Bounds on corner entanglement in quantum critical states

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The entanglement entropy in many gapless quantum systems receives a contribution from corners in the entangling surface in 2+1d. It is characterized by a universal function $a(\theta)$ depending on the opening angle θ, and contains pertinent low energy information. For conformal field theories (CFTs), the leading expansion coefficient in the smooth limit $\theta \to \pi$ yields the stress tensor 2-point function coefficient C_T. Little is known about $a(\theta)$ beyond that limit. Here, we show that the next term in the smooth limit expansion contains information beyond the 2- and 3-point correlators of the stress tensor. We conjecture that it encodes 4-point data, making it much richer. Further, we establish strong constraints on this and higher order smooth-limit coefficients. We also show that $a(\theta)$ is lower-bounded by a non-trivial function multiplied by the central charge C_T, e.g. $a(\pi/2) \geq (\pi^2 \log 2)C_T/6$. This bound for 90-degree corners is nearly saturated by all known results, including recent numerics for the interacting Wilson-Fisher quantum critical points (QCPs). A bound is also given for the Rényi entropies. We illustrate our findings using $O(N)$ QCPs, free boson and Dirac fermion CFTs, strongly coupled holographic ones, and other models. Exact results are also given for Lifshitz quantum critical points, and for conical singularities in 3+1d.

Keywords: Quantum criticality, Entanglement, Conformal field theory, Lifshitz quantum critical points, AdS/CFT

I. INTRODUCTION

In recent years, the structure of quantum entanglement in many-body systems has acquired an increasingly prominent role in diverse areas of physics, such as condensed matter,[1–8] quantum field theory,[9–14] or quantum gravity.[15–25] The interest in the subject is perhaps not surprising given that entanglement is a fundamental property of the quantum realm. This being said, its recent rise in prominence can be partially attributed to the rapid development of tools to study it in complex systems. In the context of condensed matter physics, entanglement has proven to be a powerful probe of unconventional quantum states of matter. For instance, the so-called topological entanglement entropy[1,2] in two spatial dimensions can be used to determine whether a phase is topologically non-trivial,[26] *i.e.*, whether it possesses anyonic excitations, whereas conventional local order-parameters cannot. This diagnostic was employed to analyze realistic models for quantum spin liquids.[27–29] A closely related quantity has recently been found to organize the renormalization group flow between different phases and quantum critical points described by Lorentz invariant theories.[30,31] These results have been applied to constrain the phase diagram of models with topological phases and deconfined quantum critical points.[32,33]

The entanglement entropy (EE) S and Rényi entropies[34,35] S_n are particularly useful measures of entanglement. Focusing on spatial bi-partitions, heuristically, these entropies quantify the amount of entanglement between the inside and outside of a given region. More precisely, for a spatial region V and a quantum state $|\psi\rangle$, they are defined as

$$S_n(V) = \frac{1}{1-n} \log \operatorname{Tr} \rho_V^n \,, \qquad (\text{I.1})$$

$$S(V) = \lim_{n \to 1} S_n(V) = -\operatorname{Tr}\left(\rho_V \log \rho_V\right) \,, \qquad (\text{I.2})$$

where $\rho_V = \operatorname{Tr}_{\bar{V}} |\psi\rangle\langle\psi|$ is the reduced density matrix obtained by integrating out the degrees of freedom in the complementary (outside) region, $\overline{V}$.

In this work, we shall be concerned with the entanglement and Rényi entropies of strongly interacting quantum systems described by scale invariant quantum field theories at low energy. Most, but not all, of our attention will be devoted to conformal field theories (CFTs). These possess scale and Lorentz invariance, and for example describe the quantum critical points in the Ising and XY universality classes in $d = 2 + 1$ spacetime dimensions. Generally, CFTs are strongly interacting and lack quasiparticle excitations, unless one is dealing special theories such as free bosons or free Dirac fermions. We will now see how basic entanglement measures can be used to gain insights into these complex systems. We consider the *groundstate* of a CFT in three spacetime dimensions. The corresponding Rényi entropy of a region V takes the generic form

$$S_n = B_n \frac{\ell}{\delta} - \sum_{i \text{ corners}} a_n(\theta_i) \log(\ell/\delta) + \text{const.}, \qquad (\text{I.3})$$

where ℓ is a length scale characterizing the linear size of V, and δ is a UV cut-off: in a lattice model, this would be the lattice spacing. B_n is a regulator depen-
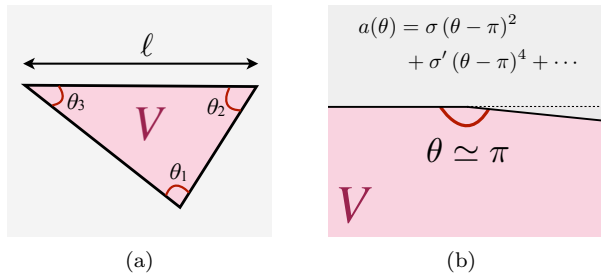
FIG. 1. **Entanglement from corners.** a) Corners in the entangling surface ∂V with opening angles θ_i. b) A corner with $\theta \simeq \pi$. The Taylor expansion of the corner function $a(\theta)$ near this smooth limit yields coefficients $\sigma, \sigma', \ldots$ that contain non-trivial information about the CFT. We have omitted the Rényi index n dependence.

dent (positive) coefficient, while the third term is finite as $\delta \to 0$ and generally depends on the shape of V. The expansion (I.3) holds in the limit $\ell \gg \delta$. While the leading divergence, corresponding to the usual "area" or boundary law, is present for general entangling regions, the subleading logarithmic term appears only when the boundary of V contains sharp corners,[12–14,36,37] each with opening angle θ_i, as shown in Fig. 1(a). This contribution is characterized by a function of the corner opening angle $a_n(\theta)$ which, as opposed to B_n, is universal (regulator-independent) and therefore encodes well-defined information about the low energy CFT. Importantly, corner contributions naturally arise in numerical calculations on lattice models, and since $a_n(\theta)$ is not polluted by UV/microscopic details it can be determined by such methods,[8,38–44] and compared with quantum field theory analysis.

It is instructive to begin by examining nearly smooth corners. On general grounds, the corner function $a_n(\theta)$ is expected to have the following expansion[12–14] near $\theta = \pi$ (see Fig. 1(b))

$$a_n(\theta \simeq \pi) = \sigma_n \, \varepsilon^2 + \sigma'_n \, \varepsilon^4 + \sigma''_n \, \varepsilon^6 + \cdots , \qquad \text{(I.4)}$$

where we have introduced the notation $\varepsilon \equiv (\pi - \theta)$, which will appear throughout. We can write the expansion more succinctly as $\sum_{p=1} \sigma_n^{(p-1)} \varepsilon^{2p}$, where $\sigma_n^{(0)} = \sigma_n$, $\sigma'_n = \sigma_n^{(1)}$, etc. Assuming $a_n(\theta)$ is smooth around $\theta = \pi$, the reflection property, $a_n(2\pi - \theta) = a_n(\theta)$ — true for pure states — implies that only even powers of ε appear in (I.4). Although a proof of the analyticity of $a(\theta)$ near π is lacking, it does hold for all the theories for which the expansion is known (see below).

In recent works[48–50] (see Refs. 8 and 51 for brief reviews), the leading order coefficient in the smooth limit, σ_n, has been studied in detail, and strong evidence[48–50,52–56] points to the simple relation:

$$\sigma_n = \frac{1}{\pi} \frac{h_n}{n - 1} , \qquad \text{(I.5)}$$

for any $n > 0$, where h_n is the conformal dimension of the so-called twist operator.[50,57,58] This is a line operator in $d = 3$, as it has support on the entangling surface, i.e., the boundary of the entangling region V. It is closely related to the swap operator used to compute the Rényi entropies with quantum Monte Carlo.[7] In the more familiar setting of 1+1d CFTs, the twist operator is a local operator[10,11] associated with the endpoint of an interval. In fact, the prefactor of the logarithmically divergent term of the Rényi entropy of a single interval[9–11] has the same form as the smooth-limit coefficient, Eq. (I.5). In the limit $n \to 1$, Eq. (I.5) reduces to[48,49]

$$\sigma = \frac{\pi^2}{24} C_T . \qquad \text{(I.6)}$$

C_T is a fundamental property of the quantum critical system (CFT in this case): it is the central charge determining the 2-point correlation function of the stress or energy-momentum tensor $T_{\mu\nu}$:

$$\langle T_{\mu\nu}(x) T_{\eta\kappa}(0) \rangle = \frac{C_T}{x^{2d}} \mathcal{I}_{\mu\nu,\eta\kappa}(x) , \qquad \text{(I.7)}$$

where x_μ is a spacetime coordinate, and $\mathcal{I}_{\mu\nu,\eta\kappa}$ is a dimensionless tensor fixed by conformal symmetry[45] (see Appendix C). Notably, $T_{00}(x)$ corresponds to the energy density of the system; Eq. (I.7) determines its autocorrelation function in the groundstate. Eq. (I.7) holds in arbitrary dimensions, including $d = 1 + 1$, where C_T is proportional to the Virasoro central charge. Remarkably, in a recent paper by Faulkner et. al.,[59] Eq. (I.6) was proven for general CFTs.

The results for the leading smooth limit coefficient, Eqs. (I.5) and (I.6), have also been generalized to CFTs in higher dimensions.[50,60] In the presence of a (hyper)conical singularity in the entangling surface ∂V, the Rényi entropy contains an analogous regulator-independent contribution characterized by a function of the cone opening angle, $a_n^{(d)}$, whose smooth limit expansion is analogous to (I.4). The leading coefficient of such an expansion has been argued to be $\sigma_n^{(d)} = g(d) \, h_n/(n - 1)$, where the constant $g(d)$ is known explicitly,[60] and h_n is the scaling dimension of a $(d-2)$-dimensional twist operator. For the special case of the EE, $n = 1$, one finds $\sigma^{(d)} = \tilde{g}(d) \, C_T$ where $\tilde{g}(d)$ is again a known d-dependent constant. The recent results of Ref. 59, combined with earlier analysis,[60,61] complete the proof of the $n = 1$ case for general CFTs in dimensions $d > 3$.
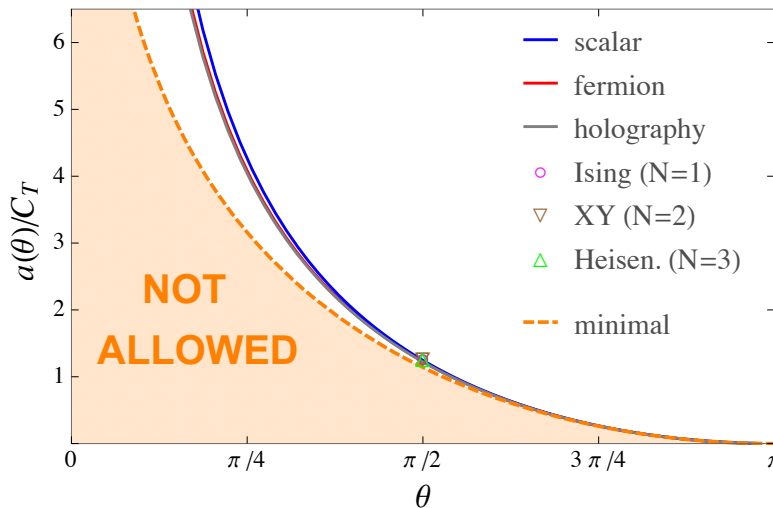
FIG. 2. **Bound on the corner function.** Corner function $a(\theta)$ normalized by the stress tensor coefficient C_T for a free scalar[12,14,45] (blue), a free Dirac fermion[12,13,45] (red), and a theory holographically dual to Einstein gravity[36,46] (gray). We also show the numerical results[39,40,43,44,47] of the $\theta = \pi/2$ values for the $N = 1, 2, 3$ O(N) Wilson-Fisher CFTs (Table I), as well as the minimal function $\mathfrak{a}_{\min}(\theta)/C_T$ (orange dashed). The shaded orange region is forbidden by the inequality $a(\theta) \geq \mathfrak{a}_{\min}(\theta)$.

