Low-temperature 1/f noise in microwave dielectric constant of amorphous dielectrics in Josephson qubits

Alexander L. Burin, Shlomi Matityahu, and Moshe Schechter

Phys. Rev. B **92**, 174201 — Published 16 November 2015

DOI: 10.1103/PhysRevB.92.174201

Low-temperature $1/f$ noise in the microwave dielectric constant of amorphous dielectrics in Josephson qubits

Alexander L. Burin
Department of Chemistry, Tulane University, New Orleans, LA 70118, USA

Shlomi Matityahu, Moshe Schechter
Department of Physics, Ben-Gurion University of the Negev, Beer Sheva 84105, Israel

The analytical solution for the low-temperature $1/f$ noise in the microwave dielectric constant of amorphous films at frequency $\nu_0 \sim 5\,\mathrm{GHz}$ due to tunneling two-level systems (TLSs) is derived within the standard tunneling model including the weak dipolar or elastic TLS-TLS interactions. The $1/f$ frequency dependence is caused by TLS spectral diffusion characterized by the width growing logarithmically with time. Temperature and field dependencies are predicted for the noise spectral density in typical glasses with universal TLSs. The satisfactory interpretation of the recent experiment by Burnett *et al.* [J. Burnett *et al.*, Nat. Commun. **5**, 4119 (2014)] in Pt capped Nb superconducting resonator is attained by assuming a smaller density of TLSs compared to ordinary glasses, which is consistent with the very high internal quality factor in those samples.

PACS numbers: 73.23.-b 72.70.+m 71.55.Jv 73.61.Jc 73.50.-h 73.50.Td

I. INTRODUCTION

$1/f$ noise exists in a variety of physical systems [1–4], and it dramatically restricts the performance of modern electronic and quantum nanodevices [5–7]. The inverse frequency dependence of the noise power spectral density, $S(f) \propto 1/f$, is a consequence of a logarithmically slow relaxation often associated with a random ensemble of fluctuators possessing a logarithmically uniform spectrum of relaxation times [2, 3, 8–10]. Such fluctuators do exist in amorphous solids in the form of universal tunneling two-level systems (TLSs, see Fig. 1) [11]. With the advent of superconducting quantum bits (qubits) based on Josephson junctions [7, 12], a comprehensive study of the noise properties due to TLSs has become crucial for the achievement of high-fidelity quantum computation. TLSs are ubiquitous, appearing in wiring dielectrics, Josephson junction barriers and other disordered insulating regions. The deleterious effects of the coupling of TLSs to the qubit are associated with the absorption of qubit energy in the microwave frequency range, and with the low-frequency noise in the microwave dielectric constant [7], resulting in qubit decoherence.

The noise in superconducting resonators has been extensively studied [13–17]. Recently, $1/f$ noise has been investigated in Pt capped Nb superconducting resonators [16]. It was found that the noise power spectral density increases with decreasing temperature as $T^{-1-\eta}$, with $\eta \approx 0.3$. This dependence was considered as being incompatible with the standard tunneling model (STM) [16, 18]. To explain this observation, the qualitative theoretical model proposed in Refs. [16] and [18] suggests an energy-dependent TLS density of states (DOS), $g(E) \approx E^{\eta}$, in contrast with the STM which assumes a constant DOS [11]. This assumption of the STM is supported by earlier experimental data in "ordinary" amorphous solids showing logarithmic temperature dependence of the dielectric constant and sound velocity [19, 20], as expected from an energy-independent DOS.

In this paper we investigate the noise in the microwave dielectric constant of amorphous insulators containing TLSs described by the STM [11, 19] including weak TLS-TLS dipolar interactions, of the form $U_{ij} \sim U_0/R_{ij}^3$. Here U_0 is the interaction constant and R_{ij} is the distance between TLSs. A general expression for the noise power spectral density, $S_y(f;T,F_{ac})$, as function of temperature (T) and external electric field at microwave frequency (F_{ac}) is derived. The noise has a $1/f$ spectral density as a consequence of the logarithmic broadening of the energy splitting of resonant TLSs with time due to spectral diffusion [21]. We then analyze the general expression obtained assuming typical amorphous dielectrics characterized by the universal value $\chi \approx 10^{-4} - 10^{-3}$ of the parameter $\chi = P_0 U_0$, where P_0 is the homogeneous DOS of TLSs. In these systems one expects the broadening of the resonance due to spectral diffusion to dominate over the broadening due to relaxation processes. At low temperatures, $T \ll \hbar\omega_0/k_{\mathrm{B}} \approx 0.2 - 0.3\,\mathrm{K}$, the zero field limit is shown to behave as $S_y(f;T,F_{ac} \to 0) \propto f^{-1}T^{-1-\eta}$, similarly to the temperature dependence observed in Ref. [16]. The inverse temperature dependence is associated with the spectral diffusion width as has been pointed out in Ref. [18]. The additional exponent $\eta \sim 0.25$ is associated with the logarithmic temperature dependence of the spectral diffusion width [21] (see Fig. 4 in Sec. IV A). However, this result cannot be used to fit the experimental data of Ref. [16] at $T \geq 0.1\,\mathrm{K}$, where the theory predicts much stronger temperature dependence. We suggest an alternative interpretation of the experiment in Sec. IV B (see Figs. 6 and 7) assuming that the relaxation rate of the TLSs is larger than their spectral width. This can be a result of an anomalously small amount of TLSs, which is consistent with the high internal quality factor of the experimental setup of Ref. [16] compared to ordinary glasses, or to an increased relax-

ation coming from interaction with conduction electrons in the Pt capping layer. Using this assumption, we show that the temperature dependence at $T \geq 0.1\,\mathrm{K}$ of the $1/f$ noise reported in Ref. [16] may be explained within the STM.

