ν=1/2 Landau level: Half-empty versus half-full

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Phys. Rev. B **93**, 085405 — Published 2 February 2016

DOI: 10.1103/PhysRevB.93.085405

The $\nu = \frac{1}{2}$ Landau level: half-empty versus half-full

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We show here that an extension of the Hamiltonian theory developed by us over the years furnishes a composite fermion (CF) description of the $\nu = \frac{1}{2}$ state that is particle-hole (PH) symmetric, has a charge density that obeys the magnetic translation algebra of the lowest Landau level (LLL), and exhibits cherished ideas from highly successful wave functions, such as a neutral quasi-particle with a certain dipole moment related to its momentum. We also a provide an extension away from $\nu = \frac{1}{2}$ which has the features from $\nu = \frac{1}{2}$ and implements the the PH transformation on the LLL as an anti-unitary operator $\mathcal{T}$ with $\mathcal{T}^2 = -1$. This extension of our past work was inspired by Son, who showed that the CF may be viewed as a Dirac fermion on which the particle-hole transformation of LLL electrons is realized as time-reversal, and Wang and Senthil who provided a very attractive interpretation of the CF as the bound state of a semion and anti-semion of charge $\pm\frac{e}{2}$. Along the way we also found a representation with all the features listed above except that now $\mathcal{T}^2 = +1$. We suspect it corresponds to an emergent charge-conjugation symmetry of the $\nu = 1$ boson problem analyzed by Read.

PACS numbers:

I. INTRODUCTION TO THE PROBLEM

Consider electrons in high magnetic field[1,2] partially filling the Lowest Landau Level (LLL) in the limit when the cyclotron frequency $\omega_c \to \infty$ is much larger than the interaction. In this limit one expects a complete description entirely in terms of the LLL states. A partially occupied band of electrons may be equally well described in terms of electrons on top of an empty band or holes depopulating the filled band. At filling factor $\nu = \frac{1}{2}$, for translationally invariant two-body interactions, the Hamiltonian has particle-hole (PH) *symmetry*[3], and one should be able to develop a treatment in which this symmetry is manifest. In addition it must be possible to relate the physics at ν to that at $1-\nu$ away from $\nu = \frac{1}{2}$ by a PH *transformation*. This had proven elusive until recent work[4–9], inspired by the work of Son[4]. It should be noted that this problem is intimately related to the surface states of strongly correlated, three-dimensional time-reversal invariant topological insulators[5,6], and that numerical work[7,8] (an early example being[10]) confirms the particle-hole symmetry of the $\nu = \frac{1}{2}$ ground state in the LLL.

A class of successful approaches to the Fractional Quantum Hall Effect (FQHE) requires flux attachment, that is, transforming the electrons into either Composite Bosons (CB)[11] by attaching an odd number of flux quanta, or Composite Fermions[12,13], by attaching an even number. At half-filling this turns the electron problem to that of CF's that see zero field on average and form a Fermi liquid[14,15], as elucidated in detail by Halperin, Lee, and Read (HLR)[15]. A similar Fermi surface arises in the problem of the hard-core bosons at $\nu = 1$ in the LLL analyzed by Read[16]. By attaching one flux quantum to each of the bosons he turns them into fermions that see zero field on the average.

When implemented in the wave function language and projected to the LLL, Composite Fermions (CFs) produce excellent parameter-free wave functions[2,12] for the Jain fractions of the type $\nu = \frac{p}{2p+1}$, and the Rezayi-Read[17] wave function for the gapless $\nu = \frac{1}{2}$ state.

There are at least two types of particles called CF's in the past fractional quantum Hall (FQH) literature. If we work in the complete Hilbert space of the electron, flux attachment in Chern-Simons (CS) theory[11,13,15] leads to a particle of charge e (the electron charge). For such a particle one can derive, at $\nu = \frac{1}{2}$, the constraint $\sigma_{xy}^{CF} = -\frac{1}{2}\frac{e^2}{h}$ following Lee, Krotov, Gan, and Kivelson[3]. We shall refer to this as the Chern-Simons CF. The other is the CF that resides entirely in the LLL and the one we will focus on. At $\nu = \frac{1}{2}$ attachment of the double-vortex (now double-zero and not just a phase 4π) drives away a charge $-e$ and leaves us with a neutral CF. This point of view was emphasized by Read[17] who also argued that the CF of momentum $\mathbf{p}$ will have a dipole moment $\hat{z} \times \mathbf{p} l^2$ where

$$l^2 = \frac{1}{eB}. \tag{1}$$

At the moment, there are conflicting claims about $\nu = \frac{1}{2}$. On the one hand are arguments from Chern-Simons theory that there are two distinct states, a particle-CF-Fermi sea and a hole-CF-Fermi sea[18]. On the other hand are numerical calculations in the wave-function language[7] or by exact diagonalization[8] which show a unique, particle-hole symmetric state at $\nu = \frac{1}{2}$.

In this work we will not attempt to resolve the controversy, but begin with the premise that the $\nu = \frac{1}{2}$ state does have particle-hole symmetry in the lowest Landau level.

Now for the main business of this paper. The work of HLR[15] which leads to a Fermi-surface for the CS-CFs is not particle-hole symmetric at $\nu = \frac{1}{2}$. One does not expect it to be since the symmetry is emergent only in the limit $\omega_0 \to \infty$ or $m_e \to 0$ which is problematic in their approach. But is there a description in which PH symmetry is manifest at $\nu = \frac{1}{2}$? Is there a way to relate the

physics at ν to that at $1-\nu$? An affirmative answer was given recently by Son[4]. The underlying physical picture was provided by Wang and Senthil (WS)[5]. Connections of this problem to the surface states of 3D time-reversal invariant topological insulators have also been elucidated in recent work[5,6].

In this paper we will re-examine and extend our Hamiltonian theory[19] in light of these developments.

The heart of our Hamiltonian approach[19] is to define the LLL problem algebraically in terms of the commutation rules of the projected electron density $\bar{\rho}(\mathbf{q})$, which alone enters the LLL Hamiltonian and obeys the magnetic translation algebra. Having defined it thus, the next step is to represent this algebra faithfully in a larger fermionic space subject to some constraints. The reason is that in this new space there is a natural Hartree-Fock state at the Jain fractions and the limiting case $\nu = \frac{1}{2}$.

When we re-examine our approach in the light of Son's work[4] we find several new results which we report here.

The first is that the most straightforward representation of a neutral CF in terms of a nonrelativistic one-component CF yields a description of an LLL system in which there is an anti-unitary operator $\mathcal{T}$ with $\mathcal{T}^2 = +1$ which plays the role of time-reversal on the CF's and charge conjugation on the physical charge. It exchanges the role of a (hard core) boson and a single vortex. Since we know that the electronic LLL problem must have $\mathcal{T}^2 = -1$[20] (confirmed by recent numerics[8]), this most probably describes an emergent charge-conjugation symmetry of the $\nu = 1$ boson problem in the LLL[16].

Secondly, after the work of Son[4] and WS[5], we realized that we could represent the magnetic translation algebra in the space of a two-component Dirac CF, whose number is always half the number of flux quanta, regardless of the number of electrons. Now we find exactly what Son did: An anti-unitary operator $\mathcal{T}$ with $\mathcal{T}^2 = -1$ which plays the role of time-reversal on the CF's and charge conjugation on the electronic charge. In addition, we obtain two representations of the physical charge density at all ν. One has a Hamiltonian with a set of constraints commuting with it, but does not manifestly show the neutrality of the Dirac CF or the PH symmetry. The other shows both symmetries in a manifest way, but ignores the constraints that limit the larger space to the LLL sector.

But for Son's work it would not have occurred to us to bring in Dirac fermions, because we were always insistent on working in a space that was adiabatically connected to the single-component electron. This was to ensure that we did not represent the problem algebraically in a space that had no bearing on the LLL problem. Furthermore, it would also not have occurred to us, who tried to implement Jain's construction using operators, to tie the number of CFs to the flux as Son did (and not the number of electrons). We thank Senthil for emphasizing the importance of this point, which in the end was what made it possible to extend the GMP algebra away from $\nu = \frac{1}{2}$ when working with Dirac fermions.