A. Main results & outline

In contrast to the leading order coefficient σ_n, it is currently not known what physical information is encoded in the higher order coefficients $\{\sigma'_n, \sigma''_n, \dots\}$, let alone in $a_n(\theta)$ at finite angles. This work sheds light on these quantities. One of our main results is a lower bound on the corner entanglement function $a(\theta)$ of any CFT, given in Eq. (II.2) of §II, and illustrated in Fig. 2. A lower bound for the Rényi case $a_{n\neq1}(\theta)$ is also given, Eq. (II.8). We further prove lower bounds for the smooth limit coefficients in §III; these do not follow from the bound on $a(\theta)$. We then unravel the asymptotic properties of the high order coefficients $\sigma_n^{(p\gg1)}$ (§IV). §V studies the smooth limit expansion in holographic theories. In §VI, we use the holographic results in conjunction with other information to shed light on the physical properties of the quartic term $\sigma'(\theta - \pi)^4$ for general theories. We show that σ' contains information beyond the 2- and 3-point functions of the stress tensor. §VII analyzes the structure of the corner function for a special class of Lifshitz quantum critical points. We also discuss the entanglement properties of conical singularities in higher dimensional CFTs (§VIII). Finally, we give a summary and outlook in §IX, where we provide consequential open questions. Four appendices provide details about the calculations.

II. LOWER BOUND ON CORNER ENTANGLEMENT

In this section we show that the strong subadditivity (SSA) property of the EE[62] imposes a non-trivial lower bound on the corner function, $a(\theta) \geq \mathfrak{a}_{\min}(\theta)$, and we give a simple closed form expression for the minimal function $\mathfrak{a}_{\min}(\theta)$. This holds for general CFTs (we are assuming *gapless* theories with a finite C_T, which excludes topological quantum field theories). We also establish a bound on the corner function for general Rényi index in §II A.

Specifically, the SSA of entanglement implies that for regions V, V', the following inequality holds for the sum of their entanglement entropies: $S(V) + S(V') \geq S(V \cup V') + S(V \cap V')$. Using this property, Hirata and Takayanagi showed[36] that $a(\theta)$ is non-negative, as well as $\partial_\theta a(\theta) \leq 0$ on $0 \leq \theta \leq \pi$ ($\partial_\theta a$ is non-negative on $[\pi, 2\pi]$). Using Lorentz invariance and SSA, Casini, Huerta and Leitao[13] then gave the following non-trivial linear constraint:

$$\partial_\theta^2 a(\theta) \geq -\frac{\partial_\theta a(\theta)}{\sin\theta}, \qquad \text{(II.1)}$$

valid for $0 \leq \theta \leq 2\pi$. The key idea to obtain a lower bound for $a(\theta)$ is to replace the inequality in Eq. (II.1) by an equality. This yields a linear second order differential equation, which can be readily solved with the appropriate boundary conditions in the smooth limit.

The solution reads

$$\mathfrak{a}_{\min}(\theta) = \frac{\pi^2 C_T}{3} \log\left[1/\sin(\theta/2)\right] . \qquad \text{(II.2)}$$

We note that $\mathfrak{a}_{\min}$ satisfies the reflection property, $\mathfrak{a}_{\min}(\theta) = \mathfrak{a}_{\min}(2\pi - \theta)$, expected for pure states. It has been normalized so as to obtain the correct leading asymptotic behavior as $\theta \to \pi$: $\pi^2 C_T(\theta - \pi)^2/24$. One of our central results is that $\mathfrak{a}_{\min}(\theta)$ provides a lower bound for the corner function of all CFTs:

$$a(\theta) \geq \mathfrak{a}_{\min}(\theta) , \qquad \text{(II.3)}$$

for all angles $0 \leq \theta \leq 2\pi$. This inequality follows from (II.1); we refer the reader to Appendix A 1 for the proof, which relies on a classic result in the theory of differential inequalities. We point out that $\mathfrak{a}_{\min}$ does not correspond to the corner function of an actual CFT since it does not have the required $1/\theta$ divergence in the sharp corner limit $\theta \to 0$. Rather, it diverges only logarithmically, as $\log(1/\theta)$. The C_T appearing in (II.2) should be interpreted as the stress tensor coefficient C_T corresponding to the theory whose $a(\theta)$ is being compared to $\mathfrak{a}_{\min}(\theta)$. We note that a cruder version of the bound can be obtained by replacing C_T by $24\sigma/\pi^2$ in Eq. (II.2).

We can explicitly verify that the lower bound Eq. (II.3) is satisfied for CFTs both at weak and strong coupling: Fig. 2 shows $a(\theta)/C_T$ for a free scalar and a free Dirac fermion,[12–14,48] and for strongly coupled holographic CFTs dual to Einstein gravity[36,63] — see next section. We also plot the lower bound $\mathfrak{a}_{\min}(\theta)/C_T$. While the bound is comfortably satisfied by the three theories for small opening angles, it becomes quite non-trivial already for $\theta \sim \pi/2$, where all theories nearly saturate it. This is a particularly important value of the opening angle, given that most numerical simulations that studied $a(\theta)$ have been performed for Hamiltonians defined on a square lattice.[8] These simulations dealt with rectangular regions V, and thus obtained $a(\pi/2)$. For 90-degree corners, our bound Eq. (II.3) becomes

$$a(\pi/2) \geq \frac{\pi^2 \log 2}{6} C_T \approx 1.1402 \, C_T . \qquad \text{(II.4)}$$

In both free and strongly coupled holographic theories, $a(\pi/2)/C_T$ exceeds this bound by only a few percent, as can be seen in Fig. 2 and Table I. That table also shows $a(\pi/2)/C_T$ for the O(N) Wilson-Fisher fixed points for $N = 1, 2, 3$. These CFTs describe the quantum critical points in quantum Ising, XY, and Heisenberg spin models, respectively. Remarkably, in all three cases the same ratio is found within error bars, $a(\pi/2)/C_T = 1.3(1)$, which lies not far above the bound Eq. (II.4). This supports the validity of the numerical data. We finally note that recently an analytic expression for the central charge, $C_T = 2(16\pi - 9\sqrt{3})/(81\pi^3)$, was ob-

tained for a strongly interacting QCP with emergent supersymmetry.[64] Such a QCP is super-conformal and can occur at the surface of a topological insulator, in which case C_T is directly proportional to the ground-state optical conductivity.[64] Combined with Eq. (II.2), one can thus obtain a simple closed-form lower bound for $a(\theta)$ at that QCP,[64] in spite of its strongly interacting nature.

A. Rényi entropy

We will now generate a lower bound for $a_n(\theta)$ valid for general Rényi index.[65] Reflection positivity of Euclidean quantum field theory leads to an infinite set of non-linear differential inequalities for the corner function:[66]

$$\det\left\{\partial_\theta^{j+k+2} a_n(\theta)\right\}_{j,k=0}^{M-1} \geq 0 , \qquad \text{(II.5)}$$

where $M \geq 1$ is an integer. Although these were originally derived for integer values of n, we will assume they hold for all $n > 0$ (by analytic continuation). Expanding the determinants, we find that the first 2 inequalities read:

$$\partial_\theta^2 a_n \geq 0 , \qquad \text{(II.6)}$$

$$\partial_\theta^2 a_n \partial_\theta^4 a_n - (\partial_\theta^3 a_n)^2 \geq 0 . \qquad \text{(II.7)}$$

In particular, the second equation can be recast in the form $\partial_\theta^4 a_n \geq G$, with $G \geq 0$, by virtue of the convexity property, Eq. (II.6). This form is suggestive of a lower bound, assuming we can somehow "integrate" the differential inequality. Indeed, we can apply the same methods that we used above in deriving the lower bound (II.3) for the EE to establish a new lower bound valid for any Rényi index (see Appendix A for the details):

$$a_n(\theta) \geq \frac{h_n}{\pi(n-1)} (\theta - \pi)^2 , \qquad \text{(II.8)}$$

where h_n is the scaling dimension of the twist operator,[50] which determines the smooth limit coefficient via Eq. (I.5). h_n is the analog of C_T for the Rényi entropies $n \neq 1$. The RHS of Eq. (II.8) plays the role of $\mathfrak{a}_{\min}(\theta)$ but for general n. Analogously to $\mathfrak{a}_{\min}$, it solves the differential equation obtained by replacing the inequality in Eq. (II.7) by an equality. Note however that for $n = 1$, $\mathfrak{a}_{\min}(\theta)$ is a stronger lower bound since it exceeds $\sigma(\theta - \pi)^2$.

The twist dimension h_n has been computed[50] for all values of n for the free scalar (boson) and fermion CFTs, as well as for strongly interacting holographic theories. For instance, a free complex scalar has[48,50,53] $h_2^{\text{cs}} = 1/(24\pi)$, implying a lower bound $a_2^{\text{cs}}(\pi/2) \geq 1/96 \approx 0.010$ for 90-degree corners. $a_2^{\text{cs}}(\pi/2)$ was nu-

merically computed[14] to be 0.0128, in agreement with our bound. In contrast, at the O(2) interacting Wilson-Fisher quantum critical point (to which the free complex scalar flows under RG), numerical lattice calculations have found $a_2(\pi/2) \approx 0.011$.[40–42] The value of h_2 for that theory is not currently known, but given the properties of C_T,[47] we expect it to be no smaller than $0.9h_2^{\text{cs}}$. This would yield a lower bound $a_2(\pi/2) \gtrsim 0.0094$, which is satisfied by the numerical calculations.

