Simple-current algebra constructions of 2+1-dimensional topological orders

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Self-consistent (non-)abelian statistics in 2+1D are classified by modular tensor categories (MTC). In recent works, a simplified axiomatic approach to MTCs, based on fusion coefficients N_k^{ij} and spins s_i, was proposed. A numerical search based on these axioms led to a list of possible (non-)abelian statistics, with rank up to $N = 7$. However, there is no guarantee that all solutions to the simplified axioms are consistent and can be realised by bosonic physical systems. In this paper, we use simple-current algebra to address this issue. We explicitly construct many-body wave functions, aiming to realize the entries in the list (*i.e.* realize their fusion coefficients N_k^{ij} and spins s_i). We find that all entries can be constructed by simple-current algebra plus conjugation under time reversal symmetry. This supports the conjecture that simple-current algebra is a general approach that allows us to construct all (non-)abelian statistics in 2+1D. It also suggests that the simplified theory based on (N_k^{ij}, s_i) is a classifying theory at least for simple bosonic 2+1D topological orders (up to invertible topological orders).

I. INTRODUCTION

We know that symmetry breaking orders[1,2] are described by group theory, which allows us to classify all different symmetry breaking orders. It is then natural to ask what mathematical theory classifies topological orders[3,4], which are beyond symmetry breaking orders. One proposal is to use the properties of topological excitations (such as their (non-)abelian statistics) to classify topological orders. This has led to the proposal that $d + 1D$ bosonic topological orders can be classified by unitary $(d + 1)$-categories with one object.[5,6] In particular, unitary $(2+1)$-categories with one object are modular tensor categories (MTC), leading to the proposal that 2+1D bosonic topological orders are classified by MTCs.[7–15] Such a classification is up to invertible topological orders, which have no nontrivial topological excitations.[5,16]

A. Simplified axiomatic approach

The papers Ref. 12,15 have formulated a simplified axiomatic approach to MTCs. This approach is based on fusion coefficients N_k^{ij} and spins s_i; it does not explicitly involve more involved data such as R- and F-matrices. The simplified axioms were used for a numerical search of simple MTCs, which led to a list of possible bosonic topological orders in 2+1D, with rank up to $N = 7$ (see Tables I - IV).

For certain special types of topological orders, the classification can be described by simpler theories. For example, topological orders with gappable edge for 2+1D interacting bosonic systems can be classified by unitary fusion categories (UFC).[17,18] For 2+1D bosonic/fermionic topological orders (with gappable or un-gappable edge) that have only abelian statistics, we can use integer K-matrices to classify them[19] and use the following $U(1)$

Chern-Simons theory to describe them[19–25]

$$\mathcal{L} = \frac{K_{IJ}}{4\pi} a_{I\mu} \partial_\nu a_{J\lambda} \epsilon^{\mu\nu\lambda}. \quad (1)$$

Such an effective theory can be realized by a multi-layer fractional quantum Hall state:

$$\prod_{I;i<j} (z_i^I - z_j^I)^{K_{II}} \prod_{I<J;i,j} (z_i^I - z_j^J)^{K_{IJ}} e^{-\frac{1}{4} \sum_{i,I} |z_i^I|^2}. \quad (2)$$

When the diagonal elements K_{II} are all even, the K-matrices classify 2+1D bosonic abelian topological orders. When some diagonal elements K_{II} are odd, the K-matrices classify 2+1D fermionic abelian topological orders.

The list produced in Ref. 12,15 gives solutions to the simplified axioms for MTCs - as such it describes possible self-consistent (non-)abelian statistics in 2+1D. However, there is no guarantee that all solutions are indeed consistent and can be realized by many-boson wavefunctions.

B. Simple-current algebra constructions

In this paper, we pursue a constructive (rather than axiomatic) approach to bosonic topological orders in 2+1D. We use simple-current algebra to construct and classify such orders, and demonstrate that simple-current algebras can produce all orders listed in Tables I - IV.

It is well-known that correlation functions in conformal field theory (CFT) can be used to construct many-body wave functions[26–32] that realize topological orders in 2+1D. In this paper we use these ideas to arrive at many-boson wavefunctions for bosonic topological orders. The main building blocks for our constructions are a set of CFT simple currents

$$\psi_I, \quad I = 1, \cdots, M . \quad (3)$$

We combine these with scalar field vertex operators to define

$$c_I = \psi_I \, \mathrm{e}^{\,\mathrm{i} \sum_\mu k_\mu^I \phi^\mu} = \psi_I \, \mathrm{e}^{\,\mathrm{i} \boldsymbol{k}^I \cdot \boldsymbol{\phi}} \tag{4}$$

and construct bosonic wave functions as

$$P(\{z_i^I\}) = \lim_{z_\infty \to \infty} \langle V(z_\infty) \prod_{i,I} c_I(z_i^I) \rangle. \tag{5}$$

We refer to section II for details and further explanation.

In this paper and in Ref. 29, we like to stress that it is misleading to state that CFT as such classifies topological orders. It is really simple-current algebra that can be used to classify 2+1D topological orders. In this paper, we show how to calculate the fusion coefficients N_k^{ij} and spins s_i of the topological excitations from simple-current algebra. This allows us to recover all entries in the Tables I - IV using simple-current algebra.

The consistency of the MTC axioms of Ref. 12,15 guarantees that all consistent orders are covered by lists such as those of Tables I - IV. In that sense those lists are an upper bound to the actual list of all consistent orders. The orders coming out of simple-current algebra constructions are consistent by construction - they thus establish a lower bound to the list of all consistent orders. In all cases considered in this paper, the two bounds agree, allowing us to conclude that both the simplified axiomatic approach and the simple-current algebra constructive approach give complete results.

II. CONSTRUCTING TOPOLOGICALLY ORDERED STATE OF A GIVEN NON-ABELIAN TYPE VIA A SIMPLE-CURRENT ALGEBRA

In this paper, we will use charged particles in multilayer system under magnetic field as a general and systematic way to realize 2+1D bosonic and fermionic topologically ordered states. We will assume all the particles are in the first Landau level. Thus the many-body wave function has a form

$$\Psi(\{z_i^I\}) = P(\{z_i^I\}) \mathrm{e}^{-\frac{1}{4} \sum_{i,I} |z_i^I|^2}, \tag{6}$$

where i labels different particles, $I = 1, \cdots, M$ labels different layers, and $P(\{z_i^I\})$ is a (anti-)symmetric polynomial (under $z_i^I \leftrightarrow z_j^I$), depending on the Bose or Fermi statistics of the particles in the Ith layer. In this paper, we are going to use such kind of systems to systematically realize non-abelian topological orders for bosons and fermions.

A. Symmetric polynomial $P(\{z_i^I\})$ as a correlation function in a simple-current algebra

Let us consider a CFT generated by simple currents $c_I(z)$, $I = 1, \cdots, M$. By definition, simple currents are operators with unit quantum dimension. The correlation function of simple currents always has one conformal block. If the simple currents $c_I(z)$ are also bosonic with integer conformal dimension or fermionic with half-integer conformal dimension, then we can use the correlation function of the simple currents $c_I(z)$ to construct the (anti-)symmetric polynomial $P(\{z_i^I\})^{26-28,33}$

$$P(\{z_i^I\}) \propto \lim_{z_\infty \to \infty} \langle V(z_\infty) \prod_{i,I} c_I(z_i^I) \rangle \tag{7}$$

where $V(z_\infty)$ represents a background to guarantee that the correlation function be non-zero. In fact $c_I(z)$ is related to the annihilation operator for the bosons in the Ith layer.

Such an approach allows us to use different simple-current CFTs to construct/label different many-boson wave function, which may correspond to different 2+1D topologically ordered states. For example, the Laughlin wave function $P(\{z_i\}) = \prod_{i<j}(z_i - z_j)^m$ can be constructed this way by choosing a Gaussian CFT and choosing

$$c(z) = \mathrm{e}^{\,\mathrm{i}\sqrt{m}\phi(z)} \tag{8}$$

as the simple-current operator. Here, the operator $\mathrm{e}^{\,\mathrm{i}a\phi(z)}$ has conformal dimension $\frac{a^2}{2}$ and the following operator product expansion (OPE)

$$\mathrm{e}^{\,\mathrm{i}a\phi(z)} \mathrm{e}^{\,\mathrm{i}b\phi(w)} =$$
$$(z-w)^{ab} \mathrm{e}^{\,\mathrm{i}(a+b)\phi(w)} + O\big((z-w)^{ab+1}\big). \tag{9}$$

In fact

$$\prod_{i<j}(z_i - z_j)^m \propto \lim_{z_\infty \to \infty} \langle \mathrm{e}^{-\,\mathrm{i}N\sqrt{m}\phi(z_\infty)} \prod_{i=1}^N \mathrm{e}^{\,\mathrm{i}\sqrt{m}\phi(z_i)} \rangle. \tag{10}$$

To construct the abelian topologically ordered states described by the K-matrix wave function (2), we can start with a Gaussian model described by ϕ^μ fields that have the following OPE

$$\mathrm{e}^{\,\mathrm{i}l_\mu \phi^\mu(z)} \mathrm{e}^{\,\mathrm{i}l'_\mu \phi^\mu(w)} = (z-w)^{l_\mu G^{\mu\nu} l'_\nu} \mathrm{e}^{\,\mathrm{i}(l_\mu + l'_\mu)\phi^\mu(w)} + \cdots \tag{11}$$

We see that $\mathrm{e}^{\,\mathrm{i}k_\mu \phi^\mu(z)}$ has a conformal dimension

$$\frac{1}{2}\boldsymbol{k} \cdot \boldsymbol{k} \equiv \frac{1}{2}\sum_{\mu\nu} k_\mu G^{\mu\nu} k_\nu, \tag{12}$$

where the inner product $\cdot$ is defined via $G^{\mu\nu}$. The metric $G^{\mu\nu}$ plays a crucial role, as a given choice of $G^{\mu\nu}$ leads to a specific set of momenta k_μ giving vertex operators with integral conformal dimension (or half-integral for fermionic theories), thereby setting the operator content of the theory. If we choose $c_I = \mathrm{e}^{\,\mathrm{i}\boldsymbol{k}^I \cdot \boldsymbol{\phi}} \equiv \mathrm{e}^{\,\mathrm{i}\sum_\mu k_\mu^I \phi^\mu}$,

TABLE I: A list of 35 bosonic topological orders in 2+1D with rank $N = 1, 2, 3, 4$ and with $\max(N_k^{ij}) \leq 3$. All $N \leq 4$ orders have $\max(N_k^{ij}) = 1$. The entries in blue are composite topological orders that can be obtained by stacking lower rank topological orders. The first column is the rank N and the central charge c (mod 8). The second column is the topological entanglement entropy $S_{top} = \log_2 D$, $D = \sqrt{\sum_i d_i^2}$. The quantum dimensions of the topological excitations in the third column are expressed in terms of $\zeta_n^m = \frac{\sin[\pi(m+1)/(n+2)]}{\sin[\pi/(n+2)]}$. The fourth column are the spins of the corresponding topological excitations. By 'type (X_l, k)' we indicate a correspondence to affine Kac-Moody current algebra $X_l^{(1)}$ at level k, and $(X_l, k)_{\frac{1}{q}}$ indicate simple-current reductions of Kac-Moody current algebra.

N_c^B	S_{top}	$d_1, d_2, \cdots$	$s_1, s_2, \cdots$	type	N_c^B	S_{top}	$d_1, d_2, \cdots$	$s_1, s_2, \cdots$	type
1_0^B	0	1	0						
2_1^B	0.5	1,1	$0,\frac14$	$U(1)_1$, $(A_1,1)$	2_{-1}^B	0.5	1,1	$0,-\frac14$	$(E_7,1)$
$2_{14/5}^B$	0.9276	$1,\zeta_3^1$	$0,\frac25$	$(G_2,1)$, $(A_1,3)_{\frac12}$	$2_{-14/5}^B$	0.9276	$1,\zeta_3^1$	$0,-\frac25$	$(F_4,1)$, $(A_2,2)_{\frac13}$
3_2^B	0.7924	1,1,1	$0,\frac13,\frac13$	$(A_2,1)$, $(A_1,4)_{\frac14}$	3_{-2}^B	0.7924	1,1,1	$0,-\frac13,-\frac13$	$(E_6,1)$
$3_{1/2}^B$	1	$1,1,\zeta_1^1$	$0,\frac12,\frac{1}{16}$	$(B_8,1)$	$3_{-1/2}^B$	1	$1,1,\zeta_1^1$	$0,\frac12,-\frac{1}{16}$	$(B_7,1)$, $(E_8,2)$
$3_{3/2}^B$	1	$1,1,\zeta_2^1$	$0,\frac12,\frac{3}{16}$	$(A_1,2)$	$3_{-3/2}^B$	1	$1,1,\zeta_2^1$	$0,\frac12,-\frac{3}{16}$	$(B_6,1)$
$3_{5/2}^B$	1	$1,1,\zeta_2^1$	$0,\frac12,\frac{5}{16}$	$(B_2,1)$	$3_{-5/2}^B$	1	$1,1,\zeta_2^1$	$0,\frac12,-\frac{5}{16}$	$(B_5,1)$
$3_{7/2}^B$	1	$1,1,\zeta_2^1$	$0,\frac12,\frac{7}{16}$	$(B_3,1)$	$3_{-7/2}^B$	1	$1,1,\zeta_2^1$	$0,\frac12,-\frac{7}{16}$	$(B_4,1)$
$3_{8/7}^B$	1.6082	$1,\zeta_5^1,\zeta_5^2$	$0,-\frac17,\frac27$	$(A_1,5)_{\frac12}$	$3_{-8/7}^B$	1.6082	$1,\zeta_5^1,\zeta_5^2$	$0,\frac17,-\frac27$	$(A_4,2)_{\frac15}$
$4_0^{B,a}$	1	1,1,1,1	$0,0,0,\frac12$	$(D_8,1)$	$4_0^{B,b}$	1	1,1,1,1	$0,0,\frac14,-\frac14$	
4_1^B	1	1,1,1,1	$0,\frac18,\frac18,\frac12$	$U(1)_2$	4_{-1}^B	1	1,1,1,1	$0,-\frac18,-\frac18,\frac12$	$(D_7,1)$
4_2^B	1	1,1,1,1	$0,\frac14,\frac14,\frac12$		4_{-2}^B	1	1,1,1,1	$0,-\frac14,-\frac14,\frac12$	$(D_6,1)$
4_3^B	1	1,1,1,1	$0,\frac38,\frac38,\frac12$	$(A_3,1)$	4_{-3}^B	1	1,1,1,1	$0,-\frac38,-\frac38,\frac12$	$(D_5,1)$
4_4^B	1	1,1,1,1	$0,\frac12,\frac12,\frac12$	$(D_4,1)$, $(A_2,3)_{\frac19}$	$4_0^{B,c}$	1.8552	$1,\zeta_3^1,\zeta_3^1,\zeta_3^1\zeta_3^1$	$0,\frac25,-\frac25,0$	
$4_{9/5}^B$	1.4276	$1,1,\zeta_3^1,\zeta_3^1$	$0,-\frac14,\frac{3}{20},\frac25$	$(A_1,3)$	$4_{-9/5}^B$	1.4276	$1,1,\zeta_3^1,\zeta_3^1$	$0,\frac14,-\frac{3}{20},-\frac25$	
$4_{19/5}^B$	1.4276	$1,1,\zeta_3^1,\zeta_3^1$	$0,\frac14,-\frac{7}{20},\frac25$		$4_{-19/5}^B$	1.4276	$1,1,\zeta_3^1,\zeta_3^1$	$0,-\frac14,\frac{7}{20},-\frac25$	$(C_3,1)$
$4_{12/5}^B$	1.8552	$1,\zeta_3^1,\zeta_3^1,\zeta_3^1\zeta_3^1$	$0,-\frac25,-\frac25,\frac15$	$(A_1,8)_{\frac14}$	$4_{-12/5}^B$	1.8552	$1,\zeta_3^1,\zeta_3^1,\zeta_3^1\zeta_3^1$	$0,\frac25,\frac25,-\frac15$	
$4_{10/3}^B$	2.1328	$1,\zeta_7^1,\zeta_7^2,\zeta_7^3$	$0,\frac13,\frac29,-\frac13$	$(A_1,7)_{\frac12}$	$4_{-10/3}^B$	2.1328	$1,\zeta_7^1,\zeta_7^2,\zeta_7^3$	$0,-\frac13,-\frac29,\frac13$	$(G_2,2)$, $(A_6,2)_{\frac17}$

where $\boldsymbol{k}^I = (k_1^I, k_2^I, \cdots)$ and $\boldsymbol{\phi} = (\phi^1, \phi^2, \cdots)$, we find that

$$P(\{z_i^I\}) = \prod_{I;i<j}(z_i^I - z_j^I)^{K_{II}} \prod_{I<J;i,j}(z_i^I - z_j^J)^{K_{IJ}}$$
$$\propto \lim_{z_\infty \to \infty} \langle V(z_\infty) \prod_{i,I} c_I(z_i^I)\rangle, \quad (13)$$

if the $\boldsymbol{k}^I$ satisfy

$$K_{IJ} = \boldsymbol{k}^I \cdot \boldsymbol{k}^J. \quad (14)$$

In order to obtain an (anti-)symmetric polynomial $P(\{z_i^I\})$, we see that K_{IJ} must be integer.

