A new kind of topological quantum order: A dimensional hierarchy of quasiparticles built from stationary excitations

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A New Kind of Topological Quantum Order: A Dimensional Hierarchy of Quasiparticles Built from Stationary Excitations

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We introduce exactly solvable models of interacting (Majorana) fermions in $d \geq 3$ spatial dimensions that realize a new kind of fermion topological quantum order, building on a model presented in ref. [1]. These models have *extensive* topological ground-state degeneracy and a hierarchy of point-like, topological excitations that are only free to move within sub-manifolds of the lattice. In particular, one of our models has fundamental excitations that are completely stationary. To demonstrate these results, we introduce a powerful polynomial representation of commuting Majorana Hamiltonians. Remarkably, the physical properties of the topologically-ordered state are encoded in an algebraic variety, defined by the common zeros of a set of polynomials over a finite field. This provides a "geometric" framework for the emergence of topological order.

Topological phases of matter are remarkable quantum states with quantized properties that are stable under local perturbations and can only be measured by non-local observables [2]. The most celebrated example is the fractional quantum Hall state, discovered more than thirty years ago [3]. The field of topological matter has now become an exciting research frontier at the cross-roads between theoretical physics, quantum information and material science.

Our theoretical understanding of topological matter is largely built on topological quantum field theory (TQFT) [4]. In this framework, the action of quantum fields in a space-time manifold is independent of its metric, but depends crucially on its topology. Canonical quantization of these fields in a multiply-connected space yields a *finite-dimensional* Hilbert space, describing the degenerate ground-states of topological matter. Wilson lines describe the world-lines of quasi-particle excitations, and the expectation value of "knotted" Wilson lines determines the quasi-particle braiding statistics. A hallmark of topologically-ordered states in two dimensions is the presence of *mobile* quasi-particles with fractional statistics, or anyons [5].

Exactly solvable models often provide ideal playgrounds and valuable insights in theoretical studies of topological phases. In the past, a wide array of non-chiral topological phases in two dimensions have been obtained in spin models [6, 7], whose universal properties are captured by topological quantum field theories. Recently, an exotic quantum phase with extensive topological ground state degeneracy was discovered by Haah in three-dimensional (3D) spin models [8]. A remarkable property of this phase is that *all* topological excitations are *strictly localized* in space, a feature which lies beyond the paradigm of topological quantum field theory.

In this work, we introduce a wide range of translationally-invariant, solvable Hamiltonians of interacting Majorana fermions that exhibit a new kind of fermion topological quantum order. These models have extensive topological degeneracy and a hierarchy of topological excitations that are only free to move within sub-manifolds of the full lattice. In one particular Hamiltonian in $d = 3$ spatial dimensions, the fundamental excitations are strictly localized, while composites of these excitations are free to move along one- and two-dimensional surfaces. The fundamental excitations are termed "fractons", as they behave as fractions of a mobile particle. Due to its fermionic nature, the topological order in our model enables an electron to break up into these immobile fractons; this appears to be the ultimate form of electron "fractionalization" in three dimensions.

To systematically search for these models, compute their ground-state degeneracy on a d-dimensional torus and study their excitations, we introduce a purely algebraic description of commuting Majorana Hamiltonians. We demonstrate that on a d-dimensional lattice with a two-site basis and a single interaction term per unit cell, an ideal Majorana Hamiltonian *generally* exhibits extensive topological degeneracy. We emphasize that each of our models may be written in terms of complex fermions by choosing appropriate pairings of Majorana fermions over the entire lattice. Our models also admit a *local* mapping to a boson model with identical topological degeneracy and a similar dimensional hierarchy of excitations, after projecting out half of the Hilbert space. We note that one of our models has similar phenomenology to a spin model studied in ref. [11, 12].

Our approach to studying ideal Majorana Hamiltonians provides a novel geometric framework for topological order, beyond topological quantum field theory. Remarkably, a commuting Majorana Hamiltonian on a torus specifies an algebraic variety – defined as the common zeros of a collection of polynomials over a finite field – that encodes all physical properties of the topologically-ordered state. While a TQFT assigns a ground-state sector to an isotopy class of smooth, closed curves on a manifold, our models associate ground-state sectors with curves based on finer equivalence relations, resulting in extensive topological degeneracy in dimensions $d \geq 3$. We emphasize that our models are distinct from the exotic phase realized by Haah's code [8] and related models [10], due to the presence of mobile topological excitations that are composites of fractons. As a related matter, separating a set of isolated fractons "optimally" only requires creating a finite number of mobile excitations during intermediate steps. Unlike Haah's code, this energy cost is independent of the *distance* of separation.

Universal features of our interacting Majorana models clearly demonstrate that they are in distinct phases from non-interacting stacks of lower dimensional systems. We consider one of our Hamiltonians – the Majorana cubic model – as a concrete example. First, in a non-interacting stack of lower-dimensional systems, *all* point-like topological excitations necessarily appear at the ends of string-like operators (Wilson lines). In contrast, the immobile fracton excitation in the cubic model can only appear in isolation at the corners of membrane-like operators. This feature alone *rigorously* establishes this model as distinct from any stack of lower-dimensional systems. Second, the topological ground-state degeneracy D for the Majorana cubic model on an $L \times L \times L$ torus satisfies $\log_2 D = 3L - 3$, for any L. The universal, subleading correction to $\log_2 D$ is a unique signature of this exotic phase that is impossible to obtain using a stack of lower-dimensional systems that respect the same lattice symmetries of our model; for example, $\log D$ must simply double as the system size doubled for a stack of lower-dimensional systems. We emphasize that both of the above features are independent of energetics. Even the low-energy effective theory of a stacked system with a similar excitation spectrum would still be describing an identifiably distinct quantum phase of matter, as these universal properties would be different. For similar reasons, the remaining Majorana models identified in our paper may not be obtained by a stacking procedure.

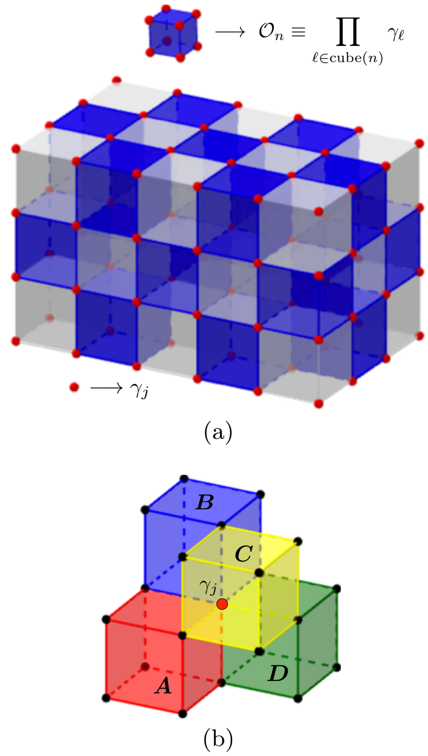


FIG. 1. **Majorana Cubic Model:** The Majorana cubic model is defined on a cubic lattice, as in (a), with a single Majorana fermion per lattice site (colored red). The operator $\mathcal{O}_n$ is the product of the 8 Majorana fermions at the vertices of a cube. The Hamiltonian is a sum of these local operators over every *other* cube (colored blue) in a checkerboard pattern. As any pair of operators either share exactly one edge or none, all operators mutually commute. We choose to label the cubic operators A, B, C, and D as shown in (b). Acting with a single Majorana operator γ_j creates these four excitations.