It is natural to ask whether the remaining infinite set of inequalities, Eq. (II.5), will yield stronger bounds, or even an *upper* bound. The answer to the former is that one can indeed get stronger bounds, but these require more information input compared to h_n, not to mention that they involve numerical solutions. With regards to the upper bounds, the answer is simple: those inequalities all yield lower bounds. This essentially follows from the fact that the reflection positivity inequalities are encoded by the determinant of the same larger and larger matrix (see Appendix A).

Eq. (II.8) immediately implies that the quartic coefficient σ_n' is positive for all n, a fact that we will now rederive, along with other bounds on the smooth-limit expansion coefficients.

III. CONSTRAINING THE EXPANSION ABOUT THE SMOOTH LIMIT

In this section we use the constraints found by Casini and Huerta on the corner functions $a_n(\theta)$ from reflection positivity[66] and SSA of EE[12–14] to establish bounds on the smooth-limit expansion coefficients $\sigma_n^{(p)}$.

A. Bounds for general Rényi index

For general Rényi index, we can use the infinite set of non-linear inequalities for the corner function $a_n(\theta)$ coming from reflection positivity,[66] Eq. (II.5), to constrain the smooth-limit coefficients. Substituting the expansion Eq. (I.4) into the first few inequalities, we find

$$\sigma_n' \geq 0\,, \tag{III.1}$$
$$\sigma_n'' \geq \frac{2}{5}\frac{(\sigma_n')^2}{\sigma_n}\,,$$
$$\sigma_n''' \geq \frac{15}{28}\frac{(\sigma_n'')^2}{\sigma_n'}\,,$$
$$\sigma_n^{(4)} \geq \frac{45(\sigma_n'')^3 - 168\sigma_n'\sigma_n''\sigma_n''' + 392\sigma_n(\sigma_n''')^2}{126[5\sigma_n\sigma_n'' - 2(\sigma_n')^2]}\,.$$

Similar inequalities can be obtained for $\sigma^{(p)}$, $p > 4$, but we omit them as they are not particularly illuminating. We have been able to show that Eq. (III.1) leads to —

see Appendix D —

$$\sigma_n, \sigma_n', \sigma_n'', \sigma_n''', \sigma_n^{(4)} \geq 0\,. \tag{III.2}$$

Namely, the first five expansion coefficients must be positive for any $n > 0$. We suspect that this holds for *all* the coefficients, however, the inequalities become sufficiently complicated at higher order, $\sigma^{(p>4)}$, that we cannot at present prove this claim. This being said, in section IV, we will argue that the coefficients $\sigma_n^{(p)}$ are strictly positive at sufficiently large p. It is not unreasonable then to expect that the positivity property extends to intermediate values of p.

B. Bounds for the entanglement entropy

If we substitute the smooth limit expansion of $a(\theta)$, (I.4), into the differential inequality obtained for the EE Eq. (II.1), we obtain the following constraint for the quartic coefficient:

$$\sigma' \geq \frac{\sigma}{24}\,. \tag{III.3}$$

We can combine this inequality with the result (I.6) relating σ and C_T to obtain:

$$\frac{\sigma'}{C_T} \geq \frac{\pi^2}{24^2} \approx 0.0171\,. \tag{III.4}$$

We can further combine this inequality with the ones obtained above, Eq. (III.1), to generate bounds for the higher order coefficients:

$$\frac{\sigma''}{C_T} \geq \frac{\pi^2}{34\,560} \approx 2.86 \times 10^{-4}\,,$$
$$\frac{\sigma'''}{C_T} \geq \frac{\pi^2}{3\,870\,720} \approx 2.55 \times 10^{-6}\,. \tag{III.5}$$

It would be interesting to see whether stronger constraints can be derived for $\sigma'', \sigma''', \ldots$, compared to what arises from the general-n inequalities (III.1). One possibility consists of looking at the smooth-limit coefficients of $\mathfrak{a}_{\min}(\theta)$. The expansion of this function near π is as follows:

$$\frac{\mathfrak{a}_{\min}(\theta)}{C_T} = \frac{\pi^2}{24}\varepsilon^2 + \frac{\pi^2}{24^2}\varepsilon^4 + \frac{\pi^2}{8640}\varepsilon^6 + \mathcal{O}\left(\varepsilon^8\right)\,. \tag{III.6}$$

We thus see that the second coefficient exactly corresponds to the lower bound for σ' obtained above. Could it be that the other coefficients provide lower bounds for σ'', σ''', etc? For instance, is it true that any CFT will have $\sigma''/C_T \geq \pi^2/8640$? Should these hold, they would be stronger lower bounds than the ones obtained above. In Appendix B, we show that all the expansion

	lower bound	Ising ($N=1$)	XY ($N=2$)	Heisen. ($N=3$)	scalar	Dirac fermion	AdS/CFT
$a(\pi/2)/C_T$	1.140	1.3(1) [39,44,47]	1.3(1) [40,47]	1.3(1) [43,47]	1.245 [12,14,45]	1.226 [12,13,45]	1.222 [36,46]
$a(3\pi/4)/C_T$	0.260	-	-	-	0.265 [12,14,45]	0.264 [12,13,45]	0.264 [36,46]

TABLE I. Ratio $a(\theta)/C_T$, with $\theta = \pi/2, 3\pi/4$, for different critical theories described by CFTs, including the Wilson-Fisher quantum critical points. The first entries are the lower bounds we derived, Eq. (II.2).
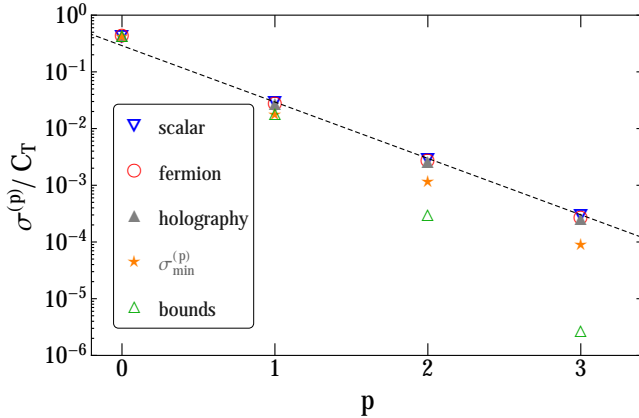

FIG. 3. **Smooth limit coefficients and bounds.** Log-linear plot of the first four smooth limit coefficients $\sigma^{(p)}$, $p \leq 3$, normalized by C_T for the free scalar, free fermion, and holographic CFTs dual to Einstein gravity. $\sigma_{\min}^{(p)}$ are the coefficients of the minimal function $\mathfrak{a}_{\min}(\theta)$. We also plot the bounds obtained in (III.4) and (III.5). The dashed line corresponds to the exponential scaling A/π^{2p}, where the properties of A are revealed in §IV and Fig. 4.

coefficients of $\mathfrak{a}_{\min}$ are positive, and give them in closed-form. The coefficients that have been computed for holographic CFTs, and free CFTs indeed lie above those of the minimal function, as shown in Fig. 3. In this figure we also observe that the smooth-limit coefficients of these theories seem to behave as $\sigma^{(p \gg 1)} \to A/\pi^{2p}$, where A is a theory-dependent constant. We analyze this behavior more closely in the following section.

IV. SMOOTH-SHARP CONNECTION

As shown in Fig. 3, the coefficients of the Taylor expansion around $\theta = \pi$ decay as π^{-2p} for growing values of p, both for free fields and holographic theories. Here we will argue that this exponential decay occurs generally for the corner function (at any Rényi index), and even for conical singularities in higher dimensions. The starting point of our discussion is, perhaps surprisingly, as far away from the smooth limit as one can go. In the sharp corner limit, the corner entanglement function

diverges as $1/\theta$:

$$a_n(\theta \to 0) = \frac{\kappa_n}{\theta}, \qquad \text{(IV.1)}$$

where $\kappa_n > 0$ is the so-called sharp limit coefficient. This implies that the Taylor expansion about π has at most a radius of convergence of π due to the divergences at $\theta = 0, 2\pi$. We claim that for any CFT, the corner function has this radius of expansion precisely equal to π. In other words, the smooth limit expansion is as well-behaved as possible. In particular, this implies that the smooth limit expansion must encode the divergence as $\theta \to 0$. It is not hard to see that the geometric series $\sum_{p=0}(\pi-\theta)^{2p}/\pi^{2p}$ yields $\pi/(2\theta)$, which has the desired divergence. Thus we deduce that the expansion coefficients of corner function will share the same asymptotics:

$$\sigma_n^{(p \gg 1)} \to \frac{2\kappa_n}{\pi^{2p+3}}, \qquad \text{(IV.2)}$$

which yields the correct weight for the pole at zero opening angle, Eq. (IV.1). In Fig. 4, we show that the asymptotic scaling given in Eq. (IV.2) already becomes a good approximation at $p = 2$ for the free scalar,[12] free Dirac fermion[12,67], and holographic theories. Indeed, the ratio $\sigma^{(p)}\pi^{2p+3}/2\kappa$ obtained for the EE deviates from unity by less than 2% for $p \geq 2$. In addition, Fig. 4 shows the coefficients for the so-called Extensive Mutual Information model,[68–70] and the cone function $a^{4d}(\theta)$ valid for *all* CFTs in 3+1 dimensions (see Section VIII).

We thus see that knowledge of the first few smooth limit coefficients can give a good estimate of the sharp limit coefficient κ via the smooth-sharp relation Eq. (IV.2). Conversely, knowledge of κ can be used to estimate the smooth limit coefficients even for relatively small values of p.

