Self-dual quantum electrodynamics as boundary state of the three-dimensional bosonic topological insulator

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Self-dual Quantum Electrodynamics as Boundary State of the three dimensional Bosonic Topological Insulator

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Inspired by the recent developments of constructing novel Dirac liquid boundary states of the $3d$ topological insulator [1–3], we propose one possible $2d$ boundary state of the $3d$ bosonic symmetry protected topological state with $U(1)_e \rtimes Z_2^T \times U(1)_s$ symmetry. This boundary theory is described by a $(2+1)d$ quantum electrodynamics (QED_3) with two flavors of Dirac fermions ($N_f = 2$) coupled with a noncompact U(1) gauge field: $\mathcal{L} = \sum_{j=1}^{2} \bar{\psi}_j \gamma_\mu (\partial_\mu - i a_\mu) \psi_j - i A^s_\mu \bar{\psi}_i \gamma_\mu \tau^z_{ij} \psi_j + \frac{i}{2\pi} \epsilon_{\mu\nu\rho} a_\mu \partial_\nu A^e_\rho$, where a_μ is the internal noncompact U(1) gauge field, A^s_μ and A^e_μ are two external gauge fields that couple to $U(1)_s$ and $U(1)_e$ global symmetries respectively. We demonstrate that this theory has a "self-dual" structure, which is a fermionic analogue of the self-duality of the noncompact CP^1 theory with easy plane anisotropy [4–6]. Under the self-duality, the boundary action takes exactly the same form except for an exchange between A^s_μ and A^e_μ. The self-duality may still hold after we break one of the U(1) symmetries (which makes the system a bosonic topological insulator), with some subtleties that will be discussed.

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— 1. Introduction

A symmetry protected topological (SPT) state may have very different boundary states without changing the bulk state, depending on the boundary Hamiltonian. As was shown in Ref. 7–11, besides the well-known boundary state, *i.e.* a single $2d$ Dirac fermion, the boundary of an interacting $3d$ topological insulator (TI) could have a topological order that respects all the symmetries of the system but cannot be realized in a $2d$ system. Very recently, the "family" of boundary states of TI has been even further expanded [1–3]: it was shown that the boundary of the $3d$ TI could be a $(2+1)d$ quantum electrodynamics (QED_3) with one single flavor of gauge-charged Dirac fermion, while the flux quantum of the U(1) gauge field carries charge 1/2 under the external electromagnetic (EM) field A_μ:

$$\mathcal{L} = \bar{\psi}\gamma_\mu(\partial_\mu - i a_\mu)\psi + \frac{1}{2}\frac{i}{2\pi}\epsilon_{\mu\nu\rho} a_\mu \partial_\nu A_\rho. \quad (1)$$

This Lagrangian without time-reversal symmetry can also be realized as a pure $2d$ theory in the half-filled Landau level [12]. This boundary state Eq. 1 is particularly interesting because it suggests a duality between interacting $2d$ single Dirac fermion and the noncompact QED_3 with one gauge-charged Dirac fermion, which is further supported by the proof of S-duality in the $3d$ bulk [13], and also the exact duality (or the mirror symmetry) between certain $(2+1)d$ supersymmetric field theories [14]. This duality is a very elegant analogue of its standard bosonic version: the duality between the $(2+1)d$ XY model and the bosonic QED_3 with one flavor of gauge-charged complex boson [15, 16]. Although the infrared fate of Eq. 1 under gauge fluctuation and fermion interaction is unclear, it was demonstrated in Ref. 1–3 that Eq. 1 can be viewed as the parent state of other well-known boundary states of $3d$ TI.