One should notice that the STM may not be applicable to TLSs in some recently developed superconducting resonators as seen, for example, from often observed anomalous field dependence of the loss tangent (see discussion in Refs. [16, 18, 22] and references therein). Therefore, we cannot exclude that deviations from the STM take place in some materials, including deviations from a homogeneous DOS [22]. Yet, our main target here is to describe the low-frequency noise in the microwave dielectric constant in the simplest possible model and attempt to fit the existing experimental data [16] without major modifications of the STM, that is by assuming a homogeneous DOS.

The paper is organized as follows. In Sec. II we define the noise, introduce the STM, review earlier results for TLS relaxation, decoherence and spectral diffusion, and describe the TLS contribution to the dielectric constant at microwave frequency. In Sec. III we derive the analytical solution for the power spectral density of the noise in the microwave dielectric constant. The frequency, temperature and field dependencies of the noise are considered in Sec. IV for ordinary glasses (Sec. IV A) and in the limit of anomalously small density of TLSs (Sec. IV B), possibly corresponding to the recent experiment of Burnett *et al.* [16]. We conclude with a brief summary of the results in Sec. V.

II. BASIC DEFINITIONS

A. Definition of noise

The noise measured in superconducting resonators [16] is given by the correlation function of fluctuations of the cavity resonance frequency at different times,

$$S_y(t) \equiv \frac{\langle \delta\omega_0(t)\delta\omega_0(0)\rangle}{\omega_0^2}, \tag{1}$$

where $\omega_0 = \langle\omega_0(t)\rangle$ is the average resonance frequency and $\delta\omega_0(t) = \omega_0(t) - \omega_0$ is its fluctuation at time t. Equation (1) can also be expressed in terms of fluctuations of the cavity dielectric constant at frequency ω_0,

$$S_y(t) = \frac{\langle \delta\epsilon(t)\delta\epsilon(0)\rangle}{\epsilon^2}, \tag{2}$$

with $\epsilon = \langle\epsilon(t)\rangle$ and $\delta\epsilon(t) = \epsilon(t) - \epsilon$. The noise spectral density of interest is determined by the Fourier transform of this correlation function at very low frequencies, $f \leq 1\,\mathrm{Hz}$, corresponding to long time $t_f \sim 1/(2\pi f)$. Specifically, for the frequency $f \sim 0.1\,\mathrm{Hz}$ studied in Ref. [16], one has $t_f \sim 1\,\mathrm{s}$. The only known excitations in amorphous solids possessing such long relaxation times are TLSs with sufficiently small tunneling amplitude Δ_0 (see Fig. 1), because the relaxation time of TLSs is proportional to Δ_0^{-2} [see Eqs. (6) and (7)]. Since TLSs possess the logarithmically uniform distribution with respect to their tunneling amplitudes [11],

$$P(\Delta, \Delta_0) = \frac{P_0}{\Delta_0}, \tag{3}$$

it is indeed expected that TLSs with small tunneling amplitude will be responsible for $1/f$ noise at low frequencies [10].

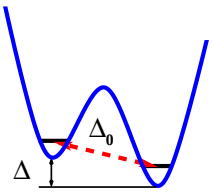

FIG. 1: Tunneling two-level system (TLS) characterized by well asymmetry Δ and tunneling amplitude Δ_0.

However, these TLSs cannot contribute directly to the fluctuations in the dielectric constant at microwave frequencies (e.g., at frequency $\nu_0 \sim 5\,\mathrm{GHz}$ as in Ref. [16]) because their relaxation time ($\sim 1\,\mathrm{s}$) is much longer than the field oscillation period. At microwave frequencies the most significant contribution to the dielectric constant is associated with resonant TLSs having $\Delta \sim \Delta_0 \sim h\nu_0 = \hbar\omega_0$ [19]. Therefore, we assume that the contribution of slowly relaxing TLSs to the dielectric constant noise is indirect; they affect resonant TLSs contributing to the dielectric constant due to their interaction capable of bringing them in and out of resonance with the external field [23, 24].

In Sec. II B we describe the resonant contribution of TLSs to the dielectric constant and then expound on the various parameters affecting this contribution, namely TLS time-dependent energy splitting due to spectral diffusion, relaxation and decoherence rates (Secs. II C-II E).

B. Time-dependent dielectric constant at high frequency

We are interested in the contribution of TLSs to the dielectric constant measured by an external electric field, $\mathbf{F} = \mathbf{e}_z F_{ac}\cos(\omega t)$, applied along the z axis (here $\mathbf{e}_z$ is a unit vector along the z direction) at a frequency ω close to the cavity resonant frequency $\omega_0 \sim 2\pi\cdot 5\,\mathrm{GHz}$. The TLSs possess dipole moments and therefore interact with the external field. The TLS-field interaction can be represented by the Hamiltonian [19, 24]

$$\hat{h} = -\Delta S^z - \Delta_0 S^x - 2p_z S^z F_{ac}\cos(\omega t), \tag{4}$$

where the operator $S^z = \pm 1/2$ describes the TLS position either in the right or in the left potential well (see

Fig. 1), and p_z is the z-axis projection of the TLS dipole moment. The energy splitting of an unperturbed TLS is $E = \sqrt{\Delta_0^2 + \Delta^2}$. Due to this interaction TLSs respond to the external AC field and contribute to the dielectric constant.

