Bulk disorder in the superconductor affects proximity-induced topological superconductivity

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Bulk disorder in the superconductor affects proximity-induced topological superconductivity

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We investigate effects of ordinary nonmagnetic disorder in the bulk of a superconductor on magnetic adatom-induced Shiba states and on the proximity-induced superconductivity in a nanowire tunnel-coupled to the bulk superconductor. Within the formalism of self-consistent Born approximation we show that, contrary to the widespread belief, the proximity-induced topological superconductivity can be adversely affected by the bulk superconducting disorder even in the absence of any disorder in the nanowire (or the superconductor-nanowire interface) when the proximity tunnel-coupling is strong. In particular, bulk disorder can effectively randomize the Shiba state energies. In the case of a proximate semiconductor nanowire, we numerically compute the dependence of the effective disorder and pairing gap induced on the wire as a function of the semiconductor-superconductor tunnel coupling. We find that the scaling exponent of the induced disorder with respect to coupling is always larger than that of the induced gap, implying that at weak coupling, the proximity induced pairing gap dominates whereas at strong coupling the induced disorder dominates. These findings bring out the importance of improving the quality of the bulk superconductor itself (in addition to the quality of the nanowire and the interface) in the experimental search for solid state Majorana fermions in proximity-coupled hybrid structures and, in particular, points out the pitfall of pursuing strong coupling between the semiconductor and the superconductor in a goal toward having a large proximity gap. In particular, our work establishes that the bulk superconductor in strongly-coupled hybrid systems for Majorana studies must be in the ultra-clean limit, since otherwise the bulk disorder is likely to completely suppress all induced topological superconductivity effect.

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I. INTRODUCTION

Majorana fermions in solid state systems [1–5] obey non-Abelian braiding statistics [6], and are a promising platform for topological quantum computation [7]. A feasible route towards realizing them utilizes a hybrid structure involving the proximitization of a semiconductor (SM) with a bulk s-wave superconductor (SC) [8–14]. With the appropriate combination of spin-orbit coupling (SOC), Zeeman spin splitting, and SC pairing terms, the proximitized system becomes topological (i.e. an effectively spinless p-wave superconductor) and localized Majorana fermions emerge at the ends of one-dimensional (nanowire) systems or at the vortices of two-dimensional systems. Their existence can then be probed by conductance measurements as quantized zero-bias peaks of height $2e^2/h$ at zero temperature associated with the perfect Andreev reflection induced by the Majorana zero energy modes [10, 15–19]. Shortly after the theoretical proposals [8–13, 20] were put forward, several experimental groups implemented different variants of the proposed Majorana experiment using nanowires in proximity to bulk superconductors [21–27]. Although the initial data reporting zero-bias tunneling conductance peaks in nanowires (albeit with conductance values below the theoretically predicted $2e^2/h$ quantized conductance) are encouraging, more theoretical and experimental work still needs to be done in order to distinguish signatures of Majorana fermions from those from other possible non-topological mechanisms as have been discussed in the literature [28–35].

The current work is on the deleterious effect of disorder on the proximity induced topological superconductivity in the hybrid system of experimental interest. The topological superconductivity induced in the nanowire, arising from a combination of s-wave superconductivity, spin-splitting, and spin-orbit coupling, is essentially equivalent to a type of an effectively spinless p-wave superconductivity [36] with triplet spin correlations [37] which is not immune to ordinary nonmagnetic disorder in the environment unlike regular s-wave spin-singlet superconductors which are protected against nonmagnetic disorder by virtue of the Anderson theorem. There have therefore been many theoretical and numerical studies [28–30, 38–54] of the effects of disorder on the topological superconductivity in this context, going back to almost 15 years ago [28]. It may appear that another theoretical study of disorder effects in this context would be redundant, but as we explain below, this is not the case here. The specific question regarding disorder effects (in the bulk superconductor itself) addressed in this paper has only been discussed three times in the literature before with the first paper [55] coming to an erroneous conclusion which was subsequently corrected [56, 57]. The conclusion we reach in our current work is of *great* importance in choosing the proper materials for the hybrid structures manifesting topological superconductivity and Majorana fermions.

In considering the effects of disorder in these hybrid SM-SC [21] (or FM-SC where FM stands for the ferromagnetic adatoms as in Ref. [27]) systems, one should distinguish between disorder in the SM and that in the bulk SC. Disorder in the SM has been extensively studied [30, 39, 40, 44, 46, 47], with the main conclusion being that the topological gap is destroyed by this type of disorder when the mean free path is comparable with or smaller than the induced coherence length in the semiconductor. It has been much emphasized in the literature [44] that the topological nanowire must be in the ballistic limit with the carrier mean free path being much larger than the proximity-induced coherence length in the nanowire for the manifestation of the Majorana zero modes, a condition which is likely (unlikely) to be satisfied in the semiconductor [21] (ferromagnetic [27]) nanowires. It has also been emphasized [58] that the applicable disorder at the nanowire-superconductor interface must be low for the induced proximity superconductivity to manifest a hard gap as has recently been apparently accomplished in the InAs-Al epitaxial core shell nanowire hybrid structures [59]. On the other hand, disorder in the bulk SC has received relatively little attention [14, 55–57], with the focus mainly on the limit where the coupling between the two materials is small (i.e. the weak-coupling limit where the SM-SC tunneling amplitude is small). In this limit, it has been found [55, 57] that the disorder in the bulk SC hardly affects the superconducting gap in the topological system, making it possible to use disordered or dirty SCs in experiments [21–25]. One consensus in the community regarding disorder effects seems to be that disorder in the nanowire itself (superconductor itself) is important (unimportant) with respect to the manifestation of proximity-induced topological superconductivity and Majorana fermions in the hybrid system. The current work directly challenges this consensus, showing that the disorder in the bulk superconductor may very well be important for the proximity induced topological superconductivity, particularly in the limit where the superconductor and the nanowire are strongly tunnel-coupled. In particular, the bulk superconductor should be in the clean limit with its elastic mean free path being much larger than the superconducting coherence length for optimal induced topological superconducting order in the hybrid structures. *Our current work indicates that having a clean superconductor with a very long mean free path is an absolute necessary condition for the realization of a large proximity-induced topological superconducting gap hosting Majorana fermions in the SM-SC and FM-SC hybrid systems.*

Two recent developments prompted us to revisit the issue of of bulk disorder in the superconductor. First, a new class of proposals [27, 60–73], utilizing Shiba states induced by magnetic adatoms on SCs to generate Majorana fermions, has emerged. This platform for using ferromagnetic adatoms on a superconducting substrate as the topological FM-SC hybrid system is in some sense the large spin-splitting limit [34] of the SM-SC hybrid structure with the spin-splitting arising intrinsically from exchange effects in the ferromagnet rather than from a Zeeman splitting induced by an external magnetic field as in the SM-SC hybrid system. This ferromagnet-superconductor hybrid system can therefore be effectively described by a Hamiltonian same as that of the SM-SC heterostructure, with a crucial difference that the tunnel coupling between the adatoms and the SC (as well as the spin splitting in the adatom chain) is much larger than the corresponding term in the SM-SC system [35], rendering the previous perturbative treatment of disorder in SC inapplicable. Second, in Ref. [59] the SM-SC structure has been grown epitaxially which drastically improved the quality of the interface between the two materials. A hard proximity-induced superconducting gap is then observed on the SM, resolving the "soft gap" issue that previous experiments found [58]. The size of the gap on SM is comparable to that on the SC, indicating strong coupling between the two materials [74]. In this limit, however, it is unclear whether disorder in the bulk SC can significantly degrade the gap on the SM, especially when a magnetic field is applied on the SM to create a Zeeman spin splitting necessary for producing the topological superconductivity in the SM wire.

