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Phys. Rev. B 92, 220416 — Published 21 December 2015
DOI: 10.1103/PhysRevB.92.220416
Self-dual Quantum Electrodynamics as Boundary State of the three dimensional Bosonic Topological Insulator

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(Dated: December 7, 2015)

Inspired by the recent developments of constructing novel Dirac liquid boundary states of the 3d topological insulator \cite{1–3}, we propose one possible 2d boundary state of the 3d bosonic symmetry protected topological state with $U(1)_s \times Z_2^3 \times U(1)_c$ symmetry. This boundary theory is described by a $(2+1)d$ quantum electrodynamics (QED$_3$) with two flavors of Dirac fermions ($N_f = 2$) coupled with a noncompact $U(1)$ gauge field: $\mathcal{L} = \sum_{j=1}^2 \bar{\psi}_j \gamma_\mu (\partial_\mu - ia_\mu) \psi_j - iA_\mu^a \bar{\psi} \gamma_\mu \gamma_5 \tau_j \psi_j + \frac{1}{2} \epsilon_{\mu\nu\rho} a_\mu \partial_\nu A_\rho$, where $a_\mu$ is the internal noncompact $U(1)$ gauge field, $A_\mu^a$ and $A_\mu^c$ are two external gauge fields that couple to $U(1)_c$ and $U(1)_e$ global symmetries respectively. We demonstrate that this theory has a “self-dual” structure, which is a fermionic analogue of the self-duality of the noncompact CP$^1$ theory with easy plane anisotropy \cite{4–6}. Under the self-duality, the boundary action takes exactly the same form except for an exchange between $A_\mu^a$ and $A_\mu^c$. The self-duality may still hold after we break one of the $U(1)$ symmetries (which makes the system a bosonic topological insulator), with some subtleties that will be discussed.

PACS numbers:

--- 1. Introduction

A symmetry protected topological (SPT) state may have very different boundary states without changing the bulk state, depending on the boundary Hamiltonian. As was shown in Ref. 7–11, besides the well-known boundary state, i.e., a single 2d Dirac fermion, the boundary of an interacting 3d topological insulator (TI) could have a topological order that respects all the symmetries of the system but cannot be realized in a 2d system. Very recently, the “family” of boundary states of TI has been even further expanded \cite{1–3}: it was shown that the boundary of the 3d TI could be a $(2+1)d$ quantum electrodynamics (QED$_3$) with one single flavor of gauge-charged Dirac fermion, while the flux quantum of the $U(1)$ gauge field carries charge 1/2 under the external electromagnetic (EM) field $A_\mu$:

$$\mathcal{L} = \bar{\psi} \gamma_\mu (\partial_\mu - ia_\mu) \psi + \frac{1}{2} \epsilon_{\mu\nu\rho} a_\mu \partial_\nu A_\rho.$$  (1)

This Lagrangian without time-reversal symmetry can also be realized as a pure 2d theory in the half-filled Landau level \cite{12}. This boundary state Eq. 1 is particularly interesting because it suggests a duality between interacting 2d single Dirac fermion and the noncompact QED$_3$ with one gauge-charged Dirac fermion, which is further supported by the proof of S-duality in the 3d bulk \cite{13}, and also the exact duality (or the mirror symmetry) between certain $(2+1)d$ supersymmetric field theories \cite{14}. This duality is a very elegant analogue of its standard bosonic version: the duality between the $(2+1)d$ XY model and the bosonic QED$_3$ with one flavor of gauge-charged complex boson \cite{15, 16}. Although the infrared fate of Eq. 1 under gauge fluctuation and fermion interaction is unclear, it was demonstrated in Ref. 1–3 that Eq. 1 can be viewed as the parent state of other well-known boundary states of 3d TI.