I. OVERVIEW

Due to the length of this paper, we begin with a detailed summary of our findings. We consider exactly solvable Hamiltonians of interacting Majorana fermions that realize exotic forms of topological order. On a d-dimensional lattice with a basis, these Hamiltonians will be the sum of a single type of local operator over all lattice sites

$$H = -\sum_m \mathcal{O}_m \tag{1}$$

so that all operators mutually commute and square to the identity, i.e.,

$$[\mathcal{O}_m, \mathcal{O}_n] = 0, \tag{2}$$

$$(\mathcal{O}_n)^2 = +1. \tag{3}$$

The operator $\mathcal{O}_n$ is required to be a product of an even number of Majorana fermions, so that the fermion parity of the entire system is conserved. A ground state $|\Psi\rangle$ of (1) will satisfy the constraint that

$$\mathcal{O}_m |\Psi\rangle = |\Psi\rangle , \tag{4}$$

for all m.

In Section II, we introduce a purely algebraic approach to systematically search for and study topological order in commuting Majorana Hamiltonians (1). A similar approach has been used previously to study topological order in commuting Pauli Hamiltonians [9]. We represent the operator $\mathcal{O}$ appearing in (1) as a set of Laurent polynomials over the field $\mathbb{F}_2$, which consists of two elements $\{0, 1\}$ with $\mathbb{Z}_2$ addition and multiplication. We derive a mathematical condition for a set of such polynomials to represent a commuting Majorana Hamiltonian with topological order. This polynomial representation enables us to analytically determine the topological ground state degeneracy on a d-dimensional torus and deduce properties of topological excitations using algebraic methods.

Using this polynomial approach, we demonstrate the following remarkable results. First, a topologically-ordered commuting Majorana Hamiltonian on a lattice with a two-site basis may be entirely specified by a single polynomial over $\mathbb{F}_2$. The ground state degeneracy for such a Hamiltonian on a d-dimensional torus of size L, which we denote by D_0, will *generally* take the asymptotic form:

$$\log_2 D_0 = c\, L^{d-2} + O(L^{d-3}) \tag{5}$$

for some constant c. We perform an exhaustive analysis and discover a class of commuting Majorana fermion models on a three-dimensional lattice with a two-site basis, which exhibit *extensive* topological degeneracy of the form (5) with $d = 3$.

Remarkably, despite being translationally invariant, our models admit fundamental point-like excitations that are strictly localized in space, and cannot move without paying a finite energy cost to create additional excitations. Composites of these fundamental excitations, however, are *topological* excitations that are free to move within sub-manifolds of the d-dimensional lattice. We term these fundamental excitations that behave as fractions of mobile particles, "fractons." Furthermore, we refer to bound states of fractons that can only move freely along an n-dimensional manifold as "dimension-n" particles. In particular, a dimension-2 particle can be an anyon with well-defined fractional statistics.

To motivate further study of ideal Majorana Hamiltonians, we now describe in detail the phenomenology of fracton excitations and their composites in the simplest of our models, the Majorana cubic model. As shown in Figure 1(a), here the operator $\mathcal{O}_n$ is the product of the eight Majorana fermions at the vertices of a cube. The Hamiltonian is simply the sum of these operators over a face-centered-cubic (fcc) array of cubes, forming a three-dimensional checkerboard. Since adjacent cubes share a common edge with two vertices, operators $\mathcal{O}_n$ on different cubes are mutually commuting, and their common eigenstate defines the ground state. For convenience in later analysis, we choose to identify four species of cube operators – A, B, C, and D – as shown in Figure 1(b).

A fundamental excitation in the Majorana cubic model is obtained when the eigenvalue of a cube operator $\mathcal{O}_n$ is flipped. The product of $\mathcal{O}_n$ over all cubes of a single type (A, B, C, or D) is equal to the fermion parity Γ of the entire system and is fixed.

$$\Gamma = \prod_{p\in A} \mathcal{O}_p = \prod_{p\in B} \mathcal{O}_p = \prod_{p\in C} \mathcal{O}_p = \prod_{p\in D} \mathcal{O}_p, \tag{6}$$

Therefore, a single cube-flip excitation cannot be created alone, and is a topological excitation. Remarkably, the fundamental cube excitation in this model is completely *immobile*, as we observe through the following physical argument. In the cubic model, acting on the ground-state with a single Majorana fermion flips the eigenvalues of four adjacent cube operators, as shown in Figure 1(b). This four-cube excitation may trivially move by acting with a Majorana bilinear. If the fundamental cube excitation were mobile, then it would be possible to move it in any arbitrary direction, as the cube operator itself preserves all lattice symmetries. In this case, the cube excitation would have well-defined (fermion or boson) statistics, and a four-cube bound-state could never be a *fermion*. Therefore, it must be the case that the fundamental cube excitation is frozen. A rigorous proof of the immobility of the fundamental excitation is given in Section IV using the polynomial representation of the ideal Majorana Hamiltonian.

Excitation	**Type**	**Statistics**	**Operator**
$ABCD$	Majorana	Fermion	γ
AA, BB, CC, DD	Dim.-2 Anyon	Boson	Pair of Adjacent Wilson Lines
AB, AC, AD, BC, BD, CD	Dim.-1 Particle	——	Single Wilson Line
A, B, C, D	Fracton	——	Membrane

TABLE I. Hierarchy of excitations in the Majorana cubic model. The fundamental cube excitation is a fracton, while two-fracton bound-states can behave as particles that are either free to move along one- or two-dimensional surfaces. The operator that creates each type of excitation is indicated.

We now analyze the fracton bound-states in the Majorana cubic model in detail, along with the mutual statistics of the excitations. Using the labeling of the cube operators shown in Figure 1(b), we find the hierarchy of quasiparticles shown in Table I in the Majorana cubic model. The fundamental fracton excitation appears at the *corners* of membrane-like operators and may only be created in groups of four. Two-fracton bound-states can form dimension-1 particles or dimension-2 anyons. Remarkably, a dimension-2 anyon has π mutual statistics with a fracton lying in its plane of motion. As a result, while the fracton is immobile, its presence may be detected by a braiding experiment. Furthermore, the exact location of a single fracton within a finite volume V may be determined by braiding dimension-2 anyons in the three mutually orthogonal planes around the boundary ∂V. In this way, the exact quasiparticle content within V is effectively encoded "holographically", and may be determined by $\sim O(\ell)$ braiding experiments, where ℓ is the linear size of a box bounding V.

We now proceed to explore the hierarchy of excitations in detail.

Dimension-1 Particle: The dimension-1 particle may be created by acting with a single Wilson line operator, defined by the product of the Majorana operators along a straight path ℓ. Up to an overall pre-factor of

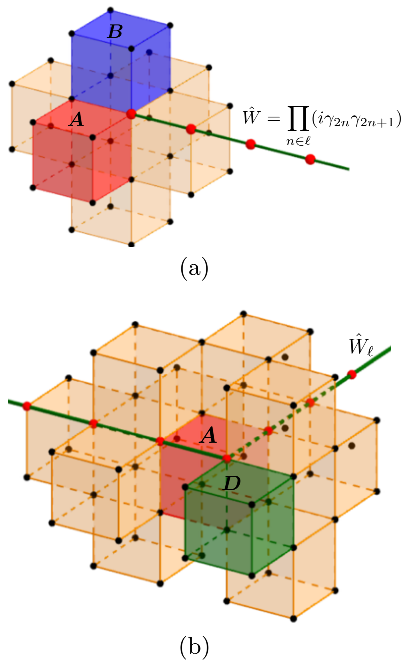


FIG. 2. **Dimension-1 Particle**: Excitations (colored) may be created by acting with Wilson line operators. In (a), a straight Wilson line creates pairs of dimension-1 particles at the endpoints. The dimension-1 particle may hop freely in the direction of the Wilson line, by acting with Majorana bilinear terms. Remarkably, the dimension-1 particle cannot hop in any other direction without creating additional excitations. Introducing a "corner" in the Wilson line, as in (b), creates an additional topological excitation localized at the corner.