Now, we are ready to construct topologically ordered states of a given non-abelian type. Let us consider a simple-current CFT generated by a set of simple currents

$$\psi_I, \quad I = 1, \cdots, M. \quad (15)$$

We assume that the ψ_I have finite orders described by an integer matrix $\boldsymbol{n} = (n_{JI})$:

$$\prod_J (\psi_J)^{n_{JI}} = 1, \quad \forall I. \quad (16)$$

Now we choose

$$c_I = \psi_I e^{i\sum_\mu k_\mu^I \phi^\mu} = \psi_I e^{i\boldsymbol{k}^I\cdot\boldsymbol{\phi}} \quad (17)$$

to construct the wave function as

$$P(\{z_i^I\}) = \lim_{z_\infty\to\infty}\langle V(z_\infty)\prod_{i,I}c_I(z_i^I)\rangle. \quad (18)$$

But in this case, in order to obtain an (anti-)symmetric polynomial $P(\{z_i^I\})$,

$$K_{IJ} \equiv \boldsymbol{k}^I\cdot\boldsymbol{k}^J = \sum_{\mu\nu}k_\mu^I G^{\mu\nu}k_\nu^J \quad (19)$$

may not be integer. In fact, introducing

$$c_{\vec a} = \prod_I c_I^{a_I} = e^{i\sum_{I,\mu}a_I k_\mu^I \phi^\mu}\prod_I \psi_I^{a_I} \quad (20)$$

and noticing that $c_{\vec a}$ and $c_{\vec b}$ must be mutually local for any integer vectors $\vec a$ and $\vec b$, we find that k_μ^I must satisfy

$$\sum_{IJ\mu\nu}a^I k_\mu^I G^{\mu\nu}k_\nu^J b^J - h_{\vec a}^{sc} - h_{\vec b}^{sc} + h_{\vec a+\vec b}^{sc}$$
$$= \sum_{IJ}a^I K_{IJ}b^J - h_{\vec a}^{sc} - h_{\vec b}^{sc} + h_{\vec a+\vec b}^{sc} \in \mathbb{N} \quad (21)$$

for any positive integer vector $\vec{a}$ and $\vec{b}$ (*i.e.* $a_I \in \mathbb{N}$ and $b_I \in \mathbb{N}$). Here $\mathbb{N} = \{0, 1, 2, \cdots\}$ and $h_{\vec{a}}^{\text{sc}}$ is the conformal dimension of $\psi_{\vec{a}} \equiv \prod_I \psi_I^{a_I}$. Since the $h_{\vec{a}}^{\text{sc}}$ are rational numbers, in general, K_{IJ} are also rational numbers. We see that, starting from a simple-current CFT, we can construct all the 2+1D topological orders of a given non-abelian type, by finding all the K-matrices that satisfy the conditions (21).

If we further require that

$$\sum_{IJ} a^I K_{IJ} a^J - 2h_{\vec{a}}^{\text{sc}} + h_{2\vec{a}}^{\text{sc}} = \text{ even}, \qquad (22)$$

for all $\vec{a}$, then we will obtain the bosonic 2+1D topological orders of a given non-abelian type. If we require that

$$\sum_{IJ} a^I K_{IJ} a^J - 2h_{\vec{a}}^{\text{sc}} + h_{2\vec{a}}^{\text{sc}} = \text{ odd}, \qquad (23)$$

for some $\vec{a}$, then we will obtain the fermionic 2+1D topological orders of a given non-abelian type.

B. Topological excitations from simple-current algebra

In the above, we have used the simple-current CFT generated by the simple currents $c_I(z)$ to obtain the ground state wave function of a 2+1D topological order. In this section, we are going to discuss how to obtain the topological excitations from the simple-current CFT.

First, we like to introduce the notion of simple-current primary field. Acting with $c_I(z)$ on to the ground state $|0\rangle$ generates the adjoint representation of the simple-current algebra. The simple-current algebra has other irreducible representations, which can be obtained by the action of $c_I(z)$ on the ground state $|\eta\rangle = \eta|0\rangle$ of a twisted sector. Thus the different irreducible representations of the simple-current algebra are labeled by η (where $\eta = 1$ corresponds the adjoint representation). The operator $\eta(z)$ that corresponds to the twisted ground state $|\eta\rangle$ under the operator-state correspondence is called a primary field of the simple-current algebra.

The primary fields $\eta(z)$ are local with respect to all the simple currents $c_I(z)$:

$$c_I(z)\eta(w) \sim (z-w)^{\alpha_{c_I,\eta}}[c_I\eta](w) + \ldots \qquad (24)$$

where $\alpha_{c_I,\eta}$ are integers. Each simple-current primary field (or each irreducible representation of the simple-current algebra) corresponds to a type of topological excitation in the corresponding topological order.

So to use CFT to study 2+1D topological order, we need to first identify the simple currents to produce the many-body wavefunction of the topological order. We then need to find the irreducible representations (or the primary fields) of the simple-current algebra to obtain the topological excitations and their properties (such as the quantum dimensions, the spins, *etc*).

In general, the simple currents $c_I(z)$ have the form

$$c_I(z) = \psi_I e^{i \boldsymbol{k}^I \cdot \boldsymbol{\phi}} = \psi_I e^{i \sum_\mu k_\mu^I \phi^\mu} \qquad (25)$$

where ψ_I are simple currents with finite order (see eqn. (16)). Let us introduce

$$\psi_{\vec{b}} \equiv \prod_I \psi_I^{b_I}, \qquad c_{\vec{b}} \equiv \prod_I c_I^{b_I}. \qquad (26)$$

Also, let us use σ_α, $\alpha = 1, 2, \cdots$, to denote the primary fields of the simple-current CFT generated by simple currents ψ_I, and use $\sigma_{\alpha;\vec{b}}$ to denote the product of σ_α and $\psi_{\vec{b}}$. $\sigma_{\alpha;\vec{b}}$ are descendent fields of the primary field σ_α and have higher conformal dimensions

$$h_{\alpha,\vec{b}}^{\text{sc}} \geq h_\alpha^{\text{sc}}, \qquad (27)$$

where h_α^{sc} is the conformal dimension of σ_α and $h_{\alpha,\vec{b}}^{\text{sc}}$ is the conformal dimension of $\sigma_{\alpha;\vec{b}}$. The OPE of $\sigma_{\alpha;\vec{b}}$ with $\psi_{\vec{a}}$ has for its leading term

$$\psi_{\vec{a}}(z)\sigma_{\alpha;\vec{b}}(w) \sim \frac{1}{(z-w)^{h_{\vec{a}}^{\text{sc}} + h_{\alpha,\vec{b}}^{\text{sc}} - h_{\alpha,\vec{a}+\vec{b}}^{\text{sc}}}} \sigma_{\alpha;\vec{a}+\vec{b}}. \qquad (28)$$

The simple-current primary field η for the original simple currents c_I is given by

$$\eta_{\alpha,\boldsymbol{l}^\alpha} = \sigma_\alpha e^{i \boldsymbol{l}^\alpha \cdot \boldsymbol{\phi}} = \sigma_\alpha e^{i \sum_\mu l_\mu^\alpha \phi^\mu}. \qquad (29)$$

The corresponding descendent fields are given by

$$\eta_{\alpha,\boldsymbol{l}^\alpha;\vec{b}} = \sigma_{\alpha,\vec{b}} e^{i \boldsymbol{l}^\alpha \cdot \boldsymbol{\phi}} e^{i \sum_I b_I \boldsymbol{k}^I \cdot \boldsymbol{\phi}} \qquad (30)$$

for all different integer vectors $\vec{b}$. Each of those operators should be mutually local with respect to $c_{\vec{a}}$. This requires $\boldsymbol{l}^\alpha$ to satisfy

$$\sum_{IJ} a^I K_{IJ} b^J + \sum_{IJ\mu\nu} a^I k_\mu^I G^{\mu\nu} l_\nu^\alpha - h_{\vec{a}}^{\text{sc}} - h_{\alpha,\vec{b}}^{\text{sc}} + h_{\alpha,\vec{a}+\vec{b}}^{\text{sc}} \in \mathbb{Z}, \qquad (31)$$

for any integer vectors $\vec{a}$ and $\vec{b}$.

To understand the above construction in more detail, let us count the number of c_I-simple-current primary fields η (which is equal to the number of topological types of the topological excitations in the corresponding topological order). First a c_I-primary field η corresponds to a pair: a ψ_I-simple-current primary field σ_α and a vector $\boldsymbol{l}^\alpha$. So the c_I-primary fields are labeled by $(\alpha, \boldsymbol{l}^\alpha)$. We have used $\eta_{\alpha,\boldsymbol{l}^\alpha}$ to denote those c_I-primary fields. $\boldsymbol{l}^\alpha$ must satisfy eqn. (31). In fact, it is enough to find rational vectors $\boldsymbol{l}^\alpha$ that satisfy

$$\sum_{IJ\mu\nu} a^I k_\mu^I G^{\mu\nu} l_\nu^\alpha - h_{\vec{a}}^{\text{sc}} - h_\alpha^{\text{sc}} + h_{\alpha,\vec{a}}^{\text{sc}} \in \mathbb{Z}, \quad \forall \vec{a}. \qquad (32)$$

For each α, we may have many solutions $\boldsymbol{l}^\alpha$ which satisfy the above equation. But two solutions $\boldsymbol{l}_1^\alpha$ and $\boldsymbol{l}_2^\alpha$ are regarded as the same if they are related by

$$\boldsymbol{l}_1^\alpha - \boldsymbol{l}_2^\alpha = \sum_{IJ} \boldsymbol{k}^I n_{IJ} a_J, \quad a_J \in \mathbb{Z}. \qquad (33)$$

TABLE II: A list of 10 bosonic rank $N = 5$ topological orders in 2+1D with $\max(N_k^{ij}) \leq 3$. The orders $5^B_{\pm 18/7}$ have $\max(N_k^{ij}) = 2$, all other $N = 5$ topological orders have $N_k^{ij} = 0, 1$.

N_c^B	S_{top}	$d_1, d_2, \cdots$	$s_1, s_2, \cdots$	type
5_0^B	1.1609	$1, 1, 1, 1, 1$	$0, \frac{1}{5}, \frac{1}{5}, -\frac{1}{5}, -\frac{1}{5}$	
5_4^B	1.1609	$1, 1, 1, 1, 1$	$0, \frac{2}{5}, \frac{2}{5}, -\frac{2}{5}, -\frac{2}{5}$	$(A_4, 1)$
$5_2^{B,a}$	1.7924	$1, 1, \zeta_4^1, \zeta_4^1, 2$	$0, 0, \frac{1}{8}, -\frac{3}{8}, \frac{1}{3}$	$(A_1, 4)$, $(U(1)_3/\mathbb{Z}_2)_{\frac{1}{2}}$
$5_2^{B,b}$	1.7924	$1, 1, \zeta_4^1, \zeta_4^1, 2$	$0, 0, -\frac{1}{8}, \frac{3}{8}, \frac{1}{3}$	$[5_2^{B,a} \otimes 4_0^{B,b}]_{\frac{1}{4}}$
$5_{-2}^{B,a}$	1.7924	$1, 1, \zeta_4^1, \zeta_4^1, 2$	$0, 0, -\frac{1}{8}, \frac{3}{8}, -\frac{1}{3}$	$(C_4, 1)$, $(A_3, 2)_{\frac{1}{2}}$
$5_{-2}^{B,b}$	1.7924	$1, 1, \zeta_4^1, \zeta_4^1, 2$	$0, 0, \frac{1}{8}, -\frac{3}{8}, -\frac{1}{3}$	$[5_{-2}^{B,a} \otimes 4_0^{B,b}]_{\frac{1}{4}}$
$5_{16/11}^B$	2.5573	$1, \zeta_9^1, \zeta_9^2, \zeta_9^3, \zeta_9^4$	$0, -\frac{2}{11}, \frac{2}{11}, \frac{1}{11}, -\frac{5}{11}$	$(F_4, 2)$, $(A_1, 9)_{\frac{1}{2}}$
$5_{-16/11}^B$	2.5573	$1, \zeta_9^1, \zeta_9^2, \zeta_9^3, \zeta_9^4$	$0, \frac{2}{11}, -\frac{2}{11}, -\frac{1}{11}, \frac{5}{11}$	$(E_8, 3)$, $(A_8, 2)_{\frac{1}{9}}$
$5_{18/7}^B$	2.5716	$1, \zeta_5^2, \zeta_5^2, \zeta_{12}^2, \zeta_{12}^4$	$0, -\frac{1}{7}, -\frac{1}{7}, \frac{3}{7}, \frac{3}{7}$	$(A_1, 12)_{\frac{1}{4}}$, $(A_2, 4)_{\frac{1}{3}}$
$5_{-18/7}^B$	2.5716	$1, \zeta_5^2, \zeta_5^2, \zeta_{12}^2, \zeta_{12}^4$	$0, \frac{1}{7}, \frac{1}{7}, -\frac{1}{7}, -\frac{3}{7}$	$(A_3, 3)_{\frac{1}{4}}$

Counting the pairs $(\sigma_\alpha, \boldsymbol{l}^\alpha)$ of inequivalent solutions will give us the number of c_I-simple-current primary fields and the number of topological types.

III. 2+1D TOPOLOGICAL ORDER FROM CHIRAL $\frac{U(1)_M}{\mathbb{Z}_2}$ ORBIFOLD CFT

In this section, we will give an example of using simple-current algebra to construct a wavefunction that realizes a 2+1D topological order. In the process, we will give a brief review on the $\frac{U(1)_M}{\mathbb{Z}_2}$ orbifold CFT, following Ref. 34.

A. Virasoro algebra

Here, we will view a CFT as a 1+1D gapless system with unit velocity $v = 1$ on a 1D ring of size 2π. The total Hilbert space $\mathcal{V}$ of the CFT can always be viewed as a sum of (irreducible) representations of the Virasoro algebra. The Virasoro algebra is generated by the energy-momentum tensor $T(z)$, whose Fourier components $T(z) = \sum_n z^{-n-2} L_n$ satisfy

$$[L_m, L_n] = (m - n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0}. \quad (34)$$

The character of a representation of the Virasoro algebra is defined as

$$\chi_{c,h}^{\text{Vir}}(q) = \text{Tr}(q^{L_0 - \frac{c}{24}}), \quad (35)$$

where $L_0 = H$ is the Hamiltonian of the CFT on the ring. So the character encodes the energy spectrum of the CFT on the ring. The irreducible representations of the Virasoro algebra are labeled by (c, h), where h is the energy of the lowest energy state in the representation. h is also the conformal dimension of the Virasoro primary field associated to the representation. The character of

the corresponding irreducible representation has the general form

$$\chi_{c,h}^{\text{Vir}}(q) = \frac{q^{\frac{1-c}{24}}}{\eta(q)}[q^h - q^{h_1} + q^{h_2} - \ldots], \quad (36)$$

where

$$\eta(q) = q^{\frac{1}{24}} \prod_{n=1}^{\infty}(1 - q^n) \quad (37)$$

and the terms $-q^{h_1} + q^{h_2} + \ldots$ represent subtractions due to null states in the Verma module with highest weight h.