V. SMOOTH LIMIT EXPANSION IN ADS/CFT

In this section, we study the corner function of strongly interacting CFTs described in terms of a higher-dimensional theory of gravity via the holographic Anti de Sitter (AdS) / CFT correspondence.[71] The results we derive for the expansion in the smooth limit $\theta \to \pi$ will be used to deduce general properties of the quartic coefficient σ' in Section VI. We focus on theories with the simplest holographic bulk description, correspond-
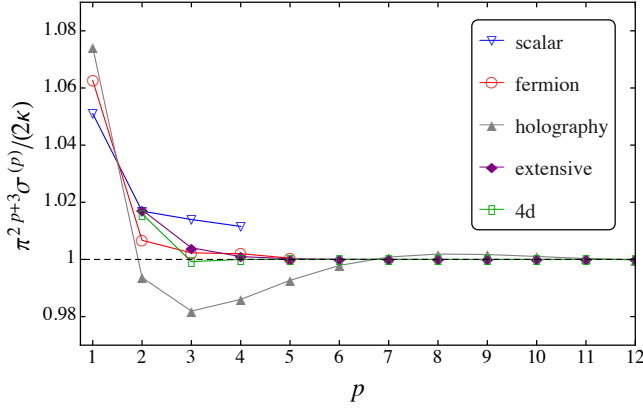
FIG. 4. **Smooth-sharp connection.** Smooth limit coefficients $\sigma^{(p)}$ normalized by the sharp limit coefficient κ (times $2/\pi^{2p+3}$). We show results for a free scalar (blue), a free Dirac fermion (red), holography dual to Einstein gravity (gray), the extensive mutual information model (purple), and the conical $d = 4$ function (green). All the ratios approach 1 as p grows, in agreement with Eq. (IV.2).

ing to pure Einstein gravity in the 3+1 dimensional AdS spacetime. The holographic minimal surface prescription of Ryu and Takayanagi[72–75] allows one to compute the EE associated with a region V in the boundary theory by extremizing the area of a surface extending from the entangling surface ∂V to the holographic bulk. In particular, one can consider a wedge-shaped region with a corner of opening angle θ. The resulting corner function was obtained implicitly via:[36,63]

$$a^E(\theta) = \frac{L^2}{2G} \int_0^\infty ds \left[1 - \sqrt{\frac{1 + h_0^2 (1 + s^2)}{2 + h_0^2 (1 + s^2)}} \right], \quad \text{(V.1)}$$

where L is the AdS radius, G is the bulk's gravitational constant, and h_0 is independent of s. We note that this integral can be evaluated exactly in terms of elliptic functions.[76,77] h_0 is implicitly determined by the corner opening angle θ via:

$$\theta = \int_0^{h_0} dh \, \frac{2h^2}{\sqrt{1 + h^2}} \frac{1}{\sqrt{h_0^2 - h^2}} \frac{1}{\sqrt{h_0^2(1 + h_0^2)^{-1} + h^2}}. \quad \text{(V.2)}$$

Heuristically, h_0 is the proportionality constant that determines how deep the minimal-area surface extends into the bulk along its spine, when moving away from the tip of the corner. In the very sharp limit $\theta \to 0$, the Ryu-Takayanagi minimal surface is very shallow, resulting in a small h_0. h_0 is plotted in Fig. 5(a), and we see that it vanishes as $\theta \to 0$, and diverges as $\theta \to \pi$, as expected. In order to obtain an expansion for $a^E(\theta)$ near π, we first need to expand the RHS of Eq. (V.2) about $h_0 = \infty$.

However, a direct Taylor expansion of the integrand of (V.2) is not well-defined. We circumvent this issue by expanding the last factor, $1/\sqrt{h_0^2(1 + h_0^2)^{-1} + h^2}$, and keeping the problematic one, $1/\sqrt{h_0^2 - h^2}$, intact. We can then perform the h integral exactly, and finally expand in powers of $1/h_0$. We find:

$$\frac{\pi - \theta}{\pi} = \frac{1}{h_0} - \frac{3}{4h_0^3} + \frac{61}{64h_0^5} - \frac{359}{256h_0^7} + \mathcal{O}\left(\frac{1}{h_0^9}\right), \quad \text{(V.3)}$$

which can be inverted to yield:

$$h_0 = \frac{\pi}{\varepsilon} - \frac{3}{4}\left(\frac{\varepsilon}{\pi}\right) - \frac{11}{64}\left(\frac{\varepsilon}{\pi}\right)^3 - \frac{17}{256}\left(\frac{\varepsilon}{\pi}\right)^5 + \mathcal{O}(\varepsilon^7), \quad \text{(V.4)}$$

where $\varepsilon = \pi - \theta$. We finally substitute this expansion into the integral for $a^E(\theta)$, Eq. (V.1), and get:

$$a^E(\theta) = \frac{L^2}{2G} \left[\frac{\varepsilon^2}{4\pi} + \frac{5\varepsilon^4}{32\pi^3} + \frac{37\varepsilon^6}{256\pi^5} + \cdots \right]. \quad \text{(V.5)}$$

Using $C_T = 3L^2/(\pi^3 G)$, we can express the first smooth-limit expansion coefficients as:

$$\frac{\sigma}{C_T} = \frac{\pi^2}{24}, \qquad \frac{\sigma'}{C_T} = \frac{5}{192},$$
$$\frac{\sigma''}{C_T} = \frac{37}{1536\pi^2}, \qquad \frac{\sigma'''}{C_T} = \frac{195}{8192\pi^4},$$
$$\frac{\sigma^{(4)}}{C_T} = \frac{3133}{131072\pi^6}, \qquad \frac{\sigma^{(5)}}{C_T} = \frac{25233}{1048576\pi^8}. \quad \text{(V.6)}$$

Note that all these are positive, and are described by the formula $r/\pi^{2(p-1)}$, where r is a rational number. We have verified that the first 20 coefficients satisfy this simple form; it would be interesting to obtain an analytical expression for all the coefficients. Fig. 3 shows that the bounds on $\sigma^{(p \leq 3)}$ are comfortably satisfied. In addition, we have verified that these coefficients lie close to the coefficients of the closed-form approximate form for $a^E(\theta)$ that we have previously derived,[50] which is further support for the latter.

As noted in the introduction, there exists strong evidence that σ is determined by C_T. It is then natural to ask what physical properties determine the other coefficients σ', σ'', etc? We now turn to this question.

VI. QUARTIC COEFFICIENT AND STRESS TENSOR CORRELATORS

The main goal of this section is to investigate what properties determine the quartic coefficient σ'. Since the leading coefficient σ is fixed by the 2-point function of the stress tensor, one might expect that the next coefficient depends on the 3-point function. The 3-point function of the stress tensor is highly constrained by con-
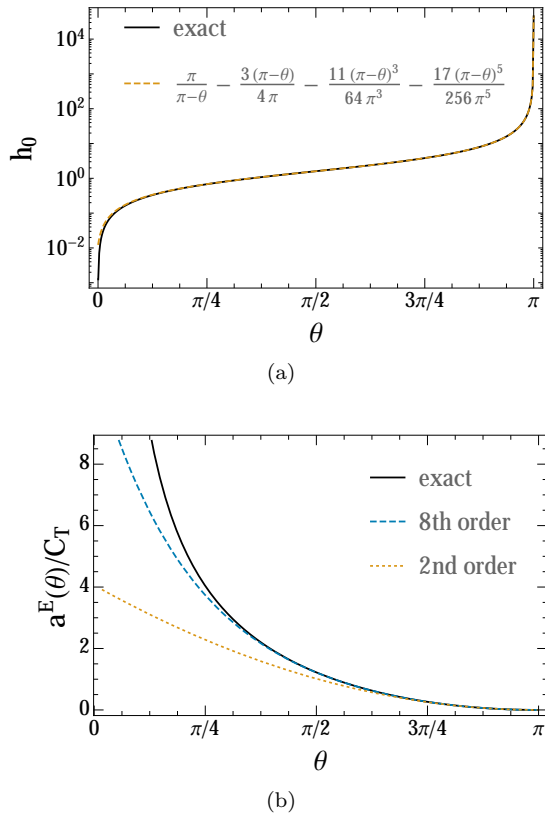
(a)

(b)

FIG. 5. **Holographic corner function.** a) Log-linear plot of the holographic depth parameter h_0 versus the corner opening angle θ. The truncated 5th order expansion near $\theta = \pi$ (dashed) accurately captures the angle dependence even away from π (deviations only become apparent as $\theta \to 0$). b) Corner function $a^E(\theta)$, normalized by C_T, and the corresponding approximations to ℓ-th order in $(\theta - \pi)^\ell$.

formal invariance and only depends on two numbers: C_T, which determines the 2-point function, and an independent constant, t_4, see appendix C. If σ' were to depend solely on C_T and t_4, it would have to be on a *linear* combination of the two constants. This is because $a(\theta)$ is additive for free CFTs, just as C_T and the product $t_4 C_T$. Indeed, for n_s complex free scalars and n_f 2-component Dirac fermions:[45,78]

$$C_T = \frac{3}{16\pi^2}(n_{\rm cs} + n_{\rm f}),$$
$$t_4 = 4\,\frac{n_{\rm cs} - n_{\rm f}}{n_{\rm cs} + n_{\rm f}}. \qquad (\text{VI.1})$$

This implies that without loss of generality we can work with an Ansatz of the form $\sigma' = c_1 + c_2 C_T + c_3 C_T t_4$. Actually, $c_1 = 0$ due to the additive property of the free field CFTs. We can also fix c_2 by virtue of our findings

in §V:

$$\frac{\sigma'}{C_T} = \frac{5}{192} + c_3\,t_4\,, \qquad (\text{VI.2})$$

where we have used the value of σ' obtained above for Einstein holography, Eq. (V.6), for which $t_4 = 0$.[78,79] For the complex scalar and Dirac fermion free CFTs, Casini and Huerta numerically found[12]

$$\sigma'_{\rm cs}/C_T = 0.0287088\,, \qquad \sigma'_{\rm f}/C_T = 0.0263940\,, \quad (\text{VI.3})$$

where the uncertainties are on the last digit. We note that both these ratios exceed the holographic one, $5/192 \approx 0.0260417$. Now, the complex scalar has a positive $t_4 = 4$, implying that $c_3 > 0$ for (VI.2) to hold. However, this is inconsistent with the fact that the fermion has a negative $t_4 = -4$ but $\sigma'_{\rm f}/C_T > 5/192$. Our analysis thus demonstrates that σ' is not fully determined by the 2- and 3-point function data of the stress tensor.

Interestingly, Ref. 52 claimed that for a class of CFTs dual to holographic bulk theories with higher derivative terms, the quartic coefficient is entirely determined by C_T and t_4, and thus given by Eq. (VI.2). As we have shown, this stands in contrast to the general case where additional information enters in σ'. The specific value of the constant c_3 in Eq. (VI.2) was not provided in Ref. 52, but it would be interesting to investigate how close these holographic theories approach the bounds we have derived in §III. Alternatively, our lower bounds would translate into bounds on the coupling of the higher derivative holographic theories.

A. 4-point data

Our analysis above showed that 2- and 3-point function data is insufficient to characterize σ'. We claim that in addition to this data, *i.e.,* C_T, t_4, information about the stress tensor 4-point function is needed to fix σ'. However, this is more difficult to show and test because the conformal symmetry is not sufficient to fix the coordinate dependence of $\langle TTTT \rangle$, in contrast to 2- and 3-point correlators. In other words, the 4-point correlator will contain undetermined functions of the conformal cross ratios (see below). These functions are theory dependent. To illustrate the point, it is sufficient to consider the 4-point function of a scalar operator $O(x)$ with scaling dimension Δ:

$$\langle O(x_1)O(x_2)O(x_3)O(x_4) \rangle = \frac{F(u,v)}{x_{12}^{2\Delta}x_{34}^{2\Delta}}\,, \qquad (\text{VI.4})$$

where $x_{ij} = x_i - x_j$, and the conformal cross ratios are $u = x_{12}^2 x_{34}^2/(x_{13}^2 x_{24}^2)$ and $v = x_{14}^2 x_{23}^2/(x_{13}^2 x_{24}^2)$. $F(u,v)$ is an arbitrary theory-dependent function.