It is our hope that the explicit representation of the electronic charge density obeying the GMP algebra, and a neutral fermion of the right dipole moment paves the way for many operator based calculations.

Now for the organization of the paper. First we will furnish a telegraphic introduction to the Hamiltonian theory[19] citing only those results germane to this paper. Next we will consider the most natural representation of the algebraic problem: in terms of a one-component nonrelativistic CF, whose number is equal to the number of electrons. This gives us a theory which has a PH symmetry with $\mathcal{T}^2 = 1$, possibly pertaining to an emergent charge-conjugation symmetry of the $\nu = 1$ hard-core boson problem[16]. Next we will demonstrate that the magnetic translation algebra can be realized in the space of a Dirac CF, whose number is half the number of flux quanta and $\mathcal{T}^2 = -1$. This representation works both at $\nu = \frac{1}{2}$ and away from it. At $\nu = \frac{1}{2}$ it provides a representation of the physical charge density that realizes the PH symmetry in a manifest way. Away from $\nu = \frac{1}{2}$, we show how the PH transformation of the LLL electrons (implemented by $\mathcal{T}$) relates $\nu \leftrightarrow 1-\nu$. A summary follows.

II. HAMILTONIAN THEORY: WHY AND HOW?

The problem of interacting electrons in the LLL is defined by the LLL-projected Hamiltonian

$$\bar{H} = \frac{1}{2}\sum_{\mathbf{q}} \bar{\rho}(\mathbf{q}) v_{ee}(\mathbf{q}) \bar{\rho}(-\mathbf{q}), \tag{2}$$

where $\bar{\rho}(\mathbf{q})$ is the electron density projected to the LLL

$$\bar{\rho}(\mathbf{q}) = \sum_j e^{-i\mathbf{q}\cdot\mathbf{R}^e_j} \tag{3}$$

where a factor $e^{-q^2 l^2/4}$ from each $\bar{\rho}(\mathbf{q})$ has been absorbed in the electron-electron potential v_{ee}, and $\mathbf{R}^e$ is the electron guiding-center coordinate obeying

$$\left[\mathbf{R}^e_x, \mathbf{R}^e_y\right] = -il^2. \tag{4}$$

As a result of Eqn.4, $\bar{\rho}(\mathbf{q})$ obeys the Girvin, MacDonald and Platzman[22] (GMP) or magnetic translation algebra:

$$[\bar{\rho}(\mathbf{q}), \bar{\rho}(\mathbf{q}')] = 2i \sin\left[\frac{l^2}{2}\mathbf{q}\times\mathbf{q}'\right] \bar{\rho}(\mathbf{q}+\mathbf{q}'). \tag{5}$$

The mathematical problem is defined by the Hamiltonian $\bar{H}$ and the GMP algebra of the projected charge density entering it. Of course, the answer could vary with the space in which we represent this algebra. (Compare the spin-$\frac{1}{2}$ and spin-1 chains.) The original electron problem in defined in the electronic LLL Hilbert space. Now the trouble with formulating the problem in the electronic space is that there is no Hartree-Fock (HF) state due to the huge degeneracy of the partially filled LLL.

Jain[12] beats this by switching to the CF which sees a weaker field and fills exactly p Landau levels for the fractions

$$\nu = \frac{p}{2p+1}. \tag{6}$$

Motivated by Jain, we will start by using the Hilbert space of a one-component non-relativistic fermion that sees just this field to represent the Hamiltonian of Eq. (2). *The number of CFs is equal to the number of electrons in this construction.*

Let us trace some of the steps along the way to the final picture. More details are provided in the Appendix. Starting with the CS theory of fermions in the full Hilbert space[19], we trade the CS gauge field (whose components are conjugate) for magnetoplasmon oscillators à la Bohm-Pines[21]. When we decouple the oscillators and freeze them in the ground state, we obtain an LLL description of the projected electron density $\bar{\rho}(\mathbf{q})$ and the constraints $\bar{\chi}(\mathbf{q})$ that pay for the collective oscillator degrees of freedom. We could only derive expressions for $\bar{\rho}(\mathbf{q})$and $\bar{\chi}(\mathbf{q})$ at small q. The series were exponentiated[19] to form expressions valid all q, with the nice feature that $\bar{\rho}(\mathbf{q})$ obeyed the GMP algebra and $\bar{\chi}(\mathbf{q})$ commuted with $\bar{\rho}(\mathbf{q})$ and hence $\bar{H}$ and closed to form a GMP-like algebra of a particle with the charge of the double-vortex.

In this connection, which evolves from the CS theory, the number of CFs equals the number of electrons ($N_{CF} = N_e$) and the (spin-polarized) fermion has only one-component.

Here is the final picture: The CF experiences a weaker magnetic field B^*, just right to fill p Landau levels, as envisaged by Jain. That is encoded in the CF velocity operator which obeys

$$\left[\mathbf{\Pi}^*_x, \mathbf{\Pi}^*_y\right] = ieB^* = ie^*B = \frac{i}{l^{*2}} = i\frac{1-c^2}{l^2} \tag{7}$$

where

$$c^2 = 2\nu = \frac{2p}{p+1}. \tag{8}$$

For example, at $\nu = \frac{1}{3}$ we have $p = 1, c^2 = \frac{2}{3}, e^* = \frac{e}{3}$ and at $\nu = \frac{1}{2}$, we have $c = 1$ and $p = \infty$.

Now we introduce in this (full fermionic) space a pair of coordinates

$$\mathbf{R}^e = \mathbf{r} - \frac{l^2}{1+c}\hat{\mathbf{z}} \times \mathbf{\Pi}^* \tag{9}$$

which obey

$$\left[\mathbf{R}^e_x, \mathbf{R}^e_y\right] = -il^2. \tag{10}$$

We recognize this as the algebra of the guiding center coordinate of the electron. This ensures that the corresponding density

$$\bar{\rho}(\mathbf{q}) = \sum_j e^{-i\mathbf{q}\cdot\mathbf{R}^e_j} \tag{11}$$

obeys the GMP algebra. The LLL-projected Hamiltonian is represented *in the CF space* by

$$\bar{H} \;=\; \frac{1}{2}\sum_{\mathbf{q}} \bar{\rho}(\mathbf{q}) v_{ee}(\mathbf{q}) \bar{\rho}(-\mathbf{q}). \tag{12}$$

Although $\bar{H}$ is now written in terms of a $\bar{\rho}(\mathbf{q})$ which obeys the same GMP algebra as the one in Eqn. 2, there is a big difference. It is now expressed in terms of CF coordinates in their Hilbert space, and *there is a unique HF state by design*: with p-filled CF Landau levels.

In the CF space there is room for another canonical pair besides $\mathbf{R}_e$

$$\mathbf{R}^v = \mathbf{r} + \frac{l^2}{c(1+c)}\hat{\mathbf{z}} \times \mathbf{\Pi}^* \tag{13}$$

We call $\mathbf{R}_v$ the guiding center coordinate of the double vortex since it has the same charge $-2\nu = -c^2$, as can be seen by the commutator

$$\left[\mathbf{R}^v_x, \mathbf{R}^v_y\right] = i\frac{l^2}{c^2}. \tag{14}$$

Finally the two conjugate pairs commute:

$$[\mathbf{R}^e, \mathbf{R}^v] \;=\; 0. \tag{15}$$

Consider the densities formed by $\mathbf{R}_v$:

$$\bar{\chi}(\mathbf{q}) \;=\; \sum_j \exp\left[-i\mathbf{q}\cdot\mathbf{R}^v_j\right]. \tag{16}$$

They obey

$$[\bar{\chi}(\mathbf{q}), \bar{\chi}(\mathbf{q}')] = -2i\sin\left[\frac{l^2}{2c^2}\mathbf{q}\times\mathbf{q}'\right]\bar{\chi}(\mathbf{q}+\mathbf{q}'). \tag{17}$$

and commute with $\bar{\rho}(\mathbf{q})$:

$$[\bar{\rho}(\mathbf{q}), \bar{\chi}(\mathbf{q}')] \equiv 0 \tag{18}$$

We see that $\bar{H}$ commutes with a huge family of operators $\bar{\chi}(\mathbf{q})$

$$\left[\bar{H}, \bar{\chi}(\mathbf{q})\right] = 0 \tag{19}$$

that do not enter $\bar{H}$ and close under commutation. The Appendix shows that the $\bar{\chi}(\mathbf{q})$ are the constraints that pay for the magnetoplasmon oscillators that were introduced à la Bohm-Pines[21] and decoupled.