B. $U(1)$ current algebra

The $U(1)$ current algebra (which is a simple-current algebra) is generated by $j = \mathrm{i}\partial\phi$. ($j$ is a simple current.) In other words, the space $\mathcal{V}_1^{U(1)}$ of the adjoint representation of the $U(1)$ current algebra is generated by $j(z)$ acting on the ground state $|0\rangle$. The corresponding primary field for the adjoint representation is the identity operator 1. This is why we use $\mathcal{V}_1^{U(1)}$ to denote the adjoint representation. The adjoint representation $\mathcal{V}_1^{U(1)}$ has a character $\chi_1^{U(1)}(q) = 1/\eta(q)$. However, the adjoint irreducible representation of the $U(1)$ current algebra is not an irreducible representation of the Virasoro algebra. Instead, it is formed by many irreducible representations $\mathcal{V}_{c=1,n^2}^{\text{Vir}}$ of the Virasoro algebra generated by the energy momentum tensor $T(z) \propto j^2(z)$. It turns out that

$$\mathcal{V}_1^{U(1)} = \bigoplus_{n \geq 0} \mathcal{V}_{c=1,n^2}^{\text{Vir}} \quad (38)$$

TABLE III: A list of 50 bosonic rank $N = 6$ topological orders in 2+1D with $\max(N_k^{ij}) \leq 2$.

N_c^B	S_{top}	D^2	$d_1, d_2, \cdots$	$s_1, s_2, \cdots$	$N_c^B \otimes \widetilde{N_{\tilde c}^B}$	type
6_1^B	1.2924	6	$1,1,1,1,1,1$	$0, \frac{1}{12}, \frac{1}{12}, -\frac{1}{4}, \frac{1}{3}, \frac{1}{3}$	$2_{-1}^B \otimes 3_2^B$	$U(1)_3$
6_{-1}^B	1.2924	6	$1,1,1,1,1,1$	$0, -\frac{1}{12}, -\frac{1}{12}, \frac{1}{4}, -\frac{1}{3}, -\frac{1}{3}$	$2_1^B \otimes 3_{-2}^B$	
6_3^B	1.2924	6	$1,1,1,1,1,1$	$0, \frac{1}{4}, \frac{1}{3}, \frac{1}{3}, -\frac{5}{12}, -\frac{5}{12}$	$2_1^B \otimes 3_2^B$	
6_{-3}^B	1.2924	6	$1,1,1,1,1,1$	$0, -\frac{1}{4}, -\frac{1}{3}, -\frac{1}{3}, \frac{5}{12}, \frac{5}{12}$	$2_{-1}^B \otimes 3_{-2}^B$	$(A_5, 1)$
$6_{1/2}^B$	1.5	8	$1,1,1,1,\zeta_2^1,\zeta_2^1$	$0, \frac{1}{4}, -\frac{1}{4}, \frac{1}{2}, -\frac{1}{16}, \frac{3}{16}$	$2_1^B \otimes 3_{-1/2}^B$	
$6_{-1/2}^B$	1.5	8	$1,1,1,1,\zeta_2^1,\zeta_2^1$	$0, \frac{1}{4}, -\frac{1}{4}, \frac{1}{2}, \frac{1}{16}, -\frac{3}{16}$	$2_1^B \otimes 3_{-3/2}^B$	
$6_{\frac{3}{2}}^B$	1.5	8	$1,1,1,1,\zeta_2^1,\zeta_2^1$	$0, \frac{1}{4}, -\frac{1}{4}, \frac{1}{2}, \frac{1}{16}, \frac{5}{16}$	$2_1^B \otimes 3_{1/2}^B$	
$6_{-3/2}^B$	1.5	8	$1,1,1,1,\zeta_2^1,\zeta_2^1$	$0, \frac{1}{4}, -\frac{1}{4}, \frac{1}{2}, -\frac{1}{16}, -\frac{5}{16}$	$2_1^B \otimes 3_{-5/2}^B$	
$6_{5/2}^B$	1.5	8	$1,1,1,1,\zeta_2^1,\zeta_2^1$	$0, \frac{1}{4}, -\frac{1}{4}, \frac{1}{2}, \frac{3}{16}, \frac{7}{16}$	$2_1^B \otimes 3_{3/2}^B$	
$6_{-5/2}^B$	1.5	8	$1,1,1,1,\zeta_2^1,\zeta_2^1$	$0, \frac{1}{4}, -\frac{1}{4}, \frac{1}{2}, -\frac{3}{16}, -\frac{7}{16}$	$2_1^B \otimes 3_{-7/2}^B$	
$6_{7/2}^B$	1.5	8	$1,1,1,1,\zeta_2^1,\zeta_2^1$	$0, \frac{1}{4}, -\frac{1}{4}, \frac{1}{2}, \frac{5}{16}, -\frac{7}{16}$	$2_1^B \otimes 3_{5/2}^B$	
$6_{-7/2}^B$	1.5	8	$1,1,1,1,\zeta_2^1,\zeta_2^1$	$0, \frac{1}{4}, -\frac{1}{4}, \frac{1}{2}, -\frac{5}{16}, \frac{7}{16}$	$2_1^B \otimes 3_{7/2}^B$	
$6_{4/5}^B$	1.7200	10.854	$1,1,1,\zeta_3^1,\zeta_3^1,\zeta_3^1$	$0, -\frac{1}{3}, -\frac{1}{3}, \frac{1}{15}, \frac{1}{15}, \frac{2}{5}$	$2_{14/5}^B \otimes 3_{-2}^B$	
$6_{-4/5}^B$	1.7200	10.854	$1,1,1,\zeta_3^1,\zeta_3^1,\zeta_3^1$	$0, \frac{1}{3}, \frac{1}{3}, -\frac{1}{15}, -\frac{1}{15}, -\frac{2}{5}$	$2_{-14/5}^B \otimes 3_2^B$	
$6_{16/5}^B$	1.7200	10.854	$1,1,1,\zeta_3^1,\zeta_3^1,\zeta_3^1$	$0, -\frac{1}{3}, -\frac{1}{3}, \frac{4}{15}, \frac{4}{15}, -\frac{2}{5}$	$2_{-14/5}^B \otimes 3_{-2}^B$	$(A_2, 2)$
$6_{-16/5}^B$	1.7200	10.854	$1,1,1,\zeta_3^1,\zeta_3^1,\zeta_3^1$	$0, \frac{1}{3}, \frac{1}{3}, -\frac{4}{15}, -\frac{4}{15}, \frac{2}{5}$	$2_{\frac{14}{5}}^B \otimes 3_2^B$	
$6_{-27/10}^B$	1.9276	14.472	$1,1,\zeta_2^1,\zeta_3^1,\zeta_3^1,\zeta_2^1\zeta_3^1$	$0, \frac{1}{2}, \frac{5}{16}, -\frac{1}{10}, \frac{2}{5}, -\frac{23}{80}$	$2_{\frac{14}{5}}^B \otimes 3_{5/2}^B$	$(E_7, 2)$
$6_{-17/10}^B$	1.9276	14.472	$1,1,\zeta_2^1,\zeta_3^1,\zeta_3^1,\zeta_2^1\zeta_3^1$	$0, \frac{1}{2}, \frac{7}{16}, -\frac{1}{10}, \frac{2}{5}, -\frac{13}{80}$	$2_{\frac{14}{5}}^B \otimes 3_{7/2}^B$	
$6_{-7/10}^B$	1.9276	14.472	$1,1,\zeta_2^1,\zeta_3^1,\zeta_3^1,\zeta_2^1\zeta_3^1$	$0, \frac{1}{2}, -\frac{7}{16}, -\frac{1}{10}, \frac{2}{5}, -\frac{3}{80}$	$2_{\frac{14}{5}}^B \otimes 3_{-7/2}^B$	
$6_{3/10}^B$	1.9276	14.472	$1,1,\zeta_2^1,\zeta_3^1,\zeta_3^1,\zeta_2^1\zeta_3^1$	$0, \frac{1}{2}, -\frac{5}{16}, -\frac{1}{10}, \frac{2}{5}, \frac{7}{80}$	$2_{\frac{14}{5}}^B \otimes 3_{-5/2}^B$	
$6_{13/10}^B$	1.9276	14.472	$1,1,\zeta_2^1,\zeta_3^1,\zeta_3^1,\zeta_2^1\zeta_3^1$	$0, \frac{1}{2}, -\frac{3}{16}, -\frac{1}{10}, \frac{2}{5}, \frac{17}{80}$	$2_{\frac{14}{5}}^B \otimes 3_{-3/2}^B$	
$6_{23/10}^B$	1.9276	14.472	$1,1,\zeta_2^1,\zeta_3^1,\zeta_3^1,\zeta_2^1\zeta_3^1$	$0, \frac{1}{2}, -\frac{1}{16}, -\frac{1}{10}, \frac{2}{5}, \frac{27}{80}$	$2_{\frac{14}{5}}^B \otimes 3_{-1/2}^B$	
$6_{33/10}^B$	1.9276	14.472	$1,1,\zeta_2^1,\zeta_3^1,\zeta_3^1,\zeta_2^1\zeta_3^1$	$0, \frac{1}{2}, \frac{1}{16}, -\frac{1}{10}, \frac{2}{5}, \frac{37}{80}$	$2_{\frac{14}{5}}^B \otimes 3_{1/2}^B$	
$6_{-37/10}^B$	1.9276	14.472	$1,1,\zeta_2^1,\zeta_3^1,\zeta_3^1,\zeta_2^1\zeta_3^1$	$0, \frac{1}{2}, \frac{3}{16}, -\frac{1}{10}, \frac{2}{5}, -\frac{33}{80}$	$2_{\frac{14}{5}}^B \otimes 3_{3/2}^B$	
$6_{27/10}^B$	1.9276	14.472	$1,1,\zeta_2^1,\zeta_3^1,\zeta_3^1,\zeta_2^1\zeta_3^1$	$0, \frac{1}{2}, -\frac{5}{16}, \frac{1}{10}, -\frac{2}{5}, \frac{23}{80}$	$2_{-14/5}^B \otimes 3_{-5/2}^B$	
$6_{17/10}^B$	1.9276	14.472	$1,1,\zeta_2^1,\zeta_3^1,\zeta_3^1,\zeta_2^1\zeta_3^1$	$0, \frac{1}{2}, -\frac{7}{16}, \frac{1}{10}, -\frac{2}{5}, \frac{13}{80}$	$2_{-14/5}^B \otimes 3_{-7/2}^B$	
$6_{7/10}^B$	1.9276	14.472	$1,1,\zeta_2^1,\zeta_3^1,\zeta_3^1,\zeta_2^1\zeta_3^1$	$0, \frac{1}{2}, \frac{7}{16}, \frac{1}{10}, -\frac{2}{5}, \frac{3}{80}$	$2_{-14/5}^B \otimes 3_{7/2}^B$	
$6_{-3/10}^B$	1.9276	14.472	$1,1,\zeta_2^1,\zeta_3^1,\zeta_3^1,\zeta_2^1\zeta_3^1$	$0, \frac{1}{2}, \frac{5}{16}, \frac{1}{10}, -\frac{2}{5}, -\frac{7}{80}$	$2_{-14/5}^B \otimes 3_{5/2}^B$	
$6_{-13/10}^B$	1.9276	14.472	$1,1,\zeta_2^1,\zeta_3^1,\zeta_3^1,\zeta_2^1\zeta_3^1$	$0, \frac{1}{2}, \frac{3}{16}, \frac{1}{10}, -\frac{2}{5}, -\frac{17}{80}$	$2_{-14/5}^B \otimes 3_{3/2}^B$	
$6_{-23/10}^B$	1.9276	14.472	$1,1,\zeta_2^1,\zeta_3^1,\zeta_3^1,\zeta_2^1\zeta_3^1$	$0, \frac{1}{2}, \frac{1}{16}, \frac{1}{10}, -\frac{2}{5}, -\frac{27}{80}$	$2_{-14/5}^B \otimes 3_{1/2}^B$	
$6_{-33/10}^B$	1.9276	14.472	$1,1,\zeta_2^1,\zeta_3^1,\zeta_3^1,\zeta_2^1\zeta_3^1$	$0, \frac{1}{2}, -\frac{1}{16}, \frac{1}{10}, -\frac{2}{5}, -\frac{37}{80}$	$2_{-14/5}^B \otimes 3_{-1/2}^B$	
$6_{37/10}^B$	1.9276	14.472	$1,1,\zeta_2^1,\zeta_3^1,\zeta_3^1,\zeta_2^1\zeta_3^1$	$0, \frac{1}{2}, -\frac{3}{16}, \frac{1}{10}, -\frac{2}{5}, \frac{33}{80}$	$2_{-14/5}^B \otimes 3_{-3/2}^B$	
$6_{1/7}^B$	2.1082	18.591	$1,1,\zeta_5^1,\zeta_5^1,\zeta_5^2,\zeta_5^2$	$0, -\frac{1}{4}, -\frac{1}{7}, -\frac{11}{28}, \frac{1}{28}, \frac{2}{7}$	$2_{-1}^B \otimes 3_{8/7}^B$	
$6_{-1/7}^B$	2.1082	18.591	$1,1,\zeta_5^1,\zeta_5^1,\zeta_5^2,\zeta_5^2$	$0, \frac{1}{4}, \frac{1}{7}, \frac{11}{28}, -\frac{1}{28}, -\frac{2}{7}$	$2_1^B \otimes 3_{-8/7}^B$	$(C_5, 1)$
$6_{15/7}^B$	2.1082	18.591	$1,1,\zeta_5^1,\zeta_5^1,\zeta_5^2,\zeta_5^2$	$0, \frac{1}{4}, \frac{3}{28}, -\frac{1}{7}, \frac{2}{7}, -\frac{13}{28}$	$2_1^B \otimes 3_{\frac{8}{7}}^B$	$(A_1, 5)$
$6_{-15/7}^B$	2.1082	18.591	$1,1,\zeta_5^1,\zeta_5^1,\zeta_5^2,\zeta_5^2$	$0, -\frac{1}{4}, -\frac{3}{28}, \frac{1}{7}, -\frac{2}{7}, \frac{13}{28}$	$2_{-1}^B \otimes 3_{-8/7}^B$	
$6_0^{B,a}$	2.1609	20	$1,1,2,2,\sqrt{5},\sqrt{5}$	$0, 0, \frac{1}{5}, -\frac{1}{5}, 0, \frac{1}{2}$		$(D_5, 2)_{\frac{1}{2}}, (U(1)_5/\mathbb{Z}_2)_{\frac{1}{2}}$
$6_0^{B,b}$	2.1609	20	$1,1,2,2,\sqrt{5},\sqrt{5}$	$0, 0, \frac{1}{5}, -\frac{1}{5}, \frac{1}{4}, -\frac{1}{4}$		$[6_0^{B,a} \otimes 4_0^{B,b}]_{\frac{1}{4}}$
$6_4^{B,b}$	2.1609	20	$1,1,2,2,\sqrt{5},\sqrt{5}$	$0, 0, \frac{2}{5}, -\frac{2}{5}, \frac{1}{4}, -\frac{1}{4}$		$(B_2, 2)$
$6_4^{B,a}$	2.1609	20	$1,1,2,2,\sqrt{5},\sqrt{5}$	$0, 0, \frac{2}{5}, -\frac{2}{5}, 0, \frac{1}{2}$		$[6_4^{B,b} \otimes 4_0^{B,b}]_{\frac{1}{4}}$
$6_{58/35}^B$	2.5359	33.632	$1,\zeta_3^1,\zeta_5^1,\zeta_5^2,\zeta_3^1\zeta_5^1,\zeta_3^1\zeta_5^2$	$0, \frac{2}{5}, \frac{1}{7}, -\frac{2}{7}, -\frac{16}{35}, \frac{4}{35}$	$2_{\frac{14}{5}}^B \otimes 3_{-8/7}^B$	
$6_{-58/35}^B$	2.5359	33.632	$1,\zeta_3^1,\zeta_5^1,\zeta_5^2,\zeta_3^1\zeta_5^1,\zeta_3^1\zeta_5^2$	$0, -\frac{2}{5}, -\frac{1}{7}, \frac{2}{7}, \frac{16}{35}, -\frac{4}{35}$	$2_{-14/5}^B \otimes 3_{8/7}^B$	
$6_{138/35}^B$	2.5359	33.632	$1,\zeta_3^1,\zeta_5^1,\zeta_5^2,\zeta_3^1\zeta_5^1,\zeta_3^1\zeta_5^2$	$0, \frac{2}{5}, -\frac{1}{7}, \frac{2}{7}, \frac{9}{35}, -\frac{11}{35}$	$2_{\frac{14}{5}}^B \otimes 3_{8/7}^B$	
$6_{-138/35}^B$	2.5359	33.632	$1,\zeta_3^1,\zeta_5^1,\zeta_5^2,\zeta_3^1\zeta_5^1,\zeta_3^1\zeta_5^2$	$0, -\frac{2}{5}, \frac{1}{7}, -\frac{2}{7}, -\frac{9}{35}, \frac{11}{35}$	$2_{-14/5}^B \otimes 3_{-8/7}^B$	
$6_{46/13}^B$	2.9132	56.746	$1,\zeta_{11}^1,\zeta_{11}^2,\zeta_{11}^3,\zeta_{11}^4,\zeta_{11}^5$	$0, \frac{4}{13}, \frac{2}{13}, -\frac{6}{13}, \frac{6}{13}, -\frac{1}{13}$		$(A_1, 11)_{\frac{1}{2}}$
$6_{-46/13}^B$	2.9132	56.746	$1,\zeta_{11}^1,\zeta_{11}^2,\zeta_{11}^3,\zeta_{11}^4,\zeta_{11}^5$	$0, -\frac{4}{13}, -\frac{2}{13}, \frac{6}{13}, -\frac{6}{13}, \frac{1}{13}$		$(A_{10}, 2)_{\frac{1}{11}}$
$6_{8/3}^B$	3.1107	74.617	$1,\zeta_7^3,\zeta_7^3,\zeta_{16}^2,\zeta_{16}^4,\zeta_{16}^6$	$0, \frac{1}{9}, \frac{1}{9}, \frac{1}{9}, \frac{1}{3}, -\frac{1}{3}$		$(A_1, 16)_{\frac{1}{4}}$
$6_{-8/3}^B$	3.1107	74.617	$1,\zeta_7^3,\zeta_7^3,\zeta_{16}^2,\zeta_{16}^4,\zeta_{16}^6$	$0, -\frac{1}{9}, -\frac{1}{9}, -\frac{1}{9}, -\frac{1}{3}, \frac{1}{3}$		$(A_2, 6)_{\frac{1}{9}}$
6_2^B	3.3263	100.61	$1, \frac{3+\sqrt{21}}{2}, \frac{3+\sqrt{21}}{2}, \frac{3+\sqrt{21}}{2}, \frac{5+\sqrt{21}}{2}, \frac{7+\sqrt{21}}{2}$	$0, -\frac{1}{7}, -\frac{2}{7}, \frac{3}{7}, 0, \frac{1}{3}$		
6_{-2}^B	3.3263	100.61	$1, \frac{3+\sqrt{21}}{2}, \frac{3+\sqrt{21}}{2}, \frac{3+\sqrt{21}}{2}, \frac{5+\sqrt{21}}{2}, \frac{7+\sqrt{21}}{2}$	$0, \frac{1}{7}, \frac{2}{7}, -\frac{3}{7}, 0, -\frac{1}{3}$		$(G_2, 3)$

