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Chirality-Protected Majorana Zero Modes at the Gapless Edge of Abelian Quantum Hall States

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We show that the $\nu = 8$ integer quantum Hall state can support Majorana zero modes at domain walls between its two different stable chiral edge phases without superconductivity. This is due to the existence of an edge phase that does not support gapless fermionic excitations; all gapless excitations are bosonic in this edge phase. Majorana fermion zero modes occur at a domain wall between this edge phase and the more conventional one that does support gapless fermions. Remarkably, due to the chirality of the system, the topological degeneracy of these zero modes has exponential protection, as a function of the relevant length scales, in spite of the presence of gapless excitations, including gapless fermions. These results are compatible with charge conservation, but do not require it. We discuss generalizations to other integer and fractional quantum Hall states, and classify possible mechanisms for appearance of Majorana zero modes at domain walls.

I. INTRODUCTION.

Majorana zero modes have been the focus of recent theoretical and experimental efforts[1,2], motivated in part by their potential applications to topological quantum information processing[3–5]. A Majorana operator γ is a self-hermitian fermionic operator, $\gamma = \gamma^\dagger$, $\gamma^2 = 1$. It is a zero mode if it commutes with the Hamiltonian H, i.e. if $[H, \gamma] = 0$. This occurs most naturally in superconductors, where the eigenstates are superpositions of particles and holes so that $\gamma = c + c^\dagger$ where c is the annihilation operator for electrons. For this reason, experimental attention[6–12] has been focussed on putative $p+ip$ superconductors[13] and on proximity-induced superconductivity in topological insulator surface states[14], semiconductor quantum wells[15], and semiconductor nanowires[16,17]. The superconductivity need not be long-range-ordered; Majorana zero modes can also occur in systems that only have quasi-long-range-ordered superconductivity[18,19]. In fact, superconductivity is not necessary at all: Ising topological order supports an analog of superconductivity[20,21] that is sufficient to support Majorana zero modes. This includes the Moore-Read state[20,22–24] and the anti-Pfaffian state[25,26], which are candidate descriptions of the $\nu = 5/2$ fractional quantum Hall state. Certain extrinsic defects in bilayer Abelian fractional quantum Hall systems[27,28] or $\mathbb{Z}_2$ toric code models[29] can also localize Majorana zero modes without any superconductivity.

Recently, it was shown that an Abelian topological phase, such as an integer quantum Hall state, can have multiple stable chiral edge phases[30]. By tuning parameters at the edge, one can drive the system through edge phase transitions that leave the bulk unaffected. One of the simplest examples of this is the $\nu = 8$ integer quantum Hall state, which has an edge phase, which we will call the I_8 phase, that is continuously connected to the edge of a non-interacting electron system, and a second edge phase, the E_8 phase, which has only bosonic excitations. This raises the question, then, of what happens when there is a domain wall at the edge, with the I_8 phase on one side and E_8 on the other, as depicted in Fig. 1. If the I_8 phase lies upstream, then a low-energy fermionic excitation cannot propagate through the domain wall to the E_8 side since all fermionic excitations are gapped in the latter region. The I_8 regions have a fermion parity that is conserved by the dynamics of the edge, so long as no electrons tunnel in from external leads or localized bulk states. Moreover, the ground state of each fermion parity has the same energy since there are no gapless excitations in either phase that can measure the fermion parity – only $hc/2e$ vortices can do that. Thus, each such region has the same Hilbert space as a pair of Majorana zero modes at the ends of a superconducting nanowire. Surprisingly, we find that even in the presence of gapless (fermionic!) degrees of freedom, the topological degeneracy is protected exponentially: the energy splitting of the ground states decays exponentially with the separations between the domain walls. The topological protection of Majorana zero modes in this scenario is attributed to the chirality of the edge modes, which require a 2D bulk phase to exist. This should be compared with the scenario considered in Refs. [18,19], where Majorana zero modes occur in number-conserving one-dimensional wires coupled to quasi-long-ranged superconductors, but are protected only algebraically.

As shown in Ref. [30], there are also fermionic fractional quantum Hall states that admit edge phases without gapless fermionic excitations. These also support Majorana zero modes at domain walls between bosonic and fermionic edge phases. Finally, there are fractional quantum Hall states with edge phases in which only a subset of the bulk quasiparticle types are gapless (the subset that braids trivially with a non-trivial bosonic quasiparticle that condenses on the edge); these, too, have a topological degeneracy associated with domain walls.

The paper is organized as follows: we begin by reviewing the edge theory of quantum Hall states and the two edge phases (E_8 and I_8) of the $\nu = 8$ integer quantum Hall state in Sec. II. In Sec. III we establish that there are Majorana zero modes at the domain walls between E_8 and I_8 edge phases by directly solving a representative model of the edge phases in the low-energy limit. Sec. V and Sec. VI address the stability of the topological degeneracy against perturbations. In Sec. IX and X we discuss generalizations to other Abelian fractional quantum Hall states. In XI, we present a synthesis of the topological degeneracy of domain walls on the edge theory.

II. EDGE PHASES OF THE $\nu = 8$ INTEGER QUANTUM HALL STATE

We begin by recalling some facts about the edge phases of the $\nu = 8$ integer quantum Hall state[30]. Low-energy edge excitations of an Abelian quantum Hall state may be described by the chiral Luttinger liquid effective action:

$$S_{LL} = \int dx\, dt \bigg(\frac{1}{4\pi} K_{IJ} \partial_t \phi^I \partial_x \phi^J - \frac{1}{4\pi} V_{IJ} \partial_x \phi^I \partial_x \phi^J + \frac{1}{2\pi} t_I \epsilon_{\mu\nu} \partial_\mu \phi^I A_\nu \bigg). \quad (1)$$

The fields in this action satisfy the periodicity condition $\phi^I \equiv \phi^I + 2\pi n^I$ for $n^I \in \mathbb{Z}$ and the equal-time commutation relation $[\phi_I(x), \partial_y \phi_J(y)] = 2\pi i K^{-1}_{IJ} \delta(x-y)$. An edge phase is characterized by an equivalence class of a positive-definite symmetric integer *K-matrix*, and integer *charge vector* t, with respect to $\mathrm{GL}(N,\mathbb{Z})$ basis transformations $\tilde{K} = W^T K W, \tilde{t} = W^T t$, with $W \in \mathrm{GL}(N,\mathbb{Z})$. Such transformations are induced by invertible changes of variables $\phi^I = W^I{}_J \tilde{\phi}^J$ that preserve the periodicity of the fields ϕ^I. The charge vector t determines the coupling to the external electromagnetic field, and the velocity matrix V_{IJ} is a real matrix that determines the velocities of the edge modes and, when the theory is not fully chiral, also determines the scaling dimensions of operators.

It is useful to characterize these phases by lattices Λ rather than equivalence classes of K-matrices. Let e^a_I be the eigenvector of K corresponding to eigenvalue λ_a: $K_{IJ} e^a_J = \lambda^a e^a_I$. We normalize e^a_J so that $e^a_J e^b_J = \delta^{ab}$ and define a metric $g_{ab} = \lambda_a \delta_{ab}$. Then, $K_{IJ} = g_{ab} e^a_I e^b_J$ or, using vector notation, $K_{IJ} = \mathbf{e}_I \cdot \mathbf{e}_J$. The metric g_{ab} defines a bilinear form on the lattice Λ – this just means we can multiply two lattice vectors $\mathbf{e}_I, \mathbf{e}_J$ together using the metric, $\mathbf{e}_I \cdot \mathbf{e}_J = e^a_I g_{ab} e^b_J$. The N vectors $\mathbf{e}_I$ define a lattice $\Lambda = \{m_I \mathbf{e}_I | m_I \in \mathbb{Z}\}$. The $\mathrm{GL}(N,\mathbb{Z})$ transformations $K \to W^T K W$ are simply basis changes of this lattice, so we can equally well describe edge phases by equivalence classes of K-matrices or by lattices Λ.

The connection of quantum Hall edge phases to lattices can be exploited more easily if we make the following change of variables, $X^a = e^a_I \phi^I$, in terms of which the action takes the form

$$S = \frac{1}{4\pi} \int dx\, dt \bigg(g_{ab} \partial_t X^a \partial_x X^b - v_{ab} \partial_x X^a \partial_x X^b . \bigg) \quad (2)$$

The variables X^a satisfy the periodicity condition $\mathbf{X} \equiv \mathbf{X} + 2\pi \mathbf{y}$ for $\mathbf{y} \in \Lambda$ and $v_{ab} \equiv V_{IJ} f^I_a f^J_b$, where f^I_a are basis vectors for the dual lattice Λ^*, satisfying $f^I_a e^a_J = e_{La} (K^{-1})^{LI} e^a_J = \delta^I_J$.