In this work we will further extend the idea of Ref. 1–3, and construct novel boundary states of 3d bosonic SPT states. We will consider bosons with a $U(1)_s \times Z_2^3 \times U(1)_c$ symmetry, where $U(1)_c$ can be viewed as the $\tilde{U}(1)$ symmetry of the electromagnetic charge, and $U(1)_e$ can be viewed as the spin symmetry generated by total spin along $z$ direction. These symmetries can be carried by a two component complex boson field $z_\alpha$, which transforms under $U(1)_c$, $U(1)_s$ and time-reversal as

$$U(1)_c : z_\alpha \to e^{i\theta} z_\alpha, \quad U(1)_s : z_\alpha \to (e^{-i\tau^3 \theta})_{\alpha\beta} z_\beta, \quad \mathcal{T} : z_\alpha \to (i\tau^0)_{\alpha\beta} z_\beta.$$  (2)

Notice that here the fact $T^2 = -1$ can be changed by a $U(1)_s$ rotation. It has been understood that for a bosonic TI, the response to an eternal gauge field (either $A_\mu^a$ or $A_\mu^c$ that couple to $U(1)_s$ and $U(1)_e$ global symmetry) contains a $\theta \mathbf{E} \cdot \mathbf{B} / (4\pi^2)$ term with $\theta = \pm 2\pi$ \cite{17}, which corresponds to an integer quantum Hall state with $\sigma_{xy} = \pm 1$ at the boundary. This is forbidden in a pure 2d bosonic system without fractionalization \cite{18}.

In this work we propose that the boundary of this bosonic SPT state constructed with $z_\alpha$ can be a noncompact QED$_3$ with two gauge-charged Dirac fermions:

$$\mathcal{L} = \sum_{j=1}^2 \bar{\psi}_j \gamma_\mu (\partial_\mu - ia_\mu) \psi_j - iA_\mu^a \bar{\psi} \gamma_\mu \gamma_5 \tau_j \psi_j + \frac{1}{2} \epsilon_{\mu\nu\rho} a_\mu \partial_\nu A_\rho^c,$$  (3)

where $\bar{\psi} = \psi^\dagger \gamma^0$, and $\gamma^0 = \sigma^y$, $\gamma^1 = \sigma^x$, $\gamma^2 = \sigma^z$. We will argue later that compared with Eq. 1, this theory with fermion flavor $N_f = 2$ has a better chance to have a stable $(2+1)d$ conformal field theory fixed point (perhaps with certain short range fermion interaction), if we ignore...
the external gauge fields $A_\mu^x$ and $A_\mu^y$. The coupling to the external gauge fields imply that the spin symmetry $U(1)_s$ is carried by the fermionic fields $\psi_j$, but the charge symmetry $U(1)_s$ is carried by the flux of the noncompact gauge field $a_\mu$, which makes $a_\mu$ a noncompact gauge field. We will also show that this theory has a nice self-dual structure, the dual theory takes exactly the same form as Eq. 3, except for an exchange between the roles of $A_\mu^x$ and $A_\mu^y$. This self-duality is reminiscent of the more familiar self-duality of the noncompact CP$^1$ theory with easy plane anisotropy [4–6], which involves two gauge charged complex bosons and one noncompact gauge field.

2. Microscopic construction of the noncompact QED$_3$ with $N_f = 2$

In this section we will give a microscopic construction of Eq. 3. The starting point of our construction is similar to Ref. 1: a 3d U(1) spin liquid state with deconfined compact internal U(1) gauge field $a_\mu$ and fermionic spinons $f_{j,\alpha}$ with gauge charge +1 that transforms as $\mathcal{T}: f_{j,\alpha} \rightarrow (i\sigma^y)_{\alpha\beta} f_{j,\beta}$. $U(1)_s: f_1 \rightarrow (\exp(i\theta^s))_{ij} f_{j,4}$ under time-reversal and $U(1)_s$ global symmetry. $j = 1, 2$ is a flavor index that the symmetry $U(1)_s$ operates on. $f_{j,\alpha}$ does not carry $U(1)_s$ charge. The $U(1)_s$ gauge symmetry and the time-reversal symmetry so-defined commute with each other, thus this spin liquid has $U(1)_s \times \mathbb{Z}_2^T$ “symmetry”, where $U(1)_s$ stands for the U(1) gauge symmetry. Now we put $f_{1,\alpha}$ and $f_{2,\alpha}$ both in a TI with topological number $n = 1$. Notice that since here $f_\alpha$ has $U(1)_s \times \mathbb{Z}_2^T$ symmetry, at the mean field level the classification of the spinon TI is $\mathbb{Z}$ [19–21], while this classification can be reduced under interaction [22, 23]. The boundary of this spin liquid is a QED$_3$ with two Dirac cones, but up until this point the internal U(1) gauge field $a_\mu$ is propagating in the entire 3d bulk.