TABLE IV: A list of 24 bosonic rank $N = 7$ topological orders in 2+1D with $\max(N_k^{ij}) \le 1$. Since $N = 7$ is a prime number, all those 24 topological orders are primitive.

N_c^B	S_{top}	D^2	$d_1, d_2, \cdots$	$s_1, s_2, \cdots$	type
$7_2^{B,a}$	1.4036	7	$1,1,1,1,1,1,1$	$0,\frac{1}{7},\frac{1}{7},\frac{2}{7},\frac{2}{7},-\frac{3}{7},-\frac{3}{7}$	
$7_{-2}^{B,a}$	1.4036	7	$1,1,1,1,1,1,1$	$0,-\frac{1}{7},-\frac{1}{7},-\frac{2}{7},-\frac{2}{7},\frac{3}{7},\frac{3}{7}$	$(A_6,1)$
$7_{9/4}^B$	2.3857	27.313	$1,1,\zeta_6^1,\zeta_6^1,\zeta_6^2,\zeta_6^2,\zeta_6^3$	$0,\frac{1}{2},\frac{3}{32},\frac{3}{32},\frac{1}{4},-\frac{1}{4},\frac{15}{32}$	$(A_1,6)$
$7_{13/4}^B$	2.3857	27.313	$1,1,\zeta_6^1,\zeta_6^1,\zeta_6^2,\zeta_6^2,\zeta_6^3$	$0,\frac{1}{2},\frac{7}{32},\frac{7}{32},\frac{1}{4},-\frac{1}{4},-\frac{13}{32}$	$[7_{9/4}^B\otimes 4_1^B]_{\frac14}$
$7_{-15/4}^B$	2.3857	27.313	$1,1,\zeta_6^1,\zeta_6^1,\zeta_6^2,\zeta_6^2,\zeta_6^3$	$0,\frac{1}{2},\frac{11}{32},\frac{11}{32},\frac{1}{4},-\frac{1}{4},-\frac{9}{32}$	$[7_{13/4}^B\otimes 4_1^B]_{\frac14}$
$7_{-11/4}^B$	2.3857	27.313	$1,1,\zeta_6^1,\zeta_6^1,\zeta_6^2,\zeta_6^2,\zeta_6^3$	$0,\frac{1}{2},\frac{15}{32},\frac{15}{32},\frac{1}{4},-\frac{1}{4},-\frac{5}{32}$	$[7_{-15/4}^B\otimes 4_1^B]_{\frac14}$
$7_{-7/4}^B$	2.3857	27.313	$1,1,\zeta_6^1,\zeta_6^1,\zeta_6^2,\zeta_6^2,\zeta_6^3$	$0,\frac{1}{2},-\frac{13}{32},-\frac{13}{32},\frac{1}{4},-\frac{1}{4},-\frac{1}{32}$	$[7_{-11/4}^B\otimes 4_1^B]_{\frac14}$
$7_{-3/4}^B$	2.3857	27.313	$1,1,\zeta_6^1,\zeta_6^1,\zeta_6^2,\zeta_6^2,\zeta_6^3$	$0,\frac{1}{2},-\frac{9}{32},-\frac{9}{32},\frac{1}{4},-\frac{1}{4},\frac{3}{32}$	$[7_{-7/4}^B\otimes 4_1^B]_{\frac14}$
$7_{1/4}^B$	2.3857	27.313	$1,1,\zeta_6^1,\zeta_6^1,\zeta_6^2,\zeta_6^2,\zeta_6^3$	$0,\frac{1}{2},-\frac{5}{32},-\frac{5}{32},\frac{1}{4},-\frac{1}{4},\frac{7}{32}$	$[7_{-3/4}^B\otimes 4_1^B]_{\frac14}$
$7_{5/4}^B$	2.3857	27.313	$1,1,\zeta_6^1,\zeta_6^1,\zeta_6^2,\zeta_6^2,\zeta_6^3$	$0,\frac{1}{2},-\frac{1}{32},-\frac{1}{32},\frac{1}{4},-\frac{1}{4},\frac{11}{32}$	$[7_{1/4}^B\otimes 4_1^B]_{\frac14}$
$7_{7/4}^B$	2.3857	27.313	$1,1,\zeta_6^1,\zeta_6^1,\zeta_6^2,\zeta_6^2,\zeta_6^3$	$0,\frac{1}{2},\frac{13}{32},\frac{13}{32},\frac{1}{4},-\frac{1}{4},\frac{1}{32}$	$(C_6,1)$
$7_{11/4}^B$	2.3857	27.313	$1,1,\zeta_6^1,\zeta_6^1,\zeta_6^2,\zeta_6^2,\zeta_6^3$	$0,\frac{1}{2},-\frac{15}{32},-\frac{15}{32},\frac{1}{4},-\frac{1}{4},\frac{5}{32}$	$[7_{7/4}^B\otimes 4_1^B]_{\frac14}$
$7_{15/4}^B$	2.3857	27.313	$1,1,\zeta_6^1,\zeta_6^1,\zeta_6^2,\zeta_6^2,\zeta_6^3$	$0,\frac{1}{2},-\frac{11}{32},-\frac{11}{32},\frac{1}{4},-\frac{1}{4},\frac{9}{32}$	$[7_{11/4}^B\otimes 4_1^B]_{\frac14}$
$7_{-13/4}^B$	2.3857	27.313	$1,1,\zeta_6^1,\zeta_6^1,\zeta_6^2,\zeta_6^2,\zeta_6^3$	$0,\frac{1}{2},-\frac{7}{32},-\frac{7}{32},\frac{1}{4},-\frac{1}{4},\frac{13}{32}$	$[7_{15/4}^B\otimes 4_1^B]_{\frac14}$
$7_{-9/4}^B$	2.3857	27.313	$1,1,\zeta_6^1,\zeta_6^1,\zeta_6^2,\zeta_6^2,\zeta_6^3$	$0,\frac{1}{2},-\frac{3}{32},-\frac{3}{32},\frac{1}{4},-\frac{1}{4},-\frac{15}{32}$	$[7_{-13/4}^B\otimes 4_1^B]_{\frac14}$
$7_{-5/4}^B$	2.3857	27.313	$1,1,\zeta_6^1,\zeta_6^1,\zeta_6^2,\zeta_6^2,\zeta_6^3$	$0,\frac{1}{2},\frac{1}{32},\frac{1}{32},\frac{1}{4},-\frac{1}{4},-\frac{11}{32}$	$[7_{-9/4}^B\otimes 4_1^B]_{\frac14}$
$7_{-1/4}^B$	2.3857	27.313	$1,1,\zeta_6^1,\zeta_6^1,\zeta_6^2,\zeta_6^2,\zeta_6^3$	$0,\frac{1}{2},\frac{5}{32},\frac{5}{32},\frac{1}{4},-\frac{1}{4},-\frac{7}{32}$	$[7_{-5/4}^B\otimes 4_1^B]_{\frac14}$
$7_{3/4}^B$	2.3857	27.313	$1,1,\zeta_6^1,\zeta_6^1,\zeta_6^2,\zeta_6^2,\zeta_6^3$	$0,\frac{1}{2},\frac{9}{32},\frac{9}{32},\frac{1}{4},-\frac{1}{4},-\frac{3}{32}$	$[7_{-1/4}^B\otimes 4_1^B]_{\frac14}$
$7_2^{B,b}$	2.4036	28	$1,1,2,2,2,\sqrt{7},\sqrt{7}$	$0,0,\frac{1}{7},\frac{2}{7},-\frac{3}{7},\frac{1}{8},-\frac{3}{8}$	$(U(1)_7/\mathbb{Z}_2)_{\frac12}$
$7_2^{B,c}$	2.4036	28	$1,1,2,2,2,\sqrt{7},\sqrt{7}$	$0,0,\frac{1}{7},\frac{2}{7},-\frac{3}{7},-\frac{1}{8},\frac{3}{8}$	$[7_2^{B,b}\otimes 4_0^{B,b}]_{\frac14}$
$7_{-2}^{B,b}$	2.4036	28	$1,1,2,2,2,\sqrt{7},\sqrt{7}$	$0,0,-\frac{1}{7},-\frac{2}{7},\frac{3}{7},-\frac{1}{8},\frac{3}{8}$	$(B_3,2),(D_7,2)_{\frac12}$
$7_{-2}^{B,c}$	2.4036	28	$1,1,2,2,2,\sqrt{7},\sqrt{7}$	$0,0,-\frac{1}{7},-\frac{2}{7},\frac{3}{7},\frac{1}{8},-\frac{3}{8}$	$[7_{-2}^{B,b}\otimes 4_0^{B,b}]_{\frac14}$
$7_{8/5}^B$	3.2194	86.750	$1,\zeta_{13}^1,\zeta_{13}^2,\zeta_{13}^3,\zeta_{13}^4,\zeta_{13}^5,\zeta_{13}^6$	$0,-\frac{1}{5},\frac{2}{15},0,\frac{2}{5},\frac{1}{3},-\frac{1}{5}$	$(A_1,13)_{\frac12}$
$7_{-8/5}^B$	3.2194	86.750	$1,\zeta_{13}^1,\zeta_{13}^2,\zeta_{13}^3,\zeta_{13}^4,\zeta_{13}^5,\zeta_{13}^6$	$0,\frac{1}{5},-\frac{2}{15},0,-\frac{2}{5},-\frac{1}{3},\frac{1}{5}$	$(A_{12},2)_{\frac{1}{13}}$

since

$$\chi_1^{U(1)}(q) = \frac{1}{\eta(q)} = \sum_{n\ge 0}\frac{q^{n^2} - q^{(n+1)^2}}{\eta(q)} = \sum_{n\ge 0}\chi_{c=1,n^2}^{\text{Vir}}.$$
(39)

The corresponding Virasoro primary fields are $1, j, j_4 = j^4 - 2j\partial^2 j + \frac{3}{2}(\partial j)^2, \cdots$.

The non-trivial representation $\mathcal{V}_k^{U(1)}$ of the $U(1)$ current algebra corresponds to the $U(1)$ primary field $e^{ik\phi}$ with conformal dimension $h = \frac{k^2}{2}$. The corresponding representations have the following fusion property

$$\mathcal{V}_k^{U(1)} \otimes \mathcal{V}_{k'}^{U(1)} = \mathcal{V}_{k+k'}^{U(1)}, \quad k,k' \in \mathbb{R}.$$
(40)

C. Extended $U(1)_M$ current algebra

The extended $U(1)_M$ current algebra $\mathcal{V}_1^{U(1)_M}$ of level M (which is another simple-current algebra) is generated by the spin-M fields $\psi_+ = e^{i\sqrt{2M}\phi}$ and $\psi_- = e^{-i\sqrt{2M}\phi}$. Note that the OPE $\psi_+\psi_- \sim 1 + j$. So the extended $U(1)_M$ current algebra is also generated by j, ψ_+, ψ_-.

The non-trivial representation $\mathcal{V}_k^{U(1)_M}$ of the extended $U(1)_M$ current algebra corresponds to the extended-$U(1)_M$ primary fields $e^{ik\phi/\sqrt{2M}}$, $k = 0, \cdots, 2M - 1$, which are local with respect to the generating fields $\psi_\pm$. The corresponding character is given by

$$\chi_k^{U(1)_M}(q) = \frac{1}{\eta(q)}\sum_{m\in\mathbb{Z}}q^{(k+2mM)^2/M}.$$
(41)

Under the modular transformation S, the characters $\chi_k^{U(1)_M}(q)$ transforms as

$$S: \chi_k^{U(1)_M}(q) \to \sum_{k'\in\mathbb{Z}_{2M}}e^{-i\pi kk'/2M}\chi_{k'}^{U(1)_M}(q).$$
(42)

The irreducible representations have the following fusion property

$$\mathcal{V}_k^{U(1)_M} \otimes \mathcal{V}_{k'}^{U(1)_M} = \mathcal{V}_{k+k'}^{U(1)_M}, \quad k,k' \in \mathbb{Z}_{2M}.$$
(43)

D. $\frac{U(1)_M}{\mathbb{Z}_2}$-orbifold simple-current algebra

The $\frac{U(1)_M}{\mathbb{Z}_2}$-orbifold simple-current algebra $\mathcal{V}_1^{\frac{U(1)_M}{\mathbb{Z}_2}}$ is generated by the spin-M simple current $\psi = \cos(\sqrt{2M}\phi)$. Note that $\mathcal{V}_1^{\frac{U(1)_M}{\mathbb{Z}_2}}$ is the $\mathbb{Z}_2$ invariant part of $\mathcal{V}_1^{U(1)_M}$, where $\mathbb{Z}_2$ acts as

$$\mathbb{Z}_2 : \phi \to -\phi. \tag{44}$$

$\mathcal{V}_1^{U(1)_M}$ contains $\mathcal{V}_1^{\mathrm{Vir}}$ generated by energy momentum tensor $T \sim j^2$ which is $\mathbb{Z}_2$ invariant. Thus $\mathcal{V}_1^{\frac{U(1)_M}{\mathbb{Z}_2}}$ also contains $\mathcal{V}_1^{\mathrm{Vir}}$, and T acts within $\mathcal{V}_1^{\frac{U(1)_M}{\mathbb{Z}_2}}$. $\mathcal{V}_1^{U(1)_M}$ also contains $\mathcal{V}_1^{U(1)}$ which contains $\mathcal{V}_{c=1,n^2}^{\mathrm{Vir}}$, $n \in \mathbb{N}$. But the states in $\mathcal{V}_{c=1,n^2}^{\mathrm{Vir}}$ transform as $|\psi\rangle \to (-)^n |\psi\rangle$ under the $\mathbb{Z}_2$. So $\mathcal{V}_1^{\frac{U(1)_M}{\mathbb{Z}_2}}$ only contains $\mathcal{V}_{c=1,n^2}^{\mathrm{Vir}}$ for n even:

$$\mathcal{V}_1^{U(1)} = \bigoplus_{n \geq 0, n \text{ even}} \mathcal{V}_{c=1,n^2}^{\mathrm{Vir}} \tag{45}$$

In particular, j_4 acts within $\mathcal{V}_1^{\frac{U(1)_M}{\mathbb{Z}_2}}$.