We now focus on the $\nu = 8$ integer quantum Hall state. There are two possible choices for (K_{IJ}, t_I) that are consistent with the same bulk phase[30]. The first is $K_{IJ} = \delta_{IJ}$ and $t_I = 1$, which is continuously connected to the edge of the $\nu = 8$ state of non-interacting electrons. We will call this the I_8 phase. The corresponding lattice is just the 8-dimensional hypercubic lattice $\mathbb{Z}^8$. For later convenience, we make the basis change $W = \mathrm{diag}(-\mathbf{1}_3, \mathbf{1}_5)$, which leaves $K_{IJ} = \delta_{IJ}$ unchanged but transforms t_I to $(-1,-1,-1,1,1,1,1,1)$. The second phase has $K_{IJ} = K^{E_8}_{IJ}$, where K^{E_8} is the Cartan matrix of E_8, given explicitly in Appendix A. The corresponding charge vector is $t = (4,-2,0,0,0,0,0,0)$. The lattice is the root lattice of the E_8 Lie algebra, hence the name.

The edge phase transition between these phases occurs when an additional non-chiral pair of modes comes down in energy and interacts with the 8 chiral modes. Such modes are normally present but, in general, some non-chiral combination of right- and left-moving modes will be gapped at low energies. The particular combination that gets gapped determines the phase of the remaining 8 gapless chiral modes. For the sake of concreteness, let us begin in the E_8 edge phase:

$$S_0 = \int dx\, dt \bigg(\frac{1}{4\pi} \left(K^{E_8} \oplus \sigma_z \right)_{IJ} \partial_t \phi^I \partial_x \phi^J - \frac{1}{4\pi} V_{IJ} \partial_x \phi^I \partial_x \phi^J + \frac{1}{2\pi} t_I \epsilon_{\mu\nu} \partial_\mu \phi^I A_\nu \bigg). \quad (3)$$

Now, $I,J = 1,\ldots,10$, and $\sigma_z = \mathrm{diag}(1,-1)$ is the K-matrix for the non-chiral pair of modes. We assume that $t_9 = t_{10} = -1$. (This corresponds to adding a non-chiral pair with $\tilde{t}_9 = \tilde{t}_{10} = 1$ to the I_8 state.) We now consider perturbations that could gap a non-chiral combination of modes. We focus on $S = S_0 + S_{\mathrm{g}}$, where

$$S_{\mathrm{g}} = \int dx\, dt \left[u_E \cos(\phi_9 + \phi_{10}) + u_I \cos(-\phi_1 + \phi_9 + 3\phi_{10}) \right] \quad (4)$$

If $u_E \gg u_I$ or is more relevant (which is determined by the matrix V_{IJ}), then ϕ_9 and ϕ_{10} will be gapped and the system will be in the E_8 phase. On the other hand, if $u_I \gg u_E$ or is more relevant, then the system will be in the I_8 phase[30], which may be seen as follows. We first note that $I_8 \oplus \sigma_z = (W^8)^T (K^{E_8} \oplus \sigma_z) W^8$, where the explicit form of W^8 is given in Appendix A. If we make the change of variables $\phi_I = W^8_{IJ} \tilde{\phi}_J$, then Eq. (3) takes the form

$$S_0 = \int dx\, dt \bigg(\frac{1}{4\pi} \left(I_8 \oplus \sigma_z \right)_{IJ} \partial_t \tilde{\phi}^I \partial_x \tilde{\phi}^J - \frac{1}{4\pi} \tilde{V}_{IJ} \partial_x \tilde{\phi}^I \partial_x \tilde{\phi}^J + \frac{1}{2\pi} \tilde{t}_I \epsilon_{\mu\nu} \partial_\mu \tilde{\phi}^I A_\nu \bigg), \quad (5)$$

where $\tilde{t} \equiv (W^8)^T t = \mathrm{diag}(-\mathbf{1}_3, \mathbf{1}_5)$ and $\tilde{V}_{IJ} \equiv (W^8)^T_{IK} V_{KL} W^8_{LJ}$. Then Eq. (4) takes the form:

$$S_{\mathrm{g}} = \int dx\, dt \left[u_E \cos(\tilde{\phi}_1 + \tilde{\phi}_2 \ldots + \tilde{\phi}_9 + 3\tilde{\phi}_{10}) + u_I \cos(\tilde{\phi}_9 + \tilde{\phi}_{10}) \right] \quad (6)$$

Thus, u_I gives a gap to $\tilde{\phi}_9, \tilde{\phi}_{10}$, leaving behind the theory with $K = I_8$. The general criterion for determining whether

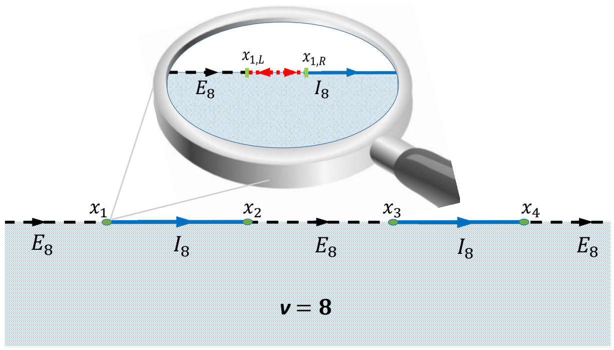

FIG. 1: The edge of a conventional $\nu = 8$ integer quantum Hall state is divided into alternating sections in the two different stable chiral edge phases: the I_8 and E_8 phases. The domain wall centered at x_i is a small interval from $x_{i,L}$ to $x_{i,R}$ in which there is a counter-propagating mode, due to edge reconstruction. As discussed in the text, this configuration has degenerate ground states, which can be interpreted as Majorana zero modes at the domain walls.

the low-energy theory is described by $K = I_8$ or K^{E_8}, given a particular gapping term, is given in Appendix B.

III. GROUND STATE DEGENERACY OF AN EDGE WITH MULTIPLE DOMAIN WALLS

We now consider a quantum Hall droplet whose edge is divided into multiple sections, with domain walls at x_1, x_2, ..., as depicted in Fig. 1. For later convenience, we will take the domain walls to have width $2a$, extending from $x_{i,L}$ to $x_{i,R}$, with $x_{i,R} - x_i = x_i - x_{i,L} = a$. The regions C_i between the domain walls lie between $x_{i,R}$ and $x_{i+1,L}$. We will assume that $C_{I_8} \equiv \bigcup_n C_{2n-1}$ is in the I_8 phase while $C_{E_8} \equiv \bigcup_n C_{2n}$ is in the E_8 phase. To arrange this, we take an effective action of the form $S = S_0 + S_{\rm g}$ with $u_E, u_I \to u_E(x), u_I(x)$, where $u_I(x) = u\chi_{\rm o}(x)$, $u_E(x) = u\chi_{\rm e}(x)$, and $\chi_{\rm o}(x) = 1$ for $x \in C_{I_8}$ and $\chi_{\rm o}(x) = 0$ otherwise while $\chi_{\rm e}(x) = 1$ for $x \in C_{E_8}$ and $\chi_{\rm e}(x) = 0$ otherwise. The parameter u is assumed to be large so that the fields are pinned to the minima of the corresponding cosines everywhere except at the domain walls, $x_{i,L} < x < x_{i,R}$.

Let us first consider one domain wall between C_1 and C_2. Since the arguments of the cosines in C_1 and C_2 do not commute, only one cosine can be pinned in a single one of its minima, while the other must be in a superposition of its minima. Suppose that we choose a basis in which $\phi_9 + \phi_{10}$ in C_1 is pinned to one of the minima of the cosine, so that $\phi_9 + \phi_{10} = 2\pi n$, for some $n \in \mathbb{Z}$. Then $-\phi_1 + \phi_9 + 3\phi_{10}$ in C_2 will be in a superposition of minima connected by the action of the operator $\cos(\phi_9 + \phi_{10})$, i.e. shifted by 4π. To be more precise, note that the sectors of the theory can be labeled $|\tilde{p}_1, \ldots, \tilde{p}_8, -\phi_1^0 + \phi_9^0 + 3\phi_{10}^0\rangle$, where $\tilde{p}_I$ is the constant part (i.e. the zero mode) of $\partial_x\tilde{\phi}_I$ (physically, it is the charge density in mode I) and $-\phi_1^0 + \phi_9^0 + 3\phi_{10}^0$ is the constant part of the corresponding combination of fields. In this basis, the ground states are $\sum_m |\mathbf{0}_8, 4\pi m\rangle$ and $\sum_m |\mathbf{0}_8, 2\pi + 4\pi m\rangle$. Here $\mathbf{0}_8$ denotes collectively the values of $\tilde{p}_1, \ldots, \tilde{p}_8$. Either of these states will satisfy both cosines. Thus, we conclude that there is a two-fold ground state degeneracy in the $u \to \infty$ limit. To generalize this analysis to arbitrary number of domain walls, it is important to take into account the global structure of the moduli space of the bosonic fields[18]. Below we will use a different, perhaps more physical, method to count the ground-state degeneracy.