In both of these experiments [27, 59], the strong tunnel coupling between the SM (or magnetic adatoms) and the SC necessitates the re-examination of the issue of disorder in the SC, as previous treatments of this problem were valid only when the coupling is small [55–57], which are not the case in these two new systems. In this paper, we investigate the effects of disorder in a SC on the spectral properties of a proximate SM in the SM-SC system and on the Shiba states in the FM-SC system. The disorder problems for the two hybrid structures (i.e. disorder effects on the Shiba states in the ferromagnetic adatom chain and on the SC/SM nanowire) are somewhat different, and we therefore study the two systems (FM/SC and SM/SC) separately so that our work applies to both experimental systems although our main emphasis in the current work is on the semiconductor-based Majorana hybrid systems since the experimental situation is better understood in such semiconductor nanowire structures. The formalism we adopt is the self-consistent Born approximation, which is valid in the limit of weak impurity scattering (specifically $k_F l \gg 1$, where k_F and l are respectively the Fermi wave number and the disorder-induced transport mean free path in the bulk SC– this condition is well-satisfied in the bulk superconductors used in the Majorana hybrid structures with the clean/dirty bulk superconductors being defined by whether $l \gg \xi$ or $l \ll \xi$, respectively where ξ is the SC coherence length). We extract the density of states (DOS) of the topological systems via their dynamic Green functions which contain the self-energy due to ensemble-averaged disorder in the bulk of the SC. Throughout the paper we shall assume the SM itself as well as the SM-SC interface is clean and only consider disorder in the bulk superconductor.

Investigating whether a strong tunnel coupling to the superconductor in hybrid systems, e.g. the inherent strong coupling in metal-on-metal ferromagnetic adatom systems or the strong coupling in epitaxial semiconductor-superconductor systems, could lead to the disorder in the bulk superconductor (quite apart from the disorder in the adatoms or the semiconductor itself) becoming relevant for the topological superconducting (and consequently Majorana fermion) properties is the goal of the current theoretical work. We ignore all disorder in the SM nanowire (or the FM chain) itself since it has already been studied extensively elsewhere and is well-understood as being detrimental to topological properties.

To qualitatively understand how the bulk superconducting disorder might be relevant for the proximity induced superconductivity in the strong tunnel coupling (which we often refer to simply as "strong coupling" in this paper) limit we refer to the simple (and approximate) formula for the proximity-induced superconducting gap in the hybrid system derived in Refs. [10, 44, 74] and used extensively:

$$\Delta_w \sim \frac{\Gamma\Delta}{\Gamma+\Delta} \tag{1}$$

where Γ is the effective coupling, Δ is the bulk gap in the superconductor, and Δ_w is the induced proximity gap in the nanowire. It has been much emphasized that in order to obtain a large induced gap one must have $\Gamma \gg \Delta$ so that $\Delta_w \sim \Delta$ which is obviously the maximum possible value of the induced gap (as achieved presumably in the epitaxial InAs-Al coreshell nanowire systems [59]). In the opposite limit of very weak coupling, $\Gamma \ll \Delta$, one gets $\Delta_w \sim \Gamma$ with a very small induced gap $\ll \Delta$ (with consequently an even smaller topological gap since the topological gap is bounded from above by Δ_w). Let us now imagine an extremely large tunnel coupling (e.g. Γ going to infinity) where there is then no discernible difference between the superconductor and the nanowire so that $\Delta_w = \Delta$ applies, and hence the nanowire has essentially become a part of the bulk superconductor as far as superconducting properties go. In such a situation, the bulk disorder in the superconductor is now a part of the disorder in the nanowire since from the perspective of superconductivity, these two have become one monolithic system. Now, if we turn on spin-orbit coupling and spin-splitting so as to convert Δ_w into a topological superconducting gap, then the disorder existing in the bulk superconductor must necessarily suppress the effectively triplet spinless topological superconductivity since it is not protected by any Anderson theorem (as time reversal invariance is explicitly broken)! We note that this argument does not apply in the weak tunneling ($\Gamma \ll \Delta$) limit where the two parts of the hybrid system (the bulk superconductor and the nanowire) are distinct, and indeed it has been explicitly shown [57] that in the weak tunneling limit (Γ going to zero), the bulk superconducting disorder does not suppress the topological superconductivity, but of course the topological gap is very small in this limit ($< \Gamma$) anyway! Although this physically motivated qualitative argument is not a proof by any means, the argument demonstrates that the strong- and weak-coupling situations could be fundamentally different with respect to disorder effects coming from the bulk superconductor, and a careful investigation is necessary to see whether the strong-coupling situation is benign or not with respect to the bulk disorder effects. While this problem is of considerable intrinsic interest itself in the context of the theory of superconducting proximity effect, the current experimental push (e.g. InAs-Al epitaxial hybrid system) to produce a hard induced gap makes our work timely in the study of Majorana fermions in solid state systems.

The paper is organized as follows. In Sec. II we consider the ferromagnet-SC hybrid system in the Shiba limit, where the coupling between the magnetic adatoms and the SC is much stronger than the inter-atomic coupling. We find that in this strong coupling regime disorder in the bulk SC has strong effects on the location of the Shiba energy in the bulk SC gap. In Sec. III we consider the effects of bulk superconducting disorder on SM-SC heterostructure, in both weak- and strong-coupling limits. We show that our results in the weak-coupling limit agree with previous works [55, 57], and highlight features specific to the strong-coupling limit, where, in contrast to the weak-coupling limit, nonmagnetic elastic disorder in the bulk superconductor invariably strongly degrades the proximity-induced topological SM superconductivity. We conclude in Sec. IV with a summary and with a brief discussion on the far-reaching implications of our findings for the future design of hybrid structures hosting Majorana fermions.

II. FERROMAGNETIC ADATOM-INDUCED SHIBA STATES IN A DISORDERED SUPERCONDUCTOR

We first consider a disordered s-wave SC strongly coupled with a magnetic impurity, described by

$$\begin{aligned} H &= \sum_{\boldsymbol{k}\sigma} \xi_{\boldsymbol{k}} a^\dagger_{\boldsymbol{k}\sigma} a_{\boldsymbol{k}\sigma} + \Delta \sum_{\boldsymbol{k}} \left(a^\dagger_{\boldsymbol{k}\uparrow} a^\dagger_{\boldsymbol{k}\downarrow} + \text{h.c.} \right) \\ &\quad - J \sum_{\sigma} \sigma a^\dagger_{\sigma}\left(\boldsymbol{r}=0\right) a_{\sigma}\left(\boldsymbol{r}=0\right) \\ &\quad + \int d\boldsymbol{r} U_{\text{dis}}\left(\boldsymbol{r}\right) \sum_{\sigma} a^\dagger_{\sigma}\left(\boldsymbol{r}\right) a_{\sigma}\left(\boldsymbol{r}\right), \end{aligned} \tag{2}$$

where $a_{\boldsymbol{k}\sigma}$ annihilates an electron with momentum $\boldsymbol{k}$ and spin σ. In the first line, $\xi_{\boldsymbol{k}}$ is the normal-state dispersion and Δ the s-wave pairing term of the SC. In the second line, J characterizes the strength of the magnetic impurity, located at the origin ($\boldsymbol{r} = 0$), which induces a local Zeeman term in the SC. The prefactor $\sigma = \pm 1$ corresponds to spin-up/down respectively. In the third line, $U_{\text{dis}}\left(\boldsymbol{r}\right)$ represents nonmagnetic elastic disorder present in the SC (which leads to a finite transport mean free path l in the bulk SC in its normal state). Below, we investigate effects of the magnetic term (J) on the DOS of the system, for both clean ($U_{\rm dis}=0$) and dirty ($U_{\rm dis}\neq 0$) SCs by generalizing the original Yu-Shiba-Rusinov theory [75–77] to include static nonmagnetic elastic disorder $U_{\rm dis}$ in the SC.