$\pm 1, \pm i$, we write the Wilson line operator as

$$\hat{W}_\ell \propto \prod_{n\in\ell}\gamma_n. \tag{7}$$

As shown in Figure 2(a), the straight Wilson line anticommutes with two cube operators at each of its endpoints; the two cube excitations at a given endpoint are of different types. As a result, $\hat{W}_\ell$ creates pairs of excitations of the form AB, AC, AD, BC, BD, or CD. Remarkably, these two-fracton bound-states are only free to move along a line, by simply extending the Wilson line operator $\hat{W}_\ell$ by acting with a Majorana bilinear along the path ℓ. If we try to move this two-fracton bound-state in a plane, we must introduce a corner in the Wilson line, which localizes an additional *topological* excitation at the corner, as shown in Figure 2(b); the excitation cannot be removed by the action of any local operator. As the pattern of excitations produced by a Wilson line $\hat{W}_\ell$ is sensitive to the *geometry* of ℓ, the two-fracton bound-states AB, AC, AD, BC, BD, and CD are restricted to move along a line and behave as dimension-1 particles. We emphasize that they cannot move in a higher-dimensional space without creating additional cube excitations.

Dimension-2 Anyon: Acting with a pair of adjacent Wilson lines $\hat{W}^{(1)}_\ell$ and $\hat{W}^{(2)}_{\ell'}$ along parallel paths ℓ and ℓ', respectively, also creates a pair of two-fracton bound-states localized at the ends, as shown in Figure 3(a). At each end of the path, however, the operator $\hat{W}^{(1)}_\ell \hat{W}^{(2)}_{\ell'}$ now creates pairs of cube excitations of the *same* type (AA, BB, CC or DD). These two-fracton bound-states, where each fracton is of the same type, are allowed to move freely in the two-dimensional plane orthogonal to the shortest line segment connecting the two paths ℓ and

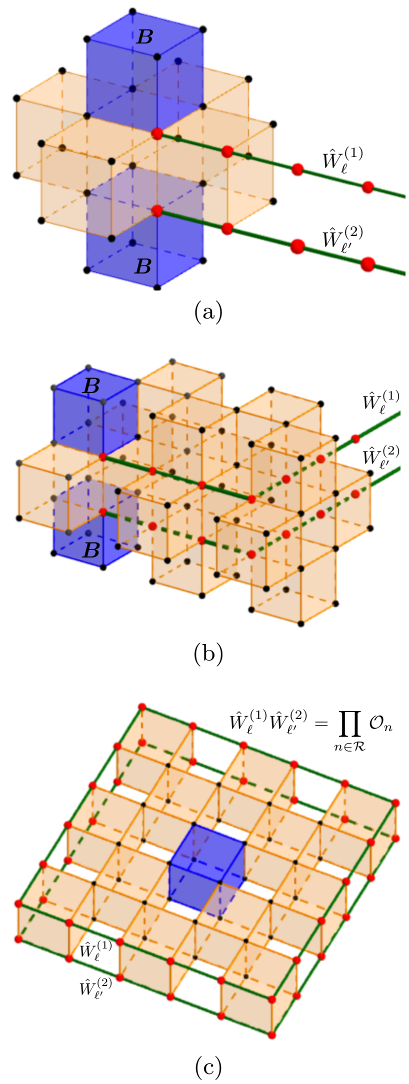


FIG. 3. **Dimension-2 Anyon**: Acting with two adjacent Wilson line operators $\hat{W}_1$ and $\hat{W}_2$ creates pairs of excitations at the endpoints of the same type (AA, BB, CC or DD). These two-fracton excitations are free to move in a two-dimensional plane orthogonal to the shortest line segment connecting the pair of Wilson lines. Furthermore, in (b) we may detect a fracton (colored blue) by braiding a dimension-2 anyon around a closed loop enclosing the fracton. As the braiding operator, a pair of closed Wilson line operators $\hat{W}_1\hat{W}_2$, is equal to the product of the enclosed cube operators as shown above. Therefore, the braiding produces an overall minus sign if an odd number of fractons are enclosed.

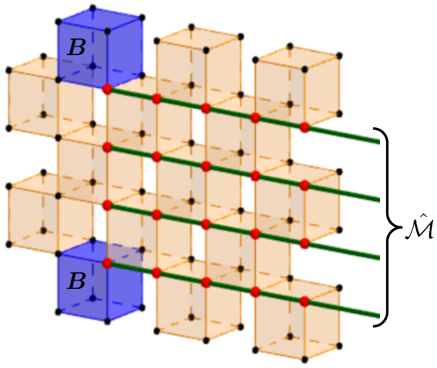


FIG. 4. **Membrane Operator & Fracton Excitations**: Acting with a product of Majorana operators on a surface Σ creates localized excitations at the corners of the boundary $\partial\Sigma$ as shown above.

ℓ' without creating additional excitations; this is shown in Figure 3(b). We note that detailed geometric features of a *single* Wilson line, such as the presence of sharp corners, determine the pattern of excitations created from the ground-state. However, when acting with an appropriate *pair* of adjacent Wilson lines, the excitations created at the sharp corners may be annihilated. Therefore, a pair of adjacent Wilson lines may be deformed in the plane with no energy cost. We conclude that the AA, BB, CC and DD two-fracton bound-states are dimension-2 anyons.

Braiding a dimension-2 anyon around a closed loop in the plane is equivalent to acting with the product of cube operators within the two-dimensional region enclosed by the loop; this is shown for a particular choice of loop in Figure 3(c). As a result, braiding a dimension-2 anyon around a closed loop enclosing a single fracton in the plane produces an overall minus sign. The ability to detect a fracton with a dimension-2 anyon produces non-trivial mutual statistics between the dimension-2 anyon and other particles in the excitation spectrum of the Majorana cubic model. First, a dimension-2 anyon has π mutual statistics with any dimension-1 particle in the same plane, as braiding the dimension-2 anyon in a closed loop will only detect one of the two fractons that make up the dimension-1 particle. Furthermore, the dimension-2 anyon has π mutual statistics with dimension-2 anyons that are free to move in adjacent, parallel planes.

Fractons and Membrane Operators: Acting with Majorana operators on a flat, two-dimensional membrane Σ creates fracton excitations at the *corners* of the boundary of Σ, as shown in Figure 4. We write the membrane operator up to an overall pre-factor of $\pm 1, \pm i$ as

$$\hat{\mathcal{M}} \propto \prod_{n\in\Sigma} \gamma_n. \tag{8}$$

For a rectangular membrane in the x-y plane, the boundary $\partial\Sigma$ is a closed, rectangular loop with dimensions ℓ_x and ℓ_y. We note that if ℓ_x and ℓ_y are both even, then the fracton excitations created at the corners of $\partial\Sigma$ will all be of the same type. Alternatively, if ℓ_x is odd and ℓ_y is even, then the pairs of fracton excitations separated in the y-direction will be of the same type, while fractons separated in the x-direction will be distinct.