First, we give a physical interpretation that explains why this degeneracy is robust. Note that, in the basis of boson fields that we have chosen, the fields $\phi_1, \ldots, \phi_8$ can only create bosonic excitations. Fermionic excitations necessarily involve ϕ_9 and ϕ_{10}, so there are no fermionic excitations in C_{E_8} in the $u_E \to \infty$ limit. Meanwhile, the coupling u_I preserves the fermion parity of ϕ_9 and ϕ_{10} in C_{I_8}. It tunnels a bosonic excitation from ϕ_9 and ϕ_{10} to $\phi_1, \ldots, \phi_8$ and can, therefore, be viewed as analogous to the pair tunneling term that couples a semiconductor wire to a superconducting wire[18]. Since the fermion parity of ϕ_9 and ϕ_{10} in C_{I_8} cannot be changed by tunneling into C_{E_8} or by tunneling to $\phi_1, \ldots, \phi_8$, it is conserved. Thus, if there are $2k$ domain walls, the total ground state degeneracy is 2^{k-1}: for fixed total fermion parity, each of the k regions in C_{I_8} can have even or odd parity. Note that states with different total fermion parity are not expected to be degenerate; we examine this point in detail in Sec VII.

As we discuss in the next paragraph, there are no terms that can be added to the Hamiltonian that would violate this low-energy conservation law without closing the energy gap to the counter-propagating modes in C_{E_8}. Moreover, as we will see in the next section, phase slips in the E_8 phase do not cause any splitting due to chirality. In summary, there are no local terms that can be added to the Hamiltonian that would cause an energy splitting between the even and odd parity ground states. Therefore, the degeneracy is robust. This can be recast in more formal terms by introducing the operators

$$
\begin{aligned}
A_j &= \exp\left[\frac{i}{2}\int_{x_{2j-1,L}}^{x_{2j,R}} \partial_x(\phi_9 + \phi_{10})\right], \\
B_j &= \exp\left[\frac{i}{2}\int_{x_{2j,L}}^{x_{2j+1,R}} \partial_x(\tilde{\phi}_9 + \tilde{\phi}_{10})\right] \\
&= \exp\left[\frac{i}{2}\int_{x_{2j,L}}^{x_{2j+1,R}} \partial_x(-\phi_1 + \phi_9 + 3\phi_{10})\right]
\end{aligned}
\tag{7}
$$

A_j operators just measure the total fermion parity of the corresponding I_8 region, while B_j measures the fermion parity stored in the $\tilde{\phi}_9$ and $\tilde{\phi}_{10}$ modes in the corresponding E_8 region and effectively tunnels a fermion across this E_8 region. These operators satisfy

$$
\begin{aligned}
A_j^2 = B_j^2 = 1, \\
\{A_j, B_j\} = \{A_j, B_{j-1}\} = 0 \\
[A_j, B_k] = 0 \text{ for } k \neq j, j-1
\end{aligned}
\tag{8}
$$

Furthermore $[H, A_j] = [H, B_j] = 0$, so these operators form an algebra over the ground state subspace. Therefore, the ground state degeneracy is 2^{k-1} if there are $2k$ domain walls. We note that since the degeneracy is 2^{k-1}, it follows that if there is only one I_8 region and one E_8 region, there is no degeneracy. While the two possible fermion parities of an I_8 region are degenerate (assuming that there are other I_8 regions), there is no degeneracy between different electron numbers of the entire droplet. (The droplet has a fixed electron number, not merely a parity, since charge is conserved.) We discuss the splitting of this degeneracy in Section VII.

Following Ref. 31, we can define Majorana fermion operators. When the coefficients of the cosines u_I, u_E are large, their arguments are 2π multiplied by integer-valued operators $\hat{m}_i$, $\hat{n}_i$:

$$2\pi\hat{m}_i = (\tilde{\phi}_9 + \tilde{\phi}_{10})_{C_{2i-1}}\ ,\ \ 2\pi\hat{n}_i = (\phi_9 + \phi_{10})_{C_{2i}} \qquad (9)$$

At the domain wall between C_0 and C_1, we can define the Majorana fermion operator:

$$\gamma_1 \equiv e^{i\pi(\hat{m}_1 - \hat{n}_0)}. \qquad (10)$$

Defining γ_2, γ_3, ... similarly in terms of the arguments of the cosines that flank the corresponding domain walls, we see that the operators introduced in the previous paragraph can be expressed as: $A_1 = i\gamma_1\gamma_2$, $B_2 = i\gamma_2\gamma_3$.

If we express γ_1 in terms of the original electron operators, however, we see that:

$$\gamma_1 = e^{-\frac{i}{2}\phi_1(x_{1,R})} \times e^{i\phi_{10}(x_{1,R})} \times e^{\frac{i}{2}[\phi_9(x_{1,R})-\phi_9(x_{1,L})]}\ e^{\frac{i}{2}[\phi_{10}(x_{1,R})-\phi_{10}(x_{1,L})]} \qquad (11)$$

From the first factor in this expression, it is apparent that the Majorana fermion operator is not local in terms of the original electron operators. In a system coupled to a 3D superconductor, we would have a similar expression, but with the second term on the right-hand-side replaced by $e^{i\theta/2}$, where θ is the phase of the superconducting order parameter, which can be treated as a classical number. However, in the present case, the fluctuating non-local expression $e^{-\frac{i}{2}\phi_1(x_{1,R})}$ is necessary to relate the Majorana fermion γ_1 to the electron operator at the domain wall, $e^{i\phi_{10}(x_{1,R})}$ (together with the local fluctuation $e^{\frac{i}{2}[\phi_9(x_{1,R})-\phi_9(x_{1,L})]}\ e^{\frac{i}{2}[\phi_{10}(x_{1,R})-\phi_{10}(x_{1,L})]}$).

One might worry that the ground state degeneracy is unstable to arbitrary perturbations, since there are gapless fermionic excitations in the I_8 regions. We show in Sec V that this is not the case and show, moreover, that the finite-size splitting between nearly-degenerate ground states decays exponentially with size.

IV. SCATTERING PROBLEM AT THE DOMAIN WALL

Now that we have shown explicitly the topological degeneracy of domain walls between E_8 and I_8 edge phases, we return to the heuristic observation made in the introduction: Intuitively, Majorana zero modes must exist at I_8-E_8 domain walls in order to absorb low-energy fermionic excitations originating within the I_8 regions. It is instructive to see how this occurs by solving the scattering problem of the fields at the domain wall, and deriving an "S matrix".

We focus on the behavior of the fields in the vicinity of the domain wall at x_2. The boundary conditions on the fields are given by:

$$\begin{aligned}
\tilde{\phi}_a(x^-_{2,L}) &= \tilde{\phi}_a(x^+_{2,L}),\ a = 1,\ldots,8\\
\phi_a(x^-_{2,R}) &= \phi_a(x^+_{2,R}),\ a = 1,\ldots,8\\
\tilde{\phi}_9(x^+_{2,L}) + \tilde{\phi}_{10}(x^+_{2,L}) &= 2\pi m\\
\phi_9(x^-_{2,R}) + \phi_{10}(x^-_{2,R}) &= 2\pi n.
\end{aligned} \qquad (12)$$

Here $x^\pm \equiv x \pm \epsilon$ with $\epsilon \to 0$.

For a narrow domain wall with $x_2 = x^+_{2,L} \approx x^-_{2,R}$, we can ignore the variation of the fields within the domain wall to find:

$$\tilde{\phi}_a(x^-_2) = \sum_{b=1}^{8}(W^8)^{-1}_{ab}\phi_b(x^+_2) - \frac{1}{2}\phi_1(x^+_2) - \pi m_1 - \pi n_1. \qquad (13)$$

for $a = 1, 2, \ldots 8$ where m_1, n_1 are defined in Eq. (9). As a result of the coefficient of $\frac{1}{2}$ in front of the second term on the right-hand side of (13), any correlation function of the form $\left\langle e^{i\sum_{a=1}^{8} n_a\tilde{\phi}_a(x<x_2)}\, e^{i\sum_{b=1}^{8} m_b\phi_b(x>x_2)}\right\rangle$ vanishes for all integers m_a and all integers n_a satisfying $\sum_{a=1}^{8} n_a \equiv 1 \pmod 2$ (i.e. with odd fermion parity). Suppose we were to add an arbitrary term of the form $e^{i\sum_{a=1}^{8} p_a\tilde{\phi}_a}$ or $e^{i\sum_{a=1}^{8} p_a\phi_a}$ to the action, either at the domain wall or in the gapped regions to either side of the domain wall. Such a term could be accounted for in perturbation theory by inserting copies of this term into the correlation function, but the correlation function will clearly still vanish since such terms cannot cancel the $\frac{1}{2}\phi_1(x^+_2)$. Thus, the 'elastic' S-matrix vanishes in all odd fermion number sectors. The only way to get a non-vanishing correlation function is to act with a fermion operator at the domain wall, e.g. $\left\langle e^{i\sum_{a=1}^{8} n_a\tilde{\phi}_a(x<x_2)}\, e^{i\sum_{b=1}^{8} m_b\phi_b(x>x_2)}\, e^{\pm i\tilde{\phi}_9(x_2)}\right\rangle \neq 0$. Therefore, when a fermionic excitation created in the I_8 region passes through the domain wall, its fermion parity gets absorbed by the Majorana zero mode and the electric charge (a superposition of charge 0 and 2) continues into the bosonic E_8 region.

