Order by disorder and by doping in quantum Hall valley ferromagnets

Akshay Kumar, S. A. Parameswaran, and S. L. Sondhi

Phys. Rev. B 93, 014442 — Published 28 January 2016

DOI: 10.1103/PhysRevB.93.014442
Order by Disorder and by Doping in Quantum Hall Valley Ferromagnets

Akshay Kumar,1,2 S. A. Parameswaran,3 and S. L. Sondhi1,2,*

1Department of Physics, Princeton University, Princeton, NJ 08544, USA
2Max-Planck-Institut für Physik Komplexer Systeme, Dresden 01187, Germany
3Department of Physics and Astronomy, University of California, Irvine, CA 92697, USA

(Dated: November 23, 2015)

We examine the Si(111) multi-valley quantum Hall system and show that it exhibits an exceptionally rich interplay of broken symmetries and quantum Hall ordering already near integer fillings \( \nu \) in the range \( \nu = 0 - 6 \). This six-valley system has a large \( [SU(2)]^3 \times D_3 \) symmetry in the limit where the magnetic length is much larger than the lattice constant. We find that the discrete \( D_3 \) factor breaks over a broad range of fillings at a finite temperature transition to a discrete nematic phase. As \( T \to 0 \) the \( [SU(2)]^3 \) continuous symmetry also breaks: completely near \( \nu = 3 \), to a residual \( [U(1)]^2 \times SU(2) \) near \( \nu = 2 \) and 4 and to a residual \( U(1) \times [SU(2)]^2 \) near \( \nu = 1 \) and 5. Interestingly, the symmetry breaking near \( \nu = 2, 4 \) and \( \nu = 3 \) involves a combination of selection by thermal fluctuations known as “order by disorder” and a selection by the energetics of Skryme lattices induced by moving away from the commensurate fillings, a mechanism we term “order by doping”.

I. INTRODUCTION

Even with the recent explosive increase in the number of interesting condensed matter systems such as topological insulators and ultracold atoms, the venerable quantum Hall effect (QHE) continues to hold its own, owing to its exceptional tunability coupled with the unique physics of QH states. The latter allows different forms of topological order to exist under a potentially infinite set of conditions involving different degrees of commensuration between particle and magnetic flux density, captured by the filling factor \( \nu \). The former allows several energy scales to be varied largely independently and — most relevant for this paper — admits the possibility of engineering multi-component QH systems which can then exhibit an interesting interplay between broken symmetries and topological order—as in the phenomenon of quantum Hall ferromagnetism[13–16].

In this paper, we revisit this interplay in the context of the QH states of multi-valley semiconductors, specifically those recently observed in two-dimensional electron gases (2DEGs) confined in Si(111) quantum wells.12,13 We find several striking new phenomena embedded in surprisingly intricate phase diagrams — even while considering only integer quantum Hall states in the lowest Landau level (LLL). This system exhibits six-fold valley degeneracy in the electronic dispersions, as shown in Figure 1. Consequently, it exhibits large symmetry group — \( [SU(2)]^3 \times D_3 \) — in the standard limit where the magnetic length \( \ell_B \) is much longer than the lattice constant \( a \). Here, \( D_n \) is the dihedral group of symmetries of a regular \( n \)-gon, and the semidirect product structure (denoted by \( \rtimes \)) reflects the fact that these discrete symmetries act upon the \( SU(2) \) axes. The rich phase structure derives from the various possibilities for breaking these symmetries, and how these manifest at different \( \nu \). For our primary example—the (111) system—we find a finite-temperature \( Z_3 \) transition into a nematic phase where the discrete factor is broken, and zero-temperature phases where the continuous \( [SU(2)]^3 \) symmetry is broken down to various subgroups. We sketch the phase diagram resulting from fitting together these possibilities in Fig. 1.

The mechanisms of symmetry breaking are also unusual. While the nematicity is driven by Hartree-Fock exchange interactions as is standard in QH ferromagnetism13, the \( T \to 0 \) ordering involves entropic selection — order by disorder14— and selection via the energetics of Skryme lattices that form in the vicinity of integer \( \nu \), a new mechanism that we term order by doping in tribute to its entropic cousin. To our knowledge, this is the first time either selection mechanism has been shown to operate in the QHE setting.

II. SILICON(111)

We begin our discussion by listing some salient features of Si(111) quantum wells relevant to understanding the QHE in
these systems. The valley degeneracy of the Si 2DEG depends on the orientation of the interface, as this choice can break the crystal symmetries responsible for the exact valley degeneracy in bulk Si. (We ignore spin.) In case of the (111) interface, effective mass theory predicts a six-fold degeneracy,\(^2\) (Fig. [1] inset). This degeneracy is quite robust — for instance, it cannot be lifted by changing the width of the confining well or by an interface potential. For the bulk of this paper, we take this degeneracy to be exact, surely an idealization; we comment on corrections to this scenario at the end.

We label valleys as shown in Fig. [1](inset). Valley \( \kappa \) is centered at \( \vec{K}_\kappa \), where \( \vec{K}_A = (\sqrt{3}K_0, K_0, 0) \), \( \vec{K}_B = (0, K_0) \) and \( \vec{K}_C = (-\sqrt{3}K_0, K_0, 0) \), with \( K_0 = K_\kappa = -K_\kappa \). Here \( K_0 \approx 1/a \) where \( a \) is the lattice constant \( K_0 = \sqrt{2/3}D_m \), where \( D_m \) is the distance in K-space within the Brillouin zone from the \( \Gamma \) point to the minimum-energy point in the conduction band. Note that in each valley the effective mass tensor is anisotropic; this is most evident in a coordinate system in which the mass tensor is diagonal. The single-particle Hamiltonian in valley \( \kappa \) (where \( \kappa = A, B, C \)) is \( H_\kappa = \sum_{i=1,2} \frac{(\vec{p}_i + \vec{A}_i - \vec{K}_\kappa) \cdot \vec{\eta}_i}{2m_i} \), where \( \vec{\eta}_A/C_1 = \frac{1}{2} (\mp 1, \sqrt{3}, 0) \), \( \vec{\eta}_{B1} = (1, 0) \) and \( \vec{\eta}_{B2} = (0, 1) \); \( H_\kappa \) is obtained by taking \( K_0 \rightarrow -K_\kappa \) in these expressions.