A. Clean Superconductor

We first briefly review the theory of Shiba states [75–77] in the absence of disorder ($U_{\rm dis}=0$) to set a context and to fix the terminology. In frequency (ω)-momentum ($\boldsymbol{k}$) space, the Green function for the system is

$$G^{(1)}_{\boldsymbol{kk'}}=G^{(0)}_{\boldsymbol{k}}\delta_{\boldsymbol{kk'}}+G^{(0)}_{\boldsymbol{k}}T_{\boldsymbol{kk'}}G^{(0)}_{\boldsymbol{k'}},\qquad(3)$$

where $G^{(0)}_{\boldsymbol{k}}(\omega)=\frac{\omega\tau_0+\xi_{\boldsymbol{k}}\tau_z+\Delta\tau_x}{\omega^2-\xi^2_{\boldsymbol{k}}-\Delta^2}$ is the Green function for a clean SC with τ_μ the Pauli matrices acting on the Nambu-Gorkov space of $\left(a_{\boldsymbol{k}\uparrow},a^\dagger_{\boldsymbol{k}\downarrow}\right)^T$. The superscript (1) in Eq. (3) indicates the presence of one magnetic impurity, but without disorder in the system. The effect of the magnetic term J is captured by the T-matrix in the second term of Eq. (3), which is given by

$$\begin{aligned}T_{\boldsymbol{kk'}}&=-\left(1+\frac{J}{V}\sum_{\boldsymbol{k}}G^{(0)}_{\boldsymbol{k}}\right)^{-1}\frac{J}{V}\\&=-\left(1-J\pi\nu_0\frac{\omega\tau_0+\Delta\tau_z}{\sqrt{\omega^2-\Delta^2}}\right)^{-1}\frac{J}{V},\end{aligned}\qquad(4)$$

where V is the volume of the system and ν_0 the normal-state DOS at the Fermi level. The pole of T in the subgap regime indicates the presence of a bound state, called the Shiba state [75–77], with the energy given by

$$\epsilon_0=\mathrm{sgn}J\frac{1-(J\pi\nu_0)^2}{1+(J\pi\nu_0)^2}\Delta.\qquad(5)$$

The local density of states (LDOS) at the position of the Shiba state is given by $\nu(\boldsymbol{r}=0)=V^{-1}\sum_{\boldsymbol{kk'}}G^{(1)}_{\boldsymbol{kk'}}$. In Fig. 1, the black lines show the the LDOS at a Shiba state with energies $\epsilon_0=0$ and $\epsilon_0=0.4\Delta$. The divergence of the LDOS at $\omega=\epsilon_0$ indicates that the Shiba states have well-defined energies. We mention the obvious fact that the Shiba state energy ϵ_0 is tuned by appropriately tuning the magnetic coupling J, and in a given experimental set-up J would typically be fixed producing a Shiba energy according to Eq. 5 above. Results corresponding to two situations with $\epsilon_0=0$ and 0.4Δ are shown in Figs. 1 and 2.

B. Disordered Superconductor

We now investigate the effects of ensemble-averaged disorder in the bulk SC ($U_{\rm dis}$) on the Shiba state energy.

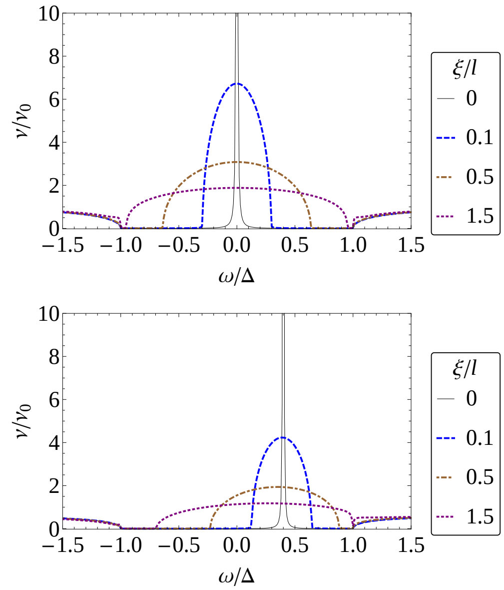

Figure 1: (Color Online) The LDOS at the position of the magnetic impurity in a clean (black solid lines) or disordered (colored lines) SC. The Shiba state energies are tuned to (a) $\epsilon_0=0$, and (b) $\epsilon_0=0.4\Delta$. A broadening of magnitude 0.001Δ is used to smear out the delta-functions for depicting the results. The elastic disorder in the system is quantified by the mean free path l which is given in the units of the clean limit coherence length ξ of the SC as shown on the right.

To this end, we assume the random quenched nonmagnetic impurities in the SC have a concentration of $n_{\rm imp}$, and each impurity has a scattering potential of the form $U_i(\boldsymbol{r})=U\delta(\boldsymbol{r}-\boldsymbol{r}_i)$, where $\boldsymbol{r}_i$ is the position of the i^{th} impurity. In the self-consistent Born approximation, the Green function is written as

$$G_{\boldsymbol{kk'}}=\left[\left(G^{(1)}_{\boldsymbol{kk'}}\right)^{-1}-\Sigma_{\boldsymbol{kk'}}\right]^{-1},\qquad(6)$$

where $G^{(1)}$ is given by Eq. (3) and the inversion here is operated on the $\boldsymbol{k}-\boldsymbol{k'}$ matrix space. The disorder-induced self-energy $\Sigma_{\boldsymbol{kk'}}$ is given (in the self-consistent Born approximation) by

$$\begin{aligned}\Sigma_{\boldsymbol{kk'}}&=n_{\rm imp}U^2\frac{1}{V}\sum_{\boldsymbol{p}}\tau_zG_{\boldsymbol{p+k},\boldsymbol{p+k'}}\tau_z,&(7)\\&\approx\frac{\tau^{-1}}{2\pi\nu_0}\delta_{\boldsymbol{kk'}}\frac{1}{V}\sum_{\boldsymbol{pq}}\tau_zG_{\boldsymbol{p},\boldsymbol{q}}\tau_z.&(8)\end{aligned}$$

where $\tau=\left(2\pi n_{\rm imp}U^2\nu_0\right)^{-1}$ is the disorder scattering time. Note that due to the lack of translational invariance, the self-energy due to disorder, in its exact form of Eq. (7), is nondiagonal in $\boldsymbol{k}-\boldsymbol{k'}$. It is easy to check that Eq. (7) reduces to the conventional form for translationally invariant systems [? ?] if $G_{\boldsymbol{kk'}}$ is proportional to $\delta_{\boldsymbol{kk'}}$.

In reaching Eq. (8), we observed that that $G^{(1)}_{\boldsymbol{kk'}}$ has the highest weight when $|\boldsymbol{k}|=|\boldsymbol{k'}|=k^{(N)}_F$, where $k^{(N)}_F$

is the Fermi momentum of the system in its normal state. Therefore, in the summation over $\boldsymbol{p}$ in Eq. (7), the summand has appreciable weights only when $|\boldsymbol{p}+\boldsymbol{k}| = |\boldsymbol{p}+\boldsymbol{k}'| = k_F^{(N)}$. In the general case where $\boldsymbol{k} \neq \boldsymbol{k}'$, this condition is satisfied only for a one-dimensional manifold of $\boldsymbol{p}$, but when $\boldsymbol{k} = \boldsymbol{k}'$, the condition reduces to $|\boldsymbol{p}+\boldsymbol{k}| = k_F^{(N)}$ and is satisfied by a two-dimensional manifold of $\boldsymbol{p}$. Thus we see that $\Sigma_{\boldsymbol{k}\boldsymbol{k}'}$ has most of its weights at $\boldsymbol{k} = \boldsymbol{k}'$, allowing us to approximate it by Eq. (8). This can be understood physically as the ensemble-averaged disorder should not introduce further translational-symmetry breaking and hence $\Sigma_{\boldsymbol{k}\boldsymbol{k}'}$ is diagonal in $\boldsymbol{k}$.