We leave for future work the investigation of the specific role that the $T_{\mu\nu}$ 4-point correlator plays in determining σ'. It is natural to expect that the higher point functions of the stress tensor describe the higher order coefficients.

VII. LIFSHITZ QUANTUM CRITICAL POINTS

Up to this point our analysis was focused on CFTs, but it is natural to ask whether our results hold for a larger class of quantum critical theories. To illustrate that this is indeed the case, we will analyze the properties of a special class of Lifshitz quantum critical points in $d = 2 + 1$. These have a dynamical exponent $z = 2$, and have scale-invariant groundstate wavefunctions.[80] In fact, their equal-time correlation functions are determined by a parent 2-dimensional CFT. For this reason they are often called "conformal" quantum critical points. The quotes appear since these theories do not possess the full $d = 3$ spacetime conformal symmetry, but rather a time-independent version. For example, this class of Lifshitz theories describes lattice models such as the quantum dimer[81] and the quantum eight-vertex[80] models. In the former case, on a square lattice, the following quantum critical theory emerges at low energy:

$$\mathcal{L} = \frac{1}{2}(\partial_\tau \varphi)^2 + \frac{\kappa^2}{2}(\partial_x^2 \varphi + \partial_y^2 \varphi)^2 \,, \qquad \text{(VII.1)}$$

where φ is a real scalar field. Eq. (VII.1) has a $d = 2$ free boson CFT describing its equal-time correlators (it has Virasoro central charge $c = 1$).

When the entangling surface contains corners, the EE has a logarithmic subleading term[82] just as for CFTs, Eq. (I.3). Owing to the spatial conformal invariance of the wavefunction, the corresponding corner function of such theories can be given in closed-form:[82]

$$a(\theta) = \frac{c}{12} \frac{(\theta - \pi)^2}{\theta(2\pi - \theta)} \,, \qquad \text{(VII.2)}$$

where c is the central charge of the parent $d = 2$ CFT. As pointed out in Ref. 50, this is the simplest θ-dependence compatible with the expected behavior in the limits $\theta \to 0, \pi$,[83] and with the reflection property of pure states, $a(2\pi - \theta) = a(\theta)$. The smooth limit expansion coefficients, Eq. (I.4), can be simply obtained by rewriting Eq. (VII.2) in the form, $\frac{c}{12}\{-1 + 1/[1 - (\theta - \pi)^2/\pi^2]\}$:

$$\sigma^{(p)} = \frac{c}{12} \frac{1}{\pi^{2p+2}} \,. \qquad \text{(VII.3)}$$

We recognize a geometric series: the $\sigma^{(p)}$ obey a pure exponential scaling A/π^{2p} for all p. We also conclude that the series has maximal radius of convergence, π.

Using our general expression for the asymptotic behavior of $\sigma^{(p)}$ at large p, Eq. (IV.2), together with Eq. (VII.3), we predict that the sharp limit coefficient is $\kappa = \pi c/24$. A direct expansion of $a(\theta)$ about $\theta = 0$ gives $\pi c/(24\theta)$, in agreement with our prediction. This class of Lifshitz theories thus obeys the smooth-sharp relation that we put forth for CFTs in §IV.

Further, we note that Eq. (VII.2) exceeds the lower bound found for CFTs, Eq. (II.2), when $\mathfrak{a}_{\min}$ is normalized to have the same leading smooth-limit coefficient σ. This follows from the fact that the expansion coefficients of $\mathfrak{a}_{\min}$ (given in Appendix B) are less than those of Eq. (VII.2), which can easily be proven by bounding the zeta function. Given that the derivation of this lower bound depended on Lorentz invariance, it is not clear how general the result is. For instance, will more general Lifshitz theories violate the CFT bound?

It would be of interest to relate the expansion coefficients given in Eq. (VII.3) to correlation functions of local operators, as in the case of CFTs, for which the leading coefficient σ is given by the stress tensor 2-point function. We leave this task for future investigation. Finally, let us mention that corner entanglement has been recently studied for other classes of non-conformal theories using holography,[84–86] where significant differences have been found with respect to the CFT results.[48,49]

VIII. HIGHER DIMENSIONS

For a conical singularity in $d = 3 + 1$, the EE of a CFT receives a subleading logarithmic correction as shown in Fig. 6. The corresponding cone function $a^{(4d)}(\theta)$ is independent of the UV regulator as for corners in $d = 3$. Interestingly, it is *entirely* fixed by C_T:[87]

$$a^{(4d)}(\theta) = \frac{\pi^4 C_T}{160} \frac{\cos^2(\theta/2)}{\sin(\theta/2)} \,, \qquad \text{(VIII.1)}$$

where θ is the cone's opening angle, see Fig. 6. This expression applies to *all* four-dimensional CFTs. In addition, the very same θ-dependence holds for the Rényi entropies, namely[60]

$$a_n^{(4d)}(\theta) = \frac{1}{4} f_b(n) \frac{\cos^2(\theta/2)}{\sin(\theta/2)} \,, \qquad \text{(VIII.2)}$$

where $f_b(n)$ depends on the Rényi index,[88,89] with $f_b(1) = \pi^4 C_T/40$. Strong evidence[60,89,90] suggests that $f_b(n)$ is determined by the twist operator scaling dimension h_n through $f_b(n) = (3\pi/2) h_n/(n - 1)$. (In $d = 4$, the twist operator is a surface operator.) The fact that h_n fully determines the entire θ-dependence is in contrast with $d = 3$, where the scaling dimension of the twist operator only determines the smooth limit coefficient.

$$S_n = \mathcal{B}_n \frac{\ell}{\delta} - a_n^{(4d)}(\theta) \left[\log(\ell/\delta)\right]^2 + \cdots$$
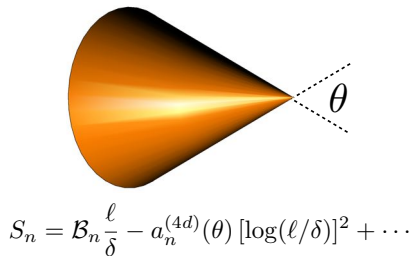
FIG. 6. **Cone entanglement.** A cone singularity in $d = 3 + 1$ contributes logarithmic term to the nth Rényi entropy, with a universal prefactor, $-a_n^{(4d)}(\theta)$.

Many similarities, however, do exist between $d = 4$ and $d = 3$, and we now discuss them in light of the findings presented above. For one, $a^{(4d)}(\theta)$ admits a smooth limit expansion about π in even powers of $(\theta - \pi)$, just as in $d = 3$, Eq. (I.4). In Appendix B, we show that this expansion has a radius of convergence of π. We conjecture that is the case for CFTs in all dimensions $d > 1 + 1$. Further, we show there that the expansion coefficients $\sigma^{(p)}$ are strictly positive, again in agreement with our general conjecture. Recall that in $d = 3$ we have been able to prove this for the first five coefficients, $\sigma_n^{(p<5)}$. In Fig. 4, we plot the first 13 coefficients (except the first one, σ) of the entanglement entropy. They vanish exponentially with p, and rapidly approach the asymptotic form (IV.2), *i.e.*, $\sigma_n^{(p\gg1)} \to 2\kappa_n/\pi^{2p+3}$, where again κ_n dictates the small angle divergence of the cone function: $a_n^{(4d)}(\theta \to 0) = \kappa_n/\theta$, with $\kappa_n = f_b(n)/2$.

For theories in higher dimensions $d > 4$, less is known about the general form of the analogous hypercone function $a_n^{(d)}(\theta)$ away from the smooth limit,[37,52,60,91] in particular for odd-dimensional theories. For the special case of even-dimensional holographic theories, one can perform the calculation, and the properties of the expansion about the smooth limit are analoguous to what we found here for $d = 3$ and 4.[92]

IX. DISCUSSION

We have studied the entanglement properties of quantum critical theories. Focusing on conformal field theories, we have established lower bounds on the corner function $a(\theta)$ in $d = 2 + 1$, Eqs. (II.2,II.3), and on its expansion coefficients in smooth limit, $\theta \to \pi$. Our bound for $a(\pi/2)$ corresponding to 90-degree corners, Eq. (II.4), is nearly saturated by all known results, including recent numerical estimates on lattice models that describe Wilson-Fisher quantum critical points. The bounds rely on fundamental properties of entanglement, such as the strong subadditivity of EE. An important extension would be to establish stronger bounds on the corner function for general Rényi index than what we obtained, Eq. (II.8). It is further natural to ask whether an *upper* bound exists for $a(\theta)$, and its expansion coefficients. The holographic correspondence could be helpful in answering this question.

With regards to the physical properties of the smooth-limit expansion coefficients, we have shown that the quartic coefficient σ' (Fig. 1(b)) contains information beyond the 2- and 3-point functions of the stress tensor, and conjectured that 4-point data is required. This potentially makes σ' much richer compared to the leading coefficient σ because 4-point functions are highly nontrivial in CFTs. It will be interesting to determine the precise physical information encoded in this and higher order coefficients. With such data in hand, the bounds we have derived for the corner entanglement function will yield bounds on the physical properties of CFTs. Another important direction for future investigation would be to examine the properties of corner entanglement in non-conformal quantum critical theories, with Lifshitz scaling for instance. Our discussion in §VII suggests that a connection exists between the latter and conformal field theories, at least for $z = 2$ theories with conformal wave functions.[80]

Appendix A: Establishing the lower bounds

1. Proving the minimality of $\mathfrak{a}_{\min}(\theta)$

We shall prove that $a(\theta) \geq \mathfrak{a}_{\min}(\theta)$, where the minimal function $\mathfrak{a}_{\min}$ is given in closed form in Eq. (II.2). To do so, we rederive a special instance of a theorem proved by S.A. Chaplygin (see page 164 of Ref. 93).