Consider the dielectric constant at high frequency $\omega \sim 2\pi \cdot 5\,\mathrm{GHz}$ corresponding to the cavity resonant microwave frequency. In this regime the TLS relaxation time $T_1 \sim 1\,\mu\mathrm{s}$ (see Eq. (6) and Refs. [25, 26]) is much larger than the field oscillation period, i.e. $\omega T_1 \gg 1$, and the TLS contribution to the dielectric constant is of resonant nature [19]. This contribution can be expressed in the form [18, 19, 27, 28]

$$\frac{\epsilon_{\mathrm{TLS}}(t)}{\epsilon} = \frac{4\pi}{V\epsilon}\sum_i \tanh\left(\frac{E_i}{2k_{\mathrm{B}}T}\right) \times \frac{(E_i(t)-\hbar\omega)\frac{\Delta_{0i}^2}{E_i^2}p_{iz}^2}{(E_i(t)-\hbar\omega)^2 + \frac{1}{T_{2i}^2}\left[1+\left(\frac{\Delta_{0i}}{E_i}\right)^2 \frac{p_{iz}^2 F_{ac}^2 T_{1i} T_{2i}}{\hbar^2}\right]}, \quad (5)$$

where V represents the sample volume and the summation is taken over all TLSs i having dipole moments $\mathbf{p}_i$, relaxation and decoherence times T_{1i} and T_{2i}, respectively, and time-dependent energies $E_i(t)$ [Eq. (8)]. The time dependence of the energies is induced by the spectral diffusion caused by dipolar or elastic TLS-TLS interactions [18, 21]. The TLS relaxation and decoherence rates are defined in Secs. II C and II E below. The time dependence of the TLS energy splitting is discussed in Sec. II D.

In general, another indirect relaxational contribution of slow TLSs to the noise can exist due to their non-resonant interactions with other TLSs or polarized vibrations. One can expect that this contributions shows a linear temperature dependence (see Ref. [10]). It can be responsible for the increase of $1/f$ noise power spectral density with temperature at $T \sim 0.5\,\mathrm{K}$ (see Figs. 6, 7, Ref. [16] and discussion in Sec. IV B).

C. Relaxation

In dielectric glasses the relaxation of TLSs is caused by phonon emission or absorption, with the relaxation rate [19, 21, 26]

$$\frac{1}{T_{1i,ph}} = A\left(\frac{\Delta_{0i}}{E_i}\right)^2\left(\frac{E_i}{k_{\mathrm{B}}}\right)^3 \coth\left(\frac{E_i}{2k_{\mathrm{B}}T}\right), \quad (6)$$

where $A = \gamma^2\left(v_l^{-5} + 2v_t^{-5}\right)/2\pi\hbar^4\rho$, with γ being the coupling of TLSs to the strain field, ρ the mass density, and v_l, v_t the longitudinal and transverse sound velocities, respectively. Note that, in general, the coupling constant γ may vary among different TLSs (and also phonon polarizations). Below we assume a single value of $\gamma \sim 1\mathrm{eV}$ for which $A \sim 10^8\,\mathrm{s^{-1}K^{-3}}$ [26].

In metallic glasses TLSs can also relax via their interaction with conduction electrons. In this case the TLS relaxation rate is [19]

$$\frac{1}{T_{1i,e}} = A_e\left(\frac{\Delta_{0i}}{E_i}\right)^2\frac{E_i}{k_{\mathrm{B}}}\coth\left(\frac{E_i}{2k_{\mathrm{B}}T}\right), \quad (7)$$

where $A_e = \pi\left[n(E_{\mathrm{F}})K\right]^2/\hbar$, with $n(E_{\mathrm{F}})$ being the electronic density of states at the Fermi energy, E_{F}, and K is the TLS-electron coupling constant. In metallic glasses this relaxation mechanism is usually much faster than the phonon-induced relaxation, Eq. (6). As shown in Sec. IV B, this relaxation mechanism may also be relevant in the experimental setup of Ref. [16], in which TLSs can interact with conduction electrons belonging to the Pt capping layer.

D. Spectral Diffusion

The spectral diffusion is the fluctuation of the TLS energy splitting E_i with time due to its interaction with neighboring TLSs. This interaction can bring the TLS in and out of resonance with the external field. The spectral diffusion theory has been developed by Black and Halperin [21] and below we briefly summarize their results which will be used later to study $1/f$ noise.

The time dependence of a TLS energy splitting due to its interaction with neighboring TLSs, enumerated by the index j, reads [21]

$$\delta E_i(t) = \frac{\Delta_i}{E_i}\sum_j \frac{\Delta_j}{E_j}U_{ij}S_j^z(t). \quad (8)$$

Here the interaction U_{ij} represents elastic or electric dipole-dipole interaction and its average absolute value can be written as $\langle|U_{ij}|\rangle = U_0/R_{ij}^3$, where U_0 is the TLS-TLS interaction constant. The neighboring TLSs j responsible for the spectral diffusion are *thermal TLSs* (sometimes called fluctuators), i.e. TLSs for which $E \lesssim k_{\mathrm{B}}T$ [21]. Such TLSs undergo random transitions due to relaxation process [Eqs. (6) and (7)], causing fluctuations of the energy E_i of the considered TLS.

