}$ chains coming from the a-type and the b-type Majorana fermions, but we focus our discussion on the case where the intra-cell FK interaction dominates for the b-type Majorana fermions and makes them inert). The FK chain model, like the dimerized spin-$\frac{1}{2}$ chain, is exactly

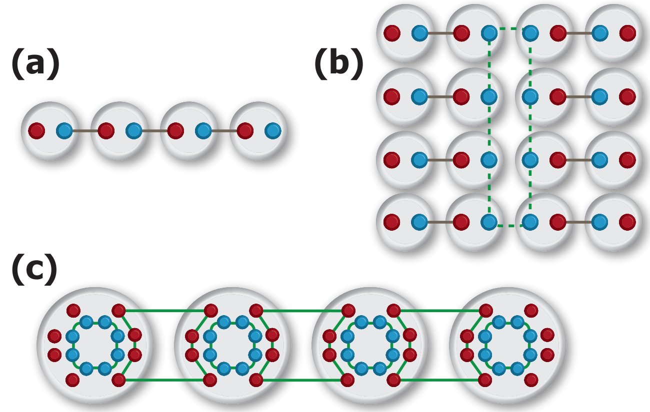

FIG. 1. (a) The Kitaev p-wave wire with time-reversal symmetry $T(T^2 = +1)$. Each large circle represents one unit cell, which contains one complex fermion. The complex fermion is split up into a-type (red) and b-type (blue) Majorana fermions. In the topological phase, Majorana modes are coupled by hopping terms (the grey lines) as described by Eq. (2.1). (b) An attempt to construct an inversion and time-reversal symmetric topological phase using a free-fermion model. This system must have four unpaired Majorana fermions of the same type on each end of the wire and this will always lead to gapless states in the bulk. The gapless states cannot be gapped out using a quadratic interaction without breaking the time-reversal symmetry and thus we need the Fidkowski-Kitaev interaction (green dotted line) to open a bulk gap. (c) The Fidkowski-Kitaev (FK)-chain model. Each unit cell (large white circle) contains eight complex fermions, which can be split into eight a-type and eight b-type Majorana fermions. We couple the eight b-type Majorana fermions in each unit cell with the quartic FK interaction Eq. (4.1)(represented by the green lines) and we also couple four of the a-type Majorana fermions in the right side of a unit cell with the four a-type Majorana fermions on the left side of the adjacent unit cell using the FK interaction.

solvable only in the completely dimerized limit. However, the topological properties of the dimerized spin-$\frac{1}{2}$ chain, for example the existence of dangling spin-$\frac{1}{2}$ end states, persist as long as the bulk energy gap stays open, which is true even away from the exactly solvable (completely dimerized) point. Since the low-energy physics of the FK-chain reduces to the dimerized spin-$\frac{1}{2}$ chain when u is large, the FK chain model possess the same stability properties as the dimerized spin-$\frac{1}{2}$ chain.

The pair of boundary spin-$\frac{1}{2}$'s on each end of the FK chain are composed of four a-type Majorana fermions and are unstable in the presence of the most general time-reversal invariant perturbations. On the left boundary, the four Majorana fermions a^1, a^2, a^3 and a^4 will be unpaired. Since these four Majorana modes transform in the same way under the action of T, the only Hermitian and time-reversal invariant term we can add to the boundary Hamiltonian is $H_{bdy} = \lambda a^1 a^2 a^3 a^4$. This term is essentially a symmetrized Hubbard-like interaction, which can be seen by defining new complex fermions $\chi_1 = \frac{1}{2}(a^1 + ia^2)$ and $\chi_2 = \frac{1}{2}(a^3 + ia^4)$. In terms of the χ_i we have

$$H_{bdy} = -\lambda(2\chi_1^\dagger\chi_1 - 1)(2\chi_2^\dagger\chi_2 - 1) = -\lambda(-1)^{F_{\chi_1,\chi_2}} \tag{4.9}$$