In Landau gauge \( \vec{A} = (0, Bx) \), the LLL eigenfunctions labeled by momentum \( \kappa_y \) are given by
\[
\phi_{\kappa, \kappa_y} = \sqrt{\frac{(f_s)^{1/4}}{(\pi/2L_y)^{1/2}}} e^{iK_x x} e^{iK_y y} E \left( f_s + \varphi_{\kappa_y} \right) \frac{(x + Ky B)^2}{2B},
\]
where \( f_s, g_B, f_C = (\sqrt{3}/\lambda^3, -\sqrt{3}/\lambda^3, \sqrt{3}/\lambda^3), \), \( f_B, B = (1/\lambda^3, 0) \), \( g_C, C = (1/\lambda^3, 0) \), \( \lambda = (m_2/m_1) \approx 3.5 \)\(^{13} \) and the magnetic length \( L_y = \frac{\hbar}{eB} \). We focus on filling fractions \( \nu < 6 \) and ignore mixing between different Landau levels (LLs). As is usual in QH ferromagnets, even if we restrict to (near-)integer filling, the exact degeneracy between the valley degrees of freedom at single-particle level is lifted by interactions, which select a ground state at each integer filling \( \nu < 6 \), and in doing so break one or more symmetries spontaneously. The question of precisely how this happens is our focus in the remainder.

A. Effective Hamiltonian

Since we are working in a degenerate manifold of the electron kinetic energy — quenched by the magnetic field — the effective Hamiltonian is comprised solely of interaction terms, that inherit the kinetic anisotropies through their dependence on the single-particle LL eigenfunctions. In the limit \( K_0/L_B \gg 1 \), the electron-electron interaction term is
\[
H = \frac{1}{2S} \sum_{\vec{q}, \kappa, \kappa'} V(\vec{q})_\kappa \rho_{\kappa\kappa'}(\vec{q}) \rho_{\kappa'\kappa}(-\vec{q})
\]
where \( S = L_x L_y \) is the total area, \( \rho_{\kappa\kappa} \) is the density operator within valley \( \kappa \) projected to the LLL and \( V(\vec{q}) = \frac{2\pi e^2}{q} \) is the matrix element of the Coulomb interaction. [A static background is omitted from [3] for clarity.]

The Hamiltonian \( \mathcal{H} \) has an approximate \( G = [SU(2)]^3 \times D_3 \) symmetry. To see this, note that \( H \) is invariant under \( SU(2) \) rotations between the two valleys \( (\kappa, \bar{\kappa}) \) in the pair, explaining the \( [SU(2)]^3 \), as well as under a \( D_6 \) discrete point-group symmetry. However, any element of \( D_6 \) that only interchanges the two valleys \( (\kappa, \bar{\kappa}) \) in a pair is equivalent to an \( SU(2) \) \( \pi \)-rotation; the \( D_6 \) elements not of this type form a \( D_3 \) subgroup that acts on the 3 \( SU(2) \) indices, leading to the semidirect product structure.\(^{12} \) Recent work on wide (001) AlGa quantum wells studied a symmetry similar to the discrete rotation above.\(^{23} \)

We now consider the cases of various integer filling fractions using the vanishing mass anisotropy limit as the starting point. (In the theoretically convenient case of \( \lambda = 1 \), \( H \) is \( SU(6) \) symmetric; qualitative pictures obtained for \( |\lambda - 1| < 1 \) remain valid even when the anisotropy is no longer small.) Note that, owing to particle-hole symmetry about \( \nu = 3 \) the problems at \( \nu = 1, 5 \) are equivalent, as are those at \( \nu = 2, 4 \). This leaves us three distinct fillings to consider. (Including spin yields identical results for \( \nu = 1, 2, 3 \), and also for \( \nu = 4, 5, 6 \) in the limit when the valley splitting is negligible compared to the Zeeman energy \( g \mu_B B \).)