Assume that at time $t = 0$ the TLS energy was equal to a certain value $E(0)$. Then according to Black and Halperin [21] [see Eq. (16) there], in the case of interaction which falls off with distance as $1/R^3$, the energy change $\delta E(t) = E(t) - E(0)$ is characterized by the Lorentzian distribution function

$$D(\delta E, t) = \frac{1}{\pi}\frac{\hbar W(t)}{\hbar^2 W^2(t) + (\delta E)^2}, \quad (9)$$

with a characteristic width $W(t) = W_0(t)|\Delta|/E$, where $W_0(t)$ is given in frequency units by

$$W_0(t) = \frac{2\pi^2\chi k_{\mathrm{B}}T}{3\hbar}\int_0^\infty \frac{d\mu}{\cosh^2\mu}\int_0^1 \frac{1 - e^{-\frac{x^2 t}{T_1(\mu)}}}{x}dx. \quad (10)$$

Here we defined the dimensionless variables $\mu = E/2k_{\rm B}T$ and $x = \Delta_0/E$, and the maximum relaxation rate $T_1^{-1}(\mu)$ is defined in accordance with Eq. (6) as $T_1^{-1}(\mu) = 8AT^3\mu^3 \coth\mu$. The dimensionless parameter $\chi = P_0U_0 \sim 5\cdot 10^{-4}$ represents the product of the TLSs DOS, P_0 in Eq. (3), and their $1/R^3$ interaction strength, U_0 [26, 29]. The product $\chi k_{\rm B}T$ in Eq. (10) represents the typical interaction of the given TLS with thermal TLSs.

Low-frequency $1/f$ noise is determined by long times $t \geq 1\,{\rm s} \gg (AT^3)^{-1}$ [16], where $(AT^3)^{-1}$ estimates the minimum relaxation time of thermal TLSs. In this limit the width of the distribution $W_0(t)$ grows logarithmically with time and can be approximated as

$$W_0(t) \approx \frac{\pi^2}{3\hbar}\chi k_{\rm B}T \ln\left(3.3\cdot AT^3 t\right). \tag{11}$$

This logarithmic time dependence is responsible for the appearance of $1/f$ noise. In addition, the logarithmic temperature dependence in Eq. (11) determines the temperature dependence of the noise power spectral density for typical glasses at small fields (see Sec. IV A). In particular, at low temperatures, $T \ll \hbar\omega/k_{\rm B}$, it gives rise to the power law dependence $S_y(f) \propto T^{-1-\eta}$, reminiscent of the $1/f$ noise temperature dependence observed in Ref. [16] and explained in terms of an energy-dependent DOS. We note that in restricted geometries deviations from the functional form of the time dependence in Eq. (11) may appear, e.g. as a result of the discreteness of the phonon spectrum [30]. Such fluctuations, however, would have only a minor quantitative effect on our results.

E. Phase decoherence

The TLS decoherence rate T_2^{-1} determines the instantaneous resonance width in Eq. (5). It is affected by both TLS relaxation, Eqs. (6) and (7), and spectral diffusion, Eq. (8) [19, 24, 26]. Thus, the decoherence rate is composed of the contributions of relaxation and pure phase decoherence,

$$T_{2i}^{-1} = \left(2T_{1i}\right)^{-1} + T_{\varphi,i}^{-1}. \tag{12}$$

The pure phase decoherence rate $T_{\varphi,i}^{-1}$ is determined by TLS spectral diffusion induced by its interaction with neighboring thermal TLSs. This rate can be written as [21, 26]

$$\frac{1}{T_{\varphi,i}} = \sqrt{40\frac{|\Delta_i|}{E_i}\frac{\chi k_{\rm B}T\cdot AT^3}{\hbar}}. \tag{13}$$

Although the use of Eq. (13) in the expression for the dielectric constant [Eq. (5)] is not well justified theoretically, its approximate relevance was demonstrated experimentally [28]. However, the limited relevance of Eq. (13) is not important for most of the consideration except for the strong AC field limit [see Eq. (24) in Sec. IV A]. Otherwise the noise power spectral density depends on TLS decoherence rate only logarithmically [see Eqs. (22) and (25)]. For the experiment of interest [16] we assume a very weak TLS-TLS interaction and the phase decoherence contribution to the decoherence rate is expected to be smaller than the contribution of the relaxation, i.e. $T_{2,i} \approx 2T_{1,i}$.