In this work we will further extend the idea of Ref. 1–3, and construct novel boundary states of $3d$ bosonic SPT states. We will consider bosons with a $U(1)_e \rtimes Z_2^T \times U(1)_s$ symmetry, where $U(1)_e$ can be viewed as the U(1) symmetry of the electromagnetic charge, and $U(1)_s$ can be viewed as the spin symmetry generated by total spin along z direction. These symmetries can be carried by a two component complex boson field z_α, which transforms under $U(1)_e$, $U(1)_s$ and time-reversal as

$$U(1)_e : z_\alpha \to e^{i\theta} z_\alpha, \quad U(1)_s : z_\alpha \to \left(e^{-i\tau^z\theta}\right)_{\alpha\beta} z_\beta,$$
$$\mathcal{T} : z_\alpha \to (i\tau^y)_{\alpha\beta} z_\beta. \quad (2)$$

Notice that here the fact $\mathcal{T}^2 = -1$ can be changed by a $U(1)_s$ rotation. It has been understood that for a bosonic TI, the response to an eternal gauge field (either A^s_μ or A^e_μ that couple to $U(1)_s$ and $U(1)_e$ global symmetry) contains a $\theta \boldsymbol{E}\cdot\boldsymbol{B}/(4\pi^2)$ term with $\theta = \pm 2\pi$ [17], which corresponds to an integer quantum Hall state with $\sigma_{xy} = \pm 1$ at the boundary. This is forbidden in a pure $2d$ bosonic system without fractionalization [18].

In this work we propose that the boundary of this bosonic SPT state constructed with z_α can be a noncompact QED_3 with two gauge-charged Dirac fermions:

$$\mathcal{L} = \sum_{j=1}^{2} \bar{\psi}_j \gamma_\mu (\partial_\mu - i a_\mu)\psi_j - i A^s_\mu \bar{\psi}_i \gamma_\mu \tau^z_{ij}\psi_j + \frac{i}{2\pi}\epsilon_{\mu\nu\rho} a_\mu \partial_\nu A^e_\rho, \quad (3)$$

where $\bar{\psi} = \psi^\dagger \gamma^0$, and $\gamma^0 = \sigma^y$, $\gamma^1 = \sigma^x$, $\gamma^2 = \sigma^z$. We will argue later that compared with Eq. 1, this theory with fermion flavor $N_f = 2$ has a better chance to have a stable $(2+1)d$ conformal field theory fixed point (perhaps with certain short range fermion interaction), if we ignore the external gauge fields A^s_μ and A^e_μ. The coupling to the external gauge fields imply that the spin symmetry $U(1)_s$ is carried by the fermionic fields ψ_j, but the charge symmetry $U(1)_e$ is carried by the flux of the noncompact gauge field a_μ, which makes a_μ a noncompact gauge field. We will also show that this theory has a nice self-dual structure, the dual theory takes exactly the same form as Eq. 3, except for an exchange between the roles of A^e_μ and A^s_μ. This self-duality is reminiscent of the more familiar self-duality of the noncompact CP^1 theory with easy plane anisotropy [4–6], which involves two gauge charged complex bosons and one noncompact gauge field.

— *2. Microscopic construction of the noncompact QED$_3$ with $N_f = 2$*

In this section we will give a microscopic construction of Eq. 3. The starting point of our construction is similar to Ref. 1: a $3d$ U(1) spin liquid state with deconfined compact internal U(1) gauge field a_μ and fermionic spinons $f_{j,\alpha}$ with gauge charge +1 that transforms as

$$\mathcal{T}: f_{j,\alpha} \to (i\sigma^y)_{\alpha\beta} f^\dagger_{j,\beta}, \quad U(1)_s : f_i \to (\exp(i\theta\tau^z))_{ij}\, f_j, \tag{4}$$

under time-reversal and $U(1)_s$ global symmetry. $j = 1, 2$ is a flavor index that the symmetry $U(1)_s$ operates on. $f_{j,\alpha}$ does not carry $U(1)_e$ charge. The U(1) gauge symmetry and the time-reversal symmetry so-defined commute with each other, thus this spin liquid has $U(1)_g \times Z^T_2$ "symmetry", where $\mathrm{U(1)}_g$ stands for the U(1) gauge symmetry. Now we put $f_{1,\alpha}$ and $f_{2,\alpha}$ both in a TI with topological number $n = 1$. Notice that since here f_α has $U(1)_g \times Z^T_2$ symmetry, at the mean field level the classification of the spinon TI is $\mathbb{Z}$ [19–21], while this classification can be reduced under interaction [22, 23]. The boundary of this spin liquid is a QED$_3$ with two Dirac cones, but up until this point the internal U(1) gauge field a_μ is propagating in the entire $3d$ bulk.