Extensive Topological Degeneracy: Using the algebraic representation of the Majorana cubic model, we compute its ground-state degeneracy D_0 to be

$$\log_2 D_0 = 3L - 3 \tag{9}$$

on an $L \times L \times L$ three-torus, with periodic boundary conditions imposed in the x, y, and z directions, with each cube having unit side length. Pairs of string-like Wilson loop operators wrapping non-trivial cycles of the torus – corresponding to tunneling dimension-2 anyons – distinguish the ground-state sectors. As the number of distinct dimension-2 anyons grows linearly with system size, the ground-state degeneracy is necessarily extensive.

We emphasize that the algebraic approach allows us to systematically search for topologically-ordered, ideal Majorana Hamiltonians, rigorously characterize the nature of excitations, and calculate the ground-state degeneracy in a wide range of Majorana models using techniques in algebraic geometry. As a result, the next two sections of this work introduce and focus on the polynomial representation of ideal Majorana Hamiltonians and draw broad conclusions based on this representation. In Section III, we present the 6 distinct three-dimensional Majorana models with nearest-neighbor interactions that are topologically-ordered. In particular, one of our models, which may naturally be written in terms of complex fermions on an fcc lattice, has a fundamental excitation that may only freely move along a line in the (1,1,1) direction.

We conclude, in Section IV, with a proof of the presence of fractons in the Majorana cubic model, and briefly outline the phenomenology of excitations in the remaining models.

II. TOPOLOGICAL ORDER IN COMMUTING MAJORANA HAMILTONIANS

In this section, we introduce a representation of the operators in the ideal Majorana Hamiltonian (1) as a vector of Laurent polynomials over the finite field $\mathbb{F}_2$. The algebraic representation provides an important starting point for studying and classifying Majorana Hamiltonians. We demonstrate that the following conditions, that

(i) All operators in the ideal Hamiltonian mutually commute, and

(ii) Degenerate ground-states of the Hamiltonian are locally indistinguishable

may be phrased entirely in the polynomial representation. The ground-state degeneracy of an ideal Majorana

Hamiltonian (1) on the torus can be computed as the dimension of a quotient ring [9].

We demonstrate that an ideal Majorana Hamiltonian obeying (i) and (ii) on a lattice with a two-site basis and a single interaction term per unit cell may be specified by a *single* polynomial over $\mathbb{F}_2$. We use this result to systematically search for and characterize commuting Majorana Hamiltonians. In $d = 3$ dimensions, we find 6 distinct, non-trivial models with nearest-neighbor interactions, extensive topological degeneracy, and a dimensional hierarchy of excitations.

A. Algebraic Representation

To study commuting Majorana Hamiltonians, we represent the operator $\mathcal{O}$ appearing in Eq. (1) as a polynomial over the field $\mathbb{F}_2$. A similar mapping has been introduced in the context of Pauli Hamiltonians [9]. Consider a d-dimensional lattice with translation operators $\{\boldsymbol{t}_1, \ldots, \boldsymbol{t}_d\}$ and an n-site unit cell. We restrict n to be an *even* integer so that there is a well-defined number of complex fermions per lattice site. We label the Majorana fermions within the unit cell at the origin as γ_j for $j = 1, 2, \ldots, n$. All other Majorana fermions on the lattice are obtained by acting with translation operators.

Any Hermitian operator acting on this lattice may be written as a sum of products of Majorana operators. Formally, we may write a summand $\mathcal{O}$ as

$$\mathcal{O} = \prod_{j=1}^{n} \prod_{\{n_i\}} (\boldsymbol{t}_1^{n_1} \cdots \boldsymbol{t}_d^{n_d} \cdot \gamma_j)^{c_j(n_1,\ldots,n_d)} \tag{10}$$

with $n_i \in \mathbb{Z}$ and $c_j(n_1, \ldots, n_d) \in \{0, 1\}$. For simplicity, we have omitted the prefactor $\pm 1, \pm i$ in the expression for $\mathcal{O}$, which plays no role in our analysis. We introduce a purely algebraic representation of this operator by noting that any product of translation operators may be written as a monomial, e.g. $\boldsymbol{t}_1^{n_1} \cdots \boldsymbol{t}_d^{n_d} \iff x_1^{n_1} \cdots x_d^{n_d}$. In this way, the action of the translation group is naturally represented by monomial multiplication.

Recall that distinct Majorana fermions anti-commute and that each Majorana operator squares to the identity. Therefore at each site within a unit cell, the identity $\mathbb{1}$ and γ under multiplication form the group $\mathbb{Z}_2$, with the two operators represented by the group elements 0 and 1, respectively. In this representation, the operator equality $\gamma^2 = \mathbb{1}$ maps to the $\mathbb{Z}_2$ group addition $1 + 1 = 0$. This simple algebra of Majorana fermions allows us to write any product of Majorana operators as the *sum* of monomials – representing the location of each Majorana operator via the action of the translation group – with $\mathbb{Z}_2$ coefficients. As an example, consider a lattice with a single site per unit cell, and the Majorana operator γ at the origin. A Majorana bilinear admits the following polynomial representation:

$$\gamma \cdot (\boldsymbol{t}_1^{m_1} \boldsymbol{t}_2^{m_2} \cdots \boldsymbol{t}_d^{m_d} \cdot \gamma) \iff 1 + x_1^{m_1} x_2^{m_2} \cdots x_d^{m_d}. \tag{11}$$

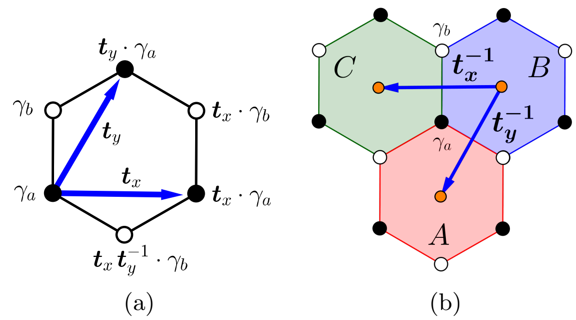


FIG. 5. The Majorana plaquette model, as studied in [1]. Consider a honeycomb lattice with a single Majorana fermion on each lattice site. We define an operator $\mathcal{O}_p$ as the product of the six Majorana fermions on the vertices of a hexagonal plaquette p, as shown in (a). The colored plaquettes in (b) correspond to the three distinct bosonic excitations (A, B, or C) that may each be created in pairs by acting with Wilson line operators.

In this notation, operator multiplication corresponds to polynomial addition with $\mathbb{Z}_2$ coefficients.

For the general case of a unit cell with n sites, we represent a product of Majorana operators as a *vector* of polynomials over $\mathbb{F}_2$, with the j-th entry of the vector representing the action of the translation group on γ_j, the j-th Majorana fermions in the unit cell at the origin. For example, the operator (10) may be written as

$$S(x_1, ..., x_d) = \sum_{\{n_i\}} x_1^{n_1} \cdots x_d^{n_d} \begin{pmatrix} c_1(n_1, ..., n_d) \\ c_2(n_1, ..., n_d) \\ \vdots \\ c_{2n}(n_1, ..., n_d) \end{pmatrix} \tag{12}$$

Adopting the terminology in Ref. [9], we refer to S as the "stabilizer map" for the remainder of this work.