III. DERIVATION OF THE NOISE SPECTRAL DENSITY

The noise is given by the time-dependent correlation function of the dielectric constant fluctuations, Eq. (2). This expression can be simplified by invoking the resonant approximation, in which one sets $E \approx \hbar\omega$ in the relaxation and decoherence rates [Eqs. (6), (7) and (13)], and in the spectral diffusion [Eq. (8)]. The resonant approximation is applicable at typical experimental temperatures $T < 1\,{\rm K}$ since all broadenings of the resonance, caused by relaxation, decoherence and spectral diffusion, are much smaller than the resonance frequency ω [31]. Setting $E = \hbar\omega$, these broadenings can be rewritten using the dimensionless parameter $x_i = \Delta_{0i}/\hbar\omega$ as

$$\begin{aligned}
\frac{1}{T_{1i,ph}} &= Ax_i^2\left(\frac{\hbar\omega}{k_{\rm B}}\right)^3 \coth\left(\frac{\hbar\omega}{2k_{\rm B}T}\right),\\
\frac{1}{T_{1i,e}} &= A_e x_i^2 \frac{\hbar\omega}{k_{\rm B}} \coth\left(\frac{\hbar\omega}{2k_{\rm B}T}\right),\\
T_{2i}^{-1} &= \left(2T_{1i}\right)^{-1} + T_{\varphi,i}^{-1},\\
\frac{1}{T_{\varphi,i}} &= \sqrt{40\sqrt{1-x_i^2}\frac{\chi k_{\rm B}T\cdot AT^3}{\hbar}},\\
W_i(t) &\approx \frac{\pi^2}{3\hbar}\chi k_{\rm B}T\ln\left(3.3\cdot AT^3 t\right)\sqrt{1-x_i^2}.
\end{aligned} \tag{14}$$

Moreover, correlations between different resonant TLSs contributing to the noise can be neglected because of the weakness of the TLS-TLS interactions (see, for example, Ref. [29]). Then the leading order contribution in the resonant approximation is a sum of average squared contributions of individual TLSs,

$$S_y(t)=\frac{(4\pi)^2}{V^2\epsilon^2}\sum_i\left(\frac{\Delta_{0i}}{\hbar\omega}\right)^4 p_{iz}^4\tanh^2\left(\frac{\hbar\omega}{2k_{\rm B}T}\right)$$
$$\times\left\langle\frac{E_i(t)-\hbar\omega}{(E_i(t)-\hbar\omega)^2+\frac{1}{T_{2i}^2}\left[1+\left(\frac{\Delta_{0i}}{E_i}\right)^2\frac{p_{iz}^2F_{ac}^2T_{1i}T_{2i}}{\hbar^2}\right]}\cdot\frac{E_i(0)-\hbar\omega}{(E_i(0)-\hbar\omega)^2+\frac{1}{T_{2i}^2}\left[1+\left(\frac{\Delta_{0i}}{E_i}\right)^2\frac{p_{iz}^2F_{ac}^2T_{1i}T_{2i}}{\hbar^2}\right]}\right\rangle. \quad (15)$$

Averaging of each term involves several integrations, including the integration over the uniform distribution of the initial energy $E_i(0)$, the integration over the energy $E_i(t)=E_i(0)+\delta E(t)$ using the Lorentzian distribution in Eq. (9) and the integrations over the dimensionless tunneling parameter $x=\Delta_0/\hbar\omega$ and dipole moment projection p_z. A further averaging over the relaxation rate parameter A should be performed if one considers its variation among different TLSs. The integration over energies can be performed analytically by expanding to partial fractions and using the identity $\int_{-\infty}^{\infty}\frac{dx}{(x+y+ia)(x-ib)}=\frac{2\pi\theta(ab)}{|a+b|-iy\,\mathrm{sign}(a)}$. Then one ends up with

$$S_y(t)=\tanh^2\left(\frac{\hbar\omega}{2k_{\rm B}T}\right)\frac{4\pi^3P_0}{\hbar V\epsilon^2}\int_0^1\frac{dx}{x(1-x^2)}\left\langle\frac{p_z^4x^4}{W_0(t)+\frac{2}{T_2(x)\sqrt{1-x^2}}\sqrt{1+\frac{(xp_zF_{ac})^2T_1(x)T_2(x)}{\hbar^2}}}\right\rangle. \quad (16)$$

The remaining average in Eq. (16) should be performed over the dipole moment projection p_z and the relaxation rate parameter A.

The noise power spectral density, $S_y(f)$, can be evaluated as a Fourier transform of Eq. (16) in the low-frequency limit $fT_1, f/W_0(t_f)\ll 1$ with $t_f=1/(2\pi f)$ being a typical $1/f$ noise measurement time. It has the pure $1/f$ spectrum if the function $S_y(t)$ depends on time as $A-B\ln|t|$, which has a Fourier transform $B/2f$ at $f\neq 0$. The correlation function $S_y(t)$ can be expanded near $|t|=t_f$ in the approximate form $S_y(|t|)\approx S_y(t_f)+\frac{dS_y(t_f)}{d\ln t_f}\ln(|t|/t_f)$ (higher order expansion terms are smaller by the factor $\ln^{-1}(AT^3/f)\approx 0.1$ for the low frequency of interest, $f\approx 0.1\,$Hz). Therefore, the noise power spectral density can be approximated as $S_y(f)\approx -(1/2f)dS_y(t_f)/d\ln t_f$, i.e.

$$S_y(f)=\frac{1}{2f}\tanh^2\left(\frac{\hbar\omega}{2k_{\rm B}T}\right)\frac{4\pi^3P_0}{\hbar V\epsilon^2}\int_0^1\frac{dx}{x(1-x^2)}\left\langle\frac{p_z^4x^4\frac{\pi^2}{3\hbar}\chi k_{\rm B}T}{\left[W_0(t_f)+\frac{2}{T_2(x)\sqrt{1-x^2}}\sqrt{1+\frac{(xp_zF_{ac})^2T_1(x)T_2(x)}{\hbar^2}}\right]^2}\right\rangle. \quad (17)$$

The noise power spectral density has approximately $1/f$ dependence which is a consequence of the logarithmic time dependence of the width $W_0(t)$ [Eq. (11)] of the energy broadening distribution [Eq. (9)]. The deviation from $1/f$ spectrum due to the logarithmic term in the denominator of Eq. (17) is small, as discussed in Sec. IV below.