If one wanted to skip the intermediate steps one could simply begin with Eqns. 7 - 12 which pose the LLL problem of electrons in the larger space of the CF, *preserving the algebra of $\bar{\rho}(\mathbf{q})$ and $\bar{H}$*. The physical sector is defined by $\bar{\chi}(\mathbf{q}) \simeq 0$ where $\simeq$ means "when $\bar{\chi}(\mathbf{q})$ appears within correlation functions".

While a HF state exists in the CF space, the result of naive HF calculations is a mixed bag. On the plus side, one sees the K-invariance of Haldane in the HF spectrum at $\nu = \frac{1}{2}$. On the minus side, $\bar{\rho}(\mathbf{q} = 0)$, the charge associated with $\bar{\rho}(\mathbf{q})$ seems to be e and not $e^* = e(1-c^2)$, there

is no evidence of the dipole at $\nu = \frac{1}{2}$, and the structure factor is $S(q) \simeq q^2$ and not q^4, as required by Kohn's theorem. However, these features can be recovered upon imposing the constraints via a conserving approximation à la Kadanoff and Baym,[23] which enforces $\bar{\chi}(\mathbf{q}) \simeq 0$. When Read[16] carried out this approximation for $\nu = 1$ bosons (also a CF Fermi liquid) he found $S(q) \simeq q^3 \log q$ and the over-damped mode of HLR. Murthy[24] found $S(q) \simeq q^4$ for Jain fractions using the conserving approximation.

III. THE PREFERRED DENSITY $\bar{\rho}_p(\mathbf{q})$.

At this point we proposed a short-cut to some of these results obtained in the conserving approximation. We argued that in an exact theory which obeyed the constraint, we could replace $\bar{\rho}(\mathbf{q})$ by $\bar{\rho}(\mathbf{q}) - \alpha\bar{\chi}(\mathbf{q})$ for any value of α. While all values of α were equal in the exact theory, the following one stood out as the *preferred density* in the HF calculation:

$$\bar{\rho}_p = \bar{\rho}(\mathbf{q}) - c^2\bar{\chi}(\mathbf{q}) \tag{20}$$

With this choice, one found, as $q \to 0$, the charge $(1-c^2)e = e^*$, $S(q) \simeq q^4$ and the correct dipole moment:

$$\bar{\rho}_p(\mathbf{q}) = \sum_j e^{-i\mathbf{q}\cdot\mathbf{r}_j}\left((1-c^2) - il^2\mathbf{q}\times\mathbf{\Pi}^*_j + \ldots\right) \tag{21}$$

At $\nu = \frac{1}{2}$, $c = 1$, the fermion becomes neutral, $\mathbf{\Pi}^*_j \to \mathbf{p}$ and we regain the dipole moment by Read's wave function analysis. In this approach using

$$\bar{H}_p = \frac{1}{2}\sum_{\mathbf{q}} \bar{\rho}_p(\mathbf{q})v_{ee}(q)\bar{\rho}_p(-\mathbf{q}) \tag{22}$$

one gets many results (at small $\mathbf{q}$) of the conserving approximation *at tree-level*. On the other hand, $\bar{\rho}_p(\mathbf{q})$ does not obey the GMP algebra except at small $\mathbf{q}$, and there is no systematic way of deriving the overdamped mode of HLR[15].

We will see shortly that this *ad hoc* recipe becomes a legitimate option in the algebraic approach *but only at* $\nu = \frac{1}{2}$.

Going forward, it should be borne in mind that *when we work with $\bar{\rho}_p$ and $\bar{H}_p$, there are no commuting constraints to select states CF space corresponding to electrons in the LLL* . Instead the role of $\bar{\chi}(\mathbf{q})$ is to represent the charge density of the double-vortex. In the spirit of a low-energy effective description, we hope, motivated by Jain[12], that despite the larger CF Hilbert space, low energy properties of the FQH states in the LLL will be correctly reproduced.

We addressed several quantitative questions[25] using $\bar{H}_p$ and found it to be a 10-20% theory for gaps, polarizations etc., as long as there was an infrared cut off in the form of a gap or temperature.

IV. THE CF-FERMI SEA FOLLOWING JAIN

Let us focus on finding a formalism which exhibits the PH symmetry at $\nu = \frac{1}{2}$. This is a special point in other ways as well. Here the electron and vortex have exactly opposite charges, the CF is electrically neutral and sees no magnetic field. But even more special is the following: *the preferred density $\bar{\rho}_p(\mathbf{q})$ itself obeys the GMP algebra*:

$$\begin{aligned}&[\bar{\rho}(\mathbf{q}) - \bar{\chi}(\mathbf{q}), \bar{\rho}(\mathbf{q}') - \bar{\chi}(\mathbf{q}')] = [\bar{\rho}(\mathbf{q}), \bar{\rho}(\mathbf{q}')] + [\bar{\chi}(\mathbf{q}), \bar{\chi}(\mathbf{q}')] \\ =\ & 2i\sin\left[\frac{l^2}{2}\mathbf{q}\times\mathbf{q}'\right](\bar{\rho}(\mathbf{q}+\mathbf{q}') - \bar{\chi}(\mathbf{q}+\mathbf{q}'))\,.\end{aligned}$$

Thus if we follow the algebraic route, *at $\nu = \frac{1}{2}$, $\bar{\rho}_p(\mathbf{q})$ is another candidate besides $\bar{\rho}(\mathbf{q})$ that satisfies the GMP algebra.* Thus the *ad hoc* introduction of $\bar{\rho}_p(\mathbf{q})$ as a short cut to the results of the conserving approximation now becomes a legitimate alternative to $\bar{\rho}(\mathbf{q})$.

In other words there are two ways to obtain a realization of the GMP algebra. The first is to find the electron guiding center coordinate in the CF space and then to exponentiate it, as in Eqns. 9 and 11. The other is to directly go for the preferred densities densities $\bar{\rho}_p = \bar{\rho}(\mathbf{q}) - \bar{\chi}(\mathbf{q})$ which are not exponentials of anything simple.

Thus, unlike $\bar{\rho}(\mathbf{q})$, which evolved adiabatically from the CS formulation, the use of $\bar{\rho}_p(\mathbf{q})$ represents a leap based entirely on algebraic considerations. There is no reason to believe it has to be realized in the space of the one-component fermion, or that if it is realized in another space, that representation has any relevance to the original LLL problem.

To begin with, let us assume that $\bar{\rho}_p(\mathbf{q})$ lives in the space of the one-component fermion and see what happens.