B. \( \nu = 1 \)

At the \( SU(6) \) symmetric point \( \lambda = 1 \), the degenerate ground states with one filled LL are \( |\psi\rangle = \prod_{k_y} d_{k_y}^\dagger |0\rangle \) where \( d_{k_y}^\dagger = \sum_{\kappa \{A, B, C\}} (\alpha_\kappa c_{\kappa k_y}^\dagger + \alpha_\kappa^* c_{\bar{\kappa} k_y}^\dagger \), with \( \sum_{\kappa} |\alpha_\kappa|^2 + |\alpha_\kappa^*|^2 = 1 \) The anisotropy splits this degeneracy. At \( |\lambda - 1| < 1 \), first-order perturbation theory yields
\[
\langle \psi | H | \psi \rangle = \sum_{\kappa, \kappa' \in \{A, B, C\}} \delta V_{\kappa \kappa'}^\sigma |(\alpha_\kappa^2 - |\alpha_\kappa^*|^2)(|\alpha_\kappa|^2 - |\alpha_\kappa^*|^2)|^2)
\]
where \( \delta V_{\kappa \kappa'}^\sigma = \frac{1}{2} |\epsilon_{\sigma \kappa \kappa'}|(V_{\sigma \sigma}^\kappa - V_{\sigma \sigma}^{\kappa'}) \), and
\[
V_{\kappa \kappa'}^\sigma = \sum_{k, k', k''} \phi_{\kappa k}^\dagger(\vec{r}) \phi_{\kappa' k''}^\dagger(\vec{r}'') V_{\kappa' k''} \phi_{\kappa'' k''}^\dagger(\vec{r}') \phi_{\kappa k'}(\vec{r}).
\]
The \( \delta V_{\kappa \kappa'}^\sigma \) are all positive and proportional to \( N \), the number of electrons. Hence the approximate new ground states in the thermodynamic limit are of the form \( |\kappa\rangle = \prod_{k_y} (\alpha_{\kappa c_{\kappa k_y}^\dagger + \alpha_{\kappa}^* c_{\bar{\kappa} k_y}^\dagger} |0\rangle \); these break the \( [SU(2)]^3 \times D_3 \) symmetry down to \( H_0 = U(1) \times [SU(2)]^2 \times D_2 \) (where the second factor refers to rotations of the unoccupied pairs), leading to a single Goldstone mode (as \( G/H_0 = S^2 \)). Working in the vicinity of \( \nu = 1 \) at \( T = 0 \), from standard energetic argument\(^{12} \) we conclude that for \( \nu \geq 1 \), skyrms are created within the occupied-valley subspace \( (\kappa, \bar{\kappa}) \); similarly, for \( \nu \leq 1 \), anti-skyrms are created.\(^{2} \) At any \( T \neq 0 \), statistical averaging over Goldstone modes restores the broken \( SU(2) \) symmetry, so that the invariance group is \( H_T = SU(2) \times [SU(2)]^2 \times D_2 \). The order parameter of the resulting phase lies in \( G/H_T = Z_2 \).\(^{19} \) We conclude that valley ferromagnetic order sets via
FIG. 2. **Possible valley-ordered states at ς = 1, 2, 3**, including representatives of Class I and II states for ς = 2, 3. Unfilled and fully-filled valleys are shown as empty and filled ellipses; valleys partially-filled due to a particular choice of SU(2) vector within the two-valley subspace are shaded.

A finite temperature Z3 transition into a nematic phase with broken orientational symmetry (Fig. 1).

C. ς = 2

We next consider the case when two LLs are filled. Again, we begin at λ = 1 where the degenerate ground states are given by \(|ψ⟩ = \prod_{i=1,2} \prod_{κ} d_{i,κ}^{-}|0⟩\) where \(d_{i,κ} = \sum_{κ∈\{A,B,C\}} (α_κc_{κ,i} + α_κ^*c_{κ,i}^+)\), and \(\sum_{κ}|α_κ|^2 + α_κ^2 = 1\). Moving away from the SU(6) point, but keeping \(|λ - 1| < 1\), the ground state manifold has two kinds of states that remain degenerate even upon inclusion of the anisotropic terms (Fig. 2). “Class I” ground states are obtained by filling both valleys in a pair, and take the form \(|κκ′⟩ = \prod_{κ} c_{κ,k}^+ (α_κc_{κ,k} + \bar{α}_κc_{κ,k}^+) |0⟩\).

“Class II” ground states on the other hand are constructed by picking two pairs and setting each to have ς = 1 by spontaneously breaking the residual SU(2) symmetry of rotations within the pair. These are of the form \(|κκ′⟩ = \prod_{κ} c_{κ,k}^+ (α_κc_{κ,k} + \bar{α}_κc_{κ,k}^+) |0⟩\). While Class I states break the \([SU(2)]^3 × D3\) symmetry down to \([SU(2)]^3\), Class II states break it down to \(U(1)^2 × SU(2)\); the order parameter spaces are \(Z_3 × S^2\) for Class I and II states, respectively,[19] which therefore host no and two Goldstone modes. This disparity leads to selection of the latter by thermal fluctuations as we discuss below.

First, however, we demonstrate that selection occurs due to charge doping — incommensuration — at \(T = 0\). To see why this is so, observe that doping Class II states to a filling \(ς \gtrsim 2\) \((ς \lesssim 2)\) proceeds by creating skyrmions (anti-skyrmions) in the two-dimensional subspaces of the occupied valley pairs. For Class I states on the other hand, the charge added or subtracted is accomodated in a conventional quasi-electron (quasi-hole) Wigner crystal. As skyrmion (anti-skyrmion) lattices have lower energy,[24] we argue that doping selects Class II states.

Turning now to \(T > 0\), we observe that the combination of a high ground state degeneracy and a disconnected ground state manifold — there is no continuous path in the set of ground states that connects a state in Class I to a state in Class II — is ideal for seeing “order by disorder”. This phenomenon, in which entropic considerations select a ground state, occurs often in frustrated spin systems.[13][16] Since there are gapless excitations about Class II states, they are selected by thermal fluctuations as the free energy of fluctuations about the degenerate ground state manifold is peaked about states with a large number of soft modes. However this mechanism comes into play above a crossover scale \(T^*\) (Fig. 1). Below \(T^*\), energetic, rather than entropic, considerations favor Class II states — the order by doping mechanism. The crossover between selection by energetic considerations and selection by entropy — that we dub the “E-S crossover” — takes place at \(T_{ES} \approx T_c\).

We observe that while Goldstone modes are responsible for order by disorder, the SU(2) symmetries remain unbroken at any \(T = 0\) so that the invariance group is \(H_T = [SU(2)]^3 × D_3\), (the \(D_3\) reappears as we may once again interchange between the filled pairs when \(SU(2)\) is restored). As \(G/H_T = Z_3\), we conclude that the system has a transition at \(T_c > 0\) described by a Z3 nematic order parameter, in which the \(D_3\) symmetry is broken (Fig. 1). Since in the experimental systems of interest the filling is tuned with field rather than by gating, we anticipate that \(T_{c} ≫ T_e\), as the relevant energy scale is the surface tension of Z3 domain walls, that depends in turn on the Coulomb energy \(e^2/ε_B\).