where $(-1)^{F_{\chi_1,\chi_2}}$ is the local fermion parity at the boundary. This local Hamiltonian has two degenerate ground states $|0\rangle_g, |1\rangle_g$, and two degenerate excited states $|0\rangle_e, |1\rangle_e$. If $\lambda > 0$ ($\lambda < 0$) the ground states both have even (odd) fermion parity and vice-versa for the excited states. As we show in the next subsection, time-reversal acts non-trivially as $T_{bdy} = i\sigma^y K$ on both the ground and excited state subspaces independently. It follows immediately from Kramers' theorem that the remaining two-fold degeneracy of the boundary states is protected against arbitrary perturbations that do not break time-reversal symmetry. Thus, even when the local fermion parity is locked by H_{bdy}, the low-energy degrees of freedom on the edge still form a projective representation of the on-site time-reversal symmetry group[20,21], and the remaining degree of freedom in the lowest energy sector is a single spin-$\frac{1}{2}$.

When H_{bdy} is turned on, the local fermion parity is fixed across the entire chain in the low-energy sector. It is in this sector, i.e., when we can safely ignore fermion-parity changing excitations, that the properties of the FK chain are similar to the gapped (Haldane) phase of a spin-1 chain protected by inversion and time-reversal symmetry[36,37]. For both systems we have: (1) a gapped bulk and gapless spin-$\frac{1}{2}$ excitations at the boundary, and (2) T acts as $T^2 = 1$ on the fundamental degrees of freedom within each unit cell, but as $T^2 = -1$ on the fractionalized degrees of freedom at the ends of an open chain. The two systems are not identical, however, as the FK chain necessarily has a larger Hilbert space due to the fermionic nature of the local degrees of freedom. We also note that a topological phase transition between a $\nu = 4$ phase and a trivial $\nu = 0$ phase can be driven by turning on intra-cell FK couplings for the a-fermions and leaving the b- fermions unmodified, while keeping local fermion parity fixed in the low-energy subspace. As we discussed earlier in this subsection, in this limit the FK chain can be mapped onto a dimerized spin-$\frac{1}{2}$. It follows that the critical theory – when the intra-cell FK interaction strength matches that of the inter-cell one– is described by the $SU(2)_1$ Wess-Zumino-Witten conformal field theory[38].

C. Proof that 1D boundary states have $T^2 = -1$

In this subsection we prove that the time-reversal operator T, which satisfies $T^2 = 1$ in the bulk of the FK chain, instead obeys the anomalous relation $T^2 = -1$ on the boundary of the chain. This same result holds for the edge of the ordinary $\nu = 4$ Majorana chain, as was proven in Refs. 21, 33, and 34. Our result in this subsection represents an alternate method for proving that result.

The complex fermion operators $\chi_1 = \frac{1}{2}(a^1 + ia^2)$, $\chi_2 = \frac{1}{2}(a^3 + ia^4)$, which are used in the definition of H_{bdy}, obey the unconventional transformation $T\chi_j T^{-1} = \chi_j^\dagger$. Now, suppose $|\tilde{0}\rangle$ is the state annihilated by χ_1 and χ_2. Then H_{bdy}, with $\lambda > 0$, has the two ground states: $|\tilde{0}\rangle$ and $|\tilde{1}\rangle \equiv \chi_1^\dagger\chi_2^\dagger|\tilde{0}\rangle$. These states transform non-trivially under the action of T. To see this, first note that since T is anti-unitary, we have $\langle T\Psi|T\Psi\rangle = \langle\Psi|\Psi\rangle^* = \langle\Psi|\Psi\rangle$ ($\langle\Psi|\Psi\rangle$ is real) for any state $|\Psi\rangle$. In particular this means that $T|\Psi\rangle \neq 0$ if $\langle\Psi|\Psi\rangle \neq 0$. We see then that $T|\tilde{1}\rangle$ can only be non-zero if $T|\tilde{0}\rangle \propto |\tilde{1}\rangle$. This is because $T|\tilde{1}\rangle = \chi_1\chi_2 T|\tilde{0}\rangle$, which is zero unless $T|\tilde{0}\rangle \propto |\tilde{1}\rangle$. We can choose the convention $T|\tilde{0}\rangle = |\tilde{1}\rangle$, which just amounts to a choice of phase since $\chi_1^\dagger$ and $\chi_2^\dagger$ anti-commute. Using this rule we also find that $T|\tilde{1}\rangle = -|\tilde{0}\rangle$. So in the basis of ground states on the edge, $|\tilde{0}\rangle$ and $|\tilde{1}\rangle$, the time-reversal operator acts non-trivially as a matrix $T_{bdy} = i\sigma^y K$ (where K is complex conjugation in this basis), such that $(T_{bdy})^2 = -1$ at the edge of our system. It follows immediately from Kramer's theorem that the remaining double degeneracy of the boundary states is protected against arbitrary perturbations that do not break time-reversal symmetry. Thus, we see that the low energy degrees of freedom on the edge form a projective representation of the on-site $\mathbb{Z}_2$ symmetry group generated by T ($T^2 = 1$) on the local degrees of freedom in the bulk of the system.