First consider the anti-unitary time-reversal operation in this representation. It is easy to see in first quantization: as $\mathbf{\Pi}^* \to \mathbf{p}$ and $c = 1$ at $\nu = \frac{1}{2}$,

$$\begin{aligned}\mathbf{R}_e &= \mathbf{r} - \frac{l^2}{1+c}\hat{\mathbf{z}}\times\mathbf{\Pi}^* = \mathbf{r} - \frac{l^2}{2}\hat{\mathbf{z}}\times\mathbf{p} \\ \mathbf{R}_v &= \mathbf{r} + \frac{l^2}{c(1+c)}\hat{\mathbf{z}}\times\mathbf{\Pi}^* = \mathbf{r} + \frac{l^2}{2}\hat{\mathbf{z}}\times\mathbf{p}\end{aligned} \tag{23}$$

Note that

$$\begin{aligned}\frac{\mathbf{R}_e + \mathbf{R}_v}{2} &= \mathbf{r}\ \text{(CF is midway between } e \text{ and } v) \\ \mathbf{R}_e - \mathbf{R}_v &= -l^2\hat{\mathbf{z}}\times\mathbf{p} \quad \text{(CF dipole moment)}\end{aligned} \tag{24}$$

Under *time-reversal* $\mathcal{T}$, we see that

$$\begin{aligned}\mathcal{T}:&\quad \mathbf{R}_e \leftrightarrow \mathbf{R}_v \\ \mathcal{T}:&\quad \bar{\rho}(\mathbf{q}) \leftrightarrow \bar{\chi}(-\mathbf{q}) \\ \mathcal{T}:&\quad \bar{\rho}_p(\mathbf{q}) \to -\bar{\rho}_p(-\mathbf{q}).\end{aligned} \tag{25}$$

The last equation informs us that $\mathcal{T}$ has effected the PH transformation on the electronic charge. The Hamiltonian being bilinear in $\bar{\rho}_p$ remains invariant.

In second quantization where

$$\bar{\rho}(\mathbf{q}) = \sum_{\mathbf{k}} d^\dagger_{\mathbf{k}-\mathbf{q}} e^{-\frac{il^2}{2}\mathbf{q}\times\mathbf{k}} d_{\mathbf{k}}$$

$$\bar{\chi}(\mathbf{q}) \;=\; \sum_{\mathbf{k}} d^{\dagger}_{\mathbf{k}-\mathbf{q}} e^{\frac{il^2}{2}\mathbf{q}\times\mathbf{k}} d_{\mathbf{k}}. \tag{26}$$

and the action of $\mathcal{T}$ is

$$\begin{aligned} \mathcal{T} d^{\dagger}_{\mathbf{k}} \mathcal{T}^{-1} &= d^{\dagger}_{-\mathbf{k}} \\ \mathcal{T} d_{\mathbf{k}} \mathcal{T}^{-1} &= d_{-\mathbf{k}} \\ \mathcal{T} i \mathcal{T}^{-1} &= -i, \end{aligned} \tag{27}$$

we have

$$\begin{aligned} \mathcal{T}\bar{\rho}(\mathbf{q})\mathcal{T}^{-1} &= \sum_{\mathbf{k}} d^{\dagger}_{-\mathbf{k}-\mathbf{q}} e^{\frac{il^2}{2}\mathbf{q}\times\mathbf{k}} d_{-\mathbf{k}} \\ &= \sum_{\mathbf{k}} d^{\dagger}_{\mathbf{k}-\mathbf{q}} e^{-\frac{il^2}{2}\mathbf{q}\times\mathbf{k}} d_{\mathbf{k}} \\ &= \bar{\chi}(-\mathbf{q}) \end{aligned} \tag{28}$$

and likewise

$$\mathcal{T}\bar{\chi}(\mathbf{q})\mathcal{T}^{-1} = \bar{\rho}(-\mathbf{q}). \tag{29}$$

Consequently

$$\begin{aligned} \mathcal{T}\bar{\rho}_p(\mathbf{q})\mathcal{T}^{-1} &= \mathcal{T}(\bar{\rho}(\mathbf{q}) - \bar{\chi}(\mathbf{q}))\mathcal{T}^{-1} \\ &= -\bar{\rho}_p(-\mathbf{q}). \end{aligned} \tag{30}$$

So this symmetry reverses the sign of the (preferred) physical charge density (represented now by $\bar{\rho}_p$), making it appropriate to call it charge-conjugation. This is reminiscent of Son's approach[4], but there is a crucial difference and we are grateful to both Son and Senthil for emphasizing this: The anti-unitary operation $\mathcal{T}$ we have proposed obeys

$$\mathcal{T}^2 = +1 \tag{31}$$

whereas the PH symmetry of the electronic LLL problem obeys[20]

$$\mathcal{T}^2 = -1. \tag{32}$$

Consequently this model cannot be a representation of the electronic $\nu = \frac{1}{2}$ LLL problem. And yet it is a model in which there is a CF-Fermi surface, which manifestly displays the GMP algebra of the charge density, and the dipolar picture. If it is not the $\nu = \frac{1}{2}$ problem of electrons, what is it? There is one obvious choice: the $\nu = 1$ hard-core boson problem analyzed by Read[16]. Indeed if we switch from the hard-core bosons to CFs by attaching one unit of statistical flux and followed our Bohm-Pines approach we would get the same expressions for $\bar{\rho}(\mathbf{q})$ and $\bar{\chi}(\mathbf{q})$ at small q. If we extended to small-q results to all q to satisfy the GMP algebra, we would get exactly the charge density and constraint algebra that Read[16] obtained for the boson problem. The dipole here is also made of charge ± 1 objects. It can be quantized as a single-component fermion given the absence of extra phase factors from the boson-vortex bound state.

If our $\bar{H}_p$ indeed describes the $\nu = 1$ boson problem, it suggests that there is an emergent charge-conjugation symmetry of the $\nu = 1$ boson problem as well. This formulation may be of relevance to the high-field superconductor-insulator transition in which indications of boson-vortex duality have been seen[26]. We have a concrete representation of the electronic charge density in $\bar{\rho}_p(\mathbf{q})$ for this case, which permits us to do many of the detailed calculations of response functions, even at $T > 0$.

V. HAMILTONIAN FORMULATION OF SON'S DIRAC CFS AT $\nu = \frac{1}{2}$

To describe the electronic $\nu = \frac{1}{2}$ problem, we need $\mathcal{T}^2 = -1$. This is impossible in the space of a single-component fermion. Having seen that the representation of $\bar{\rho}_p(\mathbf{q})$ need not be continuously connected to the primordial problem, we can seek other options. We confess we could not have made any headway till we turned to the very surprising option Son[4] provides us, of a Dirac fermion. This option is buttressed by Wang and Senthil[5] who give us nice a physical picture of why this is so: one of the double zeros must lie on the electron (by the Pauli principle) turning it into a charge $\frac{1}{2}$ semion. The remaining vortex is a charge $-\frac{1}{2}$ anti-semion. The pair is quantized as a spinor (as shown in the Appendix of WS[5]) that appears in Son's Dirac equation. Given this internal structure the phase of π due to circumnavigating the Fermi circle follows.

One may feel that where we place the two vortices (none on the electron or just one on the electron which makes it a semion) is a short distance feature, that both descriptions have the same long distance features: a net charge of zero and the same dipole moment $\simeq \hat{\mathbf{z}} \times \mathbf{p} l^2$ which appears at $\nu = \frac{1}{2}$ fermions and $\nu = 1$ bosons. However, the difference in internal structure leads to a profound difference in the topology of the Fermi surface, one with a Berry phase and one without.

Let us now implement our algebraic approach starting with a Dirac fermion, *which will be our composite fermion.* The number of these CFs, which so far equaled the number of electrons, is also equal to half the number of flux quanta penetrating the sample precisely at $\nu = \frac{1}{2}$, exactly as in Son's construction[4]. *Thus $\nu = \frac{1}{2}$ is the confluence of two approaches, the one we have always used, in which $N_{CF} = N_e$ and Son's in which $N_{CF} = \frac{1}{2}N_\phi$.* We will see that if we are to go to $\nu \neq \frac{1}{2}$ we must follow Son's assignment.

In 2+1 dimensions the noninteracting two-component Dirac equation is

$$i\hbar\partial_t \psi = \boldsymbol{\sigma} \cdot (\mathbf{p} - \mathbf{a})\ \psi \tag{33}$$

where $\mathbf{a}$ is an external gauge field. Let us initially set it $\mathbf{a} = 0$.