For each value of energy (ω), Eqs. (6) and (8) are iterated numerically until convergence. In the evaluation of the momentum integrals, we use the approximation $\frac{1}{V}\sum_{\boldsymbol{k}} \to \nu_0 \int_{-\infty}^{\infty} d\xi_{\boldsymbol{k}}$. The $\xi_{\boldsymbol{k}}$-integral is discretized on a grid with 10^4 points distributed in a way such that $\frac{1}{1+|\xi_{\boldsymbol{k}}|}$ is sampled uniformly over the interval $\left[10^{-4}, 1\right]$. The iteration converges after a few cycles for most values of ω except for those in the vicinity of the Shiba state energy (ϵ_0), which requires a few hundred of iteration cycles.

The blue dashed line in Fig. 1 shows the LDOS at the magnetic impurity when the SC is disordered with a mean free path of $l = v_F\tau = 10\xi$, where v_F and $\xi = v_F/\Delta$ are the Fermi velocity and the coherence length of the SC. The results for several other values of l are also presented. We observe that the delta-peak at $\omega/\Delta = \epsilon_0$ associated with the Shiba state is now broadened to a dome by the disorder in the SC. This can be understood as follows: the continuum states of the SC are different for each realization of disorder. The scattering phase shift due to the magnetic impurity therefore varies from one realization to another, leading to an effective disorder-induced fluctuation in J and ϵ_0 [c.f. Eq. (5)]. Ensemble-averaging the LDOS over disorder leads to a dome-like shape centered around the energy of the Shiba state in a clean SC with the dome in Fig. 1 reflecting the "spreading" in the effective Shiba energy due to disorder– it is clear that the clean SC limit with $l \gg \xi$ is necessary for the system to have a sharp Shiba energy.

We plot in Fig. 2a the effective fluctuation in J, which is defined as the standard deviation δJ of the distribution in J that would result in Shiba state energies distributed according to the subgap LDOS obtained from self-consistent Born approximation (e.g. the dome in Fig. 1). Fixing the Shiba state energy in the clean limit at $\epsilon_0 = 0$ and 0.4Δ (as in Fig. 1), we plot in Fig. 2b the width of the subgap dome of LDOS against the strength of disorder as characterized by ξ/l. In general, the Shiba subgap state is broadened by disorder, and the subgap DOS joins the continuum states when $l \lesssim \xi$, with the precise critical disorder dependent on the Shiba energy. The results presented in Figs. 1 and 2 clearly demonstrate the importance of being in the ultra-clean SC limit, i.e., $l \gg \xi$, for obtaining a Shiba state whose energy is close to that in the clean limit.

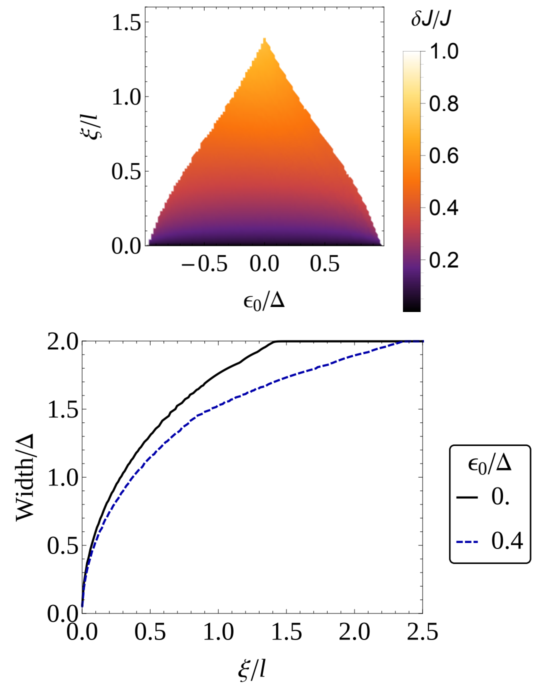

Figure 2: (Color Online) (a) The normalized fluctuation $\delta J/J$ as a function of the disorder strength ξ/l and the Shiba energy ϵ_0 for clean SC. In the white regions, the subgap dome of LDOS joins the continuum modes ($|\omega| > \Delta$) which makes δJ ill-defined. (b) The width of the subgap dome as a function of disorder strength for $\epsilon_0 = 0$ (solid line) and $\epsilon_0 = 0.4\Delta$ (dashed line).

C. Discussion

We have investigated the LDOS associated with a single magnetic impurity embedded in a disordered SC with ensemble-averaging. The result thus obtained is *not* expected to be directly applicable to a real experiment conducted with a single impurity since in reality there is only one realization of disorder, and therefore this Shiba state energy should appear as a sharp subgap LDOS peak. Our ensemble-averaged results, however, reveal that the Shiba state energy is shifted randomly from sample to sample around its value in the clean-SC limit (with the likelihood roughly proportional to the height of the subgap dome) because each specific experimental sample will have its unique disorder configuration which will differ randomly from one sample to another. Qualitatively, when the mean free path of the bulk SC is of the same order of magnitude as (or shorter than) its SC coherence length, the Shiba state energy could be anywhere within the SC gap. This is likely to have implications for a class of recent proposals which utilize the Shiba states induced in a superconductor by a chain of magnetic adatoms to generate Majorana fermions [27, 61, 64, 66, 67, 69, 72, 73]. In general, the spin-orbit-coupling strength, the lattice spacing between magnetic adatoms and the Shiba state energy all need to be fine-tuned to obtain topological superconductivity in the system [67, 72]. However, it is difficult to control the Shiba state energy even if the SC is clean, as it is determined by the strengths of the magnetic adatom and its coupling with the SC, both of which can hardly be tuned experimentally. Our finding indicates that in addition to the difficult task of fine-tuning the Shiba energy to zero, it is also necessary to use an ultra-clean SC with $l \gg \xi$ so as to ensure that the Shiba state energy for each individual atom remain close to zero. Otherwise, with Shiba state energies along the chain being shifted randomly by disorder, the system would then be fragmented into segments of topological and non-topological regions, an unfavorable situation for topological quantum computation. Thus, our current work implies that the fine-tuning problem of creating Majorana modes using Shiba states becomes substantially worse in the strong coupling situation since the bulk disorder in the superconductor now randomly shifts the Shiba state energy, leading to strong sample-to-sample variations. We mention here that the typical Shiba-induced Majorana system is a metal-on-metal system (i.e. a ferromagnetic metal chain on a superconducting metal) where the tunnel coupling is large ($\sim$eV), and the typical bulk disorder scale in the superconductor ($\sim$meV) much larger than the typical induced gap (~ 0.1 meV) leading to an intrinsically unfavorable theoretical situation for the existence of Majorana modes by virtue of the fine-tuning problem.

III. SEMICONDUCTOR NANOWIRE-SUPERCONDUCTOR SYSTEM

In Sec. II, we looked into the system in the "Shiba limit", in which the inter-atomic hopping among the magnetic adatoms is much weaker than their coupling with the bulk SC. We now turn to consider the opposite limit where the system hybridizes strongly to form a nanowire. (We mention as an aside that the two limiting situations, the Shiba limit of weak inter-wire hopping [69] and the nanowire limit of strong inter-wire hopping [70], are not separated by a quantum phase transition and are two extremes, which are applicable to different physical situations, of the same underlying physics [35, 73].) Previous work [57] has indicated that disorder in the bulk SC cannot degrade the superconducting gap in the nanowire *if* the SM-SC coupling is weak. It, however, remains unclear whether a strong SM-SC coupling could alter this conclusion qualitatively (as was already discussed in the Introduction of this paper). Therefore we now investigate this question in depth without assuming any weak SM-SC coupling. Note that the effect of disorder at the SM-SC interface is a separate issue which has been theoretically treated previously [58].