For a quantitative comparison of the theory with experiments it is convenient to introduce a volume independent parameter in a similar way to the experimentally determined Hooge's constant for $1/f$ conductivity noise in semiconductors [32]. We define this parameter as

$$\alpha_{\rm TLS}\equiv\frac{P_0Vk_{\rm B}TfS_y(f)}{\tan^2\delta}=\frac{9\pi}{8\langle p^2\rangle^2\hbar^2}\int_0^1\frac{dx}{x(1-x^2)}\left\langle\frac{p_z^4x^4\chi(k_{\rm B}T)^2}{\left[W_0(t_f)+\frac{2}{T_2(x)\sqrt{1-x^2}}\sqrt{1+\frac{(xp_zF_{ac})^2T_1(x)T_2(x)}{\hbar^2}}\right]^2}\right\rangle, \quad (18)$$

namely as the ratio of the noise power spectral density multiplied by the number of thermal TLSs, $N_T = P_0 V k_B T$, to the squared average loss tangent due to TLSs,

$$\tan\delta = \frac{\langle\epsilon''\rangle}{\epsilon} = \left(\frac{4\pi^2}{3\epsilon}\right) P_0 \langle p^2\rangle \tanh\left(\frac{\hbar\omega}{2k_B T}\right). \quad (19)$$

Here $\langle p^2\rangle$ is an average of the squared magnitude of the dipole moment.

Equations (17) and (18) contain a full description of the contribution of TLSs to the low-frequency $1/f$ noise. This is the main result of the present paper. The currently available experimental data do not permit a direct quantitative comparison of experiments with Eq. (18). Yet, we hope this result will be used in future when such data will be available together with other data needed to estimate the quantitative noise parameter $\alpha_{\rm TLS}$, particularly the TLS DOS, dipole moment and loss tangent. In Sec. IV we discuss the temperature and frequency dependence of Eqs. (17) and (18) in different regimes of interest.

IV. FIELD AND TEMPERATURE DEPENDENCE OF $1/f$ NOISE

The noise power spectral density, Eq. (17), is sensitive to the relations between the various parameters which determine the broadening of the resonance. The broadening due to the interaction with phonons can be estimated using the TLS relaxation rate, Eq. (6), whereas the interaction-induced broadening can be estimated using the spectral diffusion width $W_0(t)$, Eq. (11). This width is time dependent and we can use it at time $t = t_f = 1/(2\pi f)$ to characterize the noise at frequency f [see Eq. (17)], which we set in all estimates to be $f = 0.1\,\mathrm{Hz}$ similarly to the experiment [16]. Assuming typical parameters for amorphous solids, $A \sim 10^8\,\mathrm{s^{-1}K^{-3}}$ and $\chi \sim 5\times10^{-4}$ [26, 29], for resonant TLSs ($\Delta_0 \approx E \approx h\nu_0$ with $\nu_0 \approx 5\,\mathrm{GHz}$) one can estimate the typical broadenings as

$$\frac{1}{T_1} \approx 1.2\cdot10^6 \coth\left(\frac{0.11\mathrm{K}}{T}\right)\mathrm{s}^{-1},$$
$$W_0(t_f) \sim 2\cdot10^8 \frac{T}{0.1\mathrm{K}}\,\mathrm{s}^{-1}. \quad (20)$$

Comparing these two expressions one concludes that in a "typical" amorphous solid at temperatures exceeding 1 mK the most important contribution to the resonance broadening comes from spectral diffusion.

It should be emphasized that the regime $W_0(t_f)T_1 \gg 1$ is applicable for "typical" amorphous solids, for which $\chi \approx 10^{-3} - 10^{-4}$. Such amorphous solids are characterized by a universal value of the quality factor $Q \approx 10^3 - 10^4$ [19, 29, 33]. However, this is not the case in the experiment by Burnett *et al.* [16], where the system under investigation is characterized by a very high quality factor $Q \geq 10^6$. If this enhancement of the quality factor is a consequence of a smaller density of TLSs, P_0, or their interaction, U_0, then the resonance broadening due to spectral diffusion may be two or three orders of magnitude smaller than the estimate in Eq. (20). Moreover, the relaxation rate of TLSs can be larger than in ordinary glasses because of the contribution of conduction electrons (see Sec. IV B). Accordingly, in this case one can assume the opposite regime, that is $W_0(t_f)T_1 \ll 1$.

Below we study the frequency, temperature and field dependencies of the noise power spectral density in the two regimes $W_0(t_f)T_1 \gg 1$ (Sec. IV A) and $W_0(t_f)T_1 \ll 1$ (Sec. IV B). For $W_0(t_f)T_1 \gg 1$, a regime corresponding to "typical" amorphous solids, the results are presented in terms of the parameter $\alpha_{\rm TLS}$ [Eq. (18)]. These results can be directly compared with experimental studies to be performed in the future. In Sec. IV B we analyze the experimental data of Ref. [16] assuming the opposite regime, $W_0(t_f)T_1 \ll 1$, and show that a consistent interpretation of the experiment can be attained within the model of Sec. III.