Our next task is to confine the gauge field in the bulk, while making the gauge field at the $2d$ boundary noncompact. Ref. 1 proposed a very nice way of achieving this goal, which we will adopt here. Because $f_{1,\alpha}$ and $f_{2,\alpha}$ each forms a $n = 1$ TI, a $2\pi-$monopole of a_μ in the $3d$ bulk will acquire total polarization gauge charge +1 [39], which comes from +1/2 polarization density of f_1 and f_2 each. The quantum number of this $2\pi-$monopole is $(q = 1, Q_s = 0, 2\pi)$, where q and Q_s stand for the internal gauge charge and the $U(1)_s$ charge respectively. Now by binding this monopole with a spinon f, we obtain a gauge neutral object, and it is a boson which we call b. Depending on whether we bind the monopole with f_1 or f_2, b can carry quantum number $(q = 0, Q_s = \pm 1, 2\pi)$, thus b is a doublet boson b_α with $\alpha = 1, 2$. The bosonic statistics of b_α comes from the fermionic statistics of f_α and the mutual statistics between f_α and the monopole.

There is another way of looking at the quantum number of the boson doublet b_α. A 2π monopole of a_μ could be viewed as the source of a double-vortex of the superconductor of f, since f will view a single vortex as $\pi-$flux. Of course, we need to consider a superconductor order parameter that preserves the $U(1)_s$ symmetry. Then the source of a double vortex in this system, will acquire four Majorana fermion zero modes, or equivalently two complex fermion zero modes f^0_1 and f^0_2. Our boson doublet states $b^\dagger_1|0\rangle$ (or $b^\dagger_2|0\rangle$) corresponds to the states with filled (or unfilled) f^0_1 zero mode and unfilled (or filled) f^0_2 zero mode. Because each fermion zero mode will lead to $U(1)_s$ charge $\pm 1/2$ depending on whether it is filled or unfilled, $b^\dagger_1|0\rangle$ and $b^\dagger_2|0\rangle$ will carry $U(1)_s$ charge ± 1 respectively. As was pointed out by Ref. 23, b_α is also a Kramers doublet boson with $\mathcal{T}^2 = -1$. The fact $\mathcal{T}^2 = -1$ for boson b_α can be derived by coupling these two zero modes to a three component vector $\boldsymbol{N}$: $f^{0\dagger}\boldsymbol{\tau} f^0 \cdot \boldsymbol{N}$, after integrating out f^0_j, the effective action for $\boldsymbol{N}$ is a $(0 + 1)d$ O(3) nonlinear sigma model with a Wess-Zumino-Witten term at level-1 [24], whose ground state is a Kramers doublet with $\mathcal{T}^2 = -1$ because $\boldsymbol{N}$ is odd under time-reversal [40].