As usual, there are positive and negative energy solutions. Paying no attention to filling the Dirac sea, we can expand ψ in real space as

$$\psi^T(\mathbf{r}) = \sum_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{r}} \left[\frac{c_{\mathbf{k}}}{\sqrt{2}}(1, e^{i\theta_{\mathbf{k}}}) + \frac{d_{\mathbf{k}}}{\sqrt{2}}(1, -e^{-i\theta_{\mathbf{k}}}) \right]. \quad (34)$$

where $e^{i\theta_{\mathbf{k}}} = \frac{k_x + ik_y}{|k|}$.

The projected electron density is of course not the density of the Dirac CF, just like it was not in the previous case of the non-relativistic field. It is determined by the GMP algebra. Let us start with the same expressions for $\mathbf{R}_e$ and $\mathbf{R}_v$ as in Eqns. (23) and (23) adapted at $\nu = \frac{1}{2}$ (to fermions in zero field),

$$\begin{aligned} \mathbf{R}_e &= \mathbf{r} - \frac{l^2}{2}\hat{\mathbf{z}} \times \mathbf{p} \\ \mathbf{R}_v &= \mathbf{r} + \frac{l^2}{2}\hat{\mathbf{z}} \times \mathbf{p} \end{aligned} \quad (35)$$

Note that under time-reversal

$$\mathcal{T}: \quad \mathbf{R}_e \leftrightarrow \mathbf{R}_v. \quad (36)$$

Next we define the electron density and vortex density operators in the Hilbert space of the Dirac CF by taking matrix elements of $e^{-i\mathbf{q}\cdot\mathbf{R}_e}$ and $e^{-i\mathbf{q}\cdot\mathbf{R}_v}$ between momentum states and obtain

$$\begin{aligned} \bar{\rho}(\mathbf{q}) = \frac{1}{2}\sum_{\mathbf{k}} e^{-i\frac{1}{2}l^2\mathbf{q}\times\mathbf{k}} \Big[\Big[c^\dagger_{\mathbf{k}-\mathbf{q}} c_{\mathbf{k}} + d^\dagger_{\mathbf{k}-\mathbf{q}} d_{\mathbf{k}} \Big] (1 + e^{i(\theta_{\mathbf{k}} - \theta_{\mathbf{k}-\mathbf{q}})}) \\ + \Big[c^\dagger_{\mathbf{k}-\mathbf{q}} d_{\mathbf{k}} + d^\dagger_{\mathbf{k}-\mathbf{q}} c_{\mathbf{k}} \Big] (1 - e^{i(\theta_{\mathbf{k}} - \theta_{\mathbf{k}-\mathbf{q}})}) \Big] \end{aligned}$$

and

$$\begin{aligned} \bar{\chi}(\mathbf{q}) = \frac{1}{2}\sum_{\mathbf{k}} e^{i\frac{1}{2}l^2\mathbf{q}\times\mathbf{k}} \Big[\Big[c^\dagger_{\mathbf{k}-\mathbf{q}} c_{\mathbf{k}} + d^\dagger_{\mathbf{k}-\mathbf{q}} d_{\mathbf{k}} \Big] (1 + e^{i(\theta_{\mathbf{k}} - \theta_{\mathbf{k}-\mathbf{q}})}) \\ + \Big[c^\dagger_{\mathbf{k}-\mathbf{q}} d_{\mathbf{k}} + d^\dagger_{\mathbf{k}-\mathbf{q}} c_{\mathbf{k}} \Big] (1 - e^{i(\theta_{\mathbf{k}} - \theta_{\mathbf{k}-\mathbf{q}})}) \Big] \end{aligned}$$

One may verify that $\bar{\rho}(\mathbf{q})$ and $\bar{\chi}(\mathbf{q})$ obey the same algebra as before as before since $\mathbf{R}_e$ and $\mathbf{R}_v$ do. So this is yet another algebraically faithful representation of the LLL. Once again we will use our preferred density $\bar{\rho}_p = \bar{\rho} - \bar{\chi}$, ignoring constraints in the spirit of obtaining a low-energy theory that has all the symmetries of the original. It is $\bar{\rho}_p$ that allows us to display the PH transformation as follows.

The PH transformation of electrons is once again implemented as time-reversal on the CF. However the action of $\mathcal{T}$ naturally follows from the Dirac nature of the CF

$$\begin{aligned} \mathcal{T} c_{\mathbf{k}} \mathcal{T}^{-1} &= e^{i\theta_{\mathbf{k}}} c_{-\mathbf{k}} \\ \mathcal{T} c^\dagger_{\mathbf{k}} \mathcal{T}^{-1} &= e^{-i\theta_{\mathbf{k}}} c^\dagger_{-\mathbf{k}} \\ \mathcal{T} d_{\mathbf{k}} \mathcal{T}^{-1} &= -e^{i\theta_{\mathbf{k}}} d_{-\mathbf{k}} \\ \mathcal{T} d^\dagger_{\mathbf{k}} \mathcal{T}^{-1} &= -e^{-i\theta_{\mathbf{k}}} d^\dagger_{-\mathbf{k}} \end{aligned} \quad (37)$$

Using $\theta_{\mathbf{k}} - \theta_{-\mathbf{k}} = \pi$, one sees that $\mathcal{T}^2 = -1$. Next, one may verify that

$$\begin{aligned} \mathcal{T}\bar{\rho}(\mathbf{q})\mathcal{T}^{-1} &= \bar{\chi}(-\mathbf{q}) \\ \mathcal{T}\bar{\chi}(\mathbf{q})\mathcal{T}^{-1} &= \bar{\rho}(-\mathbf{q}) \\ \mathcal{T}\bar{\rho}_p(\mathbf{q}) i \mathcal{T}^{-1} &= -\bar{\rho}_p(-\mathbf{q}) \end{aligned} \quad (38)$$

(This is most easily seen in first-quantization by considering the action of $\mathcal{T}$ on $e^{-i\mathbf{q}\cdot\mathbf{R}_e}$ and $e^{-i\mathbf{q}\cdot\mathbf{R}_v}$: for $i\mathbf{q} \to -i\mathbf{q}$ and $\mathbf{R}_e \leftrightarrow \mathbf{R}_v$.)

Since $\mathcal{T}$ reverses the sign of the electronic charge $\bar{\rho}_p$ it is appropriate to call it a PH transformation. The Hamiltonian built out of the preferred density

$$\bar{H}_p = \frac{1}{2}\sum_{\mathbf{q}} v_{ee}(q)\bar{\rho}_p(\mathbf{q})\bar{\rho}_p(-\mathbf{q}) \quad (39)$$

is symmetric under $\mathcal{T}$.

On the other hand the Hamiltonian built out of $\bar{\rho}$

$$\bar{H} = \frac{1}{2}\sum_{\mathbf{q}} v_{ee}(q)\bar{\rho}(\mathbf{q})\bar{\rho}(-\mathbf{q}) \quad (40)$$

which does not display PH symmetry but has a huge symmetry group generated by $\bar{\chi}$ is the only way to get the overdamped mode in a conserving calculation.

(Filling the Dirac sea will lead to the replacements $d_{\mathbf{k}} \to b^\dagger_{-\mathbf{k}}$, $d^\dagger_{\mathbf{k}} \to b_{-\mathbf{k}}$, where the b, $b^\dagger$ are now hole destruction and creation operators for negative energy states in the filled Dirac sea. If one is interested in nonzero filling above the Fermi point in the Dirac problem, the d or b operators are high energy operators and can be set to zero to obtain the low-energy physics. However, the density, if projected by setting $d, d^\dagger \to 0$, will not obey the GMP algebra exactly, but only at small q.)

It should be noted that our CF differs from Son's[4] in one regard. The projected physical charge density of the *correlated* electrons is directly given in terms of our CFs but in Son's picture, the physical number density of electrons (measured from half-filling) is the curl of an emergent, minimally coupled gauge field.

One can also ask whether this theory has a sensible non-relativistic limit. At the level of the algebra the natural way to take this limit is to keep only the low-energy states around the Fermi surface. A computation reveals that while the small-q limit of the GMP algebra is correctly reproduced, one needs the "high-energy" components to reproduce the full GMP algebra.