Now let us consider irreducible representations of the $\frac{U(1)_M}{\mathbb{Z}_2}$-orbifold simple-current algebra. We note that the $\mathbb{Z}_2$ action on the irreducible representations of the extended $U(1)_M$ current algebra is given by

$$\mathcal{V}_k^{U(1)_M} \to \mathcal{V}_{-k}^{U(1)_M} = \mathcal{V}_{2M-k}^{U(1)_M}. \tag{46}$$

For $k = 1, \cdots, M-1$, the $\mathbb{Z}_2$ acts within $\mathcal{V}_k^{U(1)_M} \oplus \mathcal{V}_{2M-k}^{U(1)_M}$. The $\mathbb{Z}_2$ even part of $\mathcal{V}_k^{U(1)_M} \oplus \mathcal{V}_{2M-k}^{U(1)_M}$ forms an irreducible representation of the $\frac{U(1)_M}{\mathbb{Z}_2}$-orbifold simple-current algebra, denoted as $\mathcal{V}_{\phi_k}^{\frac{U(1)_M}{\mathbb{Z}_2}}$. The corresponding primary field is given by $\phi_k = \cos(k\phi/\sqrt{2M})$. We know that the quantum dimension of the representation $\mathcal{V}_{\pm k}^{U(1)_M}$ is equal to 1. Thus the quantum dimension for $\mathcal{V}_{\phi_k}^{\frac{U(1)_M}{\mathbb{Z}_2}}$ is equal to 2. The $\mathbb{Z}_2$ odd part of $\mathcal{V}_k^{U(1)_M} \oplus \mathcal{V}_{2M-k}^{U(1)_M}$ does not form an irreducible representation of the $\frac{U(1)_M}{\mathbb{Z}_2}$ algebra.

The $\mathbb{Z}_2$ acts within $\mathcal{V}_1^{U(1)_M}$. The $\mathbb{Z}_2$ even part of $\mathcal{V}_1^{U(1)_M}$ forms an irreducible representation of $\frac{U(1)_M}{\mathbb{Z}_2}$ algebra, denoted as $\mathcal{V}_1^{\frac{U(1)_M}{\mathbb{Z}_2}}$. The corresponding primary field is the identity 1. The $\mathbb{Z}_2$ odd part of $\mathcal{V}_1^{U(1)_M}$ also forms an irreducible representation of the $\frac{U(1)_M}{\mathbb{Z}_2}$ algebra, denoted as $\mathcal{V}_j^{\frac{U(1)_M}{\mathbb{Z}_2}}$. The corresponding primary field is the current operator j. We also have two new irreducible representations of the $\frac{U(1)_M}{\mathbb{Z}_2}$ algebra from the twisted sector that twists the current j. The corresponding representations are denoted as $\mathcal{V}_{\sigma^1}^{\frac{U(1)_M}{\mathbb{Z}_2}}$ and, $\mathcal{V}_{\tau^1}^{\frac{U(1)_M}{\mathbb{Z}_2}}$. The corresponding primary fields are denoted as σ^1 and τ^1.

TABLE V: The irreducible representations $\mathcal{V}_\alpha^{\frac{U(1)_M}{\mathbb{Z}_2}}$ of the $\frac{U(1)_M}{\mathbb{Z}_2}$-orbifold simple current algebra. The second column gives the conformal dimensions h_α of the corresponding primary fields. The third column are the quantum dimensions d_α of the representations.

label α	h_α	d_α	
1	0	1	
j	1	1	
ϕ_M^i	$M/4$	1	$i = 1, 2$
σ^i	$1/16$	$\sqrt{M}$	$i = 1, 2$
τ^i	$9/16$	$\sqrt{M}$	$i = 1, 2$
ϕ_k	$k^2/4M$	2	$k = 1, \cdots, M-1$

We have the following fusion relations for the irreducible representations

$$\mathcal{V}_j^{\frac{U(1)_M}{\mathbb{Z}_2}} \otimes \mathcal{V}_{\sigma^1}^{\frac{U(1)_M}{\mathbb{Z}_2}} = \mathcal{V}_{\tau^1}^{\frac{U(1)_M}{\mathbb{Z}_2}},$$
$$\mathcal{V}_j^{\frac{U(1)_M}{\mathbb{Z}_2}} \otimes \mathcal{V}_{\tau^1}^{\frac{U(1)_M}{\mathbb{Z}_2}} = \mathcal{V}_{\sigma^1}^{\frac{U(1)_M}{\mathbb{Z}_2}}. \tag{47}$$

Note that the irreducible representations correspond to the topological excitations. The fusion relations for the irreducible representations give rise to the fusion relations of the topological excitations.

Similarly, the $\mathbb{Z}_2$ acts within $\mathcal{V}_M^{U(1)_M}$. The $\mathbb{Z}_2$ even part of $\mathcal{V}_M^{U(1)_M}$ forms an irreducible representation of $\frac{U(1)_M}{\mathbb{Z}_2}$ algebra, denoted as $\mathcal{V}_{\phi_M^1}^{\frac{U(1)_M}{\mathbb{Z}_2}}$. The corresponding primary field is $\phi_M^1 = \cos(\sqrt{M/2}\phi)$. The $\mathbb{Z}_2$ odd part of $\mathcal{V}_M^{U(1)_M}$ also forms an irreducible representation of $\frac{U(1)_M}{\mathbb{Z}_2}$ algebra, denoted as $\mathcal{V}_{\phi_M^2}^{\frac{U(1)_M}{\mathbb{Z}_2}}$. The corresponding primary field is $\phi_M^2 = \sin(\sqrt{M/2}\phi)$. We also have two new irreducible representations of the $\frac{U(1)_M}{\mathbb{Z}_2}$ algebra from the twisted sector that twist j: $j \to -j$. The corresponding representations are denoted as $\mathcal{V}_{\sigma^2}^{\frac{U(1)_M}{\mathbb{Z}_2}}$ and, $\mathcal{V}_{\tau^2}^{\frac{U(1)_M}{\mathbb{Z}_2}}$. The corresponding primary fields are denoted as σ^2 and τ^2. We have the following fusion relations

$$\mathcal{V}_j^{\frac{U(1)_M}{\mathbb{Z}_2}} \otimes \mathcal{V}_{\sigma^2}^{\frac{U(1)_M}{\mathbb{Z}_2}} = \mathcal{V}_{\tau^2}^{\frac{U(1)_M}{\mathbb{Z}_2}},$$
$$\mathcal{V}_j^{\frac{U(1)_M}{\mathbb{Z}_2}} \otimes \mathcal{V}_{\tau^2}^{\frac{U(1)_M}{\mathbb{Z}_2}} = \mathcal{V}_{\sigma^2}^{\frac{U(1)_M}{\mathbb{Z}_2}}. \tag{48}$$

The above are all the irreducible representations of the $\frac{U(1)_M}{\mathbb{Z}_2}$-orbifold simple-current algebra. Table V summarize the result. Ref. 34 computed the full fusion rules of those irreducible representations. The fusion rules and conformal dimensions of the corresponding primary fields are obtained by studying the modular transformation properties of the characters.

It turns out that the $U_1(1)/\mathbb{Z}_2$ orbifold is the $U(1)_4$ Gaussian theory, the $U_2(1)/\mathbb{Z}_2$ orbifold is two copies

of the Ising CFT, and the $U_3(1)/\mathbb{Z}_2$ orbifold is the $\mathbb{Z}_4$ parafermion CFT of Zamolodchikov and Fateev.[35]

E. $U(1)_M$ topological orders

The $U(1)_M$ simple-current algebra is generated by a single simple-current operator $\psi = \mathrm{e}^{\mathrm{i}\sqrt{2M}\phi}$ (plus its hermitian conjugate) with conformal dimension $h = M$. The correlation function of ψ gives rise to a bosonic Laughlin wavefunction

$$\prod_{i<j}(z_i - z_j)^{2M} = \langle\psi(z_1)\psi(z_2)\psi(z_3)\cdots\rangle \qquad (49)$$

as discussed in Section II A. The simple-current primary fields $\sigma_\alpha = \mathrm{e}^{\mathrm{i}\frac{\alpha}{\sqrt{2M}}\phi}$, $\alpha = 1, \cdots, 2M-1$, produce wavefunctions that contain excitations with non-trivial topological type

$$\prod_i(\xi - z_i)^\alpha \prod_{i<j}(z_i - z_j)^{2M} = \langle\sigma_\alpha(\xi)\psi(z_1)\psi(z_2)\cdots\rangle \qquad (50)$$

We see a one-to-one correspondence between the simple-current primary fields and the topological excitations.

The above picture is valid even when M is half-integer. In this case the correlation function of ψ gives rise to a fermionic Laughlin wavefunction, and the simple-current primary fields $\sigma_\alpha = \mathrm{e}^{\mathrm{i}\frac{\alpha}{\sqrt{2M}}\phi}$, $\alpha = 1, \cdots, 2M-1$, give rise to the topological excitations in the fermionic Laughlin state.

F. $U(1)_M/\mathbb{Z}_2$-orbifold topological orders

The $U(1)_M/\mathbb{Z}_2$-orbifold simple-current algebra is generated by a single simple-current operator $\psi = \cos(\sqrt{2M}\phi)$ with conformal dimension $h = M$. We note that $\psi^2 \sim 1$ (i.e. the OPE of two ψ's produces the identity operator 1 as the leading term).

The correlation function of ψ's

$$\Psi(\{z_i\}) \propto \lim_{z_\infty \to \infty}\langle\hat{V}(z_\infty)\prod\psi(z_i)\rangle \qquad (51)$$

is single-valued (no branch cut) since the conformal dimension of ψ is integer and the OPE of ψ's only produces operators with integer conformal dimensions. Also, since ψ has an integer conformal dimension and is bosonic, the correlation function $\Psi(\{z_i\})$ is a symmetric function, which gives rise to a quantum Hall many-boson wavefunction $\Psi(\{z_i\})\mathrm{e}^{-\frac{1}{4}\sum|z_i|^2}$ with a bosonic topological order. The edge excitations of such a quantum Hall state are described by the $\frac{U(1)_M}{\mathbb{Z}_2}$-orbifold CFT, the CFT that produces the bulk wave function, as calculated in Ref. 25,29,33,36.

However, the above construction has a problem: the correlation of ψ's (i.e. $\Psi(\{z_i\})$) has poles as $z_i \to z_j$. But

this is only a technical problem that can be fixed. We may put the wave function on a lattice or add additional factors, such as $\prod|z_i - z_j|^{2M}$, to make the wave function finite.

We may also combine the $\frac{U(1)_M}{\mathbb{Z}_2}$-orbifold simple-current algebra with a Gaussian model, as described in Section II, to produce a many-body wave function without poles. We can choose the Gaussian model to have two fields $\boldsymbol{\phi} = (\phi^1, \phi^2)$ and choose $G^{\mu\nu}$ to be

$$G = \begin{pmatrix} 2M & 1 \\ 1 & 0 \end{pmatrix}. \qquad (52)$$

We choose the three simple currents as

$$c_1 = \psi\mathrm{e}^{\mathrm{i}\phi^1}, \quad c_2 = \mathrm{e}^{\mathrm{i}\phi^2}, \quad c_3 = \mathrm{e}^{\mathrm{i}\phi^1}, \qquad (53)$$

which corresponds to choosing k_μ^I as

$$k_\mu^I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}_{I\mu}. \qquad (54)$$

The $2M^{\mathrm{th}}$ order zero in the correlation function of $\mathrm{e}^{\mathrm{i}\phi^1}$ cancels the $2M^{\mathrm{th}}$ order pole in the correlation function of the $\psi(z_i)$. So the correlation functions of c_I, $I = 1, 2, 3$, are single-valued and finite, which gives rise to a triple-layer bosonic wavefunction:

$$P(\{z_i, w_i, u_i\}) = \qquad (55)$$
$$\langle c_1(z_1)c_1(z_2)\cdots c_2(w_1)c_2(w_2)\cdots c_3(u_1)c_2(u_2)\cdots\rangle.$$

To understand the topological excitations in such a triple-layer state, we note that c_I-primary fields have the form

$$\eta_{\alpha,\boldsymbol{l}^\alpha} = \sigma_\alpha\mathrm{e}^{\mathrm{i}\boldsymbol{l}^\alpha\cdot\boldsymbol{\phi}}, \quad \boldsymbol{\phi} = (\phi^1, \phi^2), \qquad (56)$$

where $\boldsymbol{l}^\alpha$ satisfies eqn. (32). Since $h^{\mathrm{sc}}_{\alpha,\vec{a}} - h^{\mathrm{sc}}_{\vec{a}} - h^{\mathrm{sc}}_\alpha$ are integers for all $\vec{a}$, we find that the $\boldsymbol{l}^\alpha$ satisfy $a^I k_\mu^I G^{\mu\nu}l_\nu^\alpha \in \mathbb{Z}$ or

$$\begin{pmatrix} 2M & 1 \\ 1 & 0 \\ 2M & 1 \end{pmatrix}\begin{pmatrix} l_1^\alpha \\ l_2^\alpha \end{pmatrix} = \begin{pmatrix} 0 \bmod 1 \\ 0 \bmod 1 \\ 0 \bmod 1 \end{pmatrix}. \qquad (57)$$

The above requires $\boldsymbol{l}^\alpha$ to be integer vectors, and all the different $\boldsymbol{l}^\alpha$ are equivalent. So we can choose $\boldsymbol{l}^\alpha = 0$.

We would like to remark that if we did not include the simple current c_3 for the third layer, $\mathrm{e}^{\mathrm{i}\phi^1}$ would correspond to a non-trivial primary field which would lead to extra topological types. With the simple current c_3, $\mathrm{e}^{\mathrm{i}\phi^1}$ will be a descendent field of the simple current algebra, and will not correspond to a new type of topological excitation. We also like to remark that there is no particle number conservation, for each layer or for all the layers. If we did have particle number conservation

for each layer, the constructed state may spontaneous break such particle-number-conservation symmetry and contain gapless Goldstone modes.

We note that $G^{\mu\nu}$ has negative eigenvalues and the corresponding purely chiral CFT is not unitary. This can be fixed by treating the part of $G^{\mu\nu}$ with negative eigenvalues as anti-holomorphic (*i.e.* producing correlations that depend on z^*). We may also remove the poles using purely chiral unitary CFT that describes the E_8 quantum Hall state, *i.e.* using eight scalar fields ϕ^i, $i = 1, \cdots, 8$ and choosing

$$G = \begin{pmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 2 \end{pmatrix}, \quad (58)$$

to form nine simple-current operators

$$c_i = e^{i\phi^i}|_{i=1,\cdots,8}, \quad c_9 = \psi e^{i\phi^1}. \quad (59)$$

This can remove the pole for the $M = 1$ case. To remove the pole for other cases with larger M, we can add several copies of E_8 quantum Hall states. The new simple-current algebra has the same topological excitations as the $\frac{U(1)_M}{\mathbb{Z}_2}$-orbifold simple-current algebra, and has the same central charge mod 8. In this paper, we will use Gaussian theory with $G^{\mu\nu}$ that may have negative eigenvalues to remove the poles. We can also choose the Gaussian theory to be several copies of E_8 states to remove the poles.

We see that the topological excitations in our triple-layer bosonic wave function are in one-to-one correspondence with the primary fields (or the irreducible representations) of the $U(1)_M/\mathbb{Z}_2$-orbifold simple-current algebra. The chiral central charge of our triple-layer bosonic state is $c = 1$ (1 from the $U(1)_M/\mathbb{Z}_2$ simple-current and $1 + (-1)$ from the Gaussian CFT). The $U(1)_M/\mathbb{Z}_2$ order is of type $N_c^B = (7 + M)_1^B$.