V. DEGENERACY SPLITTINGS

In this section we consider processes that can split the degeneracy that we found in previous sections. The worst-case scenario would be a splitting that is independent of system size, which would mean that the degeneracy that we found in Section III is not really stable against perturbations. However, even if the degeneracy is stable and exact in the thermodynamic limit, there may be a small finite-size splitting. In this section, we analyze perturbations to the effective action in Eqs. (3) and (4) in order to determine which ones can split the degeneracy and how the resulting splitting depends on the system size.

Since it depends on the conservation of fermion parity, any process that changes fermion parity in an I_8 region will cause

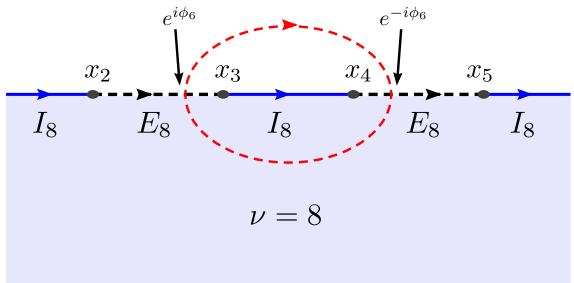

FIG. 2: A vortex tunneling process that would cause an energy splitting in a non-chiral system such as a superconducting nanowire. It does *not* cause such a splitting of the topological degeneracy associated with $I_8 - E_8$ domain walls at the $\nu = 8$ edge. As a result of the chirality of the system, this process does not measure the fermion parity of the I_8 region, as explained in Section V.

a transition between different ground states. One such process is fermion tunneling from one I_8 region to another. Since all fermions are gapped in the E_8 regions, such a process will cause an exponential splitting e^{-L/ξ_8} where L is the length of the E_8 region and ξ_8 is inversely proportional to the gap to ϕ_9 and ϕ_{10} excitations in the E_8 regions.

There are also local perturbations in the I_8 regions that do not commute with B_j operators, a simple example of which is $\cos(\tilde{\phi}_I + \tilde{\phi}_9)$ where $I = 1, 2, \cdots, 8$. Such a perturbation is gapped on either side of the domain wall, but might be present at the interface. One might worry that these perturbations completely lift the degeneracy. We postpone the discussion of this issue to Sec VI.

Another fear is that perturbations acting in the E_8 regions could cause an energy splitting between states of different fermion parities by effectively measuring the fermion parity of neighboring I_8 regions. Consider, for instance the following charge-neutral perturbation:

$$S_{\text{ps}} = \int_{C_{E_8}} dx\, dt\, \lambda(x)\, \cos(\phi_6) \tag{14}$$

The coupling $\lambda(x)$ could be uniform in C_{E_8}; it could be a sum of δ-functions acting at isolated points (e.g. due to a small number of impurities); or it could be a random function, due to a random distribution of impurities. All of these possibilites are interesting because the commutator between ϕ_6 and the I_8 gapping term is an odd multiple of $2\pi i$:

$$[\phi_6(x), (-\phi_1 + \phi_9 + 3\phi_{10})(x')] = -6\pi i, x < x'. \tag{15}$$

Hence, this perturbation can shift the u_I cosine from one minimum to another. Equivalently, $\cos\phi_6(x)$ doesn't commute with the operator B_1 if $x_{2,L} < x < x_{3,R}$. Hence, it can, in principle, give opposite-sign contributions in second-order perturbation theory to the energies of states with different fermion parities, i.e. to states with different A_2 eigenvalues. The physical picture would be that S_{ps} can tunnel a vortex across C_2. When it acts again to tunnel a vortex across C_4, it effectively causes a vortex to encircle C_3. The state of the system would then acquire a ± 1, depending on the fermion parity of C_3, as occurs in the case of a superconducting nanowire[18]. An alternate description for perturbation theory in λ is that it is an instanton gas expansion for instantons that cause the I_8 gap term $\cos(-\phi_1 + \phi_9 + 3\phi_{10})$ to tunnel from one minimum to another. All of the ϕ_a with a even share the property exhibited by ϕ_6 that $\left(K^{E_8}\right)^{-1}_{a1}$ is odd, leading to a commutation relation similar to (15). Hence, any operator $\cos(m_I \phi^I)$ in the E_8 regions could split states of different I_8 fermion parity so long as $\sum_{I=1}^{4} m_{2I}$ is odd. All such perturbations can be analyzed along similar lines, so we focus on Eq. (14) for the sake of concreteness. It happens to have the lowest possible scaling dimension for such a perturbation.

The perturbation shifts the energies of the two ground states. The energy splitting is the difference in the energy shifts of the two ground states. Consider the perturbative expansion in powers of λ. In time-dependent perturbation theory, the leading-order contribution to the ground state energy shift is equal to the ground state-to-ground state transition amplitude:

$$\begin{aligned} \Delta E_{A_2} &= -i \int_{-\infty}^{\infty} dt \int dx\, dx'\, \lambda(x)\, \lambda(x') \\ &\quad \times \langle 0_{A_2} | \mathcal{T}(\cos(\phi_6(x,t)) \cos(\phi_6(x',0))| 0_{A_2} \rangle \\ &= \sum_{n \neq 0} \frac{\left| \langle n | \int dx\, \lambda(x) \cos(\phi_6(x,0)) | 0_{A_2} \rangle \right|^2}{E_0 - E_n} \end{aligned} \tag{16}$$

Here $\mathcal{T}$ represents time-ordering and $n \neq 0$ means summing over all excited states. The subscript $A_2 = \pm 1$ labels ground states by their A_2 eigenvalues, where the operator A_2 is defined as in Eq. (7). There will be a non-zero contribution to the energy splitting when $x_{2,L} < x < x_{3,R}$ and $x_{4,L} < x' < x_{5,R}$ or vice versa, as in Fig. 2. The second equality follows by a standard spectral decomposition; the resulting expression is the second-order energy shift in time-independent perturbation theory.

To compute the desired correlation function across the intervening I_8 region, we use the boundary condition (essentially Eq. (13), but inverted to give ϕ_I in terms of $\tilde{\phi}_I$) to rewrite the field ϕ_6 in terms of the $\tilde{\phi}$ fields and the constants m_2 and n_1:

$$\phi_6(x_3^-) = 3\pi(m_2 + n_1) + \frac{1}{2}\left[\tilde{\phi}_1 + \cdots + \tilde{\phi}_7 - \tilde{\phi}_8\right]_{x_3^+} \tag{17}$$

This equation can be understood as follows: because $\frac{1}{2}(\phi_9 + \phi_{10})$ shifts the argument of the u_I cosine by 2π (it is a single fermion excitation), the combination $\phi_6 - \frac{3}{2}(\phi_9 + \phi_{10})$ commutes with the u_I cosine term and should be gapless in the I_8 region. It is also gapless in the E_8 region (since $\phi_9 + \phi_{10}$ is pinned), so $\phi_6 - \frac{3}{2}(\phi_9 + \phi_{10})$ in fact represents a gapless mode across the whole system. If the fields are taken to be very close to x_3, we can use the W^8 transformation to rewrite it in terms of the $\tilde{\phi}$ modes, which gives exactly the expression in (17). This is essentially how Eq. (13) was obtained in

Sec IV. Using (17), we find

$$\langle 0_{A_2}|\mathcal{T}(e^{i\phi_6(x_{3,L},t)}e^{-i\phi_6(x_{4,R},0)})|0_{A_2}\rangle \sim \frac{A_2}{[t-(x_3-x_4)-i\delta\,\mathrm{sgn}(t)]^2} \quad (18)$$

The points are taken near the ends x_3, x_4 of the intervening I_8 region C_3. The numerator on the right-hand-side is the fermion parity of C_3. For simplicity, we have taken all of the velocities to be equal and set them to 1. The exponent 2 on the right-hand-side of Eq. (18) must be an even integer since this is a two-point correlation function of a bosonic operator. For this particular choice of operator, we happen to find the smallest possible exponent for an operator that detects C_3 fermion parity. In addition, $(K^{-1})_{66} = 2$, so the $e^{i\phi_6}$ two-point correlation function decays with precisely the same exponent in C_{E_8} as it does across an I_8 region. Consequently, Eq. (18) holds more generally for $x_3 \in C_2$ and $x_4 \in C_4$.