Now let us take another Kramers doublet boson z_α introduced in Eq. 2 which carries both global $U(1)_e$ and $U(1)_s$ charge , and form a time-reversal singlet bound state D with b_α: $D = (z_1 b_2 - z_2 b_1)$. D carries total quantum number $(Q_e = 1, q = 0, Q_s = 0, 2\pi)$. After condensing this bound state D in the bulk, $\mathcal{T}$ is still preserved, while the $3d$ bulk is driven into a gauge confined phase, because overall speaking D carries a 2π monopole of the internal gauge field, but it carries zero gauge charge, thus all the spinons in the bulk are confined. Because the bound state D carries both the $U(1)_e$ charge and the $U(1)_g$ magnetic monopole, its condensate does not break the $U(1)_e$ global symmetry in the bulk, and the bulk remains fully gapped for all excitations, *i.e.* there is no Goldstone mode in the bulk at all [1]. Also, following the same argument as in Ref. 1, a $2\pi-$flux of a_μ at the boundary will be screened by D in the bulk, which attaches the flux with $U(1)_e$ charge 1. Thus the gauge field a_μ becomes a noncompact gauge theory at the $2d$ boundary, because its flux now carries a conserved $U(1)_e$ charge, which is precisely described by the last term of Eq. 3.

Based on the argument above, the $(2 + 1)d$ boundary of the system is described by Eq. 3 because the spinon ψ_j carries $U(1)_s$ charge, and the gauge flux of a_μ carries unit $U(1)_e$ charge. If we break the time-reversal symmetry at the boundary, ψ_j will acquire a mass term, which will generate a Chern-Simons term for both a_μ and A^s_μ at level +1. Now after integrating out a_μ, the external field A^e_μ will acquire a CS term at level −1. The full response theory at the boundary reads:

$$\mathcal{L} = \frac{i}{4\pi} A^e \wedge dA^e - \frac{i}{4\pi} A^s \wedge dA^s. \tag{5}$$

This response theory has already been derived in Ref. 17 for the boundary of $3d$ bosonic SPT states. This response theory is consistent with the physics of bosonic

TI: it is fully gapped and has no fractional excitations in the bulk, but if time-reversal symmetry is broken at the boundary, the boundary will be driven to a quantum Hall state with Hall conductivity $\sigma_{xy} = \pm 1$. This also implies that the bulk response theory to A^e_μ and A^s_μ will acquire a topological term $\theta \boldsymbol{E} \cdot \boldsymbol{B}/(4\pi^2)$ term with $\theta = \pm 2\pi$ respectively.

— 3. Self-duality of the boundary theory

Now we argue that Eq. 3 has a self-dual structure. Since in our system ψ_1 and ψ_2 each has its own U(1) global symmetry $\psi_j \to \psi_j e^{i\theta_j}$, and they come from two independent $n = 1$ TIs (let us tentatively ignore the gauge field a_μ they couple together), let us form independent superconductor Cooper pair condensate $\psi^t_j \sigma^y \psi_j \sim \Delta_j \sim \exp(i\phi_j)$, where $\sigma^y = \gamma^0$ in Eq. 3. We can destroy the superconductors by proliferating the vortices of the superconductors. But here we would like to consider the quartic vortex of ϕ_1 and ϕ_2 individually, namely vortices of ϕ_j that ψ_j would view as a 4π flux. As was shown in Ref. 1–3, the charge neutral quartic vortex in a $n = 1$ TI is a fermion. This can be understood by gauging the global U(1) symmetry of ψ_j, and consider the statistics of the $(0, 4\pi)$ monopole. The $(0, 4\pi)$ monopole is naturally a bound state of $(1/2, 2\pi)$ and $(-1/2, 2\pi)$ dyons, hence it carries angular momentum $1/2$, and it is a Kramers doublet fermion [25]. After the proliferation of these fermionic vortices, the dual boundary theory in terms of these fermionic vortices reads [1–3]:

$$\mathcal{L} = \sum_{j=1}^{2} \bar{\chi}_j \gamma_\mu (\partial_\mu - 4 i a^{(j)}_\mu) \chi_j + \cdots \tag{6}$$

Here χ_j is the dual Kramers doublet fermion that transform as $\mathcal{T} : \chi_j \to i\sigma^y \chi_j$. $a^{(j)}_\mu$ corresponds to the Goldstone mode of ϕ_j: $\partial_\mu \phi_j = \frac{1}{2\pi} \epsilon_{\mu\nu\rho} \partial_\nu a^{(j)}_\rho$.