D. ς = 3

We now examine the situation with three filled LLs. At the isotropic point λ = 1, the degenerate ground states are \(|ψ⟩ = \prod_{i=1,2,3} \prod_{κ} d_{i,κ}^{-}|0⟩\) where \(d_{i,κ} = \sum_{κ∈\{A,B,C\}} (α_κc_{κ,i} + α_κ^*c_{κ,i}^+)\). As before, the symmetry is reduced for λ \(\neq 1\), and the new ground state manifold again has two kinds of states (Fig. 2). Class I states are obtained by filling both valleys in a pair and then spontaneously breaking \(SU(2)\) in another pair, and are of the form \(|κκ′⟩ = \prod_{κ} c_{κ,k}^+ (α_κc_{κ,k} + \bar{α}_κc_{κ,k}^+) |0⟩\). While Class I states break the \([SU(2)]^3 × D3\) symmetry down to \([SU(2)]^3\), Class II states break it down to \(U(1)^2 × SU(2)\); the order parameter spaces are \(Z_3 × S^2\) for Class I and II states, respectively,[19] which therefore host no and two Goldstone modes. This disparity leads to selection of the latter by thermal fluctuations as we discuss below.

First, however, we demonstrate that selection occurs due to charge doping — incommensuration — at \(T = 0\). To see why this is so, observe that doping Class II states to a filling \(ς \gtrsim 3\) \((ς \lesssim 3)\), skyrmions (anti-skyrmions) are created about both Class I and II states. However, the structure of the resulting triangular lattices is quite different. Discussing skyrmions for specificity, for Class I states doping proceeds by making a lattice of skyrmions that live in only one two-dimensional subspace, whereas for Class II states the skyrmion lattice has a tripled unit cell, as it results from symmetrically combining three sublattices each built from skyrmions in one of the three different two-dimensional subspaces. As there is no valley Zeeman energy, the skyrmion size is set entirely by their density. In order to estimate the energies of the competing skyrmion crystals, we utilize the fact that the relevant nonlinear sigma models optimize their gradient energy for analytic (two-component) spinor solutions — non-analytic configura-
tions generically have higher energy. The simplest such analytic solution that is (quasi-)periodic with finite topological charge $Q$ per unit cell, has $Q = 2$, as all $Q = 1$ configurations with these desiderata are non-analytic and hence have higher energy.\textsuperscript{23} Such a quasi-periodic $Q = 2$ spinor solution\textsuperscript{23} gives Class I and II states identical gradient energies, so the issue turns on the Coulomb energy which is lower for Class II. We therefore conclude that doping selects Class II states.

For $T > 0$, the Goldstone mode fluctuations about Class II states restore the full $G = [SU(2)]^3 \rtimes D_3$ symmetry and hence there is no sharp finite-temperature transition owing to the lack of any broken symmetries. We nevertheless expect thermal selection of Class II states owing to the excess of Goldstone modes compared to Class I states. As in the $\nu = 2$ case this occurs above a scale $T^*_{E-S}$, below which order by doping dominates.

In combining the results for $\nu = 1, 2, 3$ we note that their distinct symmetries and doping energetics at $T = 0$ point to first-order transitions at $\nu_{12}^2$ and $\nu_{23}^3$, where $1 < \nu_{12}^2 < 2 < \nu_{23}^3 < 3$, that we expect survive to $T > 0$. Thus, we arrive at the global phase diagram of Fig. 1\textsuperscript{I}. The application of these ideas to a somewhat simpler example, of Si(110) wide quantum wells is discussed in the appendix\textsuperscript{I9}.

III. EXPERIMENTS

We expect that nematic order leads to measurable anisotropies in longitudinal conductivities $\sigma_{xx}, \sigma_{yy}$, though the orientation of the valleys with respect to the symmetry axes may present an added complication, even for samples oriented along crystallographic axes. For $\nu = 1, 2$, the anisotropy should show order-parameter onset behavior at $T_c$, typically a few kelvin at $B \approx 10T$ for systems with comparable mass anisotropy and dielectric constant.\textsuperscript{20} Class II state selection at $\nu = 2$ will be reflected by the extreme sensitivity of the activation gap to strain-induced valley Zeeman splitting absent in Class I states that lack skyrmion excitations. The selection of Class II states at $\nu = 3$ is challenging to detect as they lack nematic order, and Class I states also host skyrmions. However, restoration of orientational symmetry in going from $\nu = 2$ to $\nu = 3$ coupled with observation of the QHE would bolster this scenario. Similar considerations apply, \textit{mutatis mutandis} to the case of Si(110) quantum wells.

IV. CONCLUDING REMARKS

In closing we comment briefly on complications ignored in our discussion. Foremost among these is the neglect of various terms in\textsuperscript{22} that while suppressed by $O((a/\ell_B)^2)$ relative to the dominant Coulomb energy scale could contribute to the selection mechanisms discussed above. For $B \approx 10T$, we find that these terms split energy levels by a few millikelvin, so that there is a large window of temperature where their neglect is justified. We note that competition between quantum selection by high-order effects and thermal order-by-disorder was studied in quantum magnets\textsuperscript{25}\textsuperscript{26}; similar situations may arise here once the neglected terms become significant.

Secondly, in a more realistic situation, the six-fold valley degeneracy can be lifted due to wafer miscut and strain arising from lattice mismatch. While valley splitting due to the former mechanism is negligible compared to the cyclotron gap,\textsuperscript{29} the latter \textit{can} be more significant.\textsuperscript{29}\textsuperscript{25} Although this problem has been largely solved by working with 2DEGs on a H-terminated Si(111) surface,\textsuperscript{10,11} both mechanisms can still change the E-S crossover temperature $T^*_{E-S}$ and possibly even the Class of states that are selected below $T^*$.