Substituting the correlation function (18) into the expression for the energy shift in Eq. (16), we find an energy splitting $\delta E \equiv \Delta E_{A_2=1} - \Delta E_{A_2=-1}$ given by:

$$\begin{aligned}\delta E &= -4i\int_{C_2}dx\int_{C_4}dx'\int_{-\infty}^{\infty}dt\frac{\lambda(x)\,\lambda(x')}{[t-(x-x')-i\delta\,\mathrm{sgn}(t)]^2}\\ &= 8\pi\int_{C_2}dx\int_{C_4}dx'\,\lambda(x)\,\lambda(x')\,\delta(x-x')\\ &= 0\end{aligned} \quad (19)$$

Hence, this perturbation does not cause any splitting at all!

It is evident that the vanishing of the splitting clearly follows from the chirality of the system, and the result holds to all orders of the perturbation theory. As a comparison, if we were to replace the correlation function in this integral by a non-chiral one (e.g. in a nanowire), we would, instead find:

$$\begin{aligned}-4i\int_{C_2}dx\int_{C_4}dx'\int_{-\infty}^{\infty}dt\frac{\lambda(x)\,\lambda(x')}{[t^2-(x-x')^2-i\delta]}\\ = 4\pi\int_{C_2}dx\int_{C_4}dx'\,\frac{\lambda(x)\,\lambda(x')}{|x-x'|}\end{aligned}$$

This result can be understood more intuitively as follows. As we have explained, the only process that can measure the fermion parity (i.e. A_j) is encircling the I_8 region by an $hc/2e$ vortex (or a phase slip), which can be viewed complementarily as the virtual tunneling of a fermion from one end of the I_8 region to the other. However, fermions do not actually 'tunnel' between the two ends of an I_8 region, the reason being that the gapless fermions can only go from the upstream end to the downstream one; not back. More importantly, *every* fermion emitted by the left end *must* be absorbed by the right end. The domain walls and the C_{I_8} bulk are not weakly-coupled; instead, they are a single system with a single non-local fermion parity. To get any splitting, we would need to involve the left-moving mode in some way, e.g. to tunnel a fermion from one I_8 region to another. Such processes will contribute exponential splitting.

The final source of splitting is fermions tunneling between a metallic lead or localized states in the bulk and C_{I_8}. The corresponding terms in the action would be:

$$S_{\mathrm{F}} = \int dt\sum_{a=1}^{8}v_a\left(\Psi^\dagger(x_0,t)\,e^{i\tilde{\phi}_a(x_0,t)}+\mathrm{h.c.}\right) \quad (20)$$

where $\Psi^\dagger(x_0,t)$ creates an electron in the metallic lead/low-energy bulk state. If the spectral function of $\Psi(x_0)$ is independent of energy at low energy, i.e., if there is a constant density of states of fermions, then the lifetime of the state, τ, is given by $1/\tau \sim \sum_{a=1}^{8}v_a^2$, according to Fermi's golden rule. This leads to an exponential decay of fermion parity over time: $\langle A_i(t)A_i(0)\rangle \sim e^{-t/\tau}$, where A_i is the fermion parity of the i^{th} I_8 interval, as defined in Eq. (7).

VI. MORE GENERAL GAP-OPENING TERMS

In the previous section we noticed that local perturbations such as $\cos(\tilde{\phi}_a+\tilde{\phi}_{10})$, $a=1,2,\ldots,8$, in I_8 regions do not commute with B operators. In this section we address this issue in a more general setting. We have chosen a particular form of the cosine terms in Eq. (4) to open a gap to counter-propagating modes, but these are not the unique ways for the sytem to enter the I_8 or E_8 phases. For instance, a term such as $\cos(\tilde{\phi}_1+\tilde{\phi}_{10})$ will drive the system into the I_8 phase. Indeed, an arbitrary linear combination of $\cos(\tilde{\phi}_a+\tilde{\phi}_{10})$ terms, with $a=1,2,\ldots,9$, will also drive the system into the I_8 phase, as will more general terms that are not quadratic in the original fermionic variables. Similarly, there is a family of gap-opening terms that will drive the system into the E_8 phase. More generally, one can consider a sum of several cosine terms.

The form of the ground state generating operators given in Eq. (7) depended explicitly on the precise form of the cosine terms. Including more complicated gapping terms, we are no longer able to explicitly construct the ground state generating operators. However, we can argue that the degeneracy is unchanged by "adiabatically" deforming the Hamiltonian from the special Hamiltonian considered in Eq. (4) to any other one that leaves the C_{I_8} and C_{E_8} regions in the I_8 and E_8 phases respectively. Since the edge is gapless, the term "adiabatic deformation" means a deformation that does not close the gap to counter-propagating modes.

In order to make this argument, it is useful to distinguish between two different types of degeneracy that could occur. By "topological degeneracy", we mean states that can only be distinguished by a measurement at two distant points (e.g. two ends of an interval), while "local degeneracy" (or "accidental degeneracy") will refer to states that can be distinguished by a measurement at a single point. Then a precise statement of our claim is that the topological degeneracy remains unchanged during any deformation of the system that does not close the counter-propagating gap. The validity of this claim follows by generalizing the explicit example in Section V, which demonstrated how the chirality of the system protects the degeneracy of states that can only be distinguished by measurements at the two ends of an interval. Consequently, topological degeneracy can only be lifted when the gap to counter-propagating modes closes (apart from an exponential-in-length splitting due to virtual excitations above the gap to counter-propagating modes). Meanwhile, continuously deforming the gap-opening terms (without closing the gap to counter-propagating modes) may cause additional "local degeneracy" to develop or be lifted.

We therefore expect that the ground state generating operators also evolve with this adiabatic continuation, while preserving the algebra responsible for the degeneracy. Consequently the topological degeneracy does not depend on the particular gap-opening terms that are present in the effective action as long as they lead to the desired edge phases. However the construction of the operators in Eq. (7) is most transparent for particular effective actions with only one cosine gapping term inside each domain, such as Eq. (4).

VII. SPLITTING OF SUPERSELECTION SECTORS IN A PERIODIC SYSTEM

While we have shown that the splitting between the parity states of a given I_8 section is exponentially small in the length of the section, the same is not true of the overall fermion parity of the edge of a quantum Hall droplet, even one with alternating E_8 and I_8 regions. Note, first, that the action (4) conserves electrical charge, so the entire droplet is characterized by its electrical charge, not merely its parity. Therefore, for this model, the a more precise statement of the question is: how does the splitting between states of N and $N+1$ electrons depend on the circumference L of a droplet? On the other hand, the phenomenon described here does not depend on charge conservation, so we are free to consider models that do not conserve charge. For such a model, we could ask about the splitting between even and odd fermion parities.

We begin by again considering Eq. (13), in which the fermions on the left hand side of the equation (and the domain wall) are related to the bosonic fields on the right. We have already noted that a fermion on one side of the domain wall is completely uncorrelated with any local bosonic operator that one might write down on the other side. What, then, is the fate of a fermionic operator as it evolves along the chiral edge toward the E_8 region? The answer may be read off from a careful grouping of the r.h.s. of Eq. (13) , where we replace x_{2+} by an arbitrary location x within the E_8 region. We define the operator

$$\begin{aligned}\Phi_a(x) &= \sum_{b=1}^{8}(W^8)^{-1}_{ab}\phi_b(x) - \frac{1}{2}\phi_1(x) - \pi m_1 - \pi n_1 \\ &= \sum_{b=1}^{10}(W^8)^{-1}_{ab}\phi_b(x) + \int_{x_2}^{x}\mathrm{d}x'\partial_{x'}(\phi_9 + 2\phi_{10} - \frac{1}{2}\phi_1)\end{aligned} \tag{21}$$

It is clear that $\Phi_a(x_2^+) = \tilde{\phi}_a(x_2^-)$, so that a fermionic operator $e^{i\tilde{\phi}_a}$ becomes $e^{i\Phi_a}$ upon crossing the domain boundary. Assuming for simplicity that the bosonic fields in the E_8 region all have the same velocity (i.e. the velocity matrix is proportional to I in the E_8 basis), we may expect that x in the above expression will simply increase with time due to the chiral edge. From this rearrangement, we can see that despite the appearance of a fraction in front of ϕ_1, the fermionic fields do, in fact, evolve into allowed operators upon hitting the domain wall into the E_8 region. Instead of being a sum of local operators, however, the fermionic fields evolve into a combination of the bosonic fields along with a *non-local* charge measurement operator (the integral in the above expression). Importantly, while $e^{i\Phi_a}$ has no non-zero correlations with any bosonic operators in the E_8 region, it will continue to evolve through that region until it hits another I_8 region. Once it does, it will again be allowed to have non-zero correlations with local operators in that region.