G. Reduction to smaller N

We now establish that for M odd, the topological order $U(1)_M/\mathbb{Z}_2$ can be reduced as

$M = 4p + 3$:

$$(N, c) = (7 + M, 1) \rightarrow (N, c) = (\frac{7 + M}{2}, 2),$$

$M = 4p + 1$:

$$(N, c) = (7 + M, 1) \rightarrow (N, c) = (\frac{7 + M}{2}, 0). \quad (60)$$

This reduction is similar in spirit to the reduction $(A_1, k) \rightarrow (A_1, k)_{\frac{1}{2}}$ which we discuss in section IV B.

In the Tables II-IV we marked these reduced orders as $(U(1)_M/\mathbb{Z}_2)_{\frac{1}{2}}$.

We first consider the case $M = 3$. We already remarked that this order, with $N = 10$ primaries, precisely agrees with the $\mathbb{Z}_4$ parafermions. The dictionary reads (see section IV B 1 for notation)

$$\phi_3^1 \rightarrow \psi_1, \quad j \rightarrow \psi_2, \quad \phi_3^2 \rightarrow \psi_3,$$
$$\sigma^1 \rightarrow \Phi_1^1, \quad \tau^1 \rightarrow \Phi_5^1, \quad \sigma^2 \rightarrow \Phi_7^1, \quad \tau^2 \rightarrow \Phi_3^1,$$
$$\phi_1 \rightarrow \Phi_2^2, \quad \phi_2 \rightarrow \Phi_4^2. \quad (61)$$

Following standard practice (see section IV B) we can now combine these fields with a single scalar field so as to produce the current algebra for $SU(2)_4$ at $c = 2$. As explained in section IV this current algebra gives rise to $k + 1 = 5$ primary sectors. For example, the sector with $s = \frac{1}{8}$, $d = \zeta_4^1$ comprises the fields

$$(\sigma^1 e^{i\frac{1}{\sqrt{8}}\phi}, \ \tau^2 e^{i\frac{3}{\sqrt{8}}\phi}, \ \tau^1 e^{i\frac{5}{\sqrt{8}}\phi}, \ \sigma^2 e^{i\frac{7}{\sqrt{8}}\phi}). \quad (62)$$

We thus establish that the order $N_c^B = 5_2^{B,a}$ in Table II is generated by the CFT $(U(1)_3/\mathbb{Z}_2)_{\frac{1}{2}}$.

This construction of the order $(U(1)_3/\mathbb{Z}_2)_{\frac{1}{2}}$ is an example of a simple-current reduction of the product of two topological orders. The building blocks are the orders $(U(1)_3/\mathbb{Z}_2)$, with $N = 10$, $c = 1$, and $U(1)_4$, with $N' = 8$, $c' = 1$. In the product theory we can define the bosonic simple currents

$$1, \quad \phi_3^1 e^{i\frac{1}{\sqrt{2}}\phi}, \quad j e^{i\frac{2}{\sqrt{2}}\phi}, \quad \phi_3^2 e^{i\frac{3}{\sqrt{2}}\phi}. \quad (63)$$

Of the $N \times N' = 80$ fields in the product theory, 20 are local with respect to all bosonic simple currents. These fields organize into 5 orbits and make up a reduced order of rank $N \times N'/16 = 5$ and central charge $c + c' = 2$. In formula

$$(U(1)_3/\mathbb{Z}_2)_{\frac{1}{2}} = [U(1)_3/\mathbb{Z}_2 \otimes U(1)_4]_{\frac{1}{16}}. \quad (64)$$

Turning to $M = 5$, we can follow a similar logic, but with an important twist: the scalar field now comes with metric $G = -1$, implying that it contributes $c = -1$ to the total central charge, and that a vertex operator $e^{ia\phi}$ has conformal dimension $s = -\frac{a^2}{2}$. In section IV B we see similar minus signs in the construction of $(A_1, k)_{\frac{1}{2}}$ for $k = 4p + 1$. We can define a set of bosonic currents according to

$$1, \quad \phi_5^1 e^{i\frac{1}{\sqrt{2}}\phi}, \quad j e^{i\frac{2}{\sqrt{2}}\phi}, \quad \phi_5^2 e^{i\frac{3}{\sqrt{2}}\phi}. \quad (65)$$

With respect to these currents, the following field combinations are primary and mutually inequivalent

$$1, \quad j, \quad \phi_2, \quad \phi_1 e^{i\frac{1}{\sqrt{2}}\phi}, \quad \sigma^1 e^{i\frac{1}{\sqrt{8}}\phi}, \quad \sigma^1 e^{i\frac{3}{\sqrt{8}}\phi}, \quad (66)$$

with $s = 0, 0, \frac{1}{5}, -\frac{1}{5}, 0, \frac{1}{2}$ and $d = 1, 1, 2, 2, \sqrt{5}, \sqrt{5}$. We thus recover the $6_0^{B,a}$ topological order in Table III. The pattern for general odd M is similar and leads to the result given in (60).

IV. 2+1D TOPOLOGICAL ORDERS FROM KAC-MOODY CURRENT ALGEBRA

A rich class of simple-current algebras in CFT is provided by the affine Kac-Moody algebras $X_l^{(1)}$ at positive integer level k. To each choice (X_l, k) corresponds a unitary CFT (the level-k WZW model on the associated group manifold) whose current algebra consists of currents $J^A(z)$, with $A = 1, 2, \ldots, D$ an adjoint index of the Lie algebra X_l. The central charge of this CFT is

$$c(X_l, k) = \frac{kD}{k + g} \tag{67}$$

with D the dimension of X_l and g the dual Coxeter number. We provide some details in Appendix A, where we have also tabulated (D, g) for the simple Lie algebras X_l.

Starting from topological orders of Kac-Moody type, one may look for additional bosonic simple currents and use these to extend the bosonic simple-current algebra. In some special cases, the CFT (X_l, k) contains Kac-Moody primaries that are bosonic simple currents, and the extended current algebra leads to a novel type of topological order with reduced rank N. These orders are closely related to exceptional modular invariant partition functions (MIPF) based on these same simple currents[37]. Examples are the orders $(A_1, k)_{\frac{1}{4}}$ for $k = 4, 8, \ldots$ and $(A_2, k)_{\frac{1}{9}}$ for $k = 3, 6, \ldots$, which we present below.

A more general, but often simpler, case involves the addition of one or several scalar fields (or $U(1)$ factors) and the use of simple currents of the form

$$c_{I, \boldsymbol{k}^I} = \psi_I V_{\boldsymbol{k}^I} \tag{68}$$

where the ψ_I are simple currents in the (X_l, k) CFT and the $V_{\boldsymbol{k}^l}$ are scalar field vertex operators . Examples are reductions of type $(A_n, k)_{\frac{1}{n+1}}$ (see section IV B, IV C) and the reductions

$$T_2 : \ N_c^B \to [N_c^B \otimes 4_0^{B,b}]_{\frac{1}{4}} \tag{69}$$

$$T_8 : \ N_c^B \to [N_c^B \otimes 4_1^B]_{\frac{1}{4}} \tag{70}$$

discussed in sections IV C 4, IV B 4 below. Here $4_0^{B,b}$ and 4_1^B are the bosonic topological orders in table I. $4_0^{B,b}$ is the double-semion topological order and 4_1^B is $\nu = 1/4$ bosonic Laughline state.

We note that both the operations T_2 and T_8 do not change the number N of topological types neither the quantum dimensions d_i. The operation T_2 also does not change the central charge c. In contrast, the operation T_8 changes the central charge by $+1$. Both the operations do change the spins s_i. We also like to point out that the operation T_2 is a $\mathbb{Z}_2$ operation, while the operation T_8 is a $\mathbb{Z}_8$ operation.

Yet more general are cases where the additional bosonic simple currents contains factors in different non-Abelian orders. One example is the case

$$[(C_4, 1) \times (A_1, 1) \times (A_3, 1)]_{\frac{1}{4}} \tag{71}$$

which turns out to be equivalent to a CFT coset construction and gives rise to the $c = \frac{4}{5}$ minimal model of the Virasoro algebra, of rank $N = 10$ (see section IV C 5).

In these constructions, it is sometimes convenient to first pass from the (X_l, k) CFT to the (generalised) parafermion CFT[38] obtained by modding out $U(1)^l$, and then use the parafermions ψ_Λ, which are simple currents, as building blocks in the construction of an (extended) bosonic simple-current algebra.

We remark that the simple-current reductions that we study here correspond to the condensation of bosonic topological excitations[39–43].

We have observed that we can construct all topological orders collected in tables I-IV from orders based on Kac-Moody current algebra (X_l, k) and $U(1)$ factors if we use

- conjugation by time reversal symmetry, sending

$$c \to -c, \quad d_i \to d_i, \quad s_i \to -s_i,$$

- stacking of topological orders,

- simple-current reductions of (combinations of) topological orders.

The conformal blocks of the bosonic simple currents $c_I(z_i)$ will, in general contain both zeros and poles in the differences $(z_i - z_j)$. To define a many-body bosonic wave function, one needs to cancel the poles. This can be done by including additional scalar fields, in such a way that essential topological data (central charge and quantum dimensions and spins of all excitations) are not affected. We make this step explicit in the examples of $U(1)_M / \mathbb{Z}_2$ and (A_1, k) in sections III and IV A, and will assume that a similar step is always possible in other cases. With that, we arrive at bosonic many-body wave-functions for all cases listed in Tables I-IV.

A. $SU(2)_k$ current algebra

The case (A_1, k), commonly denoted as $SU(2)_k$, gives a CFT of central charge $c = \frac{3k}{k+2}$.

The weight and root lattices (see appendix A) have the following structure. Writing the fundamental weight as $\Lambda_1 = \frac{1}{2}\mathbf{e}_1$, the single positive root is $\alpha_1 = 2\Lambda_1$ and the Weyl group has two elements: the identity and the reflection $w_1 : \Lambda_1 \to -\Lambda_1$. A general (integral, dominant) weight is $\Lambda = l\Lambda_1$, $l \in \mathbb{N}$, so the irreducible representations are labeled by l.

At level k there are $k + 1$ irreducible representations (or primary fields) Φ_l, $l = 0, \cdots, k$, with conformal dimension (spin)

$$s_l = \frac{l^2 + 2l}{4(k + 2)}. \tag{72}$$

The modular S-matrix is found to be

$$S_{ll'} \propto \sin[\frac{\pi}{k + 2}(l + 1)(l' + 1)] \tag{73}$$

and the quantum dimensions are

$$d_l = \frac{S_{0l}}{S_{00}} = \frac{\sin[\frac{\pi}{k+2}(l+1)]}{\sin[\frac{\pi}{k+2}]} = \zeta_k^l \ . \tag{74}$$

The $SU(2)_1$ Kac-Moody algebra is generated by three simple-current operators $j^z, j^\pm$ with conformal dimension $h = 1$. In fact the $SU(2)_1$ algebra can be generated by a single simple current j^+ plus its hermitian conjugate. To obtain a many-body wave function without poles from the correlator of simple currents, we can combine the $SU(2)_1$ Kac-Moody algebra with a Gaussian model with two additional scalar fields $\boldsymbol{\phi} = (\phi^1, \phi^2)$ with metric $G^{\mu\nu}$ given by

$$G = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}, \tag{75}$$

and choose the simple-current operators as

$$\begin{aligned} c_1 &= j^z \mathrm{e}^{\mathrm{i}\phi_1}, & c_2 &= j^+ \mathrm{e}^{\mathrm{i}\phi_1}, & c_3 &= j^- \mathrm{e}^{\mathrm{i}\phi_1}, \\ c_4 &= \mathrm{e}^{\mathrm{i}\phi_2}, & c_5 &= \mathrm{e}^{\mathrm{i}\phi_1}. \end{aligned} \tag{76}$$

We note that the 2nd order pole in the j^z-j^z correlator is canceled by the 2nd order zero in the $\mathrm{e}^{\mathrm{i}\phi_1}$ correlator. The finite correlators of the c_I give rise to a (fractional) quantum Hall wavefunctions with 5 layers. We may also view the quantum Hall wavefunction as a wave function in 3 layers, where the particles in the first layer carry spin-1.

For such choice of the Gaussian model, the Gaussian model does not contribute to chiral central charge, does not change the number of topological types, and does not change the qauntum dimensions and spins of the topological excitations. The edge excitations of the constructed quantum Hall states are described by $SU(2)_k$ Kac-Moody algebra.

We see that the $N_c^B = 2_1^B$ topological order in Table I is described by $SU(2), k = 1$ Kac-Moody algebra, and we marked the entry as $(A_1, 1)$. Similarly, we marked entries (A_1, k) for orders N_c^B given by $(k+1)_{3k/k+2}^B$, $k = 2, \ldots 6$ in the corresponding tables.

B. Reductions of $SU(2)_k$ current algebra

A general affine Kac-Moody current algebra $X_l^{(1)}$ can be decomposed as a product of (generalised) parafermions times a $U(1)^l$ scalar field factor[38]. For the case of $SU(2)_k$ this gives the familiar $\mathbb{Z}_k$ parafermions with central charge $c_k = 2\frac{k-1}{k+2}$. The parafermions are simple currents, but in general they are neither bosonic nor fermionic.

In subsection IV B 1 we briefly review $\mathbb{Z}_k$ parafermions and their relation to $SU(2)_k$ current algebra.

Next we focus on orders $(A_1, k)_{\frac{1}{2}}$ for k odd, which contain half the number of fields of (A_1, k) and are realised at central charge $c = c(A_1, k) \pm 1$. Our notation follows

Ref. 12. We show how these reduced orders arise through a simple-current reduction.

In subsection IV B 3 we present the orders $(A_1, k)_{\frac{1}{4}}$ which employ a bosonic simple current that is part of the $SU(2)_k$ spectrum for $k = 4, 8, \ldots$. A subtle point is the occurrence of 'short orbits' of the simple-current action, which lead to multiplicities in the modular invariants[37,39]. The resolution of these multiplicities leads to novel modular S-matrices, which are in general not captured by Kac-Moody current algebra alone.

1. $\mathbb{Z}_k$ parafermions and (A_1, k) orders

The $\mathbb{Z}_k$ parafermion fields[35]

$$\psi_I, \quad I = 0, \cdots, k-1, \tag{77}$$

of conformal dimension $h_I = \frac{I(k-I)}{k}$, satisfy the operator algebra

$$\psi_I(z)\psi_J(w) \sim (z-w)^{s_{IJ}}\psi_{I+J} \tag{78}$$

with $s_{IJ} \equiv -\frac{2IJ}{k}$ mod 1. A general field in the parafermion theory is written as Φ_m^l, $l = 0, 1, \ldots, k$ and $m \in \mathbb{Z}$, with conformal dimension

$$s_{l,m} \equiv \frac{l(l+2)}{4(k+2)} - \frac{m^2}{4k} \quad \text{mod } 1. \tag{79}$$

The index m is periodic with period $2k$ and $m \equiv l$ mod 2. In addition we have the identification $\Phi_m^l = \Phi_{m+k}^{k-l}$. This leaves a total of $\frac{k(k+1)}{2}$ fields. All fields can be reached by acting with the parafermions $\psi_i = \Phi_{2i}^0$ on the primaries $\sigma_i = \Phi_i^i$, $i = 0, 1, \ldots k$. We also define $\epsilon_i = \Phi_0^{2i}$.

Using a single scalar field ϕ we can write the bosonic currents ($I = 0, 1, \ldots, k-1$, $j \in \mathbb{Z}$)

$$c_{I,j} = \psi_I \mathrm{e}^{\mathrm{i}k^{I,j}\phi} = \psi_I \mathrm{e}^{\mathrm{i}[I\sqrt{\frac{2}{k}}+j\sqrt{2k}]\phi}, \tag{80}$$

which have integer conformal dimension. The currents $c_{I=1,j=0}$ and $c_{I=k-1,j=-1}$ have conformal dimension 1. Together with $\mathrm{i}\partial\phi$ they generate a level-k affine Kac-Moody algebra $SU(2)_k$. With respect to the bosonic chiral algebra $c_{I,j}$ the following fields represent admissible topological excitations

$$\Phi_m^l \mathrm{e}^{\mathrm{i}[m\sqrt{\frac{1}{2k}}+j\sqrt{2k}]\phi} \tag{81}$$

with $j \in \mathbb{Z}$. The excitations with $l = m = 0, 1, \ldots k$ and $j = 0$ correspond to the highest weight states of the spin-l representations of $SU(2)_k$. They constitute a set of $k+1$ inequivalent primaries of the bosonic current algebra.