Our next task is to confine the gauge field in the bulk, while making the gauge field at the 2d boundary noncompact. Ref. 1 proposed a very nice way of achieving this goal, which we will adopt here. Because $f_{1,\alpha}$ and $f_{2,\alpha}$ each forms a $n = 1$ TI, a $2\pi$–monopole of $a_\mu$ in the 3d bulk will acquire total polarization gauge charge +1 [39], which comes from $+1/2$ polarization density of $f_1$ and $f_2$ each. The quantum number of this $2\pi$–monopole is $(q = 1, Q_s = 0, 2\pi)$, where $q$ and $Q_s$ stand for the internal gauge charge and the $U(1)_s$ charge respectively. Now by binding this monopole with a spinon $f$, we obtain a gauge neutral object, and it is a boson which we call $b$. Depending on whether we bind the monopole with $f_1$ or $f_2$, $b$ can carry quantum number $(q = 0, Q_s = \pm 1, 2\pi)$, thus $b$ is a doublet boson $b_{\alpha}$ with $\alpha = 1, 2$. The bosonic statistics of $b_{\alpha}$ comes from the fermionic statistics of $f_{\alpha}$ and the mutual statistics between $f_{\alpha}$ and the monopole.

There is another way of looking at the quantum number of the boson doublet $b_{\alpha}$. A $2\pi$ monopole of $a_\mu$ could be viewed as the source of a double-vortex of the superconductor of $f$, since $f$ will view a single vortex as $\pi$–flux. Of course, we need to consider a superconductor order parameter that preserves the $U(1)_s$ symmetry. Then the source of a double vortex in this system, will acquire four Majorana fermion zero modes, or equivalently two complex fermion zero modes $f_0$ and $f_0^\dagger$. Our boson doublet states $b_1(0)$ (or $b_1^\dagger(0)$) corresponds to the states with filled (or unfilled) $f_0$ zero mode and unfilled (or filled) $f_0^\dagger$ zero mode. Because each fermion zero mode will lead to $U(1)_s$ charge $\pm 1/2$ depending on whether it is filled or unfilled, $b_1(0)$ and $b_1^\dagger(0)$ will carry $U(1)_s$ charge $\pm 1$ respectively. As was pointed out by Ref. 23, $b_{\alpha}$ is also a Kramers doublet boson with $\mathcal{T}^2 = -1$. The fact $\mathcal{T}^2 = -1$ for boson $b_{\alpha}$ can be derived by coupling these two zero modes to a three component vector $\mathbf{N}$: $f^{0|s}\mathbf{N}$, after integrating out $f_0^\dagger$, the effective action for $\mathbf{N}$ is a $(0 + 1)d$ $O(3)$ nonlinear sigma model with a Wess-Zumino-Witten term at level-1 [24], whose ground state is a Kramers doublet with $\mathcal{T}^2 = -1$ because $\mathbf{N}$ is odd under time-reversal [40].

Now let us take another Kramers doublet boson $z_{\alpha}$ introduced in Eq. 2 which carries both global $U(1)_c$ and $U(1)_s$ charge, and form a time-reversal singlet bound state $D$ with $b_{\alpha}$: $D = (z_1b_2 - z_2b_1)$. $D$ carries total quantum number $(Q_c = 1, q = 0, Q_s = 0, 2\pi)$. After condensing this bound state $D$ in the bulk, $\mathcal{T}$ is still preserved, while the 3d bulk is driven into a gauge confined phase, because overall speaking $D$ carries a $2\pi$ monopole of the internal gauge field, but it carries zero gauge charge, thus all the spinons in the bulk are confined. Because the bound state $D$ carries both the $U(1)_c$ charge and the $U(1)_s$ magnetic monopole, its condensate does not break the $U(1)_c$ global symmetry in the bulk, and the bulk remains fully gapped for all excitations, i.e. there is no Goldstone mode in the bulk at all [1]. Also, following the same argument as in Ref. 1, a $2\pi$–flux of $a_\mu$ at the boundary will be screened by $D$ in the bulk, which attaches the flux with $U(1)_c$ charge 1. Thus the gauge field $a_\mu$ becomes a noncompact gauge theory at the 2d boundary, because its flux now carries a conserved $U(1)_c$ charge, which is precisely described by the last term of Eq. 3.