However, there is an important difference between a fermion operator that travels from I_8 region 1 to I_8 region 2 through an E_8 region and a fermion operator native to region 2. This involves the final value of the integral term once the operator has traveled through the entire E_8 region. We have already seen that at the first domain wall $\phi_9 + 2\phi_{10} - \frac{1}{2}\phi_1$ is pinned to $\pi(m_1 + n_1)$. At the second domain wall the pinned value of n for the E_8 region must be the same. However, the operator is entering a new I_8 region, which may have a new value of m. The total value of the integral is therefore $\pi(m_2 - m_1)$, and the operator that emerges into the second I_8 region is $e^{i\tilde{\phi}_a + i\pi(m_2-m_1)}$. Note that this operator is consistent with conservation of fermion parity within each I_8 region. Although $e^{i\tilde{\phi}_a}$ creates a fermion in region 2, $e^{i\pi(m_2-m_1)}$ transfers a fermion from region 2 to region 1, so that the operator as a whole always places the fermion in region 1 despite its evolution.

We now arrive at the source of power law splitting in systems with periodic boundary conditions. If we consider the edge of a quantum Hall droplet with a single E_8 and a single I_8 region along the edge, then the value of $m_2 - m_1$ defined above is necessarily zero, so $e^{i\pi(m_2-m_1)}$ becomes trivial. That is, the electron propagates coherently through the E_8 region. The energy cost of adding an electron to the edge of this system therefore has the same dependence on the length of the edge as if the entire edge were in the I_8 phase, i.e. it decays as a power law in the circumference. This fact remains true independent of the number of alternating E_8 and I_8 regions around the edge. While x is in I_8 regions, the fermion operator will have the form $e^{i\tilde{\phi}_a(x) + i\pi(m_j - m_0)}$, where the fermion originated in I_8 region 0 and x is currently within I_8 region j. Once x returns to the original I_8 region, the fermion operator becomes local once again.

Note that this effect splits only states of different *total* fermion parity of the edge, corresponding to the superselection sector of the set of zero modes formed by the domain walls. There is still an exponentially large, exponentially protected Hilbert space due to the presence of the alternating E_8 and I_8 regions, whose size goes as $2^{\frac{N}{2}-1}$ in the number N of domain walls.

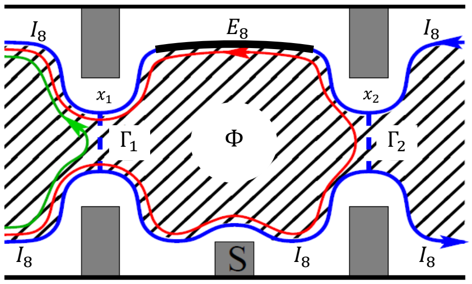

FIG. 3: Quantum Hall interferometer reading out the dual fermion parity of an E_8 region. The backscattered current oscillates as a function of flux Φ through the interferometer, which can be varied with a sidegate S that changes the area of the interference loop. If the interferometer contains a single E_8 section (black) and I_8 elsewhere (blue), then the interference pattern undergoes a π phase shift if the dual parity of the E_8 region is flipped (i.e. the eigenvalue of the corrresponding B_j operator). If the interferometer instead contains a single I_8 section (black) and E_8 elsewhere (blue), then it similarly measures the fermion parity of the I_8 region.

VIII. MEASUREMENT VIA INTERFEROMETRY

It is possible to harness the effect described in the previous section for the purpose of reading out the combined states of qubits formed by collections of $E_8 - I_8$ domain walls. To demonstrate, we place an E_8 region along one leg of a quantum Hall interferometer, as shown in Fig. VIII. We label the pinned values of the surrounding I_8 regions by m_1 and m_2, and the pinned value in the other leg by m_0. We assume that there are more E_8 regions elsewhere along the edge of the quantum Hall droplet, so that the values of m_0, m_1, and m_2 are unconstrained. We place point contacts at x_1 and x_2 along the top edge. Further, we assume that the bottom and top edges are connected at a point far to the right and that the edge has total length L, so that the bottom contacts are located at points $L - x_1$ and $L - x_2$. Our point contact Hamiltonian is therefore

$$H_\Gamma = \sum_{j=1}^{2} \sum_{a,b} \Gamma_{jab} e^{i\tilde{\phi}_a(x_j)+i\pi(m_j-m_0)} e^{-i\tilde{\phi}_b(L-x_j)} + \text{h.c.} \tag{22}$$

where Γ_{jab} is the coupling between fermion species a and b at point contact j. We can see from this Hamiltonian that the gauge invariant phase difference $\pi(m_2 - m_1)$ will appear in any measurement of the current through the interferometer. That is, the state of the qubit stored in the domain walls at the ends of the E_8 region may be read out as a π phase shift in the interferometric measurement.

One may similarly read out the fermion parity stored in an I_8 region by inverting this setup, replacing each E_8 with an I_8 and vice-versa. In this case (and in the basis we use), only half of the bosonic modes contribute to the measurement, $\phi_{2,4,6,8}$ that could measure the fermion parity in Sec. V.

IX. FRACTIONAL QUANTUM HALL STATES: THE EXAMPLE OF THE $\nu = 1/8$ BOSONIC STATE

There are several avenues along which we can generalize the preceeding construction. Perhaps the most straightforward would be other fermionic Abelian topological phases (including fractional ones) that have bosonic edge phases. Such phases are discussed in Ref. 30. Intuitively, domain walls between bosonic and fermionic chiral edge phases of a fermionic bulk state will support Majorana zero modes, as in the I_8-E_8 example discussed above, because a chiral fermion in the fermionic region cannot propagate into the bosonic region and must, therefore, be absorbed by the domain wall. This can be generalized even further to bulk Abelian topological phases that have two possible edge phases such that the set of quasiparticles that are gapless in the 'smaller' phase is a proper subset of those gapless in the other phase. Then a domain wall between the two phases must be able to absorb the quasiparticles that are 'missing' from the 'smaller' phase.

The simplest example of this is the $K = 8$ bosonic state:

$$S_{LL} = \int dx\, dt \left(\frac{8}{4\pi} \partial_t\phi\partial_x\phi - \frac{1}{4\pi} v\partial_x\phi\partial_x\phi \right). \tag{23}$$

This chiral edge has another phase that has only two primary fields. It can be accessed by considering the enlarged theory:

$$S_{LL} = \int dx\, dt \left(\frac{1}{4\pi} K_{IJ}\partial_t\phi^I\partial_x\phi^J - \frac{1}{4\pi} V_{IJ}\partial_x\phi^I\partial_x\phi^J \right). \tag{24}$$

where $I, J = 1, 2, 3$ and

$$K = \begin{pmatrix} 8 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \tag{25}$$

Now consider the following gap terms:

$$S_{\text{g}} = \int dx\, dt \left[u_8 \cos\phi_2 + u_2 \cos(8\phi_1 - 2\phi_2 + 2\phi_3) \right] \tag{26}$$

If the u_8 term dominates, then the system is in the $K = 8$ phase, as in Eq. (23). Suppose, instead, that the u_2 term dominates. The remaining gapless excitations are created by operators $\exp(im^T\phi)$ such that $m^T\phi$ commutes with the argument of the u_2 cosine. Such operators necessarily have m_1 even and, therefore, are semions or multiples of semions. Since the remaining theory is a fully chiral $c = 1$ theory in which the minimal particle is a semion, it must be the $K = 2$ theory. Thus, one edge phase has $\theta = \pi/8$ particles while the other phase has only even numbers of such particles; the domain walls between such edge phases therefore support a $\mathbb{Z}_2$ topological degeneracy associated with $\pi/8$-particle parity, generalizing the fermion parity of the I_8-E_8 case. A $\theta = \pi/8$ particle flowing downstream on the $K = 8$ edge will be absorbed by the domain wall since it cannot be transmitted into the $K = 2$ region. As in the $I_8 - E_8$ case, the splitting will be exponential in the relevant length scales since chiral perturbations, such as $\cos(\phi_2 + \phi_3)$ cannot cause any splitting, by the argument given in Section V.

This may be generalized to the $K = 2N^2$ theories, which have a $K = 2$ edge phase. The domain walls between such phases support $\mathbb{Z}_N$ zero modes.

$$S_{\rm g} = \int dxdt\,[u_1(x)\cos\phi_2 + u_2(x)\cos(N(2N\phi_1 - \phi_2 + \phi_3))]. \tag{27}$$

where $u_1(x)$ and $u_2(x)$ are non-zero in alternating intervals, $u_1(x) = u\chi_o(x)$, $u_2(x) = u\chi_e(x)$, analogous to $u_I(x), u_E(x)$ in Section III. The degeneracy corresponds to the number modulo N of $\pi/2N^2$ particles that are present in the odd regions. The ground states must represent the algebra of the operators

$$A_j = \exp\left[\frac{i}{N}\int_{x_{2j-1,L}}^{x_{2j,R}} \partial_x(2N^2\phi_1 - N\phi_2 + N\phi_3)\right],$$
$$B_j = \exp\left[\frac{i}{N}\int_{x_{2j,L}}^{x_{2j+1,R}} \partial_x(\phi_2)\right] \tag{28}$$
(29)

which satisfy $A_j^N = B_j^N = 1$, $A_jB_{j\pm1} = e^{\pm 2\pi i/N}B_{j\pm1}A_j$. This algebra implies an N-fold degeneracy associated with a pair of domain walls, generalizing the 2-fold degeneracy of Eq. (26). A two point contact interferometer can be used to measure the eigenvalues of A_j and B_j in a manner analogous to that explained in Section VIII.