Now we turn back on the original gauge field a_μ. In the superconductor phase, the low energy Lagrangian for the two superconductors that couple to a_μ is:

$$\mathcal{L} = \sum_{j=1}^{2} -t(\partial_\mu \phi_j - 2a_\mu + (-1)^j 2A^s_\mu)^2 + \frac{i}{2\pi} \epsilon_{\mu\nu\rho} A^e_\mu \partial_\nu a_\rho . \tag{7}$$

After going through the standard duality formalism, we obtain the following Lagrangian:

$$\mathcal{L} = \sum_{j=1}^{2} \frac{2i}{2\pi} \epsilon_{\mu\nu\rho} a^{(j)}_\mu \partial_\nu (a_\rho + (-1)^j A^s_\mu) + \frac{i}{2\pi} \epsilon_{\mu\nu\rho} A^e_\mu \partial_\nu a_\rho . \tag{8}$$

Integrating out a_μ will generate the following constraint:

$$2a^{(1)}_\mu + 2a^{(2)}_\mu + A^e_\mu = 0, \tag{9}$$

or in other words the photon phase of a_μ will "Higgs" and gap out the mode $2a^{(1)}_\mu + 2a^{(2)}_\mu + A^e_\mu$. This constraint can be solved by introducing a new gauge field c_μ: $4a^{(1)}_\mu = -c_\mu + A^e_\mu$, $4a^{(2)}_\mu = c_\mu + A^e_\mu$. Plugging these fields in to Eq. 6, we obtain the full dual theory of Eq. 3:

$$\begin{aligned} \mathcal{L} \ = \ & \sum_{j=1}^{2} \bar{\chi}_j \gamma_\mu (\partial_\mu - i(-1)^j c_\mu - i A^e_\mu) \chi_j \\ + \ & \frac{i}{2\pi} \epsilon_{\mu\nu\rho} c_\mu \partial_\nu A^s_\rho . \end{aligned} \tag{10}$$

We can see that χ_j are two Dirac fermions that each carries $U(1)_e$ charge $+1$. They correspond to the quartic vortex of the superconductor of ψ_j. Physically this is easy to understand: χ_j is a quartic vortex of ϕ_j, and a quartic vortex of ϕ_j carries gauge flux 2π of a_μ, which due to the last term of Eq. 3 should also carry global $U(1)_e$ charge 1. Notice that here we create quartic vortex of ϕ_1 and ϕ_2 individually, namely a vortex of ϕ_1 alone without a vortex of ϕ_2 will carry a_μ gauge flux 2π.

According to Eq. 10, the flux of c_μ carries unit $U(1)_s$ charge. Again this can be physically understood as following: a flux of c_μ is the difference between the flux number of $a^{(1)}_\mu$ and $a^{(2)}_\mu$, and based on the standard boson-vortex duality, the flux of c_μ also corresponds to the density difference between ψ_1 and ψ_2, which is a quantity that does not carry $U(1)_e$ charge, but carries $U(1)_s$ charge. Now after a particle-hole transformation $\chi_2 \to \chi^\dagger_2$, the dual boundary theory Eq. 10 takes exactly the same form as Eq. 3, with an exchanged role between A^e_μ and A^s_μ:

$$\begin{aligned} \mathcal{L}_{dual} \ = \ & \sum_{j=1}^{2} \bar{\chi}_j \gamma_\mu (\partial_\mu - i c_\mu) \chi_j \\ - \ & i A^e_\mu \bar{\chi}_i \gamma_\mu \tau^z_{ij} \chi_j + \frac{i}{2\pi} \epsilon_{\mu\nu\rho} c_\mu \partial_\nu A^s_\rho . \end{aligned} \tag{11}$$