In the absence of disorder, the system is described by the Hamiltonian $H = H_w + H_{sc} + H_T$, where $H_{w/sc}$ is the Hamiltonian for the SM wire / SC and H_T is the coupling between the two materials. Explicitly, they are given by

$$H_w = \sum_{k_z\sigma} \left[\xi^{(w)}_{k_z} c^\dagger_{k_z\sigma} c_{k_z\sigma} + B\sigma c^\dagger_{k_z\sigma} c_{k_z\sigma} + \alpha k_z c^\dagger_{k_z\bar{\sigma}} c_{k_z\sigma} \right] \tag{9a}$$

$$H_{sc} = \sum_{\boldsymbol{k}\sigma} \xi^{(s)}_{\boldsymbol{k}} a^\dagger_{\boldsymbol{k}\sigma} a_{\boldsymbol{k}\sigma} + \Delta \sum_{\boldsymbol{k}} \left(a^\dagger_{\boldsymbol{k}\uparrow} a^\dagger_{\boldsymbol{k}\downarrow} + \text{h.c.} \right) \tag{9b}$$

$$H_T = \sum_{\boldsymbol{k}\sigma} a^\dagger_{\boldsymbol{k}\sigma} c_{k_z\sigma} + \text{h.c.} \tag{9c}$$

where $\xi^{(w)}_{k_z} = k_z^2 - \mu$ and $\xi^{(s)}_{\boldsymbol{k}}$ are the dispersions of the wire and the SC, respectively. B and α are the Zeeman and SOC terms on the wire, and Δ is the s-wave pairing term on the SC. The subscript $\bar{\sigma}$ represents the spin species opposite to that of σ. The physical properties of the system are captured by its Green function, which is given by

$$\mathcal{G} = \begin{pmatrix} G_s^{(0)-1} - \Sigma_{\text{dis}} & T \\ T^\dagger & G_w^{(0)-1} \end{pmatrix}^{-1} \equiv \begin{pmatrix} G_s & \\ & G_w \end{pmatrix}. \tag{10}$$

Here, the full Green function $\mathcal{G}$ of the whole system is written in the block spinor space of $\left(\psi^{(s)}, \psi^{(w)}\right)$, where $\psi^{(s)/(w)}$ represents the Bogoliubov-de Gennes spinors for the SC / wire respectively. The off-diagonal block matrix T represents the tunneling between the two systems. Σ_{dis} is the self-energy originating from the nonmagnetic disorder, which is present in the SC only in our model. The explicit forms for these terms are

$$G_s^{(0)}\left(\omega, \boldsymbol{k}^{(s)}\right) = \frac{\omega\tau_0 + \xi^{(s)}_{\boldsymbol{k}}\tau_z + \Delta\tau_x}{\omega^2 - \xi^{(s)2}_{\boldsymbol{k}} - \Delta^2}, \tag{11a}$$

$$G_w^{(0)}\left(\omega, k_z^{(w)}\right) = \left(\omega\tau_0 - \xi^{(w)}_{k_z}\tau_z - B\sigma_z - \alpha k_z^{(w)}\sigma_x\tau_z\right)^{-1}, \tag{11b}$$

$$T\left(\boldsymbol{k}^{(s)}, k_z^{(w)}\right) = t\delta_{k_z^{(s)}, k_z^{(w)}}, \tag{11c}$$

$$\Sigma_{\text{dis}}\left(\boldsymbol{k}^{(s)}, \boldsymbol{p}^{(s)}\right) \approx \delta_{\boldsymbol{k}^{(s)}, \boldsymbol{p}^{(s)}} \frac{\tau^{-1}}{2\pi\nu_0 V} \sum_{\boldsymbol{q}_1, \boldsymbol{q}_2} G_s\left(\boldsymbol{q}_1^{(s)}, \boldsymbol{q}_2^{(s)}\right), \tag{11d}$$

where $G^{(0)}_{w/s}$ are the unperturbed Green functions of the wire / SC. The superscripts $(w)\,/\,(s)$ on the momentum variables indicate that they refer to the wire / SC. The effects of disorder in the bulk SC is captured by the self-energy Σ_{dis}, whose expression has been approximated in the same way as Eq. (8).

Inverting the matrix in Eq. (10), we get the following set of coupled equations:

$$G_w = \left(G_w^{(0)-1} - \Sigma_w\right)^{-1}, \tag{12a}$$

$$\Sigma_w = \frac{t^2}{A}\sum_{\boldsymbol{k}_\perp}\tau_z\tilde{G}_S(\boldsymbol{k})\,\tau_z, \tag{12b}$$

$$\tilde{G}_s = \left(G_s^{(0)-1} - \Sigma_{\text{dis}}\right)^{-1}, \tag{12c}$$

$$\Sigma_{\text{dis}}(\boldsymbol{k},\boldsymbol{k'}) \approx \delta_{\boldsymbol{kk'}}\frac{\tau^{-1}}{2\pi\nu_0 V}\sum_{\boldsymbol{q}_1,\boldsymbol{q}_2}\tau_z G_s(\boldsymbol{q}_1,\boldsymbol{q}_2)\,\tau_z, \tag{12d}$$

$$G_s(\boldsymbol{k},\boldsymbol{k'}) = \tilde{G}_S\delta_{\boldsymbol{kk'}} + \tilde{G}_s(\boldsymbol{k})\,t^2\tau_z \times G_w(k_z)\,\tau_z\tilde{G}_s(\boldsymbol{k'})\,\delta_{\boldsymbol{k}_z,\boldsymbol{k}'_z}, \tag{12e}$$

where $\tilde{G}_s$ is the Green function of the SC with the effects of disorder incorporated, while G_s incorporates both the effect of disorder and that of the coupling to the wire. [A in Eq. (12b) is the area required for normalization similar to the volume V normalization in the earlier equations.]

The DOS on the wire can be calculated from G_w by $\nu_w(\omega) = \frac{-1}{\pi}\text{TrIm}\int\frac{dk_z}{2\pi}G_w(k_z)$. To evaluate the momentum integrals, we use the following approximations

$$\frac{1}{A}\sum_{\boldsymbol{k}_\perp} \to \nu_{2\text{D}}\int d\xi_{\boldsymbol{k}_\perp} \tag{13a}$$

$$\frac{1}{V}\sum_{\boldsymbol{k}} \to \nu_{2\text{D}}\int_{-\infty}^{\infty}\frac{dk_z}{2\pi}\int d\xi_{\boldsymbol{k}_\perp} \tag{13b}$$

where $\nu_{2D} = \nu_0\pi/k_F$ is the density of states of a two-dimensional system and k_F is the Fermi momentum.