To illustrate the algebraic representation of operators in commuting Majorana Hamiltonians, we present a concrete example. Consider the Majorana plaquette model in Ref. [1], which is defined on a two-dimensional honeycomb lattice with one Majorana fermion per site and a Hamiltonian of the form (1) where $\mathcal{O}_p$ is the product of the six Majorana fermions at the vertex of a hexagonal plaquette p. We show a single hexagonal plaquette on the lattice in Figure 5(a), along with the Majorana fermions γ_a and γ_b within the two-site unit cell. The corresponding stabilizer map $S(x, y)$ for the six-Majorana operator is given by:

$$S(x, y) = \begin{pmatrix} 1 + x + y \\ \\ 1 + x + x\bar{y} \end{pmatrix}. \tag{13}$$

Here, we adopt the notation that $\bar{y} \equiv y^{-1}$, $\bar{x} \equiv x^{-1}$. As shown in Ref. [1], this Hamiltonian exhibits a novel form of $\mathbb{Z}_2$ topological order with fermion parity-graded excitations and exact anyon permutation symmetries.

Next, we consider the action of an arbitrary operator W on the ground state $|\Psi\rangle$ of the commuting Majorana

Hamiltonian. When W anticommutes with an operator $\mathcal{O}_n$ in the Hamiltonian, it flips its eigenvalue and thus creates an excitation. We use a polynomial to record the locations of all excitations in the state $W|\Psi\rangle$; each location is labeled by the translation vector connecting it to the origin. Specifically, for a Hamiltonian with stabilizer map $S(x_1, \cdots, x_d)$ and an arbitrary operator W with a polynomial representation $P(W)$ of the form (12), we define the "excitation map" $E(x_1, \ldots, x_d)$ so that $E(x_1, \ldots, x_d) \cdot P(W) \in \mathbb{F}_2[x_1^{\pm 1}, \cdots, x_d^{\pm 1}]$ describes the excitations created by W. In the Supplemental Material [13], we demonstrate that E is simply given from the stabilizer map as follows:

$$E(x_1, \ldots, x_d) = \overline{S(x_1, \cdots, x_d)} \tag{14}$$

where $\overline{S(x_1, \ldots, x_d)} \equiv [S(\overline{x_1}, \ldots, \overline{x_d})]^T$.

As an example, the excitation map for the Majorana plaquette model is given by $E(x, y) = (1 + \bar{x} + \bar{y},\, 1 + \bar{x} + \bar{x}y)$. Below, we show the action of the operator γ_a at the origin in the Majorana plaquette model, which creates three adjacent excitations as specified by the red points. The locations of the excitations are obtained by performing the matrix multiplication of E with the polynomial representation $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ of γ_a:

$$E(x, y) \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 1 + \bar{x} + \bar{y}. \tag{15}$$

Therefore, the action of γ_a may be represented by the polynomial $1 + \bar{x} + \bar{y}$, labeling the locations of the flipped plaquettes; here, the plaquette operator corresponding to the origin (i.e. the location "1") is to the right of γ_a, as can be seen from its polynomial representation (13).

A dictionary that summarizes the relationship between Majorana operators and polynomials is given in Table II.

B. Topological Order and Ground-State Degeneracy in the Algebraic Representation

The polynomial representation of Majorana operators serves as a starting point for constructing commuting Majorana Hamiltonians that exhibit topological orders. As we demonstrate in the Supplemental Material [13], for a translationally invariant Majorana Hamiltonian with a single operator per lattice site, *all* operators mutually commute *if and only if* its stabilizer map $S(x_1, \cdots, x_d)$ satisfies the condition

$$\overline{S(x_1, \ldots, x_d)} \cdot S(x_1, \ldots, x_d) = 0. \tag{16}$$

More generally, if the Hamiltonian contains multiple operators per lattice site $\{\mathcal{O}^{(i)}\}$, then we may define a set of stabilizer maps for each type of operator $\{S_i\}$, so that the condition $\overline{S_i(x_1, \ldots, x_d)} \cdot S_j(x_1, \ldots, x_d) = 0$ for all i, j, guarantees that all terms in the Hamiltonian commute.

Operator	**Polynomial**
Majorana Fermion γ_j [$j = 1, \ldots, n$ for each site in the unit cell]	Vector over $\mathbb{F}_2$ $\vec{e}_j$ [n-dimensional unit vector with j^{th} entry equal to 1]
Translation $\boldsymbol{t}_1^{n_1} \boldsymbol{t}_2^{n_2} \cdots \boldsymbol{t}_d^{n_d} \gamma_j$	Monomial Multiplication $x_1^{n_1} x_2^{n_2} \cdots x_d^{n_d} \vec{e}_j$
Multiplication $\gamma_j \cdot \boldsymbol{t}_k^n \gamma_\ell$	Addition in $\mathbb{F}_2[x_1, \cdots, x_d]$ $\vec{e}_j + (x_k)^n \, \vec{e}_\ell$

TABLE II. Summary of the polynomial representation of Majorana operators. An arbitrary operator in d spatial dimensions, written as the product of Majorana fermions, may be represented as a vector with entries in the (Laurent) polynomial ring $\mathbb{F}_2[x_1^{\pm 1}, \cdots, x_d^{\pm 1}]$.

We next formulate a necessary and sufficient algebraic condition for topological order in commuting Majorana Hamiltonians, which requires that any degenerate ground-states of a topologically-ordered Hamiltonian cannot be distinguished by local operators. The local indistinguishability is equivalent to the condition that, for any *local* operator M_i

$$\Pi_{\text{GS}} \, M_i \, \Pi_{\text{GS}} = c(M_i) \, \Pi_{\text{GS}} \tag{17}$$

where Π_{GS} is the projector onto a ground-state sector and $c(M_i)$ is a constant that *only* depends on the operator. For our case, consider an operator M_I that is the product of Majorana operators, and $P(M_i)$, the polynomial representation of M_i. If M_i anti-commutes with *any* term in the Hamiltonian, then M_i creates excitations when acting on the ground-state, and we have $\Pi_{\text{GS}} \, M_i \, \Pi_{\text{GS}} = 0$. If M_i commutes with the Hamiltonian, then $P(M_i) \in \ker E$, as M_i creates no excitations. In this case, the condition $\Pi_{\text{GS}} \, M_i \, \Pi_{\text{GS}} = c(M_i)\Pi_{\text{GS}}$ is guaranteed if M_i may be written as a product of operators already appearing in the Hamiltonian. More generally, any local operator M that commutes with the Hamiltonian then takes the form:

$$M = \sum_i M_i \tag{18}$$

where each term M_i is the product of operators already appearing in the Hamiltonian. This condition is necessary for distinct ground-state sectors to be locally indistinguishable.

In our polynomial representation, we enforce the condition (17) by requiring that the stabilizer and excitation maps satisfy the following condition on an infinite lattice

$$\ker E \cong \text{im } S. \tag{19}$$

Recall that the image of S is the set of all polynomial linear combinations of $S(x_1, ..., x_d)$, taking the form of

$$\sum_{\{n_i\}} x_1^{n_1} \cdots x_d^{n_d} S(x_1, \ldots, x_d), \tag{20}$$

and representing all operators that can be written as a product of the commuting operators appearing in the Hamiltonian. On the other hand, the kernel of the excitation map E is the set of all operators that do not create any excitation when acting on the ground state. The above algebraic condition (19) for topological order is thus equivalent to the statement that *any* operator that creates no excitations on a ground state on an infinite lattice is necessarily a product of operators $\{\mathcal{O}_n\}$ already appearing the Hamiltonian. In other words, there are no non-trivial, locally conserved quantities, and any degenerate ground-states of the Hamiltonian are locally indistinguishable. In summary, imposing the commutativity (16) and local indistinguishability (19) conditions on a stabilizer map produces an ideal Majorana Hamiltonian with topological order.