D. Further Discussion

An interesting property of the FK-chain model is the fact that it only contains terms which are quartic in fermion creation and annihilation operators, thus this system has no free-fermion analogue. Indeed, the complete two-particle Green function $\mathcal{G}(\omega, k)$ (i.e., the matrix of two-point functions with the regular time-ordered Green functions on the diagonal blocks and the anomalous time-ordered Green functions on the off-diagonal blocks) for this model vanishes at $\omega = 0$, which means that there is no Bogoliubov-de-Gennes (BdG) mean-field Hamiltonian that captures the physical properties of this system (recall that if $\mathcal{G}(0, k) \neq 0$ then we can construct a BdG Hamiltonian $H_{BdG}(k) \sim \mathcal{G}^{-1}(0, k)$ which defines a free fermion system from which topological phases can be determined). In terms of local complex fermions, the Hamiltonian for the FK chain contains terms of the form $\psi_n^I\psi_n^J\psi_{n+1}^I\psi_{n+1}^J + \psi_n^{I,\dagger}\psi_n^{J,\dagger}\psi_{n+1}^{I,\dagger}\psi_{n+1}^{J,\dagger}$, leading to a non-vanishing anomalous four-point function with momentum-dependence. All four-point correlation functions can be calculated exactly in this completely dimerized model, and in particular, charge conservation symmetry is broken by $4e$ tetrads such as $\langle\psi_{k_1}^1\psi_{k_2}^2\psi_{k_3}^3\psi_{-k_1-k_2-k_3}^4\rangle \sim 1 + e^{ik_1+ik_2}$. If one tunes away from this exactly solvable point (e.g., by adding quadratic tunneling or pairing terms) then one could extract a quadratic Hamiltonian from the inverse of the two-point function. However, since the two-point function is still adiabatically connected to the zero matrix, this effective Hamiltonian will be topologically trivial. The essential features of the topological phase are still contained in the four-point functions, and we suspect that a bulk topological invariant could be constructed from the momentum-dependent four-point functions analogous to Refs. 39 and 40. Although in 1D this is not necessary since the projective symmetry formalism supersedes the Green function type invariants.

The existence of non-vanishing anomalous four-point functions for the FK-chain is a direct consequence of the fact that Eq. (4.1) breaks charge-conservation symmetry. For this model to arise microscopically we would expect this symmetry to be broken spontaneously via some "mean-field" like state of an eight-body interacting Hamiltonian, or from a mechanism analogous to the charge-$4e$ superconductivity formed from a melted pair-density wave state in Ref. 41. Thus, similar to Kitaev's interpretation of the Majorana chain as a mean-field description of a spontaneously generated topological p-wave superconductor, our model can be interpreted as a topological charge $4e$ superconductor. Indeed, the single spin $1/2$ degree of freedom at each boundary can be re-interpreted in terms of the complex fermions forming the topological $4e$ superconductor as follows. When the perturbation H_{bdy} of Eq. (4.9) is added to each end, the two remaining low-energy boundary states at a single end have the same local fermion parity. However, these two degenerate states differ by the combined action of the two fermion operators χ_1 and χ_2, and hence by exactly two fermions, i.e. a Cooper pair. Thus, we find that the Majorana end modes force states with an even and odd number of *Cooper pairs* to be degenerate. We will briefly return to this point in the conclusion.