Next we turn to the nature of the ground state of the interacting Hamiltonian we have proposed.

VI. HARTREE-FOCK NATURE OF SON'S GROUND STATE

So far, all we have shown is that we can realize the GMP algebra in a Hilbert space of Dirac fermions. However, unlike the effective theory proposed by Son[4], which already comes with a "kinetic" term for the Dirac fermions, our two proposed interacting Hamiltonians

$$\begin{aligned} \bar{H} &= \frac{1}{2}\sum_{\mathbf{q}} v_{ee}(q)\bar{\rho}(\mathbf{q})\bar{\rho}(-\mathbf{q}) \\ \bar{H}_p &= \frac{1}{2}\sum_{\mathbf{q}} v_{ee}(q)\bar{\rho}_p(\mathbf{q})\bar{\rho}_p(-\mathbf{q}) \end{aligned} \quad (41)$$

have no such kinetic terms. So one may ask in what sense one can make a correspondence between Son's proposed ground state (all negative energy states filled, and positive energy states filled to some μ which guarantees the correct number of CFs).

The answer is that Son's ground state is a Hartree-Fock state of both of our interacting Hamiltonians. To see this we characterize Son's ground state in terms of the expectation values of the c and d operators defined in the previous section:

$$\begin{aligned}\langle d^\dagger_{\mathbf{k}} d_{\mathbf{k}'}\rangle &= \quad \delta_{\mathbf{k}\mathbf{k}'} \quad \forall \mathbf{k} \\ \langle c^\dagger_{\mathbf{k}} c_{\mathbf{k}'}\rangle &= \delta_{\mathbf{k}\mathbf{k}'} N_{Fc}(\mathbf{k}) \\ \langle d^\dagger_{\mathbf{k}} c_{\mathbf{k}'}\rangle &= \langle c^\dagger_{\mathbf{k}} d_{\mathbf{k}'}\rangle = 0 \end{aligned} \tag{42}$$

where $N_{Fc}(\mathbf{k}) = \Theta(k_F - k)$. One now writes the interacting Hamiltonian and reduces it to a one-body (HF) Hamiltonian by taking all possible expectation values. One can see by inspection that since translation symmetry is preserved by the ground state, the HF Hamiltonian must be of the form

$$\begin{aligned} H_{HF} = \sum_{\mathbf{k}} \big(&\epsilon_c(\mathbf{k}) c^\dagger_{\mathbf{k}} c_{\mathbf{k}} + \epsilon_d(\mathbf{k}) d^\dagger_{\mathbf{k}} d_{\mathbf{k}} \\ &+\gamma(\mathbf{k}) c^\dagger_{\mathbf{k}} d_{\mathbf{k}} + \gamma^*(\mathbf{k}) d^\dagger_{\mathbf{k}} c_{\mathbf{k}}\big) \end{aligned}$$

Each of the coefficients $\epsilon_{c,d}(\mathbf{k})$, $\gamma(\mathbf{k})$ are sums over $\mathbf{q}$. If the coefficients $\gamma(\mathbf{k})$ are not zero one generates correlations between c and d, and the ground state proposed by Son will fail to be a HF state of our interacting Hamiltonian. So the verification that Son's ground state is a HF state of our Hamiltonian reduces to verifying that $\gamma(\mathbf{k}) = 0$. A straightforward calculation shows that in the case of $\bar{H}$

$$\begin{aligned} \gamma(\mathbf{k}) = -\tfrac{i}{4} \sum_{\mathbf{q}} v_{ee}(q) \big(&\sin(\theta_{\mathbf{k}} - \theta_{\mathbf{k}-\mathbf{q}})[1 - N_{Fc}(\mathbf{k}-\mathbf{q})] \\ &+ \sin(\theta_{\mathbf{k}} - \theta_{\mathbf{k}+\mathbf{q}})[1 - N_{Fc}(\mathbf{k}+\mathbf{q})]\big) \end{aligned} \tag{43}$$

Now, we note that for every $\mathbf{k}$, $\mathbf{q}$, there is a $\mathbf{q}^*$ which is the vector $\mathbf{q}$ reflected about $\mathbf{k}$. All the terms in the expression for γ are even under the change $\mathbf{q} \to \mathbf{q}^*$, except for the prefactor $\sin(\theta_{\mathbf{k}} - \theta_{\mathbf{k}\pm\mathbf{q}})$ which changes sign. The sum is thus zero by symmetry for any rotationally invariant $v_{ee}(q)$.

A similar arguments applies for the case $\bar{H}_p$ in which we use the preferred density $\bar{\rho}_p = \bar{\rho}(\mathbf{q}) - \bar{\chi}(\mathbf{q})$, despite additional phase factors.

Examining the HF Hamiltonian in more detail reveals that for any $\mathbf{k}$, $\epsilon_d(\mathbf{k}) \le \epsilon_c(\mathbf{k})$. Equality is achieved only for $\mathbf{k} = 0$. These are also features of the noninteracting ground state of Son.

So we have established that Son's ground state is a HF ground state of our interacting Hamiltonians $\bar{H}$ and $\bar{H}_p$. We plan to use $\bar{H}$, which commutes with $\bar{\chi}$, as a starting point for a conserving calculation of response functions in the near future. As for $\bar{H}_p$, there are no obvious constraints that commute with it. However, in addition to manifestly displaying the PH symmetry, it captures many of the low energy properties of the CF at the level of naive Hartree-Fock.

VII. HAMILTIONIAN FORMULATION OF SON'S CFS AWAY FROM $\nu = \frac{1}{2}$

The construction we carried out at $\nu = \frac{1}{2}$ can be extended in a natural way away from $\frac{1}{2}$. But this requires us to change our strategy. Until recently we were of the view that $\bar{\rho}_p(\mathbf{q})$ existed as an alternative to $\bar{\rho}(\mathbf{q})$ only at $\nu = \frac{1}{2}$. This is in fact true if we insist on $N_{CF} = N_e$. However a new path opens up if we switch to $N_{CF} = \frac{1}{2} N_\phi$. Here are the details.

Recall again that the number of Son's CFs is

$$N_{CF} = \frac{N_\phi}{2} = \frac{eBA}{4\pi}. \tag{44}$$

The CF couples to electronic charge via a gauge potential whose curl is the physical charge density. The effective number of flux quanta seen by the CFs is

$$N_{\phi,CF} = N_\phi - 2N_e, \tag{45}$$

where N_e is the number of electrons. It follows that the effective magnetic field seen by the CFs is

$$B_{CF} \;=\; B(1-2\nu) \equiv B\ \theta \tag{46}$$

where

$$\theta = 1 - 2\nu. \tag{47}$$

Introducing coordinates $\mathbf{r}^{CF}$ and velocity operators $\mathbf{\Pi}^{CF}$ we demand

$$\big[\mathbf{\Pi}^{CF}_x(\theta), \mathbf{\Pi}^{CF}_y(\theta)\big] = \frac{i(1-2\nu)}{l^2} \equiv \frac{i\theta}{l^2} \tag{48}$$

where

$$\mathbf{\Pi}^{CF}(\theta) \;=\; \mathbf{p} - \mathbf{a}(\theta). \tag{49}$$

If we do a PH transformation *on electrons*, we want to map the Hamiltonians such that $\mathcal{H}(\theta) \to \mathcal{H}(-\theta)$. The PH transformation of electrons is implemented in the CF space by time-reversal. However, since the "gauge" potential representing the deviation of the electron density from $\nu = \frac{1}{2}$ is dependent only on CF coordinates, and not on CF momenta, we have

$$\mathcal{T}\mathbf{a}(\theta)\mathcal{T}^{-1} = \mathbf{a}(\theta) \tag{50}$$

This is identical to Son's formulation. Let us now consider the effect of time-reversal on $\mathbf{\Pi}^{CF}$

$$\begin{aligned} \mathcal{T}\mathbf{\Pi}^{CF}(\theta)\mathcal{T}^{-1} &= \mathcal{T}(\mathbf{p} - \mathbf{a}(\theta))\mathcal{T}^{-1} \\ &= (-\mathbf{p} - \mathbf{a}(\theta)) \\ &= -(\mathbf{p} + \mathbf{a}(\theta)) \\ &= -(\mathbf{p} - \mathbf{a}(-\theta)) \end{aligned}$$

$$= -\mathbf{\Pi}^{CF}(-\theta). \quad (51)$$

If we now conjugate Eqn. 48 we find, using $\mathcal{T} i \mathcal{T}^{-1} = -i$,

$$[\mathbf{\Pi}_x^{CF}(-\theta), \mathbf{\Pi}_y^{CF}(-\theta)] = \frac{-i\theta}{l^2} \quad (52)$$

as desired.