We may compute the ground-state degeneracy of an ideal Majorana Hamiltonian in the polynomial representation via constraint-counting. A lattice with $2M$ Majorana fermions defines a 2^M-dimensional Hilbert space. On the torus, however, fixing the eigenvalues of the commuting operators in the ideal Majorana Hamiltonian only imposes $M - k$ multiplicatively independent constraints, since the product of certain operators appearing in the Hamiltonian will yield the identity. The ground-state degeneracy is simply given by the space of states satisfying the constraints, which is precisely $2^M/2^{M-k} = 2^k$. As each ideal Majorana Hamiltonian in this work consists of exactly one term for each pair of Majorana modes, we see that k is directly equal to the number of constraints on the commuting operators appearing in the Hamiltonian.

For example, in the Majorana plaquette model, we may group the plaquette operators $\{\mathcal{O}_p\}$ into three types (A, B, and C) as shown in Figure 5(b). On the torus, the product of the A, B, and C-type operators is identical and equal to the total fermion parity [1]. This yields the following two independent constraints:

$$\prod_{p\in A} \hat{\mathcal{O}}_p \prod_{p\in B} \hat{\mathcal{O}}_p = \prod_{p\in B} \hat{\mathcal{O}}_p \prod_{p\in C} \hat{\mathcal{O}}_p = 1, \tag{21}$$

and produces a 2^2-fold degenerate ground-state on the torus. These constraints may be compactly represented using polynomials labeling the locations of the A, B and C-type plaquettes. For example, the collection of all A plaquettes is captured by the polynomial

$$p_A = (1 + xy + x^2y^2) \left(\sum_{n=0}^{L-1} x^{3n} \right) \left(\sum_{m=0}^{L-1} y^{3m} \right). \tag{22}$$

It is straightforward to expand p_A to verify that the exponents of the non-zero terms describe the positions of A plaquettes. Here, L specifies the periodic boundary conditions in the x and y directions, so that $x^L = 1,\ y^L = 1$. Similarly, the collections of all plaquettes in B and C are encoded in yp_A and $\bar{x}yp_A$, respectively. The constraints (21) arise from the fact that $(p_A + yp_A)S = 0$, using Eq. (13) and the boundary conditions.

In terms of the stabilizer map, any multiplicative constraint on the operators in the ideal Majorana Hamiltonian on the torus is in one-to-one correspondence with a solution p of the equation $p \cdot S = 0$, so that the polynomial p is an element of the kernel of S. Therefore, the number of independent relations is given by

$$k = \dim_{\mathbb{F}_2}[\ker(S)]. \tag{23}$$

We rewrite the expression (23) in a more convenient form for calculations that will also allow us to make general statements about the scaling behavior of the ground-state degeneracy with system size for an ideal Majorana Hamiltonian of the form (1). As proven in Corollary 4.5 in Ref. [9], Eq. (23) is equivalent to the dimension of the following quotient ring:

$$k = \log_2 D = \dim_{\mathbb{F}_2} \left(\frac{\mathbb{F}_2[x_1, \cdots, x_d]}{I(S) + \mathfrak{b}_L} \right). \tag{24}$$

Here, $I(S)$ the *ideal* generated by the stabilizer map; if $S^T = (s_1, \ldots, s_{2n})$ then $I(S)$ is the space of polynomials in $\mathbb{F}_2[x_1^{\pm 1}, \ldots, x_d^{\pm 1}]$ obtained as a linear combination of $\{s_i\}$:

$$I(S) \equiv \left\{ p = \sum_{i=1}^{2n} c_i s_i \;\middle|\; c_i \in \mathbb{F}_2[x_1^{\pm 1}, \ldots, x_d^{\pm 1}] \right\}. \tag{25}$$

We will denote the ideal generated by a set $\{s_1, \ldots, s_n\}$ by $\langle s_1, \ldots, s_n \rangle$. Furthermore, we define the ideal $\mathfrak{b}_L \equiv \langle x_1^L - 1, \ldots, x_d^L - 1 \rangle$. As the quotient space identifies the zero element in $\mathbb{F}_2[x_1^{\pm 1}, \ldots, x_d^{\pm 1}]$ with the generators of $I(S) + \mathfrak{b}_L$, we observe that the ideal $\mathfrak{b}_L$ is used to enforce the periodic boundary conditions on a d-dimensional torus with side-length L.

We emphasize that the ideal $I(S)$ is the space of excitations that can be created through the action of any operator on the ground-state. Therefore, the expression (24) may be physically interpreted as counting certain superselection sectors of the ideal Majorana Hamiltonian. Any $p \in \mathbb{F}_2[x_1^{\pm 1}, \cdots, x_d^{\pm 1}]$ corresponds to a virtual eigenstate of the Hamiltonian with excitations at the locations specified by the polynomial p. Certain states, however, cannot be created by acting with an operator on a ground-state $|\Psi\rangle$ due to the k constraints on the commuting operators. For example, in the Majorana plaquette model, it is impossible to obtain a state with a single plaquette excitation by acting on the ground-state, since the products of A, B and C plaquettes must satisfy (21). As $I(S)/(I(S) \cap \mathfrak{b}_L)$ is the set of excitations that can be created by the action of operators

Algebraic Expression	Physical Interpretation
$\overline{S(x_1,..,x_d)} \cdot S(x_1,..,x_d) = 0$	Commutativity condition, that all operators $\{\mathcal{O}_n\}$ appearing in the Hamiltonian mutually commute.
$\mathrm{im}(S)$	Set of operators that may be written as the product of commuting operators $\{\mathcal{O}_n\}$ in the Hamiltonian.
$\ker(E)$	Set of operators that create no excitations when acting on the ground-state $\lvert\Psi\rangle$.
$k = \dim_{\mathbb{F}_2}[\ker(S)]$	The number of independent relations among the commuting operators in the Hamiltonian, when placed on a torus. The ground state degeneracy $D = 2^k$.
$p \in \mathbb{F}_2[x_1^{\pm}, \ldots, x_d^{\pm}]$	A configuration of excitations, specified by the locations of operators $\{\mathcal{O}_n\}$ with flipped eigenvalue -1.
$q \in I(S)$	A configuration of excitations that may be created by acting with an operator on the ground-state $\lvert\Psi\rangle$.

TABLE III. Dictionary of various algebraic quantities and their physical interpretation in the context of a commuting Majorana Hamiltonian.

on the ground-state for a finite system, the quotient space $\left(\mathbb{F}_2[x_1^{\pm 1}, \cdots, x_d^{\pm 1}]/\mathfrak{b}_L\right) / \left(I(S)/(I(S) \cap \mathfrak{b}_L)\right) = \mathbb{F}_2[x_1, \cdots, x_d]/\left(\mathfrak{b}_L + I(S)\right)$ is the set of virtual eigenstates of the Hamiltonian that cannot be deformed into each other through the action of any local operator. For the Majorana plaquette model, this quotient space is

$$\frac{\mathbb{F}_2[x, y]}{\langle 1 + x + y,\, x + y + xy,\, x^L - 1,\, y^L - 1\rangle} \cong \mathbb{F}_2^2 \qquad (26)$$

when $L \bmod 3 = 0$ so that there are an equal number of A, B, and C plaquettes. In this case, the trivial vacuum (0) and a state with a single plaquette excitation (1) on A, B, or C correspond to the four superselection sectors in the quotient ring.