We are primarily interested in the spectral properties of the SM wire, which becomes topological and hosts Majorana fermions with suitable combinations of SOC, Zeeman and induced SC pairing terms. The influence of the SC on the SM wire is captured by the self-energy term Σ_w. To have a better understanding, we expand it at small frequencies as

$$\Sigma_w(\omega) \approx (\Sigma_0 + \omega\Sigma_0') + \Sigma_x\tau_x + \ldots, \tag{14}$$

where $\Sigma_\mu = \frac{1}{4}\text{Tr}\left[\Sigma_w(0)\,\tau_\mu\right]$ and $\Sigma_0' = \frac{1}{4}\text{TrRe}\Sigma_w'(0)$ are scalar numbers, and the symbol "…" represents terms proportional to other matrices. These terms renormalize the SOC, Zeeman splitting, effective mass and chemical potential of the SM wire, and do not concern us here. By substituting Eq. (14) into Eq. (12a), we obtain the following form of Green function for the wire:

$$G_w = \left[\omega - \mathcal{H}_w - (\Sigma_0 + \omega\Sigma_0') - \Sigma_x\tau_x\right]^{-1}, \tag{15}$$

$$= Z^{-1}\left(\omega + \frac{i}{\tau_w} - Z^{-1}\mathcal{H}_w - Z^{-1}\Sigma_x\tau_x\right)^{-1}, \tag{16}$$

where $Z = 1 - \Sigma_0'$ is the frequency renormalization factor, and

$$\tau_w^{-1} = -Z^{-1}\text{Im}\Sigma_0 \tag{17}$$

is the broadening induced by Σ_w. Since τ_w^{-1} has the same effect as disorder, we *define* it as the effective disorder on the SM wire which can be thought of as the proximity-induced effective disorder arising in the SM due to the presence of the SC. The term $Z^{-1}\Sigma_x$ induces a SC pairing onto the wire, but we shall not define it as the pairing term directly since Eq. (14) is an expansion at zero frequency and does not capture the frequency dependence of Σ_x. Rather, we numerically compute the spectral gap when the wire has no Zeeman splitting ($B = 0$), and identify this gap as the effective magnitude of the pairing term (Δ_w) in the SM wire.

Before presenting the full numerical results, we review the conventional treatment of the problem in the weak-coupling limit where $\Gamma \ll \Delta$, with $\Gamma = \pi\nu_{2D}t^2$ being the tunnel coupling strength between the wire and the SC. In this limit, the second term of Eq. (12e) can be neglected, and Eq. (12) has the analytic solution [78, 79]

$$G_s = \frac{\tilde{\omega}\tau_0 + \xi_{\boldsymbol{k}}\tau_z + \tilde{\Delta}\tau_x}{\tilde{\omega}^2 - \xi_{\boldsymbol{k}}^2 - \tilde{\Delta}^2} \tag{18}$$

$$\Sigma_w(k_z) = -\Gamma\frac{\omega - \Delta\tau_x}{\sqrt{\Delta^2 - \omega^2}} \tag{19}$$

where $\tilde{\omega}$ and $\tilde{\Delta}$ satisfy $\tilde{\omega} = \omega + \frac{1}{2\tau}\frac{\tilde{\omega}}{\sqrt{\tilde{\Delta}^2 - \tilde{\omega}^2}}$ and $\tilde{\Delta} = \Delta + \frac{1}{2\tau}\frac{\tilde{\Delta}}{\sqrt{\tilde{\Delta}^2 - \tilde{\omega}^2}}$. The self-energy Σ_w therefore gives a superconducting gap of size $\Gamma\,(\ll \Delta)$ on the wire. Since $\text{Im}\Sigma_w(\omega = 0) = 0$, we see that disorder is *not* induced on the wire in this limit. This is the gist of the weak-coupling result obtained earlier by Lutchyn et al. [57] establishing the immunity of the proximity-induced topological superconductivity to any disorder in the SC itself, and it is only valid for $\Gamma \ll \Delta$ when the induced gap is extremely small ($\sim \Gamma$).

In the strong-coupling limit in which the second term of Eq. (12e) is not small and cannot therefore be ignored, both superconducting gap and disorder are induced on the wire by the disordered SC. We now investigate their dependence on the strength of the bulk disorder and of the SM-SC coupling.

A. Superconducting Gap Induced on the Wire

We first investigate the superconducting gap induced on the wire which, in the absence of Zeeman term on the wire, is equal to the induced pairing potential Δ_w. For simplicity we set $B = \alpha = 0$ and $\mu = \Delta$ on the wire. (We have explicitly numerically checked that our results presented here are generic, and using different parameter values do not change the results at all qualitatively.) We note that the natural energy scale of the problem is Δ and the natural length scale is $\sqrt{k_F^{-1}\xi}$. The dimensionless parameter quantifying disorder is defined as $d = \sqrt{k_F^{-1}\xi}/l = \sqrt{\frac{\xi}{l}\frac{1}{k_F l}}$, where l is the transport mean free path of the bulk SC.

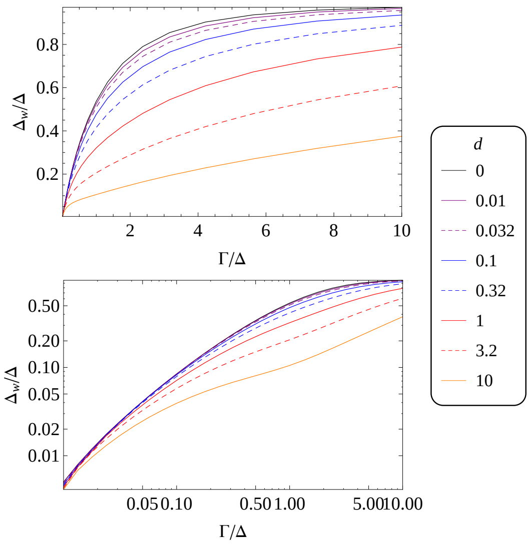

Figure 3: (Color Online) Induced superconducting gap on the wire (Δ_w) against SM-SC tunnel coupling (Γ) in (a) linear scale and (b) log scale. Different lines correspond to different strengths of disorder in the bulk of the SC $\left(d = \sqrt{k_F^{-1}\xi/l}\right)$. The parameters on the wire are chosen to be $B = \alpha = 0$ and $\mu = \Delta$.

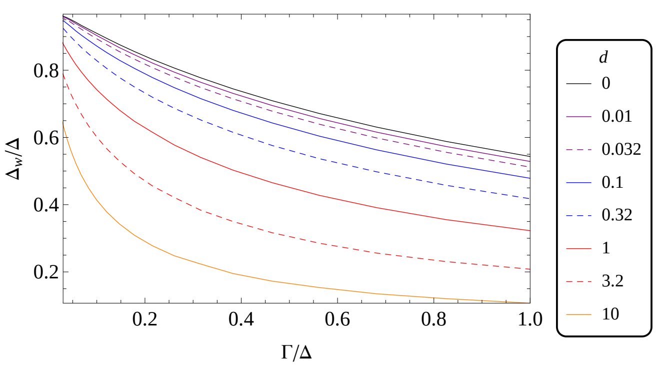

Figure 4: (Color Online) Calculated dimensionless induced gap (Δ_w/Γ) plotted against the dimensionless tunnel coupling (Δ_w/Γ) for various values of the disorder parametrized by $d = \sqrt{k_F^{-1}\xi/l}$ as shown. The parameters on the wire are chosen to be $B = \alpha = 0$ and $\mu = \Delta$.

In Fig. 3 we plot the calculated induced gap Δ_w as a function of Γ in both linear and log scales. We observe from the log-log plot that in the weak coupling and weak disorder limit, the induced gap scales linearly with coupling, i.e.,

$$\Delta_w \propto \Gamma, \text{ for } l \gg \sqrt{k_F^{-1}\xi} \text{ and } \Gamma \ll \Delta. \tag{20}$$

This is the same as previous calculations done with a clean SC [74]. Perturbative calculations [57] have also shown that at weak enough coupling Γ, disorder in the bulk SC does not change the linear scaling of Δ_w with Γ. To verify this, in Fig. 4 we plot Δ_w/Γ against Γ, where we observe that

$$\lim_{\Gamma\to 0} \frac{\Delta_w}{\Gamma} = 1, \tag{21}$$

irrespective of the strength of disorder in the SC, which is identical to the induced gap for a *clean* SC in the weak coupling limit ($\Gamma \ll \Delta$), as we pointed out in the discussion following Eq. (19). This is an indication that the induced pairing dominates over the induced disorder in the weak-coupling limit $\Gamma \ll \Delta$. In the following section, we develop a more quantitative understanding by computing the scaling behavior of disorder with respect to Γ and comparing it with Eq. (20), explicitly demonstrating that the induced pairing dominates at small Γ.