Based on the argument above, the $(2 + 1)d$ boundary of the system is described by Eq. 3 because the spinon $\psi_j$ carries $U(1)_s$ charge, and the gauge flux of $a_\mu$ carries unit $U(1)_c$ charge. If we break the time-reversal symmetry at the boundary, $\psi_j$ will acquire a mass term, which will generate a Chern-Simons term for both $a_\mu$ and $A_\mu^\alpha$ at level +1. Now after integrating out $a_\mu$, the external field $A_\mu^\alpha$ will acquire a CS term at level $-1$. The full response theory at the boundary reads:

$$\mathcal{L} = \frac{i}{4\pi} A^x \wedge dA^x - \frac{i}{4\pi} A^x \wedge dA^y. \quad (5)$$

This response theory has already been derived in Ref. 17 for the boundary of 3d bosonic SPT states. This response theory is consistent with the physics of bosonic
TI: it is fully gapped and has no fractional excitations in the bulk, but if time-reversal symmetry is broken at the boundary, the boundary will be driven to a quantum Hall state with Hall conductivity \( \sigma_{xy} = \pm 1 \). This also implies that the bulk response theory to \( A_\mu^a \) will acquire a topological term \( \theta \mathbf{E} \cdot \mathbf{B} / (4\pi^2) \) term with \( \theta = \pm 2\pi \) respectively.

3. Self-duality of the boundary theory

Now we argue that Eq. 3 has a self-dual structure. Since in our system \( \psi_1 \) and \( \psi_2 \) each has its own \( U(1) \) global symmetry \( \psi_j \to \psi_j e^{i\theta_j} \), and they come from two independent \( n = 1 \) TIs (let us tentatively ignore the gauge field \( a_\mu \) they couple together), let us form independent superconductor Cooper pair condensate \( \psi_j^\dagger \sigma^y \psi_j \sim \Delta_j \sim \exp(i\phi_j) \), where \( \sigma^y = \gamma^0 \) in Eq. 3. We can destroy the superconductors by proliferating the vortices of the superconductors. But here we would like to consider the quartic vortex of \( \phi_1 \) and \( \phi_2 \) individually, namely vortices of \( \phi_j \) that \( \psi_j \) would view as a 4\( \pi \) flux. As was shown in Ref. 1–3, the charge neutral quartic vortex in a \( n = 1 \) TI is a fermion. This can be understood by gauging the global \( U(1) \) symmetry of \( \psi_j \), and consider the statistics of the \( (0, 4\pi) \) monopole. The \( (0, 4\pi) \) monopole is naturally a bound state of \( (1/2, 2\pi) \) and \( (-1/2, 2\pi) \) dyons, hence it carries angular momentum \( 1/2 \), and it is a Kramers doublet fermion [25]. After the proliferation of these fermionic vortices, the dual boundary theory in terms of these fermionic vortices reads [1–3]:

\[
\mathcal{L} = \sum_{j=1}^2 \bar{\chi}_j \gamma_\mu (\partial_\mu - 4ia^{(j)}_\mu) \chi_j + \cdots
\]

Here \( \chi_j \) is the dual Kramers doublet fermion that transform as \( T : \chi_j \to i\sigma^y \chi_j \). \( a^{(j)}_\mu \) corresponds to the Goldstone mode of \( \phi_j \); \( \partial_\mu \phi_j = \frac{1}{2\pi} \epsilon_{\mu
u\rho} \partial_\nu a^{(j)}_\rho \).

Now we turn back on the original gauge field \( a_\mu \). In the superconductor phase, the low energy Lagrangian for the two superconductors that couple to \( a_\mu \) is:

\[
\mathcal{L} = \sum_{j=1}^2 \sum_{\nu=1}^2 \epsilon_{\mu\nu\rho} a^{(j)}_\nu \partial_\rho a_\mu - 2a_\mu + (-1)^j 2A^\mu_\mu - \frac{i}{2\pi} \epsilon_{\mu\nu\rho} A^\mu_\rho \partial_\nu a_\rho.
\]

After going through the standard duality formalism, we obtain the following Lagrangian:

\[
\mathcal{L}_{\text{dual}} = \sum_{j=1}^2 \bar{\chi}_j \gamma_\mu (\partial_\mu - i c_\mu) \chi_j - i A^\mu_\mu \bar{\chi}_j \tau^y_2 \chi_j + \frac{i}{2\pi} \epsilon_{\mu\nu\rho} c_\mu \partial_\nu A^\rho_\mu.
\]