The expression for the ground-state degeneracy (24) is convenient as the dimension of a quotient ring may be computed using algebraic techniques. Most often, we will determine a Gröbner basis for the ideal $I(S) + \mathfrak{b}_L$ in order to determine membership in the quotient ring. For a polynomial ring R, we may define a total monomial ordering (e.g. lexicographic order with $x_1 \succ x_2 \succ \ldots \succ x_d$); we denote the leading monomial in a polynomial $h \in R$ as $\mathrm{LM}(h)$ with respect to this ordering. Given an ideal $I = \langle s_1, \cdots, s_n\rangle$ of a polynomial ring, there exists a canonical choice of generators for the ideal, known as the Gröbner basis $\{g_1, \cdots, g_n\}$, with the property that for any $f \in I$, $\mathrm{LM}(f) \in \langle \mathrm{LM}(g_1), \ldots, \mathrm{LM}(g_n)\rangle$, i.e. any element of the ideal has a leading term contained in the ideal generated by the leading terms of the Gröbner basis. As a result, the dimension of the quotient ring $\dim[R/I]$ is merely given by the number of monomials that are smaller (in the monomial ordering) than all of the leading terms in the Gröbner basis. This is because any polynomial $p \in R$ may be reduced by the Gröbner basis until the leading term of the reduced polynomial satisfies $\mathrm{LM}(p_{\mathrm{red}}) < \mathrm{LM}(g_i)$ for all $i = 1, \ldots, n$. Therefore, each monomial m satisfying $m < \mathrm{LM}(g_i)$ for all i corresponds to a unique representative of the quotient ring R/I.

We note that calculations of the ground-state degeneracy for any commuting Majorana Hamiltonians presented in this work are done by determining a Gröbner basis for the ideal $I(S) + \mathfrak{b}_L$. In this way, the calculation of the degeneracy reduces to counting points in an algebraic set.

C. Unitary and Stable Equivalence

The polynomial representation of the ideal Majorana Hamiltonian contains built-in redundancies, since we may re-define the unit cell or translation operators on the d-dimensional lattice. For the stabilizer map, the translation corresponds to multiplication of any entry of $S(x_1, \ldots, x_d)$ by a monomial. In this way, a stabilizer map $S(x_1, \ldots, x_d)$ is only defined up to monomial multiplication on each of its entries. Furthermore, for an ideal Majorana Hamiltonian with longer-range interactions, we may always enlarge the unit cell. As our focus will be on Majorana models with nearest-neighbor interactions, we neglect this redundancy in the stabilizer map.

Equivalence relations, given by local unitary transformations on ideal Majorana Hamiltonians, may also be considered in the polynomial language. For instance, two ideal Majorana Hamiltonians, defined by stabilizer maps $S(x_1, \ldots, x_d)$ and $S'(x_1, \ldots, x_d)$ are *unitarily* equivalent if there exists a matrix U such that $S'(x_1, \ldots, x_d) = U \cdot S(x_1, \ldots, x_d)$ where $U \in O(n; \mathbb{F}_2)$, an orthogonal matrix over $\mathbb{F}_2$ satisfying $U^T U = 1$. This guarantees that if $\overline{S(x_1, \ldots, x_d)} \cdot S(x_1, \ldots, x_d) = 0$, then $\overline{S'(x_1, \ldots, x_d)} \cdot S'(x_1, \ldots, x_d) = 0$ as well. Finally, we take two stabilizer maps to be *stably* equivalent if we can

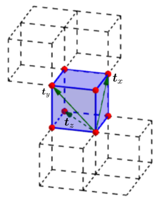
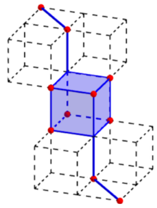
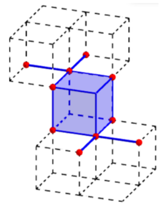
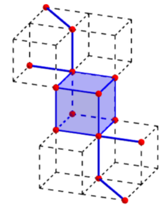
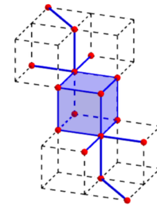
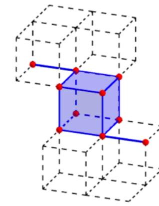
	$f_1(x,y,z)$	$f_2(x,y,z)$	$f_3(x,y,z)$	$f_4(x,y,z)$	$f_5(x,y,z)$	$f_6(x,y,z)$
f	$1+x+y+z$	$1+z+xy$ $+yz+xz$	$1+x+y$ $+yz+xz$	$1+y+z$ $+xy+yz+xz$	$1+x+y+z$ $+xy+yz+xz$	$1+x+y$ $+z+yz$
$\mathcal{O}$						
$\log_2 D$	$3L-2$	$6L-6\ (L=3n)$ $0\ (L\neq 3n)$	$4L/3\ (L=6\cdot 2^n)$ $8L/5\ (L=5\cdot 2^n)$ $\vdots$	$4L-4\ (L=2n)$ $2L-1\ (L=2n+1)$	$2L-2\ (L=2^{2n+1}-1)$ $2L-4\ (L=2^{2n}-1)$ $\vdots$	$4L/3\ (L=6\cdot 2^n)$ $8L/5\ (L=5\cdot 2^n)$ $\vdots$

TABLE IV. We find 7 distinct, topologically-ordered ideal Majorana Hamiltonians with nearest-neighbor interactions on a lattice with a two-site unit cell in $d=3$ spatial dimensions. The first model $f_0(x,y,z)=1+y+z$ (not shown) is a trivial stack of two-dimensional Majorana plaquette models, considered in Ref. [1]. For the remaining 6 models, the action of the elementary operator $\mathcal{O}$ appearing in the ideal Majorana Hamiltonian is shown above as the product of the Majorana fermions on the indicated red dots. In the depiction of the Majorana cubic model $f_1(x,y,z)$, we have also shown the choice of translation vectors $\{\boldsymbol{t}_x,\boldsymbol{t}_y,\boldsymbol{t}_z\}$ on the lattice, originating from one of the sites within the unit cell; to compute the ground-state degeneracy on an $L\times L\times L$ torus, we impose periodic boundary conditions by requiring that $\boldsymbol{t}_x^L=\boldsymbol{t}_y^L=\boldsymbol{t}_z^L=1$. The topological ground-state degeneracy (D) of each of these models is extensive. For models $f_3(x,y,z)$, $f_5(x,y,z)$, and $f_6(x,y,z)$, the ground-state degeneracy on the three-torus is a highly sensitive function of system size, and only the maximum value of the degeneracy is shown for the indicated choices of L.

obtain one from the other by attaching a trivial (dimerized) set of Majorana fermions. This is expressed as $S(x_1,\ldots,x_d)^T\sim S(x_1,\ldots,x_d)^T\oplus(0,\cdots,0,1,1)$.

III. EXTENSIVE TOPOLOGICAL DEGENERACY IN $d\geq 3$

Using the commutativity (16) and local indistinguishability (19) conditions, and the built-in redundancy in the polynomial description, we demonstrate in the Supplemental Material [13], that an ideal Majorana Hamiltonian defined on a d-dimensional lattice with a two-site basis is topologically-ordered if the stabilizer map may be written in the following form, after multiplying each entry by appropriate monomials:

$$S=\begin{pmatrix} f(x_1,\cdots,x_d) \\ \overline{f(x_1,\cdots,x_d)}\end{pmatrix} \tag{27}$$

where $f(x_1,\cdots,x_d)\in\mathbb{F}_2[x_1^{\pm 1},\cdots,x_d^{\pm 1}]$ and f and $\bar{f}$ are co-prime, i.e., f and $\bar{f}$ have no common polynomial factors. As a result, a topologically-ordered, ideal Majorana Hamiltonian with a two-site basis may be specified by a single polynomial. For example, the stabilizer map for the Majorana plaquette model takes the form $S^T=(f(x,y),\,x\cdot\overline{f(x,y)})$ with $f(x,y)=1+x+y$.