Let us consider the differential inequality:

$$\ddot{a}(\theta) \geq F(\theta, \dot{a}(\theta)), \tag{A.1}$$

where dots denote θ derivatives. We are interested in the particular case where $F = -\dot{a}/\sin\theta$, but the proof below holds more generally. We shall work on the interval $\pi \leq \theta < 2\pi$, where $\dot{a} \geq 0$. The minimal function $\mathfrak{a}_{\min}$ satisfies the differential equation obtained by replacing the inequality in (A.1) with an equality, *i.e.,*

$$\ddot{\mathfrak{a}}_{\min}(\theta) = F(\theta, \dot{\mathfrak{a}}_{\min}(\theta)). \tag{A.2}$$

Both functions satisfy the same boundary conditions at $\theta = \pi$:

$$a(\pi) = \dot{a}(\pi) = 0,$$
$$\mathfrak{a}_{\min}(\pi) = \dot{\mathfrak{a}}_{\min}(\pi) = 0. \tag{A.3}$$

Thus the leading term in the Taylor expansion around π is $\sigma(\theta - \pi)^2$, with σ taken to be the same for both functions. Subtracting (A.2) from (A.1), we get

$$\ddot{a} - \ddot{\mathfrak{a}}_{\min} - (\dot{a} - \dot{\mathfrak{a}}_{\min}) Q(\theta) \geq 0 \tag{A.4}$$

where we have defined the quotient

$$Q(\theta) = \frac{F(\theta, \dot{a}) - F(\theta, \dot{\mathfrak{a}}_{\min})}{\dot{a} - \dot{\mathfrak{a}}_{\min}}, \tag{A.5}$$

which is well-defined even at points where $\dot{a} = \dot{\mathfrak{a}}_{\min}$. The key step is to introduce the new function

$$U(\theta) = \exp\left[-\int_\pi^\theta d\vartheta \, Q(\vartheta)\right], \tag{A.6}$$

which is non-negative. We can thus rewrite (A.4) as:

$$\partial_\theta \left\{(\dot{a} - \dot{\mathfrak{a}}_{\min})U(\theta)\right\} \geq 0. \tag{A.7}$$

Integrating both sides of this inequality between π and $\theta > \pi$, and using $\dot{a}(\pi) = \dot{\mathfrak{a}}_{\min}(\pi)$, we obtain

$$\dot{a}(\theta) \geq \dot{\mathfrak{a}}_{\min}(\theta), \tag{A.8}$$

for $\pi \leq \theta < 2\pi$. (By virtue of the reflection property of $a(\theta)$ about π, this leads to $|\dot{a}(\theta)| \geq |\dot{\mathfrak{a}}_{\min}(\theta)|$ for $0 \leq \theta \leq \pi$.) Finally, we note that

$$a(\theta) = \int_\pi^\theta d\theta \, \dot{a}(\theta), \tag{A.9}$$

and the same holds for $\mathfrak{a}_{\min}$. We can thus integrate (A.8) to obtain the desired result:

$$a(\theta) \geq \mathfrak{a}_{\min}(\theta). \tag{A.10}$$

This completes our proof for the lower bound on the corner function.

2. Bounds for general Rényi index

We now give the derivation of the lower bound for $a_n(\theta)$, Eq. (II.8), following a very similar approach as in the previous subsection. We begin by re-writing the third inequality obtained from reflection positivity, Eq. (II.7):

$$\partial_\theta^4 a_n \times [\partial_\theta^2 a_n] \geq (\partial_\theta^3 a_n)^2 \,, \tag{A.11}$$

where the term in square brackets is non-negative by virtue of the second reflection positivity constraint, Eq. (II.6). This implies that we can write the inequality in the form $\partial_\theta^4 a_n \geq G$, where G depends only on the second and third derivatives of a_n. We can now apply a higher order version of Chaplygin's theorem invoked above.[93] Clearly,

$$\mathfrak{a}_n(\theta) = \sigma_n \, (\theta - \pi)^2 \,, \tag{A.12}$$

is a solution of the differential equation obtained by replacing the inequality in Eq. (A.11) by an equality. For convenience, we shall work on the interval $[\pi, 2\pi)$. The theorem guarantees that

$$a_n(\theta) \geq \mathfrak{a}_n(\theta) \,, \tag{A.13}$$

assuming they have matching initial value conditions at π:

$$\begin{aligned}
a_n(\pi) &= \mathfrak{a}_n(\pi) = 0 \,, \\
\partial_\theta a_n(\pi) &= \partial_\theta \mathfrak{a}_n(\pi) = 0 \,, \\
\partial_\theta^2 a_n(\pi) &= \partial_\theta^2 \mathfrak{a}_n(\pi) = \sigma_n \,.
\end{aligned} \tag{A.14}$$

This proves the desired lower bound, Eq. (II.8).

We now show that all the reflection positivity inequalities Eq. (II.5), labelled by an integer $M \geq 1$, lead to lower bounds, and never to an upper bound. Let us denote the Mth inequality by $I_M \geq 0$. We then have

$$\partial_\theta^{2M} a_n(\theta) \times I_{M-1} + P_M \geq 0 \,, \tag{A.15}$$

where P_M is a linear combination of products of derivatives of a_n. Since $I_{M-1} \geq 0$, we have:

$$\partial_\theta^{2M} a_n(\theta) \geq G_M \,, \tag{A.16}$$

where G_M only depends lower order derivatives of a_n. Applying Chaplygin's theorem as above, one will obtain a lower bound for a_n by solving the non-linear differential equation Eq. (A.16). In this case one needs to supply $(M-1)$ initial conditions at $\theta = \pi$. We leave the investigation of the resulting inequalities for future work.

Appendix B: Exact smooth limit expansions

We determine the smooth limit expansion in closed-form for the corner function $a_n(\theta)$ in a variety of systems. We begin with the minimal corner function $\mathfrak{a}_{\min}(\theta)$, Eq. (II.2), then analyze the Extenstive Mutual Information model for CFTs, and finally the cone function describing *all* CFTs in $d = 3+1$. In all cases, we will find that the radius of convergence of the Taylor expansion about $\theta = \pi$ is π. The series must breakdown at $\theta = 0$ since the the corner function has a $1/\theta$ divergence as $\theta \to 0$. In relation to the latter, we find that in all cases (except for $\mathfrak{a}_{\min}$, which is not the corner function of an actual CFT, but rather a bound) the expansion coefficients asymptote to

$$\sigma_n^{(p)} \xrightarrow{p \gg 1} \frac{2}{\pi^3} \frac{\kappa_n}{\pi^{2p}} \,, \tag{B.1}$$

thus decaying exponentially fast at large p, as explained in §IV. The coefficient κ_n dictates the small angle divergence,

$$a_n(\theta \to 0) = \kappa_n / \theta \,. \tag{B.2}$$

p	0	1	2	3	4	5
$\sigma_{\min}^{(p)} \, 24/(\pi^2 C_T)$	1	1/24	1/360	17/80640	31/1814400	691/479001600

TABLE II. Smooth limit coefficients $\sigma_{\min}^{(p)}$ for the minimal function $\mathfrak{a}_{\min}(\theta)$.

1. Minimal function

The minimal function, Eq. (II.2),

$$\mathfrak{a}_{\min}(\theta) = \frac{\pi^2 C_T}{3} \, \log\left[1/\sin(\theta/2)\right] , \tag{B.3}$$

has the smooth limit expansion

$$\mathfrak{a}_{\min}(\theta) = \sum_{p=1}^{\infty} \sigma_{\min}^{(p-1)} (\theta - \pi)^{2p} , \tag{B.4}$$

where

$$\sigma_{\min}^{(p-1)} = \frac{\pi^2 C_T}{3} \, \frac{(1 - 2^{-2p}) \, \zeta(2p)}{p \, \pi^{2p}} , \tag{B.5}$$

$\zeta(z)$ is Riemann's zeta function. Since $\zeta(2p)$ is strictly positive for all integers $p \geq 1$, all the coefficients are positive. We note that $\zeta(2p)/\pi^{2p}$ is a rational number related to the Bernoulli number B_{2p}: $\zeta(2p) = (-1)^{p+1}(2\pi)^{2p} B_{2p}/[2(2p)!]$. We list the leading coefficients in Table II. We can use the ratio test to check the convergence of the series. One finds

$$\lim_{p \to \infty} \left| \frac{\sigma_{\min}^{p+1} (\theta - \pi)^{2(p+2)}}{\sigma_{\min}^{p} (\theta - \pi)^{2(p+1)}} \right| = \frac{(\theta - \pi)^2}{\pi^2} , \tag{B.6}$$

so the series is convergent when $(\pi - \theta)^2/\pi^2 < 1$, i.e., for all $\theta \in (0, \pi]$. At large p, the coefficients decay to zero as

$$\sigma_{\min}^{(p)} \xrightarrow{p \gg 1} \frac{C_T}{3} \, \frac{1}{p \, \pi^{2p}} , \tag{B.7}$$

where the extra factor of p in the denominator implies that $\mathfrak{a}_{\min}(\theta)$ decays faster compared to what is expected of a CFT, Eq. (B.3).

2. Extensive Mutual Information model

We here analyze the corner function of the Extensive Mutual Information (EMI) model.[68–70] As the name suggests, it is characterized by the property that the mutual information, $I(A, B) = S(A) + S(B) - S(A \cup B)$, satisfies the extensivity property:[68,70] $I(A, B \cup C) = I(A, B) + I(A, C)$. The corner function in the EMI model reads:[68–70]

$$a_n^{\mathrm{emi}}(\theta) = \frac{3h_n}{\pi(n-1)} \, \left[1 + (\pi - \theta) \cot \theta\right] , \tag{B.8}$$

where h_n is the scaling dimension of the twist operator. This normalization was first obtained in Ref. 48. We impose on h_n all the conditions required for a CFT, in particular $h_n/(n-1) \geq 0$. This ensures that $a_n^{\mathrm{emi}}(\theta)$ is positive. Taylor expanding about $\theta = \pi$,

$$a_n^{\mathrm{emi}}(\theta) = \sum_{p=1}^{\infty} \sigma_n^{(p-1)} (\theta - \pi)^{2p} , \tag{B.9}$$

p	0	1	2	3	4	5
$\sigma_n^{(p)} \cdot \pi(n-1)/h_n$	1	1/15	2/315	1/1575	2/31185	1382/212837625

TABLE III. Smooth limit coefficients $\sigma_n^{(p)}$ for the corner function of the Extenstive Mutual Information model.

we obtain the smooth limit coefficients in closed form:

$$\sigma_n^{(p-1)} = \frac{3h_n}{\pi(n-1)} \frac{2\zeta(2p)}{\pi^{2p}} \, . \tag{B.10}$$

By virtue of the positivity of the zeta function (see previous subsection), all these coefficients are strictly positive. Further, at $n=1$ the first coefficient, σ, is equal to the one given by the minimal function $\mathfrak{a}_{\min}(\theta)$, whereas all the higher order coefficients are strictly greater:

$$\sigma^{(p)} > \sigma_{\min}^{(p)} \, , \qquad \text{for} \ \ p \geq 1 \, , \tag{B.11}$$

where we have again omitted the subscript $n=1$. We list the first few coefficients in Table III. We note that this series is convergent for all $\theta \in (0, \pi]$.

The $p \gg 1$ asymptotic behavior of the coefficients reads:

$$\sigma^{(p)} \xrightarrow{p \gg 1} \frac{C_T}{4} \frac{1}{\pi^{2p}} = \frac{2\kappa}{\pi^{2p+3}} \, , \tag{B.12}$$

vanishing exponentially fast and in agreement with Eq. (IV.2).