Third, we have ignored Landau level mixing\textsuperscript{30}\textsuperscript{32} likely to be significant in any realistic situation. We anticipate that while its inclusion will change some details, the overall picture should be largely immune to this added complication. Our confidence rests on the fact that estimates based on LLL approximations have met with noteworthy success to date — particularly in the present context of QHFM — and constitute a standard approximation in the field. We note that a full-fledged calculation of LL mixing with interactions is extremely challenging (even numerically), and as such lies well beyond the scope of this paper.

Finally, spatial disorder induces a random field acting on the nematic order parameter, which is a relevant perturbation. An infinitesimal field destroys the nematic order in the thermodynamic limit at $\nu = 1, 2$ by proliferating domain walls but the QHE survives as long as disorder is sufficiently weak.\textsuperscript{30} The interplay of disorder with the novel selection mechanisms discussed above is likely intricate and worthy of further study.

ACKNOWLEDGMENTS

We thank Roderich Moessner, Rahul Nandkishore, Joseph Maciejko, Ashvin Vishwanath and Romain Vasseur for useful discussions and especially Dmitry Kovrizhin who generously shared his calculations and code with us. We also thank Tomasz Kott and Bruce Kane for discussing their experimental data, and Dmitry Abanin and Steven Kivelson for collaboration on related work. This work was supported by NSF Grant Numbers DMR 1006608, 1311781 and PHY-1005429 (AK and SLS) and UC Irvine start-up funds (SAP).

\textsuperscript{*} afour@princeton.edu
\textsuperscript{†} sidp@uci.edu
\textsuperscript{‡} sondhi@princeton.edu
\textsuperscript{1} A. Pinczuk and S. Das Sarma, Perspectives in Quantum Hall Effects (Wiley, 1996).
On turning on the Zeeman field, we are back to results in our paper. A similar story holds for filling \( \nu \) of a fourfold valley degeneracy (Fig. 1). Order by disorder and by doping are expected to occur in this case as well, in a scenario near-perfect valley splitting, and we can safely ignore the spin degeneracy; therefore, the results for filling \( \nu \) as such lies well beyond the scope of this paper (a good example of the complexity of the relevant many-body problem is in W. Sinova, Macdonald and Girvin [Phys. Rev. B 80, 121302 (2009)]).

We note that a full-fledged calculation of LL mixing with interactions is extremely challenging (including numerically), and arguments: we do not anticipate that LL mixing will split the symmetry directly, though it could potentially lift the degeneracy. Returning to the issue of LL mixing, we anticipate that while its inclusion will change some details, the overall picture should be largely immune to this added complication. Our confidence rests on the fact that estimates based on LLL approximations in the QHFM problem.

So, our results for fillings \( \nu = 1, 2 \) and \( 3 \) still hold. Now, the results for \( \nu = 4, 5, 6 \) depend on the relative size of the spin Zeeman energy \( \Delta_Z = g \mu_B B \) and the valley splitting \( \Delta_v \). For \( \Delta_v \ll \Delta_Z \), we expect that the relevant selection mechanisms involve splitting the near-perfect valley splitting, and we can safely ignore the spin degeneracy; therefore, the results for filling 4, 5, 6 are obtained by effectively ‘particle hole conjugating’ the \( \nu = 1, 2, 3 \) results. In this limit, we see that we need not consider the spin physics in the QHFM problem.

Returning to the issue of LL mixing, we anticipate that while its inclusion will change some details, the overall picture should be largely immune to this added complication. Our confidence rests on the fact that estimates based on LLL approximations have met with noteworthy success to date — particularly in the present context of QHFM — and as such represent a standard approximation in the field. As an example, a study by Sinova, Macdonald and Girvin [Phys. Rev. B 80, 121302 (2009)] that includes disorder effects within a Hartree-Fock treatment similar to that used in our analysis, and ignores LL mixing, gives results in good qualitative agreement with experiment. A second reason is that once again, we may fall back on general symmetry arguments: we do not anticipate that LL mixing will split the symmetry directly, though it could potentially lift the degeneracy of the different ground state classes, this is an effect that we are unable to discuss in detail at present and therefore prefer to leave out. We note that a full-fledged calculation of LL mixing with interactions is extremely challenging (including numerically), and as such lies well beyond the scope of this paper (a good example of the complexity of the relevant many-body problem is in W. Bishara and C. Nayak, Phys. Rev. B 80, 121302 (2009).)

### APPENDIX

#### I. INCLUSION OF SPIN AND LANDAU LEVEL MIXING

Let us ignore LL mixing for now and add spin to the problem. In the absence of Zeeman splitting, there are two classes of degenerate ground states at filling \( \nu = 2 \): fill 2 out of set A (say) = \( A_1, A_2, A_{\uparrow}, A_{\downarrow} \) or fill 1 out of set A and 1 out of set B (say). On turning on the Zeeman field, we are back to results in our paper. A similar story holds for filling \( \nu = 3 \). So, our results for fillings \( \nu = 1, 2 \) and \( 3 \) still hold. Now, the results for \( \nu = 4, 5, 6 \) depend on the relative size of the spin Zeeman energy \( \Delta_Z = g \mu_B B \) and the valley splitting \( \Delta_v \). For \( \Delta_v \ll \Delta_Z \), we expect that the relevant selection mechanisms involve splitting the near-perfect valley splitting, and we can safely ignore the spin degeneracy; therefore, the results for filling 4, 5, 6 are obtained by effectively ‘particle hole conjugating’ the \( \nu = 1, 2, 3 \) results. In this limit, we see that we need not consider the spin physics in the QHFM problem.