A. 1/f noise in a "typical" amorphous solid

According to Eq. (20), in typical amorphous solids at temperatures above 1 mK the broadening of the resonance due to spectral diffusion exceeds other resonance widths, at least in the limit of small external field, $pF_{ac}\sqrt{T_1T_2}/\hbar \ll 1$. In this case the integral over the dimensionless tunneling parameter x in Eq. (18) can be evaluated with logarithmic accuracy by neglecting the decoherence rate term in the denominator for $\sqrt{1-x^2} > 1/[W_0(t_f)T_2(x)]$ and using the value of $x = x_c$, where $\sqrt{1-x_c^2} = 1/[W_0(t_f)T_2(x_c)]$, as the upper cutoff of the integral over x. Then we obtain

$$\alpha_{\rm TLS} \approx \frac{9\pi\chi(k_B T)^2}{16\langle p^2\rangle^2\hbar^2}\left\langle\frac{p_z^4 \ln[W_0(t_f)T_{1,\rm res}]}{W_0^2(t_f)}\right\rangle,$$
$$\frac{1}{T_{1,\rm res}} = A\left(\frac{\hbar\omega}{k_B}\right)^3 \coth\left(\frac{\hbar\omega}{2k_B T}\right). \quad (21)$$

Using the definition of the spectral diffusion width $W_0(t)$ [Eq. (11)] and performing the average over the dipole moment orientations $\langle p_z^4\rangle = \langle p^4\rangle/5$, we obtain

$$\alpha_{\rm TLS} \approx \frac{81}{80\pi^3\chi}\frac{\langle p^4\rangle}{\langle p^2\rangle^2}\left\langle\frac{\ln[W_0(t_f)T_{1,\rm res}]}{\ln^2\left(3.3\frac{AT^3}{2\pi f}\right)}\right\rangle. \quad (22)$$

The frequency dependence of the parameter $\alpha_{\rm TLS}$ is identical to the frequency dependence of the product $fS_y(f)$ [see Eq. (18)]. For a pure $1/f$ noise this product should be frequency-independent. However, the noise parameter in Eq. (22) shows a logarithmic frequency dependence. To characterize this dependence we evaluated the integral in Eq. (18) numerically, assuming a constant

dipole moment $p = 5\,\mathrm{D}$ and TLS relaxation rate parameter $A = 10^8\,\mathrm{s^{-1}K^{-3}}$ [7, 17, 25]. The TLS dipole moment is chosen as 5 D following the earlier estimates for aluminum oxide and silicon nitride glasses used in previous noise measurements [7, 17], though it may be smaller in other glasses [24]. This dependence is shown in Fig. 2 for $\chi = 5 \cdot 10^{-4}$ as in typical glasses and $\chi = 10^{-6}$ to characterize the case of small number of TLSs to be discussed in Sec. IV. The typical temperature $T = 0.1\,\mathrm{K}$ and frequency $\omega = 2\pi \cdot 5\,\mathrm{GHz}$ have been assumed.

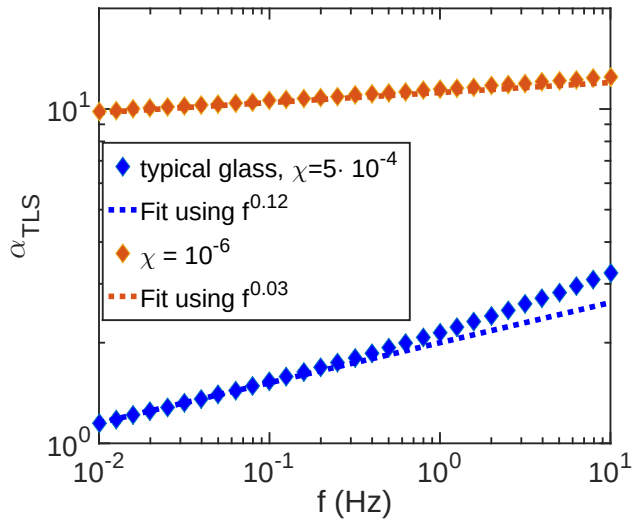

FIG. 2: (Color online) Frequency dependence of the noise parameter α_{TLS} for typical glasses with $\chi = 5 \cdot 10^{-4}$ (red) and for glasses with a smaller density of TLSs, with $\chi = 10^{-6}$ (blue). The curves with diamond markers describe the numerical evaluation of the integral in Eq. (18) with $T = 0.1\,\mathrm{K}$, $\omega = 2\pi \cdot 5\,\mathrm{GHz}$, $p = 5\,\mathrm{D}$ and $A = 10^8\,\mathrm{s^{-1}K^{-3}}$. Dashed lines are fit to $\alpha_{\mathrm{TLS}} \propto f^{\beta}$.

At the frequency of interest, $f \approx 0.1\,\mathrm{Hz}$, the frequency dependence of the noise parameter α_{TLS} is very weak and estimated as $\alpha_{\mathrm{TLS}} \propto f^{0.12}$ for typical glasses and $\alpha_{\mathrm{TLS}} \propto f^{0.03}$ in the case of a small number of TLSs. The corresponding noise power spectral density is $S_y^{\mathrm{typ}} \propto f^{-0.88}$ and $S_y^{\mathrm{sm}} \propto f^{-0.97}$, respectively. Thus, the theory predicts an experimentally recognizable $1/f$ noise spectrum.