B. Disorder Induced on the Wire

Apart from the proximity-induced superconducting gap, the wire also inherits disorder from the SC. In the topological phase, the superconducting gap protecting the Majorana fermions could be destroyed by the induced disorder if its strength becomes comparable to the gap [44]. It is therefore important to study the dependence of the induced disorder on the coupling strength so as to compare its strength with that of the gap. The disorder on the wire has been defined in Eq. (17), which when written out in full is

$$\tau_w^{-1} = \frac{\mathrm{Tr}\mathrm{Im}\Sigma_w\,(\omega=0)}{\mathrm{Tr}\mathrm{Re}\Sigma_w'\,(\omega=0)-4}. \tag{22}$$

We remark here that τ_w^{-1} could in general be defined as a frequency-dependent quantity, but for our purpose of investigating its dependence on coupling strength we take only its value at zero frequency. Also, from Eqs. (12) we see that $\mathrm{Im}\Sigma_w\,(\omega)$ is nonzero only if the band dispersion of the wire crosses the energy ω. Therefore in order to produce a nonzero τ_w^{-1}, we choose the Zeeman term on the wire to be $B = 5\Delta$ while keeping $\alpha = 0$ and $\mu = \Delta$, noting that the results depend only weakly on the parameters.

In Fig. 5 we plot τ_w^{-1} against Γ in linear and log scales. It shows that τ_w^{-1} scales quadratically with Γ when $\tau_\mathrm{w}^{-1} \ll \Gamma$. However when τ_w^{-1} is comparable with Γ, the dependence changes to linear. Similarly, from the plot of τ_w^{-1} against l^{-1} in Fig. 6 we see that it scales linearly when l^{-1} is small. In summary, we have

$$\tau_w^{-1} \propto \begin{cases} l^{-1}\Gamma^2, & \tau_\mathrm{w}^{-1} \ll \Gamma, \\ \Gamma, & \tau_\mathrm{w}^{-1} \lesssim \Gamma. \end{cases} \tag{23}$$

This result could be compared with previous results obtained from perturbative treatments which apply in the limit of very small induced disorder [55, 57]. We note that in addition to recovering the quadratic scaling at weak coupling, our result also shows a crossover to linear scaling at intermediate coupling strength, which

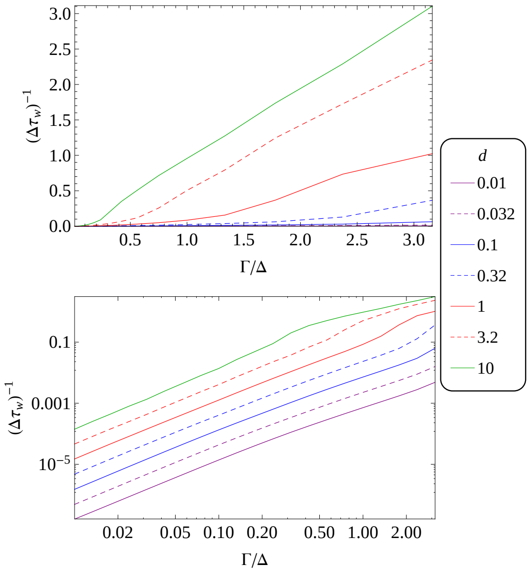

Figure 5: (Color Online) Effective disorder strength on the wire $\left(\tau_{\mathrm{w}}^{-1}\right)$ as a function of SM-SC coupling (Γ) in (a) linear scale and (b) log scale. The parameters on the wire are chosen as $\alpha = 0$, $\mu = \Delta$, and $B = 5\Delta$. Different lines represent different disorder strength defined by $d = \sqrt{k_F^{-1}\xi}/l$.

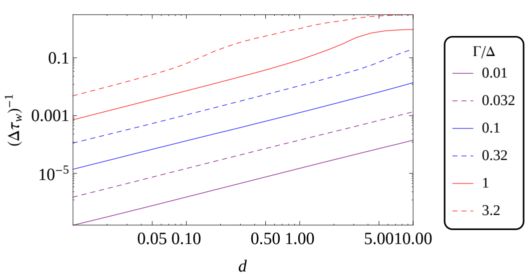

Figure 6: (Color Online) Effective disorder on the wire $\left(\tau_{\mathrm{w}}^{-1}\right)$ against disorder in bulk SC $\left(l^{-1}\right)$ in log scale. The parameters on the wire are chosen as $\alpha = 0$, $\mu = \Delta$, and $B = 5\Delta$. The disorder strengths are defined by $d = \sqrt{k_F^{-1}\xi}/l$.

cannot be captured by perturbative approaches. This qualitatively new linear scaling behavior of the induced disorder has important consequences for the Majorana-carrying SC-SM hybrid nanowire systems as discussed below.

C. Suppression of Topological Gap by Disorder

For practical reasons, the calculations in Sec. III B and Sec. III A have been performed for wires with and without Zeeman terms, respectively. It is natural to ask whether there is a system in which both induced SC gap and induced disorder are at play. The Majorana nanowires, with nonzero SOC and Zeeman terms, are such systems. If its normal-state dispersion crosses zero energy, the dis-

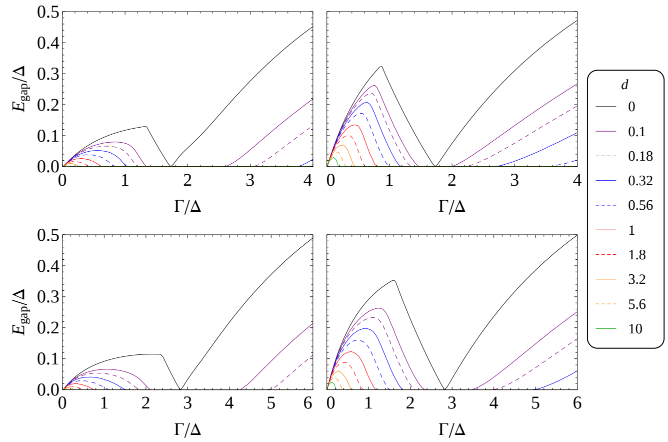

Figure 7: Superconducting gap on the SM wire (E_{gap}) as a function of coupling strength (Γ) for various values of disorder strengths in the bulk SC $\left(d = \sqrt{k_F^{-1}\xi/l}\right)$. The parameters on the wire are chosen to be $\mu = \Delta$ and (a) $B = 2\Delta$, $\alpha = 0.3\sqrt{\frac{\Delta}{2m}}$, (b) $B = 2\Delta$, $\alpha = \sqrt{\frac{\Delta}{2m}}$, (c) $B = 3\Delta$, $\alpha = 0.3\sqrt{\frac{\Delta}{2m}}$, (d) $B = 3\Delta$, $\alpha = \sqrt{\frac{\Delta}{2m}}$.

order term as defined by Eq. (17) is nonzero. On the other hand, with the SOC term present, the induced pairing can produce a spectral gap even in the presence of the Zeeman term. Therefore a comparison between the scaling behaviors of induced pairing and induced disorder with respect to the SC-SM tunnel coupling is in order.