We now briefly discuss the case of Si(110) where in the presence of a strong interface potential, effective mass theory predicts a fourfold valley degeneracy (Fig. 1). Order by disorder and by doping are expected to occur in this case as well, in a scenario

---


19 See appendix for application to Si(110), group-theoretic analysis of symmetry structures, explicit construction of order parameter spaces, and discussion of the role of spin and Landau level mixing.


22 There is also a factor of \( 1/2 \) from the fact that at fixed electron number only half of the electrons enter the domain wall energetics at \( \nu = 2 \).


that closely parallels the six-valley case of Si(111) described in the main text.

The valleys centered at $\vec{K}_A = -\vec{K}_A = (K, 0)$ and $\vec{K}_B = -\vec{K}_B = (0, K)$. In the Landau gauge $\vec{A} = (0, Bx)$, the LLL eigenfunctions are given by Equation 1 of the main text with $(f, g)_{A, \vec{A}} = (\sqrt{A}, 0)$ and $(f, g)_{B, \vec{B}} = (\frac{1}{\sqrt{A}}, 0)$. The interaction Hamiltonian has $[SU(2)]^2 \rtimes D_2$ symmetry where the $SU(2)$s are independent rotations in the pair subspaces ($A, \vec{A}$) and ($B, \vec{B}$) and the $D_2$ symmetry interchanges these pairs. A group-theoretic analysis of this symmetry structure is provided below.

For $|\lambda - 1| \ll 1$, the ground states at $\nu = 1$ are given by $|\psi⟩ = \prod_{k_y}(\alpha_k e_{\nu,k_y} + \alpha_k e_{\nu,-k_y})|0⟩$ and $(|\alpha_k|^2 + |\alpha_k|^2) = 1$ (Fig. 1). This case resembles $\nu = 1$ for Si(111) and we expect analogous results. The anisotropy in longitudinal conductivities should show up in the form of order-parameter onset behavior at $T_c$.

At $\nu = 2$ the ground state manifold supports two types of states (Fig. 1). Class I states have the form $|κ\vec{k}⟩ = \prod_{k_y} e_{κ,k_y} e_{κ,k_y}^\dagger |0⟩$. Class II states are of the form $|AB⟩ = \prod_{k_y} \prod_{κ=A,\vec{A}} (α_k e_{κ,k_y} + α_k e_{κ,-k_y})|0⟩$ where $|α_k|^2 + |α_k|^2 = 1$. Though the $T = 0$ mode counting — zero versus a pair of Goldstone modes — is similar to $\nu = 2$ for Si(111), at $T > 0$ the full $[SU(2)]^2 \rtimes D_2$ symmetry is restored to Class II states by thermal averaging, analogous to the Si(111) $\nu = 3$ Class II state. Moreover unlike Class II states, the Class I states lack skyrmion excitations. Therefore, we expect selection of Class II by thermal fluctuations and charge doping, but no finite-temperature transition about $\nu = 2$.

III. GROUP-THEORETIC ANALYSIS OF SYMMETRY BREAKING

We discuss the simpler four-valley case first as a warm-up, before moving to the six-valley example. In each case, we first discuss the high-temperature symmetry-group $G$, the finite-temperature invariance subgroup of the broken-symmetry states $H_T$, and finally its zero-temperature counterpart $H_0$. The nonlinear sigma models (NL$\sigma$M) governing the $T > 0$ and $T = 0$ transitions have order-parameter spaces given by the group manifolds $G/H_T$ and $H_T/H_0$, respectively. Valley indices are as described in the main text.

A. Four-Valley Case

We first show that high-temperature valley symmetry group is $[SU(2)]^2 \rtimes D_2$, where the $D_2$ interchanges the two $SU(2)$ axes. To see this, we note that the valley Hamiltonian (after including the anisotropy terms) has the following symmetries (refer to Fig. 3 of the main text for valley labeling): two distinct $SU(2)$ symmetries that each act within a valley pair ($A, \vec{A}$) and ($B, \vec{B}$), and the dihedral group of symmetries of the square, that we denote $D_4$. The full symmetry group $G$ is obtained by combining these symmetries. Clearly $N = [SU(2)]^2$ is a subgroup of $G$. Turning to $D_4$, we recall that this is generated by two operations, a permutation $r = (AB\bar{A}B\bar{A})$ and a swap $\rho = (AA\bar{A})$, where we use conventional notation to describe the action of finite groups: for instance $(a_1 \ldots a_m)(b_1 \ldots b_n)$ denotes the cyclic permutations $a_1 \rightarrow a_2 \rightarrow \ldots \rightarrow a_m \rightarrow a_1$ and $b_1 \rightarrow b_2 \rightarrow \ldots \rightarrow b_n \rightarrow b_1$ with all other ‘letters’ left invariant. Now, it is clear that $r^2 = e$, the identity; also, we note that $r^2$ simply interchanges the two valleys in a pair, and is therefore equivalent to a $π$ rotation within each valley pair, i.e. $r^2 \in N$. Therefore it follows that the coset $r^2 N = N = N r^2$. A similar argument reveals that $N = N = N r$, since $r$ corresponds to a $π$-rotation in the pair ($A\bar{A}$) coupled with an identity operation in the other valley pair. Finally, we observe that since $r$ preserves the valley pair structure, transforming valley indices by $r$, performing independent $SU(2)$ rotations within each pair of valleys, and undoing the index transformation, must be equivalent to a product of independent rotations within each pair, so that $r^{-1} N r = N$. Since $D_4 = \{e, r, r^2, r^3, \rho, r\rho, r^2 \rho, r^3 \rho\}$, we see that full list of cosets is $\{N, rN\}$. Thus, (i) $N$ is a normal subgroup of $G, N \triangleleft G$ and (ii) $G = \{N, rN\}$, so that $G/N \cong D_4$. We therefore conclude that $G = N \rtimes D_2 = [SU(2)]^2 \rtimes D_2$, where in identifying the $D_2$ structure we used the fact $r^2 \sim e$ in the coset space since $r^2 N = N = eN$.