At $\theta \neq 0$, we define two sets of conjugate coordinates $\mathbf{R}^e$ and $\mathbf{R}^v$ as follows

$$\begin{aligned} \mathbf{R}^e(\theta) &= (1+\tfrac{\theta}{4})\mathbf{r}^{CF} - \tfrac{1}{2}l^2\hat{z}\times\mathbf{\Pi}^{CF}(\theta) \\ \mathbf{R}^v(\theta) &= (1-\tfrac{\theta}{4})\mathbf{r}^{CF} + \tfrac{1}{2}l^2\hat{z}\times\mathbf{\Pi}^{CF}(\theta). \end{aligned} \quad (53)$$

It can easily be checked that *for all* θ,

$$[R_x^e, R_y^e] = -il^2, \quad [\mathbf{R}^e, \mathbf{R}^v] = 0, \quad [R_x^v, R_y^v] = il^2. \quad (54)$$

There are several pleasing features of these sets of coordinates. Firstly, under time-reversal in the Dirac world, since $\mathcal{T}\mathbf{\Pi}(\theta)\mathcal{T}^{-1} = -\mathbf{\Pi}(-\theta)$ we find

$$\begin{aligned} \mathcal{T}\mathbf{R}^e(\theta)\mathcal{T}^{-1} &= (1+\frac{\theta}{4})\mathbf{r}^{CF} + \frac{1}{2}l^2\hat{z}\times\mathbf{\Pi}^{CF}(-\theta) \\ &= \mathbf{R}^v(-\theta) \\ \mathcal{T}\mathbf{R}^v(\theta)\mathcal{T}^{-1} &= (1-\frac{\theta}{4})\mathbf{r}^{CF} - \frac{1}{2}l^2\hat{z}\times\mathbf{\Pi}^{CF}(-\theta) \\ &= \mathbf{R}^e(-\theta). \end{aligned} \quad (55)$$

Secondly, the position coordinate of the Dirac CF is still the average of $\mathbf{R}^e$ and $\mathbf{R}^v$, as at $\nu = \frac{1}{2}$.

Finally we define $\bar{\rho}$ and $\bar{\chi}$ in the Hilbert space of the Dirac CFs exactly as before, by taking the matrix elements of $e^{-i\mathbf{q}\cdot\mathbf{R}_e}$ and $e^{-i\mathbf{q}\cdot\mathbf{R}_v}$ between momentum states of the Dirac fermion.

Because the commutation relations of $\mathbf{R}^e$ and $\mathbf{R}^v$ are identical to those at $\nu = \frac{1}{2}$, we can once again choose to represent the physical charge density in two distinct ways, either as $\bar{\rho}$ or as $\bar{\rho}_p = \bar{\rho} - \bar{\chi}$.

If we choose to represent the Hamiltonian in terms of $\bar{\rho}$ it will commute with the set of $\bar{\chi}(\mathbf{q})$, and will thus be amenable to a conserving approximation[23]. Of course, the physical charge e^* of the quasiparticles and the PH mapping will not be manifest.

If, on the other hand, we choose to represent the physical charge density as $\bar{\rho}_p$ then the PH transformation (implemented by $\mathcal{T}$ with $\mathcal{T}^2 = -1$) can be explicitly realized as follows. Given Eqn. 55 and 55:

$$\begin{aligned} \mathcal{T}\bar{\rho}(\mathbf{q},\theta)\mathcal{T}^{-1} &= \bar{\chi}(-\mathbf{q},-\theta) \\ \mathcal{T}\bar{\chi}(\mathbf{q},\theta)\mathcal{T}^{-1} &= \bar{\rho}(-\mathbf{q},-\theta) \\ \mathcal{T}\bar{\rho}_p(\mathbf{q},\theta)\mathcal{T}^{-1} &= -\bar{\rho}_p(-\mathbf{q},-\theta). \end{aligned} \quad (56)$$

The Hamiltonian, quadratic in $\bar{\rho}_p(\mathbf{q})$ will respond as follows:

$$\bar{H}_p(\theta) \to \bar{H}_p(-\theta). \quad (57)$$

An important point to note: the *ad hoc* combination $\bar{\rho}_p = \bar{\rho}(\mathbf{q}) - c^2\bar{\chi}(\mathbf{q})$ stood for a particle of charge $e^* = e(1-2\nu)$, with $e^* = 0$ only at $\nu = \frac{1}{2}$. In the present approach mirroring Son's, $\bar{\rho}_p$ always describes a neutral particle ($\bar{\rho}_p(\mathbf{q}=0) = 0$). This is actually too much of a good thing, since the actual charge of the quasiparticle at long wavelengths for $\nu \neq \frac{1}{2}$ should be $e^* \neq 0$. This is a problem we hope to resolve in future work.

Thus, we have been able to find a representation of the GMP algebra in terms of the neutral Dirac CFs of Son[4] for all ν. Presumably, this could be the starting point for calculations of gaps, response functions, etc, as in our previous work on Jain's CFs.

VIII. SUMMARY

This paper explores the Son's[4] recent solution for displaying the PH *transformation* of electrons in the LLL and the PH *symmetry* of the $\nu = \frac{1}{2}$ problem, within the framework of our Hamiltonian formalism. In our approach we map the algebraic problem of the LLL projected charge $\bar{\rho}(\mathbf{q})$ (which obeys the GMP algebra) and the projected Hamiltonian $\bar{H}(\bar{\rho}(\mathbf{q}))$, from the electronic space (plagued with ground state degeneracy) to a different space which permits a unique HF ground state. When realized in the CF space of a single-component fermion which saw the weaker field mandated by Jain[12], we also found a closed algebra of constraints $\bar{\chi}(\mathbf{q})$ that commuted with $\bar{H}$, delineated the physical sector, and formed the GMP like algebra of an object with the charge -2ν of the double vortex. These results could be derived at small q, as indicated in the Appendix.

Recently we realized that *at and only at* $\nu = \frac{1}{2}$, a preferred charge density $\bar{\rho}_p(\mathbf{q}) = \bar{\rho}(\mathbf{q}) - \bar{\chi}(\mathbf{q})$ also obeyed the GMP algebra and could equally well represent the projected, correlated electron density. The role of this isolated second option escaped us until recently, as did the importance of the PH transformation of the LLL. We now see that it allows us to realize the PH transformation as an anti-unitary operator $\mathcal{T}$ with $\mathcal{T}^2 = +1$ in the space of a one-component fermion. Since one wants $\mathcal{T}^2 = -1$ for electrons[8,20], we conjecture this describes an emergent symmetry of the $\nu = 1$ boson problem studied by Read[16]. Such a theory could potentially be relevant to the high-field superconductor-insulator transition, where indications of boson-vortex duality[26] have been seen.