V. HIGHER DIMENSIONAL EXAMPLES

A. Two-Dimensional Interaction-Enabled Topological Crystalline Phase

We now discuss an example of a 2D interaction-enabled TCI in the BDI class with additional translation and discrete rotation symmetry. Two-dimensional TCS phases in the BDI class with translation and discrete rotation symmetries were discussed in Refs. 5–19. Generically these TCS's can carry a number of different non-trivial topological invariants, each of which is stable in the presence of a certain subset of the symmetries of the model. We are interested in the *weak invariant*; an invariant which is stabilized by translation symmetry[42]. Heuristically, the weak invariant in 2D is a topological vector generated by stacking 1D topological wires into 2D, and is thus necessarily anisotropic. The stacks of topological wires define a 2D lattice with reciprocal lattice vectors $\mathbf{b}_1$ and $\mathbf{b}_2$ and the weak invariant takes the form

$$\mathbf{G}_\nu = \frac{\nu_1}{2}\mathbf{b}_1 + \frac{\nu_2}{2}\mathbf{b}_2 \ , \tag{5.1}$$

where ν_1 and ν_2 are integers for the BDI class, i.e., they match the BDI 1D topological invariant. As long as translation symmetry is protected, then two phases with different weak invariants cannot be adiabatically connected without either closing the gap or breaking a symmetry.

Just as for the 1D case, we want to require additional spatial symmetries. To be explicit let us choose C_4 rotation symmetry, which implies $\mathbf{b}_1 = \frac{2\pi}{a}\hat{\mathbf{x}}$ and $\mathbf{b}_2 = \frac{2\pi}{a}\hat{\mathbf{y}}$ (with lattice spacing a). Just like the case of ν under inversion in 1D, $\mathbf{G}_\nu$ transforms non-trivially (i.e., as a vector) under C_4 symmetry. Enforcing the symmetry constrains the weak index to satisfy $G^x_\nu = G^y_\nu$ and $G^y_\nu = -G^x_\nu$. Since $\nu_1, \nu_2 \in \mathbb{Z}$, the only solution is $\nu_1 = \nu_2 = 0$[12]. However, just as above, if we allow for interactions then $\nu_1, \nu_2 \in \mathbb{Z}_8$, which means that $\nu_1 = \nu_2 = 4$ is also a valid possibility, but one that requires strong interactions. In Fig. 2 we show a model realizing this non-trivial 2D state. The model is constructed out of orthogonally crossed 1D FK chains, and each unit cell contains 16 complex fermions $\psi^J_{\mathbf{n}}$, $J = 1, \ldots, 16$, where $\mathbf{n}$ now labels a site on the square lattice. This model exhibits a non-trivial weak invariant $\mathbf{G}_\nu = \frac{1}{2}(4\mathbf{b}_1 + 4\mathbf{b}_2)$, and could represent a 2D topological charge-$4e$ superconductor, using the same interpretation discussed for the 1D model. We note that an almost identical discussion could be had for C_2 rotation or a reflection symmetry with, for example, $\mathbf{G}_\nu = \frac{1}{2}(4\mathbf{b}_1)$ or $\frac{1}{2}(4\mathbf{b}_2)$.

A non-trivial $\mathbf{G}_\nu$ of this form implies that in a system with open boundary conditions, any unit cell on a boundary normal to $\mathbf{b}_1$ or $\mathbf{b}_2$ will contain four unpaired Majorana modes of the same type. Then, adding H_{bdy} reduces the low energy degrees of freedom in each boundary unit cell to a single spin-$1/2$. When time-reversal and translation symmetry are preserved, these boundary spin-$1/2$'s will generically form a gapless system with the same low energy description as the critical FK chain discussed above. However, the boundary theory of our 2D system cannot be realized in a 1D electronic system with the same symmetries, and hence the boundary is anomalous. To see this, note that the boundary of our 2D system consists solely of Majorana fermions of a single type (say a-type). But any 1D electronic system with T-symmetry consists of an equal number of a- and b- type Majorana modes. Suppose we take a 1D electronic system with four a- and b-type Majorana modes per unit cell, and try to gap the b-modes to make it identical to the anomalous edge. This procedure will be successful only if we break time-reversal or translation symmetry. For example, we could add quadratic tunneling terms between b-type Majorana modes within the same unit cell, but this would break time-reversal. A second possibility would be to couple the four b-type Majorana modes in unit cell $2n - 1$ with the four b-type Majorana modes in