There is one important subtlety that we did not face in the $N = 2$ case. The operator B_j does not commute with operators such as $\cos\phi_3$ at the domain wall. However, such an operator opens a gap to the counter-propagating modes in any gapless region. Hence, if it is present, we can view it as follows. The size of the domain wall is shrunk due to this additional term. Meanwhile, the $K = 2N^2$ region is expanded and has a spatially-varying gap-opening term: $\cos\phi_2$ in most of the region and $\cos\phi_3$ in a small part that has been reclaimed from the domain wall. There is no domain wall between these two parts of the $K = 2N^2$ region for the reasons described in Section VI.

For an alternative perspective on the innocuousness of a $\cos\phi_3$ perturbation, note that it would cause an energy splitting between different numbers modulo N of $\pi/2N^2$ particles in the odd regions (i.e. an energy splitting between A_j eigenstates with different eigenvalues). The same effect can result from a perturbation $\cos(\phi_2 + \phi_3)$ acting in the even regions flanking the odd region under consideration. However, such a chiral perturbation causes no splitting by the arguments in Section V.

Further generalizations will be discussed elsewhere[32].

X. DOMAIN WALLS BETWEEN REGIONS OF THE SAME EDGE PHASE

We have seen two nontrivial examples of domain walls carrying topological degeneracy on the edge of an Abelian quantum Hall state. Both examples share a common feature: the edge theory is fully chiral and the particle content of the gapless edge modes do not match on the two sides of the domain wall. Such chirality-protected zero modes occur in Abelian states which contain at least one topologically nontrivial bosonic quasiparticle and in fermionic systems which admit both bosonic and fermionic edge phases. In both of these cases, we can identify the topological degeneracy directly from the mismatch of the quasiparticles in the edge theory.

One might wonder what happens at domain walls between regions of the edge that are described by the same edge phase, but with distinct gapping terms. An example is the $K^{E_8} \oplus \sigma_z$ edge theory with the gapping terms $u_1(x)\cos(\phi_9 + \phi_{10}) + u_2(x)\cos(\phi_9 - \phi_{10})$. This theory is exactly a bosonic E_8 edge decoupled from a one-dimensional system of spinless fermions with p-wave pairing, which hosts localized Majorana zero modes at the domain walls between superconducting and insulating regions. We now argue that this is the most generic situation when no other symmetries are present. Since the bulk is short-ranged entangled, there are no fractionalized excitations. The only way to protect zero modes on the edge is through the conservation of fermion parity. We now consider the $K = K^{E_8} \oplus \sigma_z$ edge with the most general gapping term:

$$u\cos(\phi_9 + \phi_{10}) + u'\cos(n_I\phi_I). \tag{30}$$

Let us assume n is a primitive vector. Since we are interested in the case where $\cos(n_I\phi_I)$ gaps the edge to the E_8 phase, the criteria in Appendix B requires that n_9 and n_{10} must be odd and $n_{1,\ldots,8}$ are all even. We construct the following ground-state generating operators:

$$A_j = \exp\left[\frac{i}{2}\int_{x_{2j-1,L}}^{x_{2j,R}} \partial_x(\phi_9 + \phi_{10})\right]$$
$$B_j = \exp\left[\frac{i}{2}\int_{x_{2j,L}}^{x_{2j+1,R}} n_I\partial_x\phi_I\right]. \tag{31}$$

A_j and B_j are chosen to satisfy $A_j^2 = B_j^2 = 1$. We can easily check that all fermion-parity-conserving local operators commute with them. Their commutation algebra is

$$A_jB_j = (-1)^{\frac{n_9 - n_{10}}{2}}B_jA_j. \tag{32}$$

If $(n_9 - n_{10})/2$ is odd, there is topological degeneracy from Majorana zero modes at the domain walls. Otherwise the two phases should be regarded as the same, and the two gapping terms can be continuously deformed into each other, as described in VI.

We can similarly calculate the topological degeneracy when the gapping terms are $\cos(\phi_9 - \phi_{10})$ and $\cos n_I\phi_I$: it is determined by the parity of $(n_9 + n_{10})/2$. Because n_9 and n_{10} are both odd, $(n_9 \pm n_{10})/2$ have opposite parities. Thus, $\cos n_I\phi_I$ is continuously connected to either $\cos(\phi_9 + \phi_{10})$ or $\cos(\phi_9 - \phi_{10})$. We conclude that there are only two distinct ways to gap the edge.

For the quantum Hall systems we are considering, it is natural to impose U(1) charge conservation symmetry: without loss of generality, take the charge vector to be $(2t, 1, 1)$ where t is an integer vector. We now prove that if the gapping terms $\cos(n_I\phi_I)$ and $\cos(\phi_9 + \phi_{10})$ both yield the E_8 phase and conserve charge, they cannot host the Majorana degeneracy (32): the charge of the gapping term is given by

$$Q = 2\left(\sum_{a=1}^{8} n_a (K^{E_8})^{-1} t_a + \frac{n_9 - n_{10}}{2}\right). \quad (33)$$

Because the first term in the parentheses above is even, $Q = 0$ requires $(n_9 - n_{10})/2$ to also be even. Hence, the A and B operators commute, and there is no degeneracy.

We now present an example of exponentially-protected Majorana zero modes at domain walls on a chiral charge-conserving edge without gapless chiral fermions. They have the further virtue that they are impervious to coupling to low-energy or out-of-equilibrium fermions (except at the domain walls), unlike in the case of the $E_8 - I_8$ edge. We take the bulk phase to be the $\nu = \frac{1}{2}$ strong-pairing state, in which electrons pair up into charge $2e$ Cooper pairs, which then form a $1/8$ bosonic Laughlin state. This edge theory can be described by $K = (8) \oplus \sigma_z$ with charge vector $t = (2, 1, 1)$. We label the bosonic modes as ϕ_1, ϕ_2, and ϕ_3 and consider the gapping term:

$$S_{\rm g} = \int dx dt \left[u \cos(\phi_2 + \phi_3) + u' \cos(8\phi_1 + \phi_2 + 3\phi_3)\right]. \quad (34)$$

Both gapping vectors are null and charge-conserving. In regions dominated by $u\cos(\phi_2 + \phi_3)$, we are left with the low-energy $K = 8$ edge mode. When $u' \cos(8\phi_1 + \phi_2 + 3\phi_3)$ dominates, we perform a basis change $\phi = W\tilde{\phi}$, with

$$W = \begin{pmatrix} -3 & 0 & -1 \\ 0 & 1 & 0 \\ 8 & 0 & 3 \end{pmatrix}, \quad (35)$$

to rewrite the gapping term as $8\phi_1 + \phi_2 + 3\phi_3 = \tilde{\phi}_2 + \tilde{\phi}_3$ (and $\phi_2 + \phi_3 = 8\tilde{\phi}_1 + \tilde{\phi}_2 + 3\tilde{\phi}_3$); the K matrix is invariant:

$$W^T \begin{pmatrix} 8 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} W = \begin{pmatrix} 8 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \quad (36)$$

Thus, the low-energy theory is still $K = 8$ (in the $\tilde{\phi}$ basis), and there are no gapless fermions in either case.

We now proceed to analyze the domain walls. The coefficients of $\phi_{2,3}$ in (34) are identical to those of $\phi_{9,10}$ in the I_8-E_8 case, so the ground-state generating operators take the same form as (7), with $9, 10$ replaced by $2, 3$. There are 2^{k-1} degenerate ground states associated to $2k$ domain walls. Let us now consider the length-dependence of the energy splitting. Since fermions are gapped everywhere, splitting due to single fermion tunneling is exponentially suppressed. Chiral perturbations – the only perturbations available at low energies – will not give any splitting, as described in Section V (although, since all of the terms that are left in the low-energy theory take the form $e^{ik\phi_1}$ or $e^{ik\tilde{\phi}_1}$, and such fermion-parity-conserving terms commute with both A and B, we need not have worried about these perturbations in this case.)

Further insight comes from the expression of the Majorana zero mode operator (for clarity we neglect the uninteresting local density fluctuations at the domain wall):

$$\gamma_j = e^{4i\phi_1(x_{j,R})}\, e^{i\phi_3(x_{j,R})} \quad (37)$$

Notice that $e^{4i\phi_1(x_{j,R})}$ operator creates a charge-e excitation, which corresponds to the nontrivial $\mathbb{Z}_2$ boson in the bulk. The existence of this boson allows the Majorana zero mode to be a charge-neutral fermion. Such an operator is only present in a topologically ordered state. In general, we may expect that charge-conserving, exponentially protected Majorana zero modes can emerge on the edge of chiral Abelian quantum Hall states which contain a nontrivial $\mathbb{Z}_2$ boson carrying an odd number of electric charges among the quasiparticles. The $\nu = 1/2$ strong pairing state turns out to be the simplest Abelian FQH state with this property.