Because the dual fermion χ_j transforms as $\mathcal{T} : \chi_j \to i\sigma^y \chi_j$ under time-reversal, if we ignore the gauge field c_μ in Eq. 10, the symmetry for χ_j is $U(1)_e \rtimes Z^T_2$, which is the symmetry of the ordinary $3d$ TI [26–28], and as is well-known, it has an $\mathbb{Z}_2$ classification. This implies that without c_μ, there is a mass term that is allowed by all the symmetries: $m\chi^\dagger_i \sigma^y \otimes \tau^y_{ij} \psi_j$. However, because χ_1 and χ_2 carry opposite gauge charge under c_μ in Eq. 10, this mass term is forbidden by the gauge symmetry. Thus this dual boundary theory Eq. 10 cannot be trivially gapped out without breaking symmetry or gauge symmetry.

If we explicitly break either the $U(1)_e$ or $U(1)_s$ symmetry, then the $3d$ bulk can be called a bosonic TI. But in this case a_μ or c_μ will become a compact gauge field, because their fluxes will no longer carry conserved quantities, hence the instanton monopole process is allowed in the $(2 + 1)d$ space-time. And inspired by Ref. 14, we conjecture that the duality we propose here may have an analogue in supersymmetric field theories.

— 4. Relation to other possible boundary states

There are many possible boundary states of this system, for example states with different spontaneous symmetry breaking. We are most interested in boundary states that do not break any symmetry. Ref. 29 gave us another way of looking at QED_3 with $N_f = 2$: Eq. 3 and Eq. 11 can both be mapped to a O(4) nonlinear sigma model with a topological $\Theta-$term at $\Theta = \pi$. Here we reproduce the discussion in Ref. 29. First we couple Eq. 3 to a three component dynamical unit vector field $\boldsymbol{N}(x,\tau)$:

$$\mathcal{L} = \sum_{j=1}^{2} \bar{\psi}_j \gamma_\mu (\partial_\mu - i a_\mu)\psi_j + m\bar{\psi}\boldsymbol{\tau}\psi \cdot \boldsymbol{N}, \quad (12)$$

introducing this slow moving vector $\boldsymbol{N}$ is equivalent to turning on certain four fermion interaction for ψ_j, and $\boldsymbol{N}$ could be introduced through Hubbard-Stratonovich transformation.

Now following the standard $1/m$ expansion of Ref. 24, we obtain the following action after integrating out the fermion ψ_j:

$$\mathcal{L}_{eff} = \frac{1}{g}(\partial_\mu \boldsymbol{N})^2 + i\pi \mathrm{Hopf}[\boldsymbol{N}] + i a_\mu J^T_\mu + \frac{1}{e^2} f^2_{\mu\nu}, \quad (13)$$

where $1/g \sim m$. $J^T_0 = \frac{1}{4\pi}\epsilon_{abc} N^a \partial_x N^b \partial_y N^c$ is the Skyrmion density of $\boldsymbol{N}$, thus J^T_μ is the Skyrmion current. The second term of Eq. 13 is the Hopf term of $\boldsymbol{N}$ which comes from the fact that $\pi_3[S^2] = \mathbb{Z}$.

Now if we introduce the CP^1 field $z_\alpha = (z_1, z_2)^t = (n_1 + in_2, n_3 + in_4)^t$, the Hopf term becomes precisely the $\Theta-$term for the O(4) vector $\boldsymbol{n}$ with $\Theta = \pi$:

$$i\pi \mathrm{Hopf}[\boldsymbol{N}] = \frac{i\pi}{2\pi^2}\epsilon_{abcd} n^a \partial_x n^b \partial_y n^c \partial_\tau n^d. \quad (14)$$