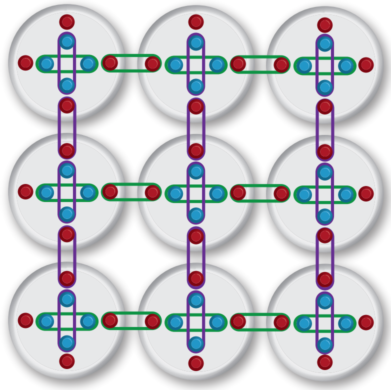

FIG. 2. A 2D model of a time-reversal invariant, C_4-symmetric topological crystalline superconductor with four unpaired Majorana fermions in each boundary unit cell. This model is made of crossed vertical and horizontal FK chains, and so each unit cell (large white circles) contains 16 complex (32 Majorana) fermions. To reduce clutter in the Fig., each red circle represents four a-type Majorana fermions and each blue circle represents four b-type Majorana fermions. The green lines indicate a FK interaction in the horizontal wires and the purple lines indicate a FK interaction in the vertical wires.

unit cell $2n$ with the FK interaction, but this would produce a dimerized pattern which breaks translation symmetry. Thus, the edge of our 2D system is anomalous, and this is a direct consequence of the fermionic nature of our system.

In addition to non-trivial boundary states, the crystalline symmetry-protected topology gives rise to topological qubits (i.e., non-Abelian excitations) localized on (semiclassical) lattice defects. Based on the work of Refs. 10,43–45 we can determine by inspection that a dislocation with Burgers vector $\mathbf{B}$ will have $\frac{1}{\pi}\mathbf{B}\cdot\mathbf{G}_\nu$ Majorana bound states at the core. Additionally, a vertex-type disclination with Frank angle $\Omega = \pm\pi/2$ will also trap a tetrad of unpaired a-type Majorana bound states, while a plaquette-type disclination with $\Omega = \pm\pi/2$ will not trap any unpaired Majorana modes. Each non-trivial defect Σ, of either kind, thus binds a decoupled tetrad of a-type Majorana fermions. Adding the local quartic perturbation (c.f. H_{bdy}) reduces the Majorana tetrad to a single spin-1/2 degree of freedom, which is identical to an end of the topological FK-chain. Therefore each of these defects carries a quantum dimension $d_\Sigma = 2$, which signifies their non-Abelian nature and ability to store quantum information non-locally in space.

Let us further consider the fusion properties of defects in our 2D interaction-enabled model of a weak topological superconductor. Consider two non-trivial defects (say a pair of dislocations) in the 2D TCS. On each defect the stable degree of freedom is an effective single spin-1/2. Analogous to the tensor product of a pair of spins $[\frac{1}{2}]\otimes[\frac{1}{2}] = [0]\oplus[1]$, a pair of separated defects Σ_1, Σ_2 is associated to a fourfold degeneracy seen by the defect fusion

$$\Sigma_1 \times \Sigma_2 = 1 + \psi_1\psi_2 + \psi_1\psi_3 + \psi_1\psi_4 \ . \qquad (5.2)$$

The vacuum channel 1 is the ground state if the dislocation pair is coupled by the FK interaction. It corresponds to the singlet channel for the pair of spins. The other three are time-reversal breaking ground states when the dislocations are coupled by Eq. (4.1) but with a reversed sign in front of the sum over even permutations (the sum over $\sigma \in A_4$). This corresponds to the ferromagnetic Heisenberg interaction and the three states are the tensor products $|\uparrow\uparrow\rangle_X, |\uparrow\uparrow\rangle_Y, |\uparrow\uparrow\rangle_Z$ with respect to spin-up in the x, y, z directions. It is convenient to choose these non-orthogonal, but still linearly independent, basis vectors. For example the $\psi_1\psi_2$ channel has a non-trivial vacuum expectation value for the Cooper pair $\langle\psi_1\psi_2\rangle = -\langle\psi_3\psi_4\rangle = i$. In a defect-less $4e$ superconductor, Cooper pairs are gapped excitations and are not responsible for transport at low temperatures. The presence of these non-Abelian defects could provide pinned sites between which Cooper pairs can teleport. A non-vanishing charge $2e$ tunneling between two normal BCS superconductor leads in contact with a $4e$ superconductor would therefore be a signature for the non-trivial topology. We note that since these defects are extrinsic/semiclassical, their projective braiding properties can be determined, but we leave this for future work.