XI. CLASSIFICATION OF DOMAIN WALLS ON THE EDGE OF AN ABELIAN QUANTUM HALL STATE

In the previous sections, we have seen a number of examples of zero modes and protected degeneracy at domain walls in gapless fully chiral edges of topological phases. From these examples, we can distill the following general picture.

Zero modes and protected topological degeneracy can only occur when one or more edge phases have fewer gapless quasiparticle types than there are bulk (gapped) quasiparticle types. (If we also count the gapped quasiparticle types at the edge, then the edge must have precisely the same quasiparticle types as the bulk, but not not all of them must be gapless.) Though necessary, this is not a sufficient condition; there are two different possible sufficient conditions that we give below.

Let us first consider bosonic states. To state the second of these conditions, it is useful to call the two chiral edge phases 1 and 2. Moreover, it is useful to consider a configuration with at least 4 different alternating regions of phases 1 and 2, separated by domain walls. The two possible sufficient conditions are:

1. The two edge phase phases on either side of a domain wall have different sets of gapless particle types.

2. The two edge phases on either side of a domain wall have the same gapless particle types. There is at least one gapped particle type a such that there is an operator A that transfers an a particle between two phase 1 regions, and an operator B that transfers a b particle between two phase 2 regions. The A and B operators should commute with the edge Hamiltonian at low energy (i.e. commute with the argument of the dominant cosine gapping term), and by construction they satisfy $AB = R^{ab}_{a\times b} R^{ba}_{a\times b} BA$. Here $R^{ab}R^{ba} \neq \mathbb{1}$ represents the full braid phase between a and b. Physically, we can view B operators as measuring the number of a particles in region 1 by braiding.

Case 1 generalizes the I_8-E_8 domain wall and the domain wall between $K = 2N^2$ and $K = 2$: there is topological degeneracy due to the mismatch in particle types and the chirality. Case 2 generalizes the topological degeneracy of parafermionic zero modes in Abelian quantum Hall states[27,31,33–35].

For fermionic systems, however, there is an additional possibility. Physical fermions can "pair condense" on the edge, in which case a and b are just the fermions. This can lead to Majorana zero modes, as those showing up at domain walls discussed at the end of Section X and also the domain walls between the topological superconducting phase and the insulating phase of a nanowire. The commutation algebra between A and B operators, however, do not correspond to the full braiding phase. Heuristically, because full braiding between physical fermions is trivial one has to exchange the fermions to count the fermion parity.

XII. DISCUSSION

We have found two surprises in this paper: neither superconductivity nor even its analogues are necessary for Majorana zero modes; and, in spite of the presence of gapless charged excitations in this system, the splitting between the nearly-degenerate ground states of a collection of these Majorana zero modes decays exponentially as a function of the relevant length scales.

This means that Majorana zero modes might be observable in a system composed entirely of GaAs or graphene, without any need for a junction with a second type of material, namely a superconductor. It is possible that we have simply traded one difficulty – the problem of inducing superconductivity in a semiconductor system – for another – the problem of tuning the edge of the system between phases at will. However, it is alternatively possible that the latter can be accomplished purely through electrostatic gate control of the edge of the system. Determining the edge phase in realistic experimental devices is, thus, an important problem for future research.

In order to transform the states associated with multiple domain walls, we need to braid these zero modes. This may, perhaps, be accomplished by creating corner junctions, as discussed in Ref. 1 in the context of the edges of 2D topological insulators with proximity-induced superconductivity. Alternatively, a measurement-only scheme[36] can be used to manipulate quantum information. Some of the necessary measurements can be performed via interferometry, as described in Section VIII. However, it will also be necessary to measure the party of a pair of non-consecutive zero modes. The design of such a measurement is an important problem for the future.

Although we have focused here on the $\nu = 8$ integer quantum Hall state, many fermionic quantum Hall states have edge phases without gapless fermions[30]. Integer quantum Hall states in this class must have $\sigma_{xy} = 8n\frac{e^2}{h}$. However, there are also many fractional quantum Hall examples, such as the $\nu = 3 + \frac{1}{5}$ state, which is an observed bulk state[37] – the edge phase that it exhibits in experiments is, at present, unknown. Thus, Majorana zero modes can be created at the edge of all of these states as well.

By folding the field-theoretic description of the edge about a domain wall, we can view the zero modes discussed here as a conformally-invariant boundary condition for a conformal field theory in which the right-moving fields are governed by the I_8 theory while the left-moving fields are governed by the E_8 theory. It may be possible, by such a mapping, to connect the quantum information contained in these Majorana zero modes to the boundary entropy of this conformal boundary condition.

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Appendix A: Useful Matrices

In this appendix we define the matrices refered to in Sec. II:

$$K^{E_8} = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & -1 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix} \tag{A1}$$

$$W^8 = \begin{pmatrix} 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 8 & 16 \\ 10 & 10 & 10 & 9 & 9 & 9 & 9 & 9 & 15 & 30 \\ 8 & 8 & 8 & 8 & 7 & 7 & 7 & 7 & 12 & 24 \\ 6 & 6 & 6 & 6 & 6 & 5 & 5 & 5 & 9 & 18 \\ 4 & 4 & 4 & 4 & 4 & 4 & 3 & 3 & 6 & 12 \\ 2 & 2 & 2 & 2 & 2 & 2 & 2 & 1 & 3 & 6 \\ 7 & 7 & 6 & 6 & 6 & 6 & 6 & 6 & 10 & 20 \\ 4 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 5 & 10 \\ -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -3 & -4 \\ 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 4 & 7 \end{pmatrix} \tag{A2}$$

Appendix B: Criterion for gapping to E_8

Consider the action of Eq. (3). A gapping term will take the form $\cos(n_I \phi_I)$, where the n_I satisfy $n_I (K^{E_8} \oplus \sigma_z)^{-1}_{IJ} n_J = 0$. Here we show how to determine the Lagrangian for the remaining low-energy fields, which will be described by Eq. (1) with some K-matrix, K_{eff}. Because the low-energy theory has eight gapless modes and satisfies $|\det(K_{\text{eff}})| = 1$, there are only two options: $K_{\text{eff}} \sim K^{E_8}$ or $K_{\text{eff}} \sim \mathbb{I}_8$, where $\sim$

denotes equality up to $\mathrm{GL}(8,\mathbb{Z})$ transformations. We can distinguish these theories by the absence or presence of quasiparticles with fermionic statistics.

Let 2^g be the largest power of 2 which divides n_9 and n_{10}. We now show that if either n_9 or n_{10} has another factor of 2, then $K_{\text{eff}} \sim \mathbb{I}_8$. Otherwise, if 2^{g+1} is a factor of n_I for all $I \leq 8$ then $K_{\text{eff}} \sim K^{E_8}$, while if not, $K_{\text{eff}} \sim \mathbb{I}_8$. In the latter case, the greatest common factor of all n_I is 2^g, so the gapping vector is not primitive unless $g = 0$.

A quasiparticle in the theory described by K_{eff} is labelled by an integer vector m_I, which satisfies $m_I(K^{E_8} \oplus \sigma_z)^{-1}_{IJ} n_J = 0$. We consider three cases: first, consider the case where 2^{g+1} divides n_9. Then a valid quasiparticle in the theory described by K_{eff} has $m_{I\leq 8} = 0, m_9 = 2^{-g} n_{10}, m_{10} = 2^{-g} n_9$; since m_9 is odd and m_{10} is even, this quasiparticle is a fermion. If we had chosen 2^{g+1} to divide n_{10} instead of n_9, we could have made a similar construction. Thus, whenever 2^{g+1} divides n_9 or n_{10}, $K_{\text{eff}} \sim \mathbb{I}_8$.

Second, consider the case where 2^{g+1} does not divide n_9 or n_{10}, but 2^{g+1} divides $(K^{E_8} \oplus \sigma_z)^{-1}_{IJ} n_J$ for all $I \leq 8$. Then for every m which describes a quasiparticle in the K_{eff} theory, 2^{g+1} divides $m_9 n_9 - m_{10} n_{10}$, which requires $m_9 = m_{10} \mod 2$. Hence, every m describes an even quasiparticle and $K_{\text{eff}} \sim K^{E_8}$.

Third, consider the remaining case, where 2^{g+1} does not divide n_9 or n_{10} and for some I_0, $2^{g_0} \leq 2^g$ is the largest power of 2 which divides $(K^{E_8} \oplus \sigma_z)^{-1}_{I_0 J} n_J$. Then a valid quasiparticle in the K_{eff} theory has $m_{I_0} = 2^{-g_0} n_{10}$, $m_{10} = 2^{-g_0} (K^{E_8})^{-1}_{I_0 J} n_J$ and all other $m_I = 0$. Such an m describes a fermion. Hence, $K_{\text{eff}} \sim \mathbb{I}_8$ in this case.

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