Following Son[4] and the work of Wang and Senthil[5] we then cast the algebraic formulation in the space of Dirac fermion. By following Son's approach of equating the number of CFs to half the number of flux quanta (and not the number of electrons) we were able to extend the operator approach to all ν. In this version the CF is always neutral. The commutation relations for $\bar{\rho}(\mathbf{q})$ and $\bar{\chi}(\mathbf{q})$ are the same at $\nu = \frac{1}{2}$. As always, we have two options. One is to use $\bar{\rho}(\mathbf{q})$ as the electronic charge density and $\bar{\chi}(\mathbf{q})$ as the algebra of constraints that specifies the physical LLL sector. The constraints are then to be enforced in a conserving approximation[23], which would yield the overdamped mode at $\nu = \frac{1}{2}$. The other option is to use $\bar{\rho}_p(\mathbf{q})$, in which the PH transformation properties are transparent. However, unlike the $\bar{\rho}_p(\mathbf{q})$ of the one-component (Jain) CFs, the quasiparticle charge does

not come out correctly at tree level. Perhaps there is an even better representation in which all the algebraic and symmetry properties of the CF are manifest.

One can ask whether the particle-hole symmetry can be realized with two flavors of *nonrelativistic* CFs, say two spin flavors[27]. One can write down such a Hamiltonian theory in which one spin flavor gets mapped into the other under $\mathcal{T}$, and it will even have the desired $\mathcal{T}^2 = -1$. However, an undesirable feature is that it will have two Fermi surfaces, and thus this state is not a good candidate for the spin-polarized $\nu = \frac{1}{2}$ state.

There are a number of future directions we would like to pursue. The first is to carry out a conserving calculation at $\nu = \frac{1}{2}$ in the new formulation in terms of Dirac CFs. We have already established the first necessary step, that Son's ground state is a HF ground state of our interacting Hamiltonian. The structure factor should vanish as q^4 to be in compliance with Kohn's theorem. We should also recover the overdamped mode of HLR[15], and in the presence of disorder we should be able to see the suppression of backscattering[8]. Presumably, we should be able to extend this kind of treatment to ν away from half as well. At the moment, we have a realization of the GMP algebra away from $\frac{1}{2}$ that does show the mapping from $\nu \to 1-\nu$. However, the density operator does not have the correct quasiparticle charge. We would like to find a representation in which all the algebraic, symmetry, and physical properties of the quasiparticles of the problem are transparently visible.

Another direction leading towards more realistic systems is to incorporate the breaking of PH symmetry[27]. Recall that under $\mathcal{T}$ the preferred density changes sign. This suggests that the most straightforward way to incorporate PH symmetry breaking is by introducing a small three-body term. Such a term (generated by Landau-level mixing) was shown to be crucial in stabilizing the anti-Pfaffian[28] state over the Pfaffian[29] in *GaAs* systems.

We are grateful to Son for extended discussions at the Aspen Center for Physics. We thank Senthil for sharing with us his work with Wang and the physical picture in terms of semions. His input, through extensive email and phone communications, was crucial to our understanding of the central issues. RS thanks Ashvin Viswanath for conversations. We thank the Aspen Center for Physics, where this work was conceived and completed, and which is supported by National Science Foundation grant PHY-1066293. Murthy also acknowledges partial support from the NSF via DMR-1306897 (GM) and from the US-Israel Binational Science Foundation via grant no. 2012120.

IX. APPENDIX

In our earliest work we began with the CS theory and adapted the strategy of Bohm and Pines. Collective charge degrees of freedom were represented by magnetoplasmon oscillators $A(\mathbf{q})$ of cyclotron energy ω_c and to pay for them some constraints $\chi(\mathbf{q})$ were imposed. At the end the fermions and oscillators were decoupled in the small q limit to yield the following results for Jain fractions:

$$\begin{aligned}
H_{osc} &= \sum_{\mathbf{q}} A^\dagger(\mathbf{q}) A(\mathbf{q}) \omega_c \\
\mathbf{j}_e(\mathbf{q}) &= \hat{\mathbf{q}}(A(\mathbf{q}) + A^\dagger(\mathbf{q})) \\
\rho_e(\mathbf{q}) &= q(A(\mathbf{q}) + A^\dagger(\mathbf{q})) + \bar{\rho}(\mathbf{q}) \quad \text{where} \\
\bar{\rho}(\mathbf{q}) &= \sum_j \varepsilon^{-i\mathbf{q}\cdot\mathbf{r}_j} \left(1 - \frac{il^2}{1+c}\mathbf{q}\times\mathbf{\Pi}_j^* + \dots\right) \\
\bar{\chi}(\mathbf{q}) &= \sum_j \varepsilon^{-i\mathbf{q}\cdot\mathbf{r}_j} \left(1 + \frac{il^2}{c(1+c)}\mathbf{q}\times\mathbf{\Pi}_j^* + \dots\right) \\
0 &= \bar{\chi}(\mathbf{q})|\text{physical state}\rangle \quad \text{(constraint)}
\end{aligned} \tag{58}$$

where

$$c^2 = 2\nu = \frac{2p}{2p+1}, \tag{59}$$

and $\mathbf{\Pi}_j^*$ is the canonical momentum of CF number j which experiences the right field to satisfy Jain's condition

$$\left[\mathbf{\Pi}_x^*, \mathbf{\Pi}_y^*\right] = \frac{i(1-c^2)}{l^2} \equiv \frac{i}{l^{*2}}. \tag{60}$$

Notice that the current is carried only by the oscillators at every ν. When the charge ρ_e is coupled to an external scalar potential $\Phi(\mathbf{q})$, the resultant Hall current gives the correct σ_{xy}. Our CF, restricted the LLL makes no contribution since the current has no leading matrix elements within the LLL.

Following this, it was conjectured by RS that the two terms in the expression for $\bar{\rho}(\mathbf{q})$ and $\bar{\chi}(\mathbf{q})$ were the beginnings of the following exponentials

$$\begin{aligned}
\bar{\rho}(\mathbf{q}) &= \sum_j \exp\left[-i\mathbf{q}\cdot\mathbf{R}^e j\right] \\
\bar{\chi}(\mathbf{q}) &= \sum_j \exp\left[-i\mathbf{q}\cdot\mathbf{R}_j^v\right] \quad \text{where} \\
\mathbf{R}^e &= \mathbf{r} - \frac{l^2}{1+c}\hat{\mathbf{z}}\times\mathbf{\Pi}^* \\
\mathbf{R}^v &= \mathbf{r} + \frac{l^2}{c(1+c)}\hat{\mathbf{z}}\times\mathbf{\Pi}^*
\end{aligned} \tag{61}$$

In this "all $\mathbf{q}$" formalism, the coordinates $\mathbf{R}^e$ and $\mathbf{R}^v$ were named thus because they have the following commutation relations:

$$\begin{aligned}
\left[\mathbf{R}_x^e, \mathbf{R}_y^e\right] &= -il^2 \\
\left[\mathbf{R}_x^v, \mathbf{R}_y^v\right] &= i\frac{l^2}{c^2} \\
\left[\mathbf{R}^e, \mathbf{R}^v\right] &= 0.
\end{aligned} \tag{62}$$

We recognize the commutation rules of $\mathbf{R}^e$ as that of the guiding center-coordinate of the electron and $\mathbf{R}^v$ as describing the guiding center coordinate of the double vortex. This ensures that the density corresponding to $\mathbf{R}^e$,

$$\bar{\rho}(\mathbf{q}) = \sum_j e^{-i\mathbf{q}\cdot\mathbf{R}_j^e} \tag{63}$$

obeys the GMP algebra. The density formed from the vortex coordinate obeys

$$[\bar{\chi}(\mathbf{q}), \bar{\chi}(\mathbf{q}')] = -2i \sin\left[\frac{l^2}{2c^2}\mathbf{q}\times\mathbf{q}'\right]\bar{\chi}(\mathbf{q}+\mathbf{q}'). \quad (64)$$

and commutes with $\bar{\rho}(\mathbf{q})$:

$$[\bar{\rho}(\mathbf{q}), \bar{\chi}(\mathbf{q}')] \equiv 0 \quad (65)$$

and therefore with $\bar{H}(\bar{\rho}(\mathbf{q}))$.

The careful reader will note that these results apply equally well to the $\nu = 1$ boson problem after the fluxes are attached to the bosons.

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[27] We thank one of our anonymous referees for asking us to consider this question.

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