B. Three-Dimensional Interaction-Enabled Topological Crystalline Phase

Our final example is an application of our mechanism to 3D time-reversal symmetric topological superconductors in the DIII class, with an additional reflection symmetry. Unlike class BDI in 1D, in this class the time-reversal symmetry operator acts as $T^2 = (-1)^F$, where $(-1)^F$ is the fermion parity. Non-interacting systems in class DIII are classified by an integer $\nu \in \mathbb{Z}$, however, interactions have been shown to reduce the classification to $\mathbb{Z}_{16}$[46–48]. In addition, it was shown in Ref. 13 that for systems without interactions, the addition of a certain reflection symmetry could reduce the classification from $\mathbb{Z}$ to 0 (i.e., only a trivial phase exists). This reflection operation has the effect of sending $\nu \to -\nu$, so according to our general mechanism, 3D TSC's in class DIII with the additional reflection symmetry should admit an interaction-enabled phase with $\nu = 8$. Let us now provide a more explicit discussion.

Consider a slab geometry of a 3D TSC in this class, in which the bulk of the material extends to infinity in the x and z directions, but with surfaces at the locations $y = \pm y_0$. The physical meaning of the topological invariant ν is that in such a geometry, the top surface will host $|\nu|$ gapless two-component Majorana (real) fermion fields $\chi_j(\mathbf{k})$, where $\mathbf{k} = (k_x, k_z)$ is momentum on the surface, with the Hamiltonian

$$H_{top} = \int \frac{d^2\mathbf{k}}{(2\pi)^2} \sum_{j=1}^{|\nu|} \chi_j^T(\mathbf{k})(k_x\sigma^z + \mathrm{sgn}(\nu)k_z\sigma^x)\chi_j(\mathbf{k}) \ , \tag{5.3}$$

while the bottom surface hosts another $|\nu|$ gapless Majorana fields $\tilde{\chi}_j(\mathbf{k})$ with Hamiltonian

$$H_{bottom} = \int \frac{d^2\mathbf{k}}{(2\pi)^2} \sum_{j=1}^{|\nu|} \tilde{\chi}_j^T(\mathbf{k})(k_x\sigma^z - \mathrm{sgn}(\nu)k_z\sigma^x)\tilde{\chi}_j(\mathbf{k}) \ . \tag{5.4}$$

Because of the combined time-reversal and charge-conjugation symmetries, the fermions on the top and bottom surfaces have a well-defined psuedo-chirality/winding and they are the opposite of each other (e.g., the sign of the k_z term in the kinetic energy is opposite). Time-reversal symmetry acts on these fields as $T\chi_j(\mathbf{k})T^{-1} = i\sigma^y\chi_j(-\mathbf{k})$, and likewise for the fields $\tilde{\chi}_j(\mathbf{k})$, and forbids quadratic terms which could gap out the fermions on one of the surfaces. However, it is possible to gap the fermions by "gluing" the top of the slab to the bottom by introducing symmetry mass terms of the form $i\tilde{\chi}_j(\mathbf{k})\sigma^x\chi_j(\mathbf{k}) - i\chi_j(\mathbf{k})\sigma^x\tilde{\chi}_j(\mathbf{k})$.

Now let us see the consequences of an additional reflection symmetry R_y which negates the y coordinate. In the simplest case R_y could act on the fields χ_j and $\tilde{\chi}_j$ as $R_y\chi_j(\mathbf{k})R_y^{-1} = \tilde{\chi}_j(\mathbf{k})$ and vice-versa. It is immediately clear that R_y sends $\nu \to -\nu$, so that the free fermion classification of systems in class DIII with additional R_y symmetry must be zero. As we know from previous examples, this is not the end of the story. Since interactions reduce the classification of systems in DIII to $\mathbb{Z}_{16}$, there should be an interaction-enabled TSC phase with $\nu = 8$ which possesses the reflection symmetry R_y. We now briefly discuss the construction of this state.