2. The orders $(A_1, k)_{\frac{1}{2}}$ with k odd

In the $SU(2)_k$ theory, the field Φ_k is a simple current with fusion rules

$$\Phi_k \Phi_l = \Phi_{k-l}, \quad l = 0, 1, \ldots k \ . \tag{82}$$

This simple current can be used for a number of simple current reductions of the order (A_1, k).

First assume that k is odd and of the form $k = 4p + 3$. We can form a product with $U(1)_1 \sim SU(2)_1$, and consider the bosonic simple currents

$$\Phi_0 e^{i\frac{(2j)}{\sqrt{2}}\phi'}, \quad \Phi_k e^{i\frac{(2j+1)}{\sqrt{2}}\phi'}, \quad j, j' \in \mathbb{Z}. \quad (83)$$

The primary sectors with respect to these currents are

$$\Phi_l e^{i\frac{(l+2j)}{\sqrt{2}}\phi'}, \quad l = 0, 1, \ldots \frac{k-1}{2}, \quad j \in \mathbb{Z}. \quad (84)$$

They form the excitations of the reduced order $(A_1, k)_{\frac{1}{2}}$ at $c = 2\frac{2k+1}{k+2}$ and $N = (k+1)/2$. In formula we have

$$(A_1, k)_{\frac{1}{2}} = [(A_1, k) \otimes U(1)_1]_{\frac{1}{4}}, \quad k = 3, 7, \ldots \quad (85)$$

For k of the form $k = 4p + 1$ one needs instead a factor $U(1)_1^* \sim SU(2)_1^*$ with $c = -1$ and non-trivial primary at $s = -\frac{1}{4}$,

$$(A_1, k)_{\frac{1}{2}} = [(A_1, k) \otimes U(1)_1^*]_{\frac{1}{4}}, \quad k = 5, 9, \ldots \quad (86)$$

It is instructive to re-examine these same reductions starting from $\mathbb{Z}_k$ parafermions ψ_I, $I = 0, \ldots k - 1$, and the two scalar fields ϕ, ϕ'. For $k = 3$ and with respect to the basis

$$\phi_1 = \sqrt{\frac{2}{3}}\phi, \quad \phi_2 = \sqrt{\frac{1}{6}}\phi + \sqrt{\frac{1}{2}}\phi' \quad (87)$$

the metric becomes

$$G^{(3)} = \begin{pmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} \end{pmatrix}. \quad (88)$$

Writing $V_{\boldsymbol{k}}$ for $e^{i\boldsymbol{k}\cdot\boldsymbol{\phi}}$, we can write bosonic currents

$$c_{I,\boldsymbol{k}^I} = \psi_I V_{\boldsymbol{k}^I} \quad (89)$$

where k_1^I, k_2^I are integers satisfying $2k_1^I + k_2^I \equiv 2I \mod 3$. The admissible topological excitations become

$$\Phi_m^1 V_{\boldsymbol{k}} \quad \text{with} \quad 2k_1 + k_2 \equiv m \mod 3. \quad (90)$$

Note that $\Phi_1^1 = \sigma_1$, $\Phi_3^1 = \epsilon_1$ and $\Phi_5^1 = \sigma_2$. The fields $\Phi_m^1 V_{\boldsymbol{k}}$ form a single primary sector, with conformal dimension $s = \frac{2}{5}$ and quantum dimension $d = \zeta_3^1$, and we recover the order $(A_3, 1)_{\frac{1}{2}} = 2_{\frac{14}{5}}^B$.

For general $k = 4p + 3$, the 2-scalar metric becomes

$$G^{(k)} = \begin{pmatrix} \frac{2}{k} & \frac{1}{k} \\ \frac{1}{k} & \frac{1+k}{2k} \end{pmatrix}. \quad (91)$$

The bosonic currents are

$$c_{I,\boldsymbol{k}^I} = \psi_I V_{\boldsymbol{k}^I} \quad (92)$$

with $2k_1^I + k_2^I \equiv 2I \mod k$ and the primaries are

$$\Phi_m^l V_{\boldsymbol{k}} \quad \text{with} \quad 2k_1 + k_2 \equiv m \mod k, \quad (93)$$

with $l = 1, 2, \ldots, \frac{k-1}{2}$ and quantum dimensions $d = \zeta_k^l$.

In this notation, the underlying $SU(2)_k \times SU(2)_1$ current algebra is formed by

$$\psi_1 V_{(1\ 0)}, \ \psi_{k-1} V_{(-1\ 0)} \ ; \quad V_{(-1\ 2)}, \ V_{(1\ -2)}; \quad (94)$$

together with the fields $i\partial\phi$. Odd-l primaries under $SU(2)_k$ are doublets under the $SU(2)_1$, while even-l primaries are singlets.

For $k = 3$ there is even more symmetry. The following currents have conformal dimension equal to 1

$$\psi_1 V_{(1\ 0)}, \ \psi_1 V_{(-1\ 1)}, \ \psi_1 V_{(0\ -1)},$$
$$\psi_2 V_{(-1\ 0)}, \ \psi_2 V_{(1\ -1)}, \ \psi_2 V_{(0\ 1)},$$
$$V_{(2\ -1)}, \ V_{(1\ 1)}, \ V_{(-1\ 2)},$$
$$V_{(1\ -2)}, \ V_{(-1\ -1)}, V_{(-2\ 1)}. \quad (95)$$

Together with $i\partial\boldsymbol{\phi}$ these form the (14-dimensional) current algebra of $G_2^{(1)}$. The excitations, all of conformal dimension $s = \frac{2}{5}$,

$$\epsilon_1, \ \sigma_1 V_{(-1\ 0)}, \ \sigma_1 V_{(1\ -1)}, \ \sigma_1 V_{(0\ 1)},$$
$$\sigma_2 V_{(1\ 0)}, \ \sigma_2 V_{(-1\ 1)}, \ \sigma_2 V_{(0\ -1)} \quad (96)$$

form the 7-dimensional representation of G_2. Thus, the $(G_2, 1)$ simple current algebra can also produce the topological order $2_{\frac{14}{5}}^B$.

For $k = 4p + 1$ the 2-scalar metric can be picked as

$$G'^{(k)} = \begin{pmatrix} \frac{2}{k} & \frac{1}{k} \\ \frac{1}{k} & \frac{1-k}{2k} \end{pmatrix}. \quad (97)$$

Note that the metric $G'^{(k)}$ has determinant $\det G'^{(k)} = -\frac{1}{k}$, whereas $\det G^{(k)} = \frac{1}{k}$. This implies that for $k = 4p + 1$ the 2-scalar sector adds $1 + (-1) = 0$ to the total central charge, in agreement with eq. (86). The currents

$$V_{(-1\ 2)}, \ V_{(1\ -2)}, \quad (98)$$

have conformal dimension -1 and generate the algebra $SU(2)_1^*$.

3. The orders $(A_1, k)_{\frac{1}{4}}$, $k = 4, 8, \ldots$

For $k = 4p$ the simple current Φ_k is bosonic and can be added to the currents of the $SU(2)_1$ Kac Moody algebra. In this situation, there exists a modular invariant partition function, labeled as $\mathcal{D}_{\frac{k}{2}+2}$, which only features the even-l primaries (see e.g. Ref. 44)

$$\mathcal{D}_{\frac{k}{2}+2}: \quad Z_k = \sum_{l=0,2,\ldots}^{\frac{k-4}{2}} |\chi_l + \chi_{k-l}|^2 + 2|\chi_{\frac{k}{2}}|^2. \quad (99)$$

Corresponding to this partition function is a bosonic topological order with $N = \frac{k}{4} + 2$, which we denote as $(A_1, k)_{\frac{1}{4}}$.

The quantum dimensions of the fields Φ_l, $l = 0, 2, \ldots \frac{k-4}{2}$ are simply ζ_k^l. The theory features two fields $\Phi_{\frac{k}{2}}^{(1)}$ and $\Phi_{\frac{k}{2}}^{(2)}$, which need to be 'resolved' in the modular S-matrix[37,39]. The result is that the two fields share the total quantum dimension $\zeta_k^{\frac{k}{2}}$, leading to twice a value $\frac{1}{2}\zeta_k^{\frac{k}{2}}$.

This construction for $k = 4$ reproduces the abelian order at $N_c^B = 3_2^B$, while for $k = 8$ we reproduce the order at $N_c^B = 4_{12/5}^B$ (we used $\zeta_8^2 = (\zeta_3^1)^2$ and $\zeta_8^4 = 2\zeta_3^1$).

The case $\mathcal{D}_8$ at $k = 12$ gives $5_{18/7}^B$ with

$$d = 1, \ \frac{1}{2}\zeta_{12}^6, \ \frac{1}{2}\zeta_{12}^6, \ \zeta_{12}^2, \ \zeta_{12}^4$$
$$s = 0, \ -\frac{1}{7}, \ -\frac{1}{7}, \ \frac{1}{7}, \ \frac{3}{7}. \tag{100}$$

Using $\zeta_5^2 = \frac{1}{2}\zeta_{12}^6$ we find a perfect match with the entry in the Table II.

Similarly, the entry at $6_{8/3}^B$ in Table III is found to agree with the order $(A_1, 16)_{\frac{1}{4}}$. Note that $\zeta_7^3 = \frac{1}{2}\zeta_{16}^8 = \zeta_{16}^2$, revealing a triple degeneracy in the primary sectors. This hints at an alternative interpretation, which we obtain in section IV C 3.

4. The $\mathbb{Z}_8$ operation T_8 for (A_1, k), $k = 2, 6, \ldots$

Inspecting the case $k = 2, 6, \ldots$, we find that the simple current Φ_k gives rise to yet another type of simple-current reduction. In this case, an appropriate scalar field factor is $U(1)_2$, which is the order 4_1^B. Constructing the order

$$[(A_1, k) \otimes U(1)_2]_{\frac{1}{4}}, \quad k = 2, 6, \ldots \tag{101}$$

we arrive at $N = k+1$, $c = \frac{3k}{k+2} + 1$, whereas the starting point (A_1, k) corresponded to $N = k + 1$, $c = \frac{3k}{k+2}$. This reduction is thus an example of the operation T_8, which we defined in more general terms in the equation (70).

C. Affine Kac-Moody algebras of higher rank

We can repeat the analysis for the $SU(2)$ case for the affine Kac-Moody extension $X_l^{(1)}$ of all simple Lie algebras. As is well known, these have been classified as four regular series A_l, B_l, C_l, and D_l, $l = 1, 2, \ldots$ plus five exceptional algebras E_6, E_7 and E_8, F_4 and G_2. This leads to many more examples of bosonic orders of low rank, which we have marked in the tables. Note that $C_2 \sim B_2$, $D_2 \sim A_1 \times A_1$, $D_3 \sim A_3$ and . In the tables we have displayed c modulo 8 and conformal dimensions s_i modulo 1.

A tentative list of simple-current primaries in the $X_l^{(1)}$ Kac-Moody current algebras has been given in[37]. As for the $SU(2)_k$ case, these give rise to a variety of simple-current reductions of the order (X_l, k).

For (A_n, k) a reduction by a factor $\mathbb{Z}_{n+1}$ is possible if g.c.d.$(n + 1, k) = 1$ (see Ref. 12), leading to orders $(A_n, k)_{\frac{1}{n+1}}$. Below we discuss the cases with $n = 2$ and the general case with level $k = 2$ and n even. We remark that other reductions involving additional abelian factors are possible, such as a reduction $(A_3, 2)_{\frac{1}{2}}$ which leads to the order $5_{-2}^{B,a}$.

A second class are reductions based on bosonic simple-current primaries. Below we present the case of $(A_2, k)_{\frac{1}{9}}$.

1. $(A_2, k)_{\frac{1}{3}}$ for $k = 2, 4, 5, 7, \ldots$

For $k = 3p + 2$ this reduction can concisely be written as

$$(A_2, k)_{\frac{1}{3}} = [(A_2, k) \otimes (A_2, 1)]_{\frac{1}{9}}, \quad k = 2, 5, \ldots \tag{102}$$

For $k = 2$ this reduces the order $(A_2, 2) = 6_{16/5}^B$ to $(A_2, 2)_{\frac{1}{3}} = 2_{-14/5}^B$.

One can re-examine this reduction in terms of the $SU(3)_2$ parafermions and four scalar fields. For $k = 3p+2$ the scalar field metric reads, in a convenient basis

$$G^{(k)} = \frac{1}{k}\begin{pmatrix} 2 & 1 & 2 & 2 \\ 1 & 2 & 2 & 3 \\ 2 & 2 & 4+2p & 4+p \\ 2 & 3 & 4+p & 6+2p \end{pmatrix}. \tag{103}$$

The currents

$$\Phi_{(\pm 2 \ \mp 1)}^{(0 \ 0)} V_{(\pm 1 \ 0 \ 0 \ 0)},$$
$$\Phi_{(\mp 1 \ \pm 2)}^{(0 \ 0)} V_{(0 \ \mp 1 \ 0 \ 0)},$$
$$\Phi_{(\pm 1 \ \pm 1)}^{(0 \ 0)} V_{(\pm 1 \ \mp 1 \ 0 \ 0)}, \tag{104}$$

together with two scalars $i\partial\phi$, form an $SU(3)_k$ current algebra. In addition,

$$V_{(\mp 1 \ 0 \ \pm 2 \ \mp 1)}, \ V_{(0 \ \mp 2 \ \mp 1 \ \pm 2)}, \ V_{(\mp 1 \ \mp 2 \ \pm 1 \ \pm 1)} \tag{105}$$

together with the other two scalars form an $SU(3)_1$.

For $k = 2$ the lattice defined by the matrix $G^{(2)}$ admits a total of 8 'short' integral vectors $\boldsymbol{k}_i^S$, with $\boldsymbol{k}_i^S \cdot \boldsymbol{k}_i^S = 1$, as well as 24 'long' integral vectors $\boldsymbol{k}_i^L$, with $\boldsymbol{k}_i^L \cdot \boldsymbol{k}_i^L = 2$. In fact, one recognizes in $G^{(2)}$ the metric of the $SO(9)$ weight lattice. Combining the integral vectors with a single Ising fermion (which is the parafermion for $SO(9)_1$), one can write a total of $24 + 8 + 4 = 36$ bosonic currents, which form the $SO(9)_1$ Kac-Moody current algebra. Combining these same vectors with the $SU(3)_2$ parafermions, which include three fields of conformal dimension $s = \frac{1}{2}$, leads to a total of $24 + 3 \times 8 + 4 = 52$

bosonic currents, which form the Kac-Moody algebra for $F_4^{(1)}$ at level 1. Combining these same vectors with the $SU(3)_2$ parafermion spin fields, one can construct 26 fields of dimension $s = \frac{3}{5}$, which form an irreducible representation under F_4 and together constitute the single non-trivial primary sector of the topological order $N_c^B = 2_{-14/5}^B$.

For $k = 3p + 1$, the reduction becomes

$$(A_2, k)_{\frac{1}{3}} = [(A_2, k) \otimes (A_2, 1)^*]_{\frac{1}{9}}, \quad k = 4, 7, \ldots \quad (106)$$

We checked that for $k = 4$ the quantum dimensions and spins of this reduced order match with the entry $N_c^B = 5_{18/7}^B$ in Table II.

2. $(A_n, 2)_{\frac{1}{n+1}}$ for $n = 2, 4, \ldots$

For $n = 2, 6, \ldots$, this reduction can be implemented as

$$(A_n, 2)_{\frac{1}{n+1}} = [(A_n, 2) \otimes (\phi_1, \phi_2)]_{\frac{1}{n+1}}, \quad n = 2, 6, \ldots \quad (107)$$

with the scalar field metric given by (91) with $k = n + 1$. The field content becomes

$$\phi_{(l_1 \ l_2 \ \ldots \ l_n)} V_{\mathbf{k}} \quad (108)$$

where the l_j are the Dynkin labels of the A_n representation carried by $\phi_{(l_1 \ l_2 \ \ldots \ l_n)}$ and

$$2k_1 + k_2 \equiv \sum_{j=1}^{n} j l_j \mod n + 1. \quad (109)$$

This reduction adds $+2$ to the central charge. For $n = 4, 8, \ldots$, one uses instead the metric (97) and the central charge remains unchanged.

We observe that there is a duality between the orders $(A_1, k)_{\frac{1}{2}}$ and $(A_{k-1}, 2)_{\frac{1}{k}}$, in the sense that they form a pair (N_c^B, N_{-c}^B) with identical quantum dimensions d_i and opposite spins s_i. This duality is a manifestation of the well-known level-rank duality between $SU(2)_k$ and $SU(k)_2$.