3. Cones in $4d$ CFTs

As we argued in §VIII, in three spatial dimensions ($d=4$), a conical singularity in the entangling surface gives rise to the following universal function:[60,87]

$$a_n^{(4d)}(\theta) = \frac{3\pi h_n}{8(n-1)} \frac{\cos^2(\theta/2)}{\sin(\theta/2)} \, , \tag{B.13}$$

where the smooth limit again corresponds to $\theta = \pi$. Just as above, h_n is the scaling dimension of the twist operator (which has support on two dimensional surfaces in $4d$). In that limit $a^{(4d)}$ has the Taylor expansion

$$a_n^{(4d)}(\theta) = \sum_{p=1}^{\infty} \sigma_n^{(p-1)} (\theta - \pi)^{2p} \, . \tag{B.14}$$

The coefficients are given by

$$\sigma_n^{(p-1)} = \frac{3\pi h_n}{8(n-1)} \frac{(-1)^p (E_{2p} - 1)}{4^p (2p)!} \, , \tag{B.15}$$

where $p \geq 1$, and E_{2p} is the Euler number. We note that $E_{2p} = (-1)^p A_{2p}$, where A_{2p} is the so-called zig-zag Euler number, which is a positive integer with some combinatorical significance. We thus conclude that all the expansion coefficients are strictly positive. Further, just as in all the cases studied above, the radius of convergence of the series is π, namely the expansion converges on $(0, 2\pi)$. To establish this one needs to use the fact that $|E_{2(p+1)}/E_{2p}| \to 16p^2/\pi^2$ as $p \to \infty$, which can be readily derived from the asymptotic behavior of E_{2p} at large p: $E_{2p} \to (-1)^p 8\sqrt{p/\pi} \left(\frac{4p}{\pi e}\right)^{2p}$.

	scalar	fermion	AdS/CFT
C_T	$3/(16\pi^2)$	$3/(16\pi^2)$	$3L^2/(\pi^3 G)$
t_4	4	-4	0

TABLE IV. Stress tensor 2- and 3-point function parameters $\{C_T, t_4\}$ in $d = 3$ for a complex scalar, Dirac fermion, and CFTs holographically dual to pure Einstein gravity.

The large p behavior of the coefficients is

$$\sigma_n^{(p)} \xrightarrow{p \gg 1} \frac{6h_n}{\pi^2(n-1)} \frac{1}{\pi^{2p}} = \frac{2\kappa_n}{\pi^{2p+3}} \,, \tag{B.16}$$

where we again find the exponential decay given by $1/\pi^{2p}$.

Appendix C: Stress tensor 2- and 3-point functions

The stress or energy-momentum tensor $T_{\mu\nu}(x)$ has scaling dimension d, equal to the spacetime dimension. In a CFT, its groundstate 2- and 3-point functions are strongly constrained by conservation of energy and momentum, and by the conformal symmetry. In this appendix, we review their basic properties. The 2-point function reads[45]

$$\langle T_{\mu\nu}(x) T_{\eta\kappa}(0) \rangle = \frac{C_T}{x^{2d}} \mathcal{I}_{\mu\nu,\eta\kappa}(x) \,, \tag{C.1}$$

where

$$\mathcal{I}_{\mu\nu,\eta\kappa}(x) = \frac{1}{2} \left(I_{\mu\eta}(x) I_{\nu\kappa}(x) + I_{\mu\kappa}(x) I_{\nu\eta}(x) \right) - \frac{1}{d} \delta_{\mu\nu} \delta_{\eta\kappa} \,, \tag{C.2}$$

where we have defined the dimensionless tensor $I_{\mu\nu}(x) = \delta_{\mu\nu} - 2x_\mu x_\nu / x^2$, and we are working in Euclidean time. From Eq. (C.1), we see that the 2-point function is entirely determined by a single real number in all dimensions, $C_T \geq 0$. In Table IV, we list the value of C_T in free boson and free Dirac fermion CFTs, and CFTs holographically dual to pure Einstein gravity.

The stress tensor 3-point function $\langle T_{\mu\nu}(x_1) T_{\eta\kappa}(x_2) T_{\sigma\rho}(0) \rangle$ is more complex, and its explicit form can be found in Ref. 45. The main point is that it is entirely characterized by only two parameters in $d = 3$.[45] (In $d \geq 4$, a third parameter is needed.[45]) A Ward identity[45] relating the 2- and 3-point functions of the stress tensor implies that one of these parameters can be chosen to be C_T, encountered above. We choose the second parameter to be t_4.[78] Table IV gives the value of t_4 for the free and holographic CFTs. For general CFTs, C_T and t_4 are related to the couplings defined by Osborn and Petkou,[45] $\hat{a}, \hat{b}, \hat{c}$, appearing in $\langle TTT \rangle$ as follows:

$$C_T = 4S_d \frac{(d-2)(d+3)\hat{a} - 2\hat{b} - (d+1)\hat{c}}{d(d+2)} \,,$$
$$t_4 = -\frac{(d+1)(d+2)\left[3(d-1)(2d+1)\hat{a} + 2d^2\hat{b} - d(d+1)\hat{c}\right]}{d\left[(d-2)(d+3)\hat{a} - 2\hat{b} - (d+1)\hat{c}\right]} \,, \tag{C.3}$$

where $S_d = 2\pi^{d/2}/\Gamma(d/2)$. The first equation follows from a Ward identity of the stress tensor. We emphasize that $C_T t_4$ is a *linear* combinations of the couplings $\hat{a}, \hat{b}, \hat{c}$. This is important because these latter couplings are additive for free bosons and fermions,[45] meaning that $C_T t_4$ is additive for these theories. For $d = 3$, Eq. (C.3) reduces to:

$$C_T = \frac{32\pi}{15}(3\hat{a} - \hat{b} - 2\hat{c}) \,,$$
$$t_4 C_T = -\frac{128\pi}{3}(7\hat{a} + 3\hat{b} - 2\hat{c}) \,. \tag{C.4}$$

Using these equations we can for instance solve for the Osborn-Petkou parameters for CFTs holographically dual to pure Einstein gravity:

$$\frac{\hat{a}}{C_T} = -\frac{27}{256\pi}\,, \qquad \frac{\hat{b}}{C_T} = -\frac{3}{256\pi}\,, \qquad \frac{\hat{c}}{C_T} = -\frac{99}{256\pi}\,. \tag{C.5}$$

Appendix D: Proof of the inequality $\sigma_n^{(4)} \geq 0$

In §III we showed that on general grounds,

$$\sigma_n^{(4)} \geq \frac{45(\sigma_n'')^3 - 168\sigma_n'\sigma_n''\sigma_n''' + 392\sigma_n(\sigma_n''')^2}{126[5\sigma_n\sigma_n'' - 2(\sigma_n')^2]}\,. \tag{D.1}$$

The denominator is always positive by virtue of the second equation in (III.1), so we only need to prove the positivity of the numerator. Using the second and third inequalities in (III.1), we define two quantities, k_1 and k_2,

$$k_1 = \sigma_n'' - \frac{2}{5}\frac{(\sigma_n')^2}{\sigma_n} \geq 0\,, \tag{D.2}$$

$$k_2 = \sigma_n''' - \frac{15}{28}\frac{(\sigma_n'')^2}{\sigma_n'} \geq 0\,, \tag{D.3}$$

which are non-negative. Substituting σ_n'' and σ_n''' in terms of these constants in (D.1), we find

$$45(\sigma_n'')^3 - 168\sigma_n'\sigma_n''\sigma_n''' + 392\sigma_n(\sigma_n''')^2 = 392k_2^2\sigma_n +$$
$$\frac{3k_1}{10\sigma_n^2(\sigma_n')^2}\left[5k_1\sigma_n + 2(\sigma_n')^2\right]\left[75k_1^2\sigma_n^2 + 60k_1\sigma_n(\sigma_n')^2 + 280k_2\sigma_n^2\sigma_n' + 12(\sigma_n')^4\right]\,, \tag{D.4}$$

which is a sum of non-negative terms. This implies $\sigma_n^{(4)} \geq 0$.

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[1] A. Kitaev and J. Preskill, Phys. Rev. Lett. **96**, 110404 (2006), arXiv:hep-th/0510092 [hep-th].

[2] M. Levin and X.-G. Wen, Physical Review Letters **96**, 110405 (2006), cond-mat/0510613.

[3] A. Hamma, R. Ionicioiu, and P. Zanardi, Physics Letters A **337**, 22 (2005), quant-ph/0406202.

[4] H. Li and F. D. M. Haldane, Physical Review Letters **101**, 010504 (2008), arXiv:0805.0332.

[5] S. T. Flammia, A. Hamma, T. L. Hughes, and X.-G. Wen, Physical Review Letters **103**, 261601 (2009), arXiv:0909.3305 [cond-mat.str-el].

[6] M. A. Metlitski, C. A. Fuertes, and S. Sachdev, Phys. Rev. B **80**, 115122 (2009), arXiv:0904.4477 [cond-mat.stat-mech].

[7] M. B. Hastings, I. González, A. B. Kallin, and R. G. Melko, Physical Review Letters **104**, 157201 (2010), arXiv:1001.2335 [cond-mat.str-el].

[8] N. Laflorencie, ArXiv e-prints (2015), arXiv:1512.03388 [cond-mat.str-el].

[9] C. Holzhey, F. Larsen, and F. Wilczek, Nuclear Physics B **424**, 443 (1994), hep-th/9403108.

[10] P. Calabrese and J. L. Cardy, J. Stat. Mech. **0406**, P06002 (2004), arXiv:hep-th/0405152 [hep-th].

[11] P. Calabrese and J. L. Cardy, *Workshop on Quantum Entanglement in Physical and Information Sciences Pisa, Italy, December 14-18, 2004*, Int. J. Quant. Inf. **4**, 429 (2006), arXiv:quant-ph/0505193 [quant-ph].

[12] H. Casini and M. Huerta, J. Phys. **A42**, 504007 (2009), arXiv:0905.2562 [hep-th].

[13] H. Casini, M. Huerta, and L. Leitao, Nucl. Phys. **B814**, 594 (2009), arXiv:0811.1968 [hep-th].

[14] H. Casini and M. Huerta, Nucl. Phys. **B764**, 183 (2007), arXiv:hep-th/0606256 [hep-th].

[15] M. Van Raamsdonk, (2009), arXiv:0907.2939 [hep-th].

[16] M. Van Raamsdonk, Gen. Rel. Grav. **42**, 2323 (2010), [Int. J. Mod. Phys.D19,2429(2010)], arXiv:1005.3035 [hep-th].

[17] E. Bianchi and R. C. Myers, Class. Quant. Grav. **31**, 214002 (2014), arXiv:1212.5183 [hep-th].

[18] V. Balasubramanian, B. D. Chowdhury, B. Czech, J. de Boer, and M. P. Heller, Phys. Rev. **D89**, 086004 (2014), arXiv:1310.4204 [hep-th].

[19] R. C. Myers, J. Rao, and S. Sugishita, JHEP **06**, 044 (2014), arXiv:1403.3416 [hep-th].