In the slab geometry we are considering, the interaction-enabled $\nu = 8$ state of class DIII should have eight gapless Majorana fermion fields on the top and bottom surface, but with the fields on the top and bottom surface having the *same* pseudo-chirality. One way to construct such a system is to take a slab with $\nu = -8$ and glue its top surface to the bottom of a slab with $\nu = 8$. At the interface we have 16 gapless Majorana fields with the same pseudo-chirality, and a method is needed to gap out this surface without breaking time-reversal symmetry. Remarkably, such a method was described in Ref. 47, and it requires strong interactions in a fundamental way. Therefore a non-trivial $\nu = 8$ phase in class DIII with reflection symmetry R_y does exist, but its construction requires strong interactions.

VI. CONCLUSION

Let us briefly discuss some possible experimental consequences. We have shown that our models can be interpreted as "mean-field" topological $4e$ superconductors. Here we briefly discuss a possible experimental signature of a topological $4e$ superconductor which would be interesting to explore in future work. Unlike an ordinary BCS superconductor, Cooper pairs are finite energy excitations in a gapped $4e$ superconductor since they are not the fundamental bosons in the condensate. It follows that in a generic charge $4e$ superconductor, states with even or odd numbers of Coopers pairs are not degenerate since there is a gap to create a Cooper pair in the system. However, in the FK chain model that we constructed, ground states with even and odd numbers of Cooper pairs are degenerate, and that degeneracy is a direct consequence of the presence of the four unpaired Majorana fermions at each end of an open FK chain. Just as a single Majorana end state allows for single-electron teleportation in a mesoscopic sample by forcing the ground states with even and odd fermion parity to be degenerate[49], the boundaries of the FK chain allow for Cooper pair teleportation since the ends force the ground states with an even and odd number of *Cooper pairs* to be degenerate.

In conclusion, we have shown that interactions can allow for a general mechanism to produce interacting topological phases that have no free-fermion description. The boundaries of these systems, and bulk topological defects, can trap non-Abelian excitations which could be used for the robust, non-local storage of quantum information. Our construction is easily extended to any system where $\mathbb{Z} \to \mathbb{Z}_{2n}$, even, as we discussed in Sec. V, 3D time-reversal invariant interacting topological superconductors ($\mathbb{Z} \to \mathbb{Z}_{16}$)[46–48] when extra reflection symmetries are imposed.

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[1] M. Z. Hasan and C. L. Kane, Rev. Mod. Phys. **82**, 3045 (2010).
[2] A. Kitaev, in *AIP Conference Proceedings*, Vol. 1134 (2009).
[3] S. Ryu, A. P. Schnyder, A. Furusaki, and A. W. Ludwig, New J. Phys. **12**, 065010 (2010).
[4] X.-L. Qi, T. L. Hughes, and S.-C. Zhang, Phys. Rev. B **78**, 195424 (2008).