The dimension of the quotient ring (24) scales as the dimension of the space of the zeros of the ideal $I(S)$ over the field extension $\mathbb{F}_{2^m}$ when $L=2^m-1$. As a result, for an ideal Majorana Hamiltonian (1) with a two-site unit cell, the space of solutions to

$$f(x_1,\ldots,x_d)=0,\qquad \overline{f(x_1,\ldots,x_d)}=0 \tag{28}$$

generally defines an $(d-2)$-dimensional variety, so that the ground-state degeneracy scales on the d-dimensional torus with side-length L as $\log_2 D=cL^{d-2}+\cdots$ for some constant c. We emphasize that this produces a class of ideal Majorana models with extensive topological degeneracy in $d=3$ dimensions. Remarkably, while our models have a two-dimensional Hilbert space and a single interaction term per lattice site, this only constrains the full Hilbert space up to extensive topological degeneracy.

We have exhaustively searched for distinct, ideal Majorana Hamiltonians with a two-site basis and nearest-neighbor interactions in $d=2$ and $d=3$ spatial dimensions. This is straightforward as the orthogonal group $O(2;\mathbb{F}_2)=\{\mathbb{1}_{2\times 2},\,\sigma^x\}$ so that the space of local unitary transformations between these ideal Majorana Hamiltonians is trivial. In $d=2$ spatial dimensions, the only such Hamiltonian is the Majorana plaquette model with

$$f(x,y)=1+x+y. \tag{29}$$

In $d=3$ dimensions, however, we find 7 distinct Majorana models with a two-site basis and nearest-neighbor interactions. The first model has the polynomial representation $f_0(x,y,z)=1+y+z$ and is a trivial stack of two-dimensional Majorana plaquette models. The polynomial representations of the remaining models, along with their ground-state degeneracies on a torus of side-length L are shown in Table IV. For simplicity, we have imposed periodic boundary conditions by requiring that $\boldsymbol{t}_x^L = \boldsymbol{t}_y^L = \boldsymbol{t}_z^L = 1$ for the translation vectors $\{\boldsymbol{t}_x, \boldsymbol{t}_y, \boldsymbol{t}_z\}$ shown in the representation of the Majorana cubic model $f_1(x, y, z) = 1 + x + y + z$ in Table IV. Each of the models shown exhibits extensive topological degeneracy and admits at least one topological excitation that is free to move in a sub-manifold of the full lattice.

IV. FRACTON EXCITATIONS AND DIMENSION-n ANYONS

A remarkable feature of these Majorana models is the presence of fundamental excitations that are either perfectly immobile or only free to move in a sub-manifold of the lattice; attempting to move these excitations by acting with any local operator will necessarily create additional excitations. A bound-state of these immobile excitations, however, forms a particle that can freely move along a higher-dimensional sub-manifold.

The existence of a fracton fundamental excitation may be shown rigorously in the polynomial representation of the Majorana models. An element $p \in I(S)$ of the ideal defined by the stabilizer map corresponds to a set of excitations that may be created by acting on the ground-state. The fundamental excitation is mobile if and only if it is possible to create an isolated pair of such excitations. Therefore, an ideal Majorana model admits fracton excitations if the stabilizer ideal contains no binomial terms, i.e.

$$1 + x_1^{n_1} x_2^{n_2} \cdots x_d^{n_d} \notin I(S) \tag{30}$$

for any $n_i \in \mathbb{Z}$.

We now apply the polynomial criterion for fracton excitations to the Majorana cubic model and to the model $f_5(x, y, z) = 1 + x + y + z + xy + yz + xz$, both shown in Table IV.

A. Fractons in the Majorana Cubic Model

We consider the Majorana cubic model, specified by the single polynomial $f(x, y, z) = 1+x+y+z$, so that the stabilizer map is given by $S = \left(f(x, y, z), \overline{f(x, y, z)}\right)^T$. We wish to prove that the ideal generated by the stabilizer map $I(S)$ contains no binomial terms, so that the fundamental cube excitation is a fracton. This may be shown by considering the zero-locus of the ideal, i.e., the solutions to the zeros of the generators of the ideal:

$$1 + x + y + z = 0 \tag{31}$$
$$xyz + xy + yz + xz = 0. \tag{32}$$

A polynomial p belong to $I(S)$ only if p vanishes on the zero-locus of the ideal. Note that solutions to (31) take the form $(x, y, z) = (1, \alpha, \alpha)$, $(\alpha, 1, \alpha)$ or $(\alpha, \alpha, 1)$, where α is an arbitrary element in the extension of $\mathbb{F}_2$. However, we see that the binomial $1 + x^n y^m z^\ell$ vanishes on this space of solutions only if $n = m = \ell = 0$, in which case the binomial is zero. Therefore, we conclude that

$$1 + x^n y^m z^\ell \notin I(S). \tag{33}$$

As a result, there is no way to create the fundamental cube excitation in the Majorana cubic model in pairs. Therefore, the cube excitation is an immobile fracton; a single cube excitation cannot be moved without creating additional excitations.

B. Dimension-1 Fundamental Excitations in $f_5(x, y, z)$

Now, we consider the isotropic model $f_5(x, y, z) = 1 + x + y + z + xy + yz + xz$, with stabilizer map defined by $S(x, y, z) = \left(f_5(x, y, z),\, xyz \cdot \overline{f_5(x, y, z)}\right)$. From the excitation map $E(x, y, z) \equiv \overline{S(x, y, z)}$, we find that the Majorana bilinear along the (1,1,1) direction creates a pair of fundamental excitations:

$$E(x, y, z) \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 1 + \overline{xyz}. \tag{34}$$

Therefore, the fundamental excitation in this model is clearly not a fracton. We now demonstrate that the fundamental excitation may *only* hop freely along the (1,1,1) direction, without creating additional excitations. Consider the variety $V(I)$ defined by the stabilizer ideal $I(S) = \langle 1 + x + y + z + xy + yz + xz,\, xyz + x + y + z + xy + yz + xz\rangle$, i.e. the zero-locus of the generators of the ideal over an extension of $\mathbb{F}_2$. The following is a point on the variety:

$$(x, y, z) = \left(t,\, \frac{1}{1+t},\, \frac{t+1}{t}\right) \tag{35}$$

with t in an extension of $\mathbb{F}_2$. As a result, if $1 + x^n y^m \in I(S)$, we must have from (35) that $t^n = (1 + t)^m$ for infinitely many t. This can only be true if $n = m = 0$. As a result, the fundamental excitation cannot hop freely in the xy-plane. As the generators of the ideal are symmetric under exchanging any pair of variables (e.g. $x \longleftrightarrow y$), we conclude that $1 + y^n z^m$, $1 + x^n z^m \in I(S)$ only if $n = m = 0$, so that the fundamental excitation cannot freely hop in the yz- or xz-planes. From these results, we are led to the conclusion that

$$1 + x^n y^m z^\ell \notin I(S) \tag{36}$$

when n, m and ℓ are distinct. Therefore, we have shown that the fundamental excitation in the model defined by $f_5(x, y, z)$ is restricted to hop along the (1,1,1) direction of the cubic lattice without creating additional excitations.

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