[20] B. Czech, X. Dong, and J. Sully, JHEP **11**, 015 (2014), arXiv:1406.4889 [hep-th].

[21] M. Headrick, R. C. Myers, and J. Wien, JHEP **10**, 149 (2014), arXiv:1408.4770 [hep-th].

[22] S. Ryu and T. Takayanagi, JHEP **08**, 045 (2006), arXiv:hep-th/0605073 [hep-th].

[23] T. Nishioka, S. Ryu, and T. Takayanagi, J. Phys. **A42**, 504008 (2009), arXiv:0905.0932 [hep-th].

[24] T. Takayanagi, Class. Quant. Grav. **29**, 153001 (2012), arXiv:1204.2450 [gr-qc].

[25] S. Ryu and T. Takayanagi, Phys. Rev. Lett. **96**, 181602 (2006), arXiv:hep-th/0603001 [hep-th].

[26] X. G. Wen, Int. Journal of Modern Phys. B **04**, 239 (1990), http://www.worldscientific.com/doi/pdf/10.1142/S0217979290000139.

[27] S. V. Isakov, M. B. Hastings, and R. G. Melko, Nature Physics **7**, 772 (2011), arXiv:1102.1721 [cond-mat.str-el].

[28] H.-C. Jiang, Z. Wang, and L. Balents, Nature Physics **8**, 902 (2012), arXiv:1205.4289 [cond-mat.str-el].

[29] S. Depenbrock, I. P. McCulloch, and U. Schollwöck, Phys. Rev. Lett. **109**, 067201 (2012).

[30] R. C. Myers and A. Sinha, Phys. Rev. D **82**, 046006 (2010), arXiv:1006.1263 [hep-th].

[31] H. Casini and M. Huerta, Phys. Rev. D **85**, 125016 (2012), arXiv:1202.5650 [hep-th].

[32] B. Swingle and T. Senthil, Phys. Rev. B **86**, 155131 (2012), arXiv:1109.3185 [cond-mat.str-el].

[33] T. Grover, Phys. Rev. Lett. **112**, 151601 (2014), arXiv:1211.1392 [hep-th].

[34] A. Renyi, Proceedings of the 4th Berkeley Symposium on Mathematics, Statistics and Probability, vol. 1, (Berkeley, CA), p. 547, U. of California Press (1961).

[35] A. Renyi, Rev. Int. Stat. Inst. 33 , no. 1 (1965).

[36] T. Hirata and T. Takayanagi, JHEP **02**, 042 (2007), arXiv:hep-th/0608213 [hep-th].

[37] R. C. Myers and A. Singh, JHEP **09**, 013 (2012), arXiv:1206.5225 [hep-th].

[38] L. Tagliacozzo, G. Evenbly, and G. Vidal, Phys. Rev. B **80**, 235127 (2009), arXiv:0903.5017 [cond-mat.str-el].

[39] A. B. Kallin, K. Hyatt, R. R. P. Singh, and R. G. Melko, Physical Review Letters **110**, 135702 (2013), arXiv:1212.5269 [cond-mat.stat-mech].

[40] E. M. Stoudenmire, P. Gustainis, R. Johal, S. Wessel, and R. G. Melko, Phys. Rev. **B90**, 235106 (2014).

[41] J. Helmes and S. Wessel, Phys. Rev. B **89**, 245120 (2014), arXiv:1403.7395 [cond-mat.str-el].

[42] T. Devakul and R. R. P. Singh, Phys. Rev. B **90**, 064424 (2014), arXiv:1406.0185 [cond-mat.stat-mech].

[43] A. B. Kallin, E. M. Stoudenmire, P. Fendley, R. R. P. Singh, and R. G. Melko, J. Stat. Mech. **1406**, P06009 (2014), arXiv:1401.3504 [cond-mat.str-el].

[44] S. Sahoo, E. Miles Stoudenmire, J.-M. Stéphan, T. Devakul, R. R. P. Singh, and R. G. Melko, ArXiv e-prints (2015), arXiv:1509.00468 [cond-mat.stat-mech].

[45] H. Osborn and A. C. Petkou, Annals Phys. **231**, 311 (1994), arXiv:hep-th/9307010 [hep-th].

[46] H. Liu and A. A. Tseytlin, Nucl. Phys. **B533**, 88 (1998), arXiv:hep-th/9804083 [hep-th].

[47] F. Kos, D. Poland, and D. Simmons-Duffin, JHEP **06**, 091 (2014), arXiv:1307.6856 [hep-th].

[48] P. Bueno, R. C. Myers, and W. Witczak-Krempa, Phys. Rev. Lett. **115**, 021602 (2015), arXiv:1505.04804 [hep-th].

[49] P. Bueno and R. C. Myers, JHEP **08**, 068 (2015), arXiv:1505.07842 [hep-th].

[50] P. Bueno, R. C. Myers, and W. Witczak-Krempa, JHEP **09**, 091 (2015), arXiv:1507.06997 [hep-th].

[51] P. Bueno, R. C. Myers, and W. Witczak-Krempa, Fortschritte der Physik , n/a (2015).

[52] R.-X. Miao, (2015), 10.1007/JHEP10(2015)038, arXiv:1507.06283 [hep-th].

[53] H. Elvang and M. Hadjiantonis, Phys. Lett. **B749**, 383 (2015), arXiv:1506.06729 [hep-th].

[54] J. S. Dowker, (2015), arXiv:1509.00782 [hep-th].

[55] J. S. Dowker, (2015), arXiv:1510.08378 [hep-th].

[56] L. Bianchi, M. Meineri, R. C. Myers, and M. Smolkin, ArXiv e-prints (2015), arXiv:1511.06713 [hep-th].

[57] L.-Y. Hung, R. C. Myers, M. Smolkin, and A. Yale, JHEP **12**, 047 (2011), arXiv:1110.1084 [hep-th].

[58] L.-Y. Hung, R. C. Myers, and M. Smolkin, JHEP **10**, 178 (2014), arXiv:1407.6429 [hep-th].

[59] T. Faulkner, R. G. Leigh, and O. Parrikar, (2015), arXiv:1511.05179 [hep-th].

[60] P. Bueno and R. C. Myers, (2015), arXiv:1508.00587 [hep-th].

[61] M. Mezei, Phys. Rev. **D91**, 045038 (2015), arXiv:1411.7011 [hep-th].

[62] M. Nielsen and I. Chuang, *Quantum Computation and Quantum Information: 10th Anniversary Edition* (Cambridge University Press, 2010).

[63] N. Drukker, D. J. Gross, and H. Ooguri, Phys. Rev. **D60**, 125006 (1999), arXiv:hep-th/9904191 [hep-th].

[64] W. Witczak-Krempa and J. Maciejko, ArXiv e-prints (2015), arXiv:1510.06397 [cond-mat.str-el].

[65] We thank Horacio Casini for suggesting this posibility.

[66] H. Casini and M. Huerta, JHEP **11**, 087 (2012), arXiv:1203.4007 [hep-th].

[67] Regarding the free scalar and Dirac fermion, $\sigma^{(p)}$ are given in Ref. 12 for $0 \le p \le 3$. We are grateful to Horacio Casini for sharing with us the unpublished results for higher p, as shown in Fig. 4.

[68] H. Casini and M. Huerta, JHEP **03**, 048 (2009), arXiv:0812.1773 [hep-th].

[69] B. Swingle, (2010), arXiv:1010.4038 [quant-ph].

[70] H. Casini, C. D. Fosco, and M. Huerta, J. Stat. Mech. **0507**, P07007 (2005), arXiv:cond-mat/0505563 [cond-mat].

[71] J. M. Maldacena, Adv.Theor.Math.Phys. **2**, 231 (1998), arXiv:hep-th/9711200 [hep-th].

[72] S. Ryu and T. Takayanagi, JHEP **08**, 045 (2006), arXiv:hep-th/0605073 [hep-th].

[73] S. Ryu and T. Takayanagi, Phys. Rev. Lett. **96**, 181602 (2006), arXiv:hep-th/0603001 [hep-th].

[74] T. Nishioka, S. Ryu, and T. Takayanagi, J. Phys. **A42**, 504008 (2009), arXiv:0905.0932 [hep-th].

[75] T. Takayanagi, Class. Quant. Grav. **29**, 153001 (2012), arXiv:1204.2450 [gr-qc].

[76] P. Fonda, L. Giomi, A. Salvio, and E. Tonni, JHEP **02**, 005 (2015), arXiv:1411.3608 [hep-th].

[77] P. Fonda, D. Seminara, and E. Tonni, (2015), arXiv:1510.03664 [hep-th].

[78] A. Buchel, J. Escobedo, R. C. Myers, M. F. Paulos, A. Sinha, and M. Smolkin, Journal of High Energy Physics **3**, 111 (2010), arXiv:0911.4257 [hep-th].

[79] D. M. Hofman and J. Maldacena, Journal of High Energy Physics **5**, 012 (2008), arXiv:0803.1467 [hep-th].

[80] E. Ardonne, P. Fendley, and E. Fradkin, Annals of Physics **310**, 493 (2004), cond-mat/0311466.

[81] D. S. Rokhsar and S. A. Kivelson, Phys. Rev. Lett. **61**, 2376 (1988).

[82] E. Fradkin and J. E. Moore, Physical Review Letters **97**, 050404 (2006), cond-mat/0605683.

[83] We point out the vanishing as $(\theta - \pi)^2$ in the smooth limit, and $1/\theta$ divergence in the sharp limit, are expected for CFTs. The fact that these properties hold in other scale invariant theories is not unreasonable.

[84] D.-W. Pang, JHEP **09**, 133 (2015), arXiv:1506.07979 [hep-th].

[85] M. Alishahiha, A. F. Astaneh, P. Fonda, and F. Omidi, JHEP **09**, 172 (2015), arXiv:1507.05897 [hep-th].

[86] M. R. M. Mozaffar, A. Mollabashi, and F. Omidi, (2015), arXiv:1511.00244 [hep-th].

[87] I. R. Klebanov, T. Nishioka, S. S. Pufu, and B. R. Safdi, JHEP **07**, 001 (2012), arXiv:1204.4160 [hep-th].

[88] D. V. Fursaev, JHEP **05**, 080 (2012), arXiv:1201.1702 [hep-th].

[89] J. Lee, L. McGough, and B. R. Safdi, Phys. Rev. **D89**, 125016 (2014), arXiv:1403.1580 [hep-th].

[90] A. Lewkowycz and E. Perlmutter, JHEP **01**, 080 (2015), arXiv:1407.8171 [hep-th].

[91] B. R. Safdi, JHEP **12**, 005 (2012), arXiv:1206.5025 [hep-th].

[92] P. Bueno, R. C. Myers, and W. Witczak-Krempa, (In preparation).

[93] M. Cloud, B. Drachman, and L. Lebedev, *Inequalities: With Applications to Engineering*, SpringerLink : Bücher (Springer, 2014).