Here $\Theta = \pi$ is protected by time-reversal symmetry. This is because $\boldsymbol{N}$ is odd under time-reversal, and hence z_α is a Kramers doublet boson. Simple algebra shows that Eq. 14 changes sign under time-reversal. In the CP^1 formalism, the Skyrmion current $J^T_\mu = \frac{1}{2\pi}\epsilon_{\mu\nu\rho}\partial_\nu \alpha_\rho$, where α_μ is the gauge field that the CP^1 field z_α couples to. Due to the coupling $a_\mu J^T_\mu = \frac{i}{2\pi}\epsilon_{\mu\nu\rho} a_\mu \partial_\nu \alpha_\mu$, after integrating out a_μ, α_μ is Higgsed and gapped, and z_α becomes a complex boson that does not couple to any gauge field, and its transformation $z_\alpha \to e^{i\theta} z_\alpha$ becomes the physical $U(1)_e$ symmetry, due to the mutual Chern-Simons coupling between a_μ and α_μ. Thus z_α now carries both the $U(1)_e$ and $U(1)_s$ quantum numbers, and if we start with the dual theory Eq. 11, the same O(4) NLSM with $\Theta-$term in Eq. 14 can be derived.

The phase diagram of the O(4) NLSM with a $\Theta-$term was discussed in Ref. 30, and it was proposed that in the large g (small m) disordered phase, $\Theta = \pi$ is the quantum critical point (quantum phase transition) between stable fixed points $\Theta = 0$ and $\Theta = 2\pi$, which is consistent with the conjecture made in Ref. 29 that the quantum disordered phase of the O(4) NLSM with $\Theta = \pi$ could be a gapless paramagnet (a 2+1d CFT). Recently this conjecture was confirmed numerically in Ref. 31, 32, and the sign-problem-free simulation in both Ref. 31, 32 strongly suggest that the quantum disordered phase of the O(4) NLSM with $\Theta = \pi$ is indeed a strongly coupled CFT, sandwiched between two fully gapped quantum disordered phases controlled by fixed points $\Theta = 0$ and 2π (Fig.4 in Ref. 32, and discussion therein).

The $1/m$ expansion above is certainly valid for large m (small g), which corresponds to the ordered phase of the O(4) CP^1 field $\boldsymbol{n}$ and three component vector $\boldsymbol{N}$. The usual expectation of QED_3 with $N_f = 2$ is that it leads to spontaneous chiral symmetry breaking at low energy [33–36], which precisely corresponds to the order of vector $\boldsymbol{N}$. However, if a proper four fermion interaction term is turned on in Eq. 3 and Eq. 11 that prevents the chiral symmetry breaking, it may remain a CFT that corresponds to the disordered phase of O(4) NLSM with $\Theta = \pi$.

Other possible boundary states can be constructed through the O(4) NLSM with $\Theta = \pi$, as was discussed in Ref. 17, 37. For example, let us break the $U(1)_s$ symmetry, and keep the following time-reversal symmetry $\mathcal{T} : z_\alpha \to (\tau^x)_{\alpha\beta} z_\beta = (\tau^y \exp(-i\tau^z \pi/2))_{\alpha\beta}\, z_\beta$, then one can see that this O(4) NLSM model with $\Theta = \pi$ precisely correspond to the boundary of the bosonic TI with $U(1)_e \rtimes Z_2^T$ symmetry. And following the discussion in Ref. 17, this $\Theta-$term can drive the boundary into the so called $eCmC$ Z_2 topological order, namely its e and m anyons with mutual semion statistics both carry half $U(1)_e$ charge.

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[39] The magnetic monopole of a_μ is invariant under time-reversal symmetry, while the polarization charge is odd under time-reversal, thus whether this polarization charge of the $2\pi-$monopole is $+1$ or -1 depends on how time-reversal symmetry is broken on the boundary, which we have confirmed numerically. But no matter how time-reversal symmetry is broken at the boundary, we can always construct the Kramers doublet boson b_α monopole we need with quantum numbers $(q = 0, Q_s = \pm 1, 2\pi)$.
[40] More precisely, $\boldsymbol{N}$ is odd under the effective time-reversal symmetry introduced in Ref. 23, which is a combination of $\mathcal{T}$ and a $\pi-$rotation of the pairing order parameter.