We now turn to the breaking of symmetries at different fillings. We will discuss the symmetry breaking in two stages: first, we will determine the residual symmetry group $H_T$ for $T > 0$, where the Mermin-Wagner theorem precludes the breaking of...
continuous symmetries; the corresponding NLσM has target space \( G/H_T \). Then, we will discuss how \( H_T \) is further broken down to \( H_0 \) at \( T = 0 \), described by a NLσM with target space \( H_T/H_0 \).

At \( \nu = 1 \), and finite temperature we choose to fill a single valley pair while leaving the other unfilled. The invariant subgroup \( H_T \) of the resulting state is \( SU(2)^2 \) (corresponding to rotating in the filled and unfilled pairs – since for \( T > 0 \) Goldstone modes lead to averaging over all possible superpositions of valleys within the filled pair), but the semidirect product structure does not survive as we can distinguish the filled and unfilled pairs and therefore their interchange is not a symmetry. The corresponding NLσM target space is \( G/H_T = \langle [SU(2)^2 \times D_2]/[SU(2)^2] = D_2 \cong Z_2 \rangle \), consistent with our argument that the symmetry is broken via a finite-temperature Ising transition. As \( T \to 0 \) the Goldstone mode fluctuations responsible for the finite-temperature restoration of \( SU(2) \) symmetry (of rotating between valleys \( (\kappa, \bar{\kappa}) \) in the filled pair) are suppressed, and this symmetry is broken by a specific choice of \( SU(2) \) vector in the \( (\kappa, \bar{\kappa}) \) subspace. This leaves a residual \( U(1) \) phase, but the \( SU(2) \) symmetry between the unfilled valleys is still preserved, and therefore we have the residual symmetry group \( H_0 = U(1) \times SU(2) \), giving the target space \( H_T/H_0 = [SU(2) \times SU(2)]/[U(1) \times SU(2)] = SU(2)/U(1) = S^2 \).

For Class II states at \( \nu = 2 \), we fill both valley pairs to \( \nu = 1 \), but as before the \( SU(2) \) symmetry between the two valleys in each pair is restored at any finite temperature. As a consequence, we can still interchange \( SU(2) \) axes in this case, so we have \( H_T = G = [SU(2)^2 \times D_2] \), and so there is no finite-temperature phase transition as \( G/H_T \) is the trivial group. We may parameterize any state in this base space \( \vec{z} \in H_T \) by \( \vec{z} = g \cdot \vec{z}_0 \), where

\[
g = \begin{pmatrix} g_1 & 0 \\ 0 & g_2 \end{pmatrix},
\]

with \( g_1, g_2 \in SU(2) \), and \( \vec{z}_0 = (1, 0, 1, 0)^T \) is a reference spinor; note that this implicitly respects the semidirect product structure of \( H_T \). As \( T \to 0 \), we break each of the two \( SU(2) \) symmetries down to \( U(1) \). This remaining \( U(1) \) invariance within each valley pair is generated by matrices \( h \in H_0 \), where

\[
h = \begin{pmatrix} h_1 & 0 \\ 0 & h_2 \end{pmatrix},
\]

with \( h_1, h_2 \in U(1) \). Using the equivalence relation on \( \vec{z}, \vec{z} \) given by \( \vec{z} \sim \vec{z} \iff \vec{z} = h \cdot g \vec{z} \) for \( h \in H_0 \), we see that \( H_T/H_0 \cong S^2 \times S^2 \). For Class I states at \( \nu = 2 \), we fill both valleys in a pair, and therefore the residual symmetry group is \( H_T = SU(2) \times SU(2) \) as this corresponds to rotating within the filled and unfilled pairs – we cannot interchange between the pairs. So, there is a putative finite-temperature transition possible for Class I states, as \( G/H_T = D_2 \cong Z_2 \). However, as \( T \to 0 \), we observe that there is no additional structure that emerges as the full \( SU(2) \times SU(2) \) symmetry is preserved. In contrast, Class II states have Goldstone modes that will lead to their selection over Class II states as \( T \to 0 \). Therefore starting from \( T = 0 \) and restoring symmetry in stages we see that Class I states do not emerge in the phase diagram.