Comparing Eq. (20) and Eq. (23), we see that the scaling exponent (2) of disorder $\left(\tau_w^{-1}\right)$ with respect to coupling strength (Γ) is larger than that (1) of the pairing term (Δ_w). This implies that Δ_w dominates at smaller Γ while τ_w^{-1} dominates at larger Γ. Therefore, increasing the coupling strength does not always lead to a larger proximity-induced topological gap since the induced effective disorder increases faster. Rather, the optimal value of Γ at which the topological gap is maximum is dependent on the parameters of the system, and one expects some intermediate value of Γ, which is neither too small (so that the intrinsic topological gap itself is not too small) nor too large (so that the induced disorder is not too strong overwhelming the induced gap), to be the optimal choice. Purely on dimensional ground, the optimal value of the tunnel coupling is expected to be $\Gamma \sim \Delta$ in the relatively clean SC limit, but with increasing SC disorder we expect the optimal value of Γ to decrease so as to keep induced disorder effects small.

Fig. 7 shows the superconducting gap on the wire (E_{gap}) as a function Γ for several values of B and α. For $0 < \Gamma < \sqrt{B^2 - \mu^2}$, the system is in topological regime since the s-wave pairing term is smaller than the Zeeman term [11, 13]. When α is nonzero and in the absence of disorder (black line in Fig. 7), a topological gap exists and a zero-energy Majorana fermion is present at each end of the nanowire. When Γ increases to values near

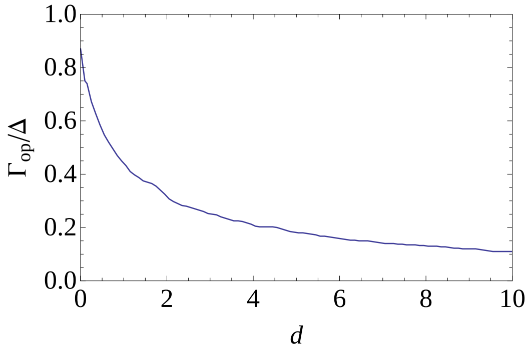

Figure 8: The optimal value of Γ at which the topological gap on the wire is maximum (Γ_{op}), as a function of disorder in the bulk SC$\left(d = \sqrt{k_F^{-1}\xi/l}\right)$. The parameters of the wire are equal to those of Fig. 7(b), i.e. $\mu = \Delta$, $B = 2\Delta$, and $\alpha = \sqrt{\frac{\Delta}{2m}}$.

the Zeeman energy $|B|$, the topological gap shrinks and closes completely at $\Gamma = \sqrt{B^2 - \mu^2}$, indicating the onset of the topological phase transition. The gap then reopens at $\Gamma > \sqrt{B^2 - \mu^2}$ where the system is non-topological.

The gap on the wire is suppressed by disorder in the bulk SC, but the degree of this suppression is dependent on Γ. In particular, we see from Fig. 7 that the effect of disorder vanishes when $\Gamma \to 0$, and a topological gap (albeit small) exists in that limit. This is consistent with our results in Sec. III A and Sec. III B on the scaling of Δ_w and τ_w^{-1} with respect to Γ [Eq. (20) and Eq. (23) respectively]: since at small Γ, Δ_w scales as Γ while τ_w^{-1} scales as Γ^2, the pairing term eventually dominates over disorder at small Γ, producing a superconducting gap.

Fig. 8 shows the dependence of the optimal values of tunnel coupling for which the topological gap on the wire is largest, as a function of the disorder in the bulk SC. We see that with a more disordered SC, it is actually more favorable to have a *smaller* coupling between the SM and the SC in order to generate a larger topological gap. We believe this non-intuitive finding (Fig. 8) to be an important new result for the fabrication of optimal SC-SM hybrid structures for the realization of Majorana fermions– the tunnel coupling could be strong for an ultra-clean SC (where $l \gg \xi$), but for dirty SCs, one is far better off (as shown in Fig. 8) having a rather small SC-SM tunnel coupling!

D. Discussion

We have analyzed, within the framework of self-consistent Born approximation, the effect of disorder residing solely in the bulk of the SC on the spectral properties of the proximity induced topological superconductivity in the SC-SM hybrid system. The dependence of the induced pairing gap and the induced disorder on the coupling strength is theoretically explored. Crucially, we find that the topological gap induced on a SM wire with SOC and Zeeman splitting can be very susceptible to the disorder in the bulk SC when the SC-SM tunnel coupling is strong. While the specific optimal coupling strength depends on the details of the system, in general with high disorder in the SC a weak SM-SC coupling is preferable.

These results have implications for the on-going experimental efforts to generate Majorana fermions by proximitizing a spin-orbit-coupled SM nanowire under magnetic field in contact with a SC. Although it is important to improve the interface quality between the two materials so as to generate a hard gap on the SM [58, 59], one should also so be aware that a strong SC-SM tunnel coupling induces stronger disorder on the SM wire, if the SC is diffusive. Thus it is necessary to either use an ultra-clean SC or to introduce a barrier between the SM and the SC so as to effectively reduce the coupling strength.

IV. CONCLUSION

In this paper we examined the effect of disorder in the bulk SC on a Shiba state or a proximate SM in the context of the current search for Majorana fermions in hybrid superconducting systems. In both cases we found that this type of disorder can have significant detrimental impact, and could be an obstacle to create topological superconductivity in the hybrid systems. In particular, disorder in the bulk SC can randomly shift the energy of the Shiba states in the ferromagnet-superconductor hybrid system, which is unfavorable for the existence of Majorana modes since realizing a Majorana-carrying topological system requires the fine-tuning of the Shiba state energy. (We mention that a complementary model [34, 70] of the ferromagnetic adatom chain on the superconductor system, which is adiabatically connected [35, 73] to the Shiba model [69], assumes the system to be equivalent to the semiconductor nanowire on the superconductor structure except for the spin splitting in the adatom chain being extremely large so that the system is completely spin-polarized– in such a spin-polarized nanowire model of the ferromagnetic chain, the effect of bulk disorder is qualitatively similar in the semiconductor and the ferromagnetic chain system with disorder being detrimental in the strong coupling situation for reasons discussed in Section III above.) In the case of semiconductor-superconductor structure, the SM inherits both superconducting pairing and disorder from the SC through the proximity effect. We find that the scaling exponent of inherited disorder with respect to the coupling strength between the two materials is always larger than that of the inherited pairing. This implies that while the pairing term can dominate over disorder and produce a spectral gap on the SM at small coupling, upon increasing the coupling strength the inherited disorder will eventually dominate and destroy the induced SC gap. While the precise optimal value of the relevant coupling strength for producing the strongest topological superconductivity depends on the particular details of various parameter values, the key message of our theory for the choice of the most suitable topological materials parameters is that one should use ultra-clean bulk superconductors with extremely large normal state low-temperature mean free path and tune the tunnel coupling to a suitable value lower than the bulk superconducting gap energy. We note that the disorder in the bulk superconductor enters our theory through the dimensionless combination $d = \sqrt{\frac{1}{k_F l}\frac{\xi}{l}}$, which implies that increasing either $k_F l$ or l/ξ in the parent superconductor should help to keep the disorder effects weak in the system. This leads to our conclusion that among the commonly used parent superconductors in the experimental hybrid systems probably Nb (Al) is the best (worst) choice with Pb being somewhere in between since typically Nb (Al) has the shortest (longest) coherence length, thus making it easy (difficult) to satisfy $l \gg \xi$ condition. The detailed choice for the superconductor requires careful materials preparation with the longest (shortest) possible values of the mean free path (coherence length) in the system. The precise prediction of our theory is simple: Choose a superconductor with the smallest possible value of the dimensionless disorder parameter "d", and given this value of "d", tune the tunnel coupling so that it equals $\Gamma_{\rm op}$ shown in our Fig. 8.

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