Because the dual fermion \( \chi_j \) transforms as \( T : \chi_j \to i\sigma^y \chi_j \) under time-reversal, if we ignore the gauge field \( c_\mu \) in Eq. 10, the symmetry for \( \chi_j \) is \( U(1)_c \times Z_2^f \), which is the symmetry of the ordinary 3d TI [26–28], and as is well-known, it has a \( Z_2 \) classification. This implies that without \( c_\mu \), there is a mass term that is allowed by all the symmetries: \( m \bar{\chi}_j \sigma^y \chi_j \). However, because \( \chi_1 \) and \( \chi_2 \) carry opposite gauge charge under \( c_\mu \) in Eq. 10, this mass term is forbidden by the gauge symmetry. Thus this dual boundary theory Eq. 10 cannot be trivially gapped without breaking symmetry or gauge symmetry.

If we explicitly break either the \( U(1)_c \) or \( U(1)_s \) symmetry, then the 3d bulk can be called a bosonic TI. But in this case \( a_\mu \) or \( c_\mu \) will become a compact gauge field, because their fluxes will no longer carry conserved quantities, hence the instanton monopole process is allowed in the \( (2 + 1)d \) space-time. And inspired by Ref. 14, we conjecture that the duality we propose here may have an analogue in supersymmetric field theories.

4. Relation to other possible boundary states

There are many possible boundary states of this system, for example states with different spontaneous symmetry breaking. We are most interested in boundary
states that do not break any symmetry. Ref. 29 gave us another way of looking at QED with $N_f = 2$: Eq. 3 and Eq. 11 can both be mapped to a O(4) nonlinear sigma model with a topological $\Theta$–term at $\Theta = \pi$. Here we reproduce the discussion in Ref. 29. First we couple Eq. 3 to a three component dynamical unit vector field $\mathbf{N}(x, \tau)$:

$$\mathcal{L} = \frac{1}{2} \sum_{j=1}^{2} \bar{\psi}_j \gamma_\mu (\partial_\mu - i a_\mu) \psi_j + m \bar{\psi} \tau \psi \cdot \mathbf{N},$$  \hspace{1cm} (12)

introducing this slow moving vector $\mathbf{N}$ is equivalent to turning on certain four fermion interaction for $\psi_j$, and $\mathbf{N}$ could be introduced through Hubbard-Stratonovich transformation.

Now following the standard $1/m$ expansion of Ref. 24, we obtain the following action after integrating out the fermion $\psi_j$:

$$\mathcal{L}_{eff} = \frac{1}{4g} (\partial_\mu \mathbf{N})^2 + i \pi \text{Hopf}[\mathbf{N}] + ia_\mu J^T_\mu + \frac{1}{e^2} J^{\mu \nu \rho \sigma},$$  \hspace{1cm} (13)

where $1/g \sim m$. $J^T_\mu = \frac{1}{2\pi} \epsilon^{abcd} N^a \partial_x N^b \partial_y N^c$ is the Skyrmion density of $\mathbf{N}$, thus $J^T_\mu$ is the Skyrmion current. The second term of Eq. 13 is the Hopf term of $\mathbf{N}$ which comes from the fact that $\pi_3 [S^2] = Z$.

Now if we introduce the CP$^1$ field $z_\alpha = (z_1, z_2)^t = (n_1 + in_2, n_3 + i n_4)^t$, the Hopf term becomes precisely the $\Theta$–term for the O(4) vector $\mathbf{n}$ with $\Theta = \pi$:

$$i \pi \text{Hopf}[\mathbf{N}] = \frac{i \pi}{2\pi^2} \epsilon^{abcd} n^a \partial_x n^b \partial_y n^c \partial_z n^d.$$  \hspace{1cm} (14)