B. Six-Valley Case

In the six-valley case, we begin by observing that the high-temperature symmetry group is \( G = [SU(2)]^3 \times D_3 \), where \( D_3 \) is the dihedral group of symmetries of an equilateral triangle (isomorphic to \( S_3 \), the symmetric group) that acts on the three \( SU(2) \) axes. To see this, first observe that the symmetries of the valley Hamiltonian are (i) three \( SU(2) \) symmetries that rotate between the two valleys in each pair \((AA), (BB), (CC)\) and (ii) the dihedral group \( D_6 \) of symmetries of a regular hexagon. Clearly, \( N = SU(2)^3 \) is a subgroup. Turning next to \( D_6 \), we observe that it is generated by the sixfold rotation \( r = (ABC\bar{A}B\bar{C}) \) and the reflection \( \rho = (A\bar{A})(B\bar{C})(\bar{B}C) \), so that we may write \( D_6 = \{e, r, r^2, r^3, r^4, r^5, \rho, \rho r, \rho r^2, \rho r^3, \rho r^4, \rho r^5\} \). [Note that, unlike the swap in the four-valley case, here we cannot write \( \rho \) as equivalent to a rotation; while it is indeed a rotation in the indices \( (\bar{A}A) \), it is \( \rho \) not on the remaining indices.] In listing cosets, we first observe that left and right cosets must be equivalent, \( i.e. \ s^{-1}Ns = N \) for any \( s \in D_6 \), following the same logic as in the four-valley case: performing a discrete transformation \( s \) on the valley indices, performing independent \( SU(2) \) rotations in each valley pair and then transforming back to the original valley indices using \( s^{-1} \) should be simply equivalent to three independent \( SU(2) \) rotations, as long as \( s \) preserves the pairing of the valleys, and it is clear this is satisfied by \( r, \rho \), and hence by any combination of their powers. From this reasoning, we conclude that \( N \) is a normal subgroup. In addition, we note that \( \rho^3 = e \), and furthermore \( r^3 = (AA)(BB)(CC) \), corresponding to \( \pi \)-rotations in each valley pair, whence \( r^3N = N \). Putting these arguments together, we find the list of cosets to be \{\( N, rN, r^2N, \rho N, \rho rN, \rho^2N \}\}. Now, the fact that left and right cosets are equivalent means that the coset space has a group structure, obtained by simply writing \( g_1N g_2N = g_1g_2N \). We can verify that the operations \( r, \rho \) satisfy the identity \( \rho^{-1}r^{-1}pr = r^2 \), so that under the coset multiplication rule \( (r\rho)^{-1}prN = r^2N \neq N \) \( i.e. \) the quotient group \( G/N \) is non-Abelian. Since the unique non-Abelian group of order 6 is \( D_3 \), we see that \( G/N \cong D_3 \), from which the structure \( G = [SU(2)^3] \times D_3 \) follows immediately.

We now discuss symmetry breaking using the same conventions as previously. At \( \nu = 1 \) and \( T > 0 \), we fill a single valley, breaking the \( D_3 \) structure, but thermal fluctuations of Goldstone modes restore the \( SU(2) \) symmetry within the filled pair. We
still have the ability to perform $SU(2)$ rotations within the unfilled pairs as well as swap their axes; we argue that this leads to an $SU(2)^2 \times D_2$ structure within the unfilled subspace, so that in total we have $H_T \cong SU(2) \times [SU(2)^2 \times D_2]$. From this, we see that the NLsM target space is given by $G/H_T = [SU(2)^3 \times D_3]/[SU(2) \times [SU(2)^2 \times D_2)] \cong \mathbb{Z}_3$. At $T = 0$, we argue in analogy with the 4-valley case that $SU(2)$ in the filled pair is broken down to $U(1)$ so that the residual symmetry $H_0 = U(1) \times [SU(2)^2 \times D_2]$, so that the target space is $H_T/H_0 \cong S^2$.

Turning now to $\nu = 2$ and $T > 0$, we first consider class II states where we fill a pair of valleys breaking the $S^3$ structure but preserving $SU(2)$ via thermal restoration of symmetry, so that $H_T = [SU(2)^2 \times D_2] \times SU(2)$ again (but the role of the filled and unfilled valleys are interchanged), and once again $G/H_T \cong \mathbb{Z}_3$. At $T = 0$, the situation is similar to $\nu = 2$ for the four-valley case: the residual symmetry is $U(1)^2 \times SU(2)$, yielding the target space $H_T/H_0 \cong S^2 \times S^2$. For Class I states at $\nu = 2$, we fill both valleys in a pair, leading to residual symmetry group $H_T = SU(2) \times [SU(2)^2 \times D_2]$; since $G/H_T \cong \mathbb{Z}_3$, it appears that there is an alternative finite-temperature transition. However, proceeding to $T = 0$, we see that there is no additional symmetry breaking, i.e. $H_0 = H_T$, leading to a lack of Goldstone modes, and therefore near $T = 0$ Class II states are selected. Since the thermal fluctuations of the Goldstone modes about class II states restores $[SU(2)^2 \times D_2] \times SU(2)$ symmetry, which returns to the full symmetry $G$ via a finite temperature transition, we never access Class I states.

At $\nu = 3$, for Class II states we fill each of the three valley pairs to $\nu = 1$, and and $T > 0$ we have $H_T = G = [SU(2)]^3 \times S^3$; once again $G/H_T$ is trivial, reflecting the fact that there is no finite-temperature transition. To examine further symmetry breaking we may parametrize this new base space $H_T$ similarly to the four-valley case: we write any spinor in this space as $\tilde{z} = g \tilde{z}_0$ with

$$g = \begin{pmatrix} g_1 & 0 & 0 \\ 0 & g_2 & 0 \\ 0 & 0 & g_3 \end{pmatrix},$$

(3)

with $g_1, g_2, g_3 \in SU(2)$, and $\tilde{z}_0 = (1, 0, 1, 0, 1, 0)^T$ is a reference spinor. As $T \to 0$, each $SU(2)$ breaks to $U(1)$, so once again we consider states $\tilde{z}, \tilde{z}'$ that are connected by a product of $U(1)$ rotations in each valley to be equivalent, yielding target space $H_T/H_0 \cong S^2 \times S^2 \times S^2$. For Class I states, we fill both members of one valley pair to $\nu = 1$, while filling one of the other pairs at $\nu = 1$, and leaving the third pair unfilled. The corresponding symmetry is simply $H_T = SU(2)^3$, and therefore $G/H_T \cong D_3$, suggesting a potential finite-temperature transition. However, as $T \to 0$, we break the $SU(2)$ of the filled pair down to $U(1)$, so that $H_0 = U(1) \times SU(2)^2$, and $H_T/H_0 \cong S^2$; since the corresponding theory has one Goldstone mode compared to three about Class II states, $T = 0$ state selection favors Class II states, and therefore we do not see Class I states as thermal fluctuations for $T \neq 0$ restore the full symmetry $G$ immediately.