Other manifestations of level-rank duality are the pair $(A_1, 4)$ and $(A_3, 2)_{\frac{1}{2}}$ and the pair $(A_2, 4)_{\frac{1}{3}}$ and $(A_3, 3)_{\frac{1}{4}}$, both with rank $N = 5$.

3. $(A_2, k)_{\frac{1}{9}}$ for $k = 3, 6, \ldots$

For $k = 3p$, the $SU(3)_k$ primaries with weight $(k0)$ and $(0k)$ are bosonic simple currents. They lead to an exceptional modular invariant, labeled $\mathcal{D}_k$ in the classification of Ref. 45. These exceptional invariants only include fields with triality zero, $l_1 + 2l_2 \equiv 0 \mod 3$.

For $k = 3$ the partition function is

$$Z_3 = |\chi_{(00)} + \chi_{(30)} + \chi_{(03)}|^2 + 3|\chi_{(11)}|^2. \quad (110)$$

The corresponding order has 4 fields: the identity and 3 fields originating from $\phi_{(11)}$, with $d_i = 1$, $s_i = \frac{1}{2}$. The value $d_i = 1$ arises via equal distribution of the quantum dimension $d[\phi_{(11)}] = 3$. We recognize the entry $N_c^B = 4_4^B$.

For $k = 6$, $c = \frac{16}{3}$, the modular invariant reads

$$
\begin{aligned}
Z = &|\chi_{(00)} + \chi_{(60)} + \chi_{(06)}|^2 \\
&+ |\chi_{(11)} + \chi_{(41)} + \chi_{(14)}|^2 \\
&+ |\chi_{(33)} + \chi_{(30)} + \chi_{(03)}|^2 + 3|\chi_{(22)}|^2.
\end{aligned} \quad (111)
$$

The weight $(2\,2)$, with $s = -\frac{1}{9}$, comes in with multiplicity 3 and quantum dimension $3\zeta_7^3$ - after resolution into 3 primaries this leads to the values $d_i = \zeta_7^3$. The data for the other sectors are

$$(0\,0), \ (6\,0), \ (0\,6) : s = 0, \ d = 1$$

$$(3\,3), \ (3\,0), \ (0\,3) : s = -\frac{1}{3}, \ d = \frac{\sin[\frac{4\pi}{7}]\sin[\frac{4\pi}{7}]}{\sin[\frac{\pi}{7}]\sin[\frac{2\pi}{7}]} = \zeta_{16}^4$$

$$(1\,1), \ (4\,1), \ (1\,4) : s = \frac{1}{3}, \ d = \frac{\sin[\frac{2\pi}{7}]\sin[\frac{4\pi}{7}]}{\sin[\frac{\pi}{7}]\sin[\frac{\pi}{7}]} = \zeta_{16}^6, \quad (112)$$

all in agreement with the data for the entry $6_{-8/3}^B$.

For general $k = 3p$, the rank of the order $(A_2, k)_{\frac{1}{9}}$ is $N = (k^2 + 3k)/18 + 3$.

4. The $\mathbb{Z}_2$ operation T_2

Inspecting the table II of rank-5 orders, we observe that $5_{\pm 2}^{B,a}$ derive directly from Kac-Moody current algebra, but $5_{\pm 2}^{B,b}$ do not. We remark that the orders $5_{\pm 2}^{B,b}$ arise through a simple-current reduction of the product of $5_{\pm 2}^{B,a}$ with $4_0^{B,b}$,

$$5_{\pm 2}^{B,b} = [5_{\pm 2}^{B,a} \otimes 4_0^{B,b}]_{\frac{1}{4}}. \quad (113)$$

This is a special case of the operation T_2 defined in eqn. (69). Similar doublets under the action of T_2 are $(6_0^{B,a}, 6_0^{B,b})$, $(6_4^{B,a}, 6_4^{B,b})$, and $(7_{\pm 2}^{B,b}, 7_{\pm 2}^{B,c})$.

5. More general reductions

We already mentioned that simple-current reductions of products of non-abelian orders are possible. While these are not needed to reproduce the $N \leq 7$ orders that we list in this paper, they are needed to cover such cases as minimal models of the Virasoro or $\mathcal{W}_n$ algebras, which are understood via a coset construction[44,46]. The idea is that a coset G/H is viewed as $G \times H^{-1}$ and that the corresponding order can be obtained as a simple-current

reduction of the product of orders G and H^*. As a concrete example, consider the coset

$$\frac{SU(2)_3 \times SU(2)_1}{SU(2)_4}, \tag{114}$$

which describes the $c = \frac{4}{5}$ unitary minimal model of the Virasoro algebra, of rank $N = 10$. Inspecting Table II, we see that the role of $(A_4, 1)^*$ can be played by $(C_4, 1)$. We therefore consider the product

$$(C_4, 1) \otimes (A_1, 3) \otimes (A_1, 1) \tag{115}$$

and pick as additional bosonic simple current the field

$$\Phi_{(0001)} \times \Phi_3 \times e^{i \frac{\phi}{\sqrt{2}}}. \tag{116}$$

Of the $5 \times 4 \times 2 = 40$ fields in the product theory, 20 are primary with respect to the extended simple-current algebra, and these organize into orbits of length 2. We thus recover the $N = 10$ primary sectors of the minimal model.

V. SUMMARY

In this paper, we use simple-current algebra to construct many-body wave functions for 2+1D bosonic topological orders. We found that simple-current algebra can produce all the simple topological orders. This supports the conjecture that all the (non-)abelian statistics described by MTC can be realized by bosonic systems. It also suggests that, in a certain sense, simple-current algebra can be classified by MTC.

The simple-current reduction is an important tool in our constructions. Such reductions correspond to the condensation of bosonic topological excitations.[39–43] So the simple-current reduction is also a tool to study the condensation of bosonic topological excitations and the induced topological phase transition between the original topological order and the reduced topological order.

Appendix A: CFT of Kac-Moody current algebra

The starting point for the construction of a CFT based on Kac-Moody current algebra is a simple Lie algebra X_l plus a positive integer k (which is called the level of the Kac-Moody current algebra). In this appendix we briefly review the connection between CFT and Kac-Moody current algebra and specify some of the data needed to identify key properties of the CFT.

1. Root and weight lattices of finite dimensional Lie algebras

In the structure theory of simple Lie algebras, it is common to choose a Cartan-Weyl basis $\{h^i, e_\alpha\}$, where the h^i, $i = 1, \ldots, l$, form a basis of the Cartan sub-algebra $\mathcal{H}$, and the e_α are ladder operators for the roots $\alpha = (\alpha^1, \ldots, \alpha^l)$,

$$[h^i, e_\alpha] = \alpha^i e_\alpha. \tag{A1}$$

The Killing form

$$K^{ab} = \text{Tr}(\text{ad}(J^a)\text{ad}(J^b)), \tag{A2}$$

leads to an inner product in the root space $\mathcal{H}^*$

$$(\alpha, \beta) = \sum_{ij} K_{ij} \alpha^i \beta^j, \quad \sum_j K_{ij} K^{jk} = \delta_{ik}. \tag{A3}$$

Integral linear combinations of the roots α form the so-called root-lattice associated with X_l. For a choice of simple roots α_i, which form a basis of the root lattice, one defines the Cartan matrix is

$$A_{ij} = 2\frac{(\alpha_i, \alpha_j)}{(\alpha_j, \alpha_j)}. \tag{A4}$$

Dual to the root lattice is the weight lattice, which plays a crucial role in a systematic description of the irreducible representations of X_l. Its elements Λ can be characterised by the Dynkin labels

$$l_i = 2\frac{(\Lambda, \alpha_i)}{(\alpha_i, \alpha_i)}. \tag{A5}$$

The weight is then written as a linear combination of fundamental weights $\Lambda = \sum_i l_i \Lambda_i$, where the fundamental weights have inner product

$$(\Lambda_i, \Lambda_j) = G_{ij} \tag{A6}$$

with

$$G_{ij} = (A^{-1})_{ij} \frac{(\alpha_j, \alpha_j)}{2}. \tag{A7}$$

2. Primaries of Kac-Moody current algebra

The CFT associated with Lie algebra X_l and level k is characterised by a larger symmetry algebra, which is the so-called affine Kac-Moody extension or current algebra $X_l^{(1)}$ of X_l at level k. The central charge of this CFT can be expressed as

$$c = \frac{kD}{k+g} \qquad (A8)$$

where D is the dimension of X_l and g is the dual Coxeter number. In table VI we list these data for the simple Lie algebras X_l.

The primary sectors of the current algebra CFT are labeled by particular weights Λ - the so-called dominant integral weights. Their Dynkin labels satisfy $l_j \geq 0$ and

$$\sum_{j=1}^{l} l_j \, a_j^\vee \leq k, \qquad (A9)$$

where $a_j^\vee$ is the comark (or dual Kac label) to the root α_j[47].

The conformal dimension (spin) of the primary sector labeled by Λ is given by

$$s_\Lambda = \frac{(\Lambda, \Lambda + 2\rho)}{2(k+g)} \qquad (A10)$$

where $\rho = \sum_i \Lambda_i$ is the sum of the fundamental weights.

The S-matrix is given by

$$S_{\Lambda\Lambda'} \propto \sum_{w \in W} \mathrm{sign}(w) \, \mathrm{e}^{-\frac{2\pi i}{k+g}(w(\Lambda+\rho),\Lambda'+\rho)} \qquad (A11)$$

where the summation is over the Weyl group of X_l. Via the relation

$$d_i = \frac{S_{0i}}{S_{00}} \qquad (A12)$$

this S-matrix fixes the quantum dimensions d_i.

We refer to Ref. 44 for further details. Here, for the sake of illustration, we present such details for the rank-2 algebras A_2 (or $su(3)$), B_2 (or $so(5)$) and G_2.

3. The rank 2 simple Lie algebras

a. The algebra A_2

For this Lie algebra the weight-lattice metric G_{ij} is given by

$$G = \begin{pmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} \end{pmatrix}. \qquad (A13)$$

TABLE VI: Dimension D and dual Coxeter number g of the simple Lie algebras X_l.

algebra		D	g
A_l	$l \geq 1$	$l(l+2)$	$l+1$
B_l	$l \geq 2$	$l(2l+1)$	$2l-1$
C_l	$l \geq 3$	$l(2l+1)$	$l+1$
D_l	$l \geq 4$	$l(2l-1)$	$2l-2$
E_6		78	12
E_7		133	18
E_8		248	30
F_4		52	9
G_2		14	4

With respect to an othonormal basis $\mathbf{e}_i$, the fundamental weights can be written as

$$\Lambda_1 = \sqrt{\frac{2}{3}} \mathbf{e}_2, \quad \Lambda_2 = \frac{1}{\sqrt{2}} \mathbf{e}_1 + \frac{1}{\sqrt{6}} \mathbf{e}_2, \qquad (A14)$$

and the positive roots are

$$\alpha_1 = -\Lambda_1 + 2\Lambda_2, \quad \alpha_2 = 2\Lambda_1 - \Lambda_2,$$
$$\alpha_{12} = \Lambda_1 + \Lambda_2, \qquad (A15)$$

which we write as $(-1\,2)$, $(2\,-1)$ and $(1\,1)$, respectively. The Weyl group has 6 elements, the orbit of $\rho = (1\,1)$ is

$$\mathrm{sign}(w) = +1: \quad (1\,1), \ (1\,-2), \ (-2\,1)$$
$$\mathrm{sign}(w) = -1: \quad (2\,-1), \ (-1\,2), \ (-1\,-1). \quad (A16)$$

Dominant integral weights at level k satisfy $l_1 + l_2 \leq k$, their number is $N_k = (k+1)(k+2)/2$. The conformal and quantum dimensions for the primary $(l_1\,l_2)$ are given by

$$s_{(l_1\,l_2)} = \frac{l_1^2 + l_2^2 + l_1 l_2 + 3l_1 + 3l_2}{3(k+3)},$$

$$d_{(l_1\,l_2)} = \frac{\sin[\frac{\pi(l_1+1)}{k+3}] \sin[\frac{\pi(l_2+1)}{k+3}] \sin[\frac{\pi(l_1+l_2+2)}{k+3}]}{\sin[\frac{\pi}{k+3}] \sin[\frac{\pi}{k+3}] \sin[\frac{2\pi}{k+3}]}. \quad (A17)$$

The central charges are $c = \frac{8k}{k+3}$ for the $SU(3)_k$ CFT and $c_k = 6\frac{k-1}{k+3}$ for the corresponding parafermions.

b. The algebra B_2

For this Lie algebra the weight-lattice metric B_{ij} is given by

$$G = \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}. \qquad (A18)$$

With respect to an othonormal basis $\mathbf{e}_i$, the fundamental weights can be written as

$$\Lambda_1 = \frac{1}{\sqrt{2}}\mathbf{e}_1 + \frac{1}{\sqrt{2}}\mathbf{e}_2, \quad \Lambda_2 = \mathbf{e}_1. \quad \text{(A19)}$$

The simple roots are

$$\alpha_1 = -\Lambda_1 + 2\Lambda_2, \quad \alpha_2 = -2\Lambda_1 + 2\Lambda_2. \quad \text{(A20)}$$

The four positive roots are

$$-\Lambda_1 + 2\Lambda_2, \ -2\Lambda_1 + 2\Lambda_2, \Lambda_1, \ 2\Lambda_2 \ . \quad \text{(A21)}$$

The Weyl group has 8 elements, the orbit of $\rho = (1\ 1)$ is

$$\text{sign}(w) = +1: \ (1\ 1),\ (2\ -3),\ (-1\ -1),(-2\ 3)$$
$$\text{sign}(w) = -1: \ (-1\ 3),\ (-2\ 3),\ (1\ -3),(2\ -3)\ . \quad \text{(A22)}$$

Dominant integral weights at level k satisfy $l_1 + l_2 \leq k$. Their number is $N_k = (k+1)(k+2)/2$ and the conformal dimensions are given by

$$s_{(l_1\ l_2)} = \frac{2l_1^2 + l_2^2 + 2l_1 l_2 + 6l_1 + 4l_2}{4(k+3)} \ . \quad \text{(A23)}$$

The central charges are $c = \frac{10k}{k+3}$ for the $SO(5)_k$ CFT and $c_k = 2\frac{k-3}{k+3}$ for the corresponding parafermions.

c. The algebra G_2

The weight-lattice metric G_{ij} is given by

$$G = \begin{pmatrix} 2 & 1 \\ 1 & \frac{2}{3} \end{pmatrix}. \quad \text{(A24)}$$

With respect to an othonormal basis $\mathbf{e}_i$, the fundamental weights can be written as

$$\Lambda_1 = \sqrt{2}\mathbf{e}_2, \quad \Lambda_2 = \frac{1}{\sqrt{6}}\mathbf{e}_1 + \frac{1}{\sqrt{2}}\mathbf{e}_2, \quad \text{(A25)}$$

and the simple roots are

$$\alpha_1 = -\Lambda_1 + 2\Lambda_2, \quad \alpha_2 = 2\Lambda_1 - 3\Lambda_2. \quad \text{(A26)}$$

The six positive roots are

$$-\Lambda_1 + 2\Lambda_2,\ 2\Lambda_1 - 3\Lambda_2,$$
$$\Lambda_1,\ \Lambda_2,\ \Lambda_1 - \Lambda_2,\ -\Lambda_1 + 3\Lambda_2\ . \quad \text{(A27)}$$

The Weyl group has 12 elements. The orbit of $\rho = (1\ 1)$ is

$$w = +1: \ (1\ 1),\ (-2\ 5),\ (-3\ 4),$$
$$\qquad\qquad (-1\ -1),\ (2\ -5),\ (3\ -4)$$
$$w = -1: \ (2\ -1),\ (-1\ 4),\ (-3\ 5),$$
$$\qquad\qquad (-2\ 1),\ (1\ -4),\ (3\ -5). \quad \text{(A28)}$$

Dominant integral weights at level k satisfy $2l_1 + l_2 \leq k$. Their conformal dimensions are given by

$$s_{(l_1\ l_2)} = \frac{3l_1^2 + l_2^2 + 3l_1 l_2 + 9l_1 + 5l_2}{3(k+4)}. \quad \text{(A29)}$$

The central charges are $c = \frac{14k}{k+4}$ for the $(G_2)_k$ WZW model and $c_k = 4\frac{3k-2}{k+4}$ for the corresponding parafermions.

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