Here $\Theta = \pi$ is protected by time-reversal symmetry. This is because $\mathbf{N}$ is odd under time-reversal, and hence $z_\alpha$ is a Kramers doublet boson. Simple algebra shows that Eq. 14 changes sign under time-reversal. In the CP$^1$ formalism, the Skyrmion current $J^T_\mu = \frac{1}{4\pi} \epsilon^{abcd} n^a \partial_x n^b \partial_y n^c \partial_z n^d$. $\alpha_\mu$ is the gauge field that the CP$^1$ field $z_\alpha$ couples to. Due to the coupling $a_\mu J^T_\mu = \frac{i}{4\pi} \epsilon^{abcd} n^a \partial_x n^b \partial_y n^c \partial_z n^d$, after integrating out $a_\mu$, $\alpha_\mu$ is Higgsed and gapped, and $z_\alpha$ becomes a complex boson that does not couple to any gauge field, and its transformation $z_\alpha \rightarrow e^{i \phi} z_\alpha$ becomes the physical $U(1)_c$ symmetry, due to the mutual Chern-Simons coupling between $a_\mu$ and $\alpha_\mu$. Thus $z_\alpha$ now carries both the $U(1)_c$ and $U(1)_T$ quantum numbers, and if we start with the dual theory Eq. 11, the same O(4) NLSM with $\Theta$–term in Eq. 14 can be derived.

The phase diagram of the O(4) NLSM with a $\Theta$–term was discussed in Ref. 30, and it was proposed that in the large $g$ (small $m$) disordered phase, $\Theta = \pi$ is the quantum critical point (quantum phase transition) between stable fixed points $\Theta = 0$ and $\Theta = 2\pi$, which is consistent with the conjecture made in Ref. 29 that the quantum disordered phase of the O(4) NLSM with $\Theta = \pi$ could be a gapless paramagnet (a 2+1d CFT). Recently this conjecture was confirmed numerically in Ref. 31, 32, and the sign-problem-free simulation in both Ref. 31, 32 strongly suggest that the quantum disordered phase of the O(4) NLSM with $\Theta = \pi$ is indeed a strongly coupled CFT, sandwiched between two fully gapped quantum disordered phases controlled by fixed points $\Theta = 0$ and $2\pi$ (Fig.4 in Ref. 32, and discussion therein).

The $1/m$ expansion above is certainly valid for large $m$ (small $g$), which corresponds to the ordered phase of the O(4) CP$^1$ field $\mathbb{n}$ and three component vector $\mathbf{N}$. The usual expectation of QED with $N_f = 2$ is that it leads to spontaneous chiral symmetry breaking at low energy [33–36], which precisely corresponds to the order of vector $\mathbf{N}$. However, if a proper four fermion interaction term is turned on in Eq. 3 and Eq. 11 that prevents the chiral symmetry breaking, it may remain a CFT that corresponds to the disordered phase of O(4) NLSM with $\Theta = \pi$.

Other possible boundary states can be constructed through the O(4) NLSM with $\Theta = \pi$, as was discussed in Ref. 17, 37. For example, let us break the $U(1)_c$ symmetry, and keep the following time-reversal symmetry $T : z_\alpha \rightarrow (\tau^a)_{\alpha \beta} z_\beta = (\tau^a \exp(-i\pi/2))_{\alpha \beta} z_\beta$, then one can see that this O(4) NLSM model with $\Theta = \pi$ precisely correspond to the boundary of the bosonic T$	ext{I}$ with $U(1)_c \times Z^2_T$ symmetry. And following the discussion in Ref. 17, this $\Theta$–term can drive the boundary into the so called $eCmC Z^2_T$ topological order, namely its $e$ and $m$ anyons with mutual semion statistics both carry half $U(1)_c$ charge.

The authors are supported by the David and Lucile Packard Foundation and NSF Grant No. DMR-1151208. The authors are grateful to Chong Wang, T. Senthil for very helpful discussions. We also acknowledge a related unpublished work we learned through private communication [38].

[38] A. C. Potter, M. A. Metlitski, and A. Vishwanath, In preparation (private communication).
[39] The magnetic monopole of $a_{\mu}$ is invariant under time-reversal symmetry, while the polarization charge is odd under time-reversal, thus whether this polarization charge of the $2\pi -$monopole is $+1$ or $-1$ depends on how time-reversal symmetry is broken on the boundary, which we have confirmed numerically. But no matter how time-reversal symmetry is broken at the boundary, we can always construct the Kramers doublet boson $\mathcal{b}_\alpha$ monopole we need with quantum numbers $(q = 0, Q_s = \pm 1, 2\pi)$.
[40] More precisely, $N$ is odd under the effective time-reversal symmetry introduced in Ref. 23, which is a combination of $T$ and a $\pi -$rotation of the pairing order parameter.