[5] J. C. Y. Teo, L. Fu, and C. L. Kane, Phys. Rev. B **78**, 045426 (2008).
[6] L. Fu, Phys. Rev. Lett. **106**, 106802 (2011).
[7] T. L. Hughes, E. Prodan, and B. A. Bernevig, Phys. Rev. B **83**, 245132 (2011).
[8] A. M. Turner, Y. Zhang, R. S. Mong, and A. Vishwanath, Phys. Rev. B **85**, 165120 (2012).
[9] C. Fang, M. J. Gilbert, and B. A. Bernevig, Phys. Rev. B **86**, 115112 (2012).
[10] J. C. Teo and T. L. Hughes, Phys. Rev. Lett. **111**, 047006 (2013).
[11] R.-J. Slager, A. Mesaros, V. Juričić, and J. Zaanen, Nat. Phys. **9**, 98 (2012).
[12] W. A. Benalcazar, J. C. Teo, and T. L. Hughes, arXiv preprint arXiv:1311.0496 (2013).
[13] T. Morimoto and A. Furusaki, Phys. Rev. B **88**, 125129 (2013).
[14] C.-K. Chiu, H. Yao, and S. Ryu, Phys. Rev. B **88**, 075142 (2013).
[15] C. Fang, M. J. Gilbert, and B. A. Bernevig, Phys. Rev. B **87**, 035119 (2013).
[16] T. L. Hughes, H. Yao, and X.-L. Qi, arXiv:1303.1539 (2013).
[17] Y. Ueno, A. Yamakage, Y. Tanaka, and M. Sato, Phys. Rev. Lett. **111**, 087002 (2013).
[18] F. Zhang, C. Kane, and E. Mele, Phys. Rev. Lett. **111**, 056403 (2013).
[19] P. Jadaun, D. Xiao, Q. Niu, and S. K. Banerjee, Phys. Rev. B **88**, 085110 (2013).
[20] X. Chen, Z.-C. Gu, and X.-G. Wen, Phys. Rev. B **83**, 035107 (2011).
[21] X. Chen, Z.-C. Gu, and X.-G. Wen, Phys. Rev. B **84**, 235128 (2011).
[22] M. Levin and Z.-C. Gu, Phys. Rev. B **86**, 115109 (2012).
[23] A. Vishwanath and T. Senthil, Phys. Rev. X **3**, 011016 (2013).
[24] M. A. Metlitski, C. Kane, and M. P. Fisher, Phys. Rev. B **88**, 035131 (2013).
[25] Y.-M. Lu and A. Vishwanath, Phys. Rev. B **86**, 125119 (2012).
[26] X. Chen, Z.-C. Gu, Z.-X. Liu, and X.-G. Wen, Phys. Rev. B **87**, 155114 (2013).
[27] C. Wang, A. C. Potter, and T. Senthil, Science **343**, 629 (2014).
[28] We will usually not distinguish between SPTs protected by local symmetries, and crystalline SPTs protected by possibly non-local point-group symmetries.
[29] T. Neupert, C. Chamon, C. Mudry, and R. Thomale, Phys. Rev. B **90**, 205101 (2014).
[30] A. Y. Kitaev, Phys. Usp. **44**, 131 (2001).
[31] A. Altland and M. R. Zirnbauer, Phys. Rev. B **55**, 1142 (1997).
[32] L. Fidkowski and A. Kitaev, Phys. Rev. B **81**, 134509 (2010).
[33] L. Fidkowski and A. Kitaev, Phys. Rev. B **83**, 075103 (2011).
[34] A. M. Turner, F. Pollmann, and E. Berg, Phys. Rev. B **83**, 075102 (2011).
[35] S. Tewari and J. D. Sau, Phys. Rev. Lett. **109**, 150408 (2012).
[36] F. D. M. Haldane, Phys. Lett. A **93**, 464 (1983).
[37] I. Affleck, T. Kennedy, E. H. Lieb, and H. Tasaki, Commun. Math. Phys. **115**, 477 (1988).
[38] I. Affleck and F. Haldane, Phys. Rev. B **36**, 5291 (1987).
[39] V. Gurarie, Phys. Rev. B **83**, 085426 (2011).
[40] S. R. Manmana, A. M. Essin, R. M. Noack, and V. Gurarie, Phys. Rev. B **86**, 205119 (2012).
[41] E. Berg, E. Fradkin, and S. A. Kivelson, Nat. Phys. **5**, 830 (2009).
[42] L. Fu and C. L. Kane, Phys. Rev. B **76**, 045302 (2007).
[43] Y. Ran, Y. Zhang, and A. Vishwanath, Nat. Phys **5**, 298 (2009).
[44] J. C. Y. Teo and C. L. Kane, Phys. Rev. B **82**, 115120 (2010).
[45] S. Gopalakrishnan, J. C. Teo, and T. L. Hughes, Phys. Rev. Lett. **111**, 025304 (2013).
[46] L. Fidkowski, X. Chen, and A. Vishwanath, Phys. Rev. X **3**, 041016 (2013).
[47] M. A. Metlitski, L. Fidkowski, X. Chen, and A. Vishwanath, arXiv preprint arXiv:1406.3032 (2014).
[48] C. Wang and T. Senthil, Phys. Rev. B **89**, 195124 (2014).
[49] L. Fu, Phys. Rev. Lett. **104**, 056402 (2010).