The temperature dependence of the noise parameter α_{TLS} at frequency $f = 0.1\,\mathrm{Hz}$ for typical glasses is shown in Fig. 3. The temperature dependence is remarkable even though it is logarithmic [Eq. (22)]. As $F_{ac} \to 0$, it can be approximated by a power law $T^{-\eta}$ with $\eta \approx 0.25$ at temperatures $T \lesssim 0.05\,\mathrm{K}$. As a result, the temperature dependence of the noise power spectral density $S_y(f)$ is predicted to be $S_y(f) \propto \alpha_{\mathrm{TLS}}/T \propto T^{-1-\eta}$ at low temperatures, $T \ll \hbar\omega/k_{\mathrm{B}}$. Interestingly, this is the same temperature dependence observed in Ref. [16]. However, this result is not applicable to the measurements of Ref. [16] which are reported at temperatures $T \sim \hbar\omega/k_{\mathrm{B}}$ (that is, above 0.1K). As we show in Sec. IV B, the temperature dependence of the $1/f$ noise observed in Ref. [16] may be explained within the STM by assuming a smaller density of TLSs than in typical glasses, as suggested by the much higher quality factor of the resonator.

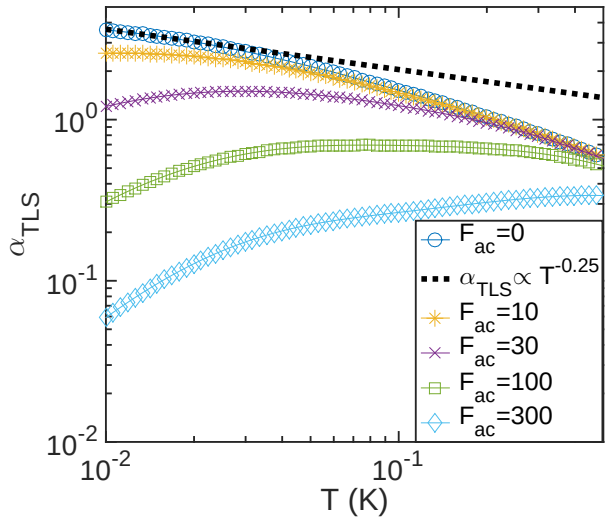

FIG. 3: (Color online) Temperature dependence of the noise parameter α_{TLS} in typical glasses ($\chi = 5 \cdot 10^{-4}$) at frequency $f = 0.1\,\mathrm{Hz}$ for various electric fields. Other parameters are the same as in Fig. 2. Dashed line is a fit of the zero field curve to $\alpha_{\mathrm{TLS}} \propto T^{-\eta}$. The values of the field F_{ac} are in units of V/m.

Finally, consider the field dependence of the noise parameter α_{TLS} in a typical glass. Based on Eq. (22) one can specify three different regimes of F_{ac} with distinguishable behaviors. The pure linear regime takes place for $F_{ac} \ll F_1$, where F_1 is the non-linear absorption threshold defined by the condition $pF_1\sqrt{T_1T_2}/\hbar \approx 1$ [19]. At $T < 0.1\,\mathrm{K}$ both relaxation times are of the order of $1\,\mu\mathrm{s}$ and one can estimate the critical field using the typical TLS dipole moment $p \sim 5\,\mathrm{D}$ [25] as

$$F_1 = \frac{\hbar}{p\sqrt{T_1T_2}} \approx 10\,\mathrm{V/m}. \tag{23}$$

For $F_{ac} \ll F_1$ the field dependence is negligible.

The field dependence remains relatively weak for $F_{ac} > F_1$ until the associated contribution to the resonance broadening, $(pF_{ac}/\hbar)\sqrt{T_1/T_2}$, becomes comparable to the spectral diffusion width $W_0(t_f)$ [Eq. (11)]. This occurs at $F_{ac} \approx F_2$ where the second threshold field, F_2, can be estimated as

$$F_2 \approx \frac{\pi^2\chi k_{\mathrm{B}}T}{3p} \ln\left(3.3 \cdot \frac{AT^3}{2\pi f}\right). \tag{24}$$

At $T = 0.1\,\mathrm{K}$ one can estimate this field as $F_2 \approx 200\,\mathrm{V/m}$. It decreases approximately linearly with decreasing temperature.

At intermediate fields, $F_1 < F_{ac} < F_2$, one can estimate the TLS noise parameter within the same logarith-

mic accuracy as in Eq. (22) by

$$\alpha_{\rm TLS} \approx \frac{81}{80\pi^3\chi}\frac{\langle p^4\rangle}{\langle p^2\rangle^2}\left\langle \frac{\ln[\hbar W_0(t_f)/(pF_{ac})]}{\ln^2\left(3.3\frac{AT^3}{2\pi f}\right)}\right\rangle. \quad (25)$$

At large fields, $F_2 \ll F_{ac}$, the spectral diffusion width can be approximately neglected in the denominator of Eq. (18) and one can estimate the noise parameter as

$$\alpha_{\rm TLS} \approx \frac{9\pi\chi(k_{\rm B}T)^2}{32\langle p^2\rangle F_{ac}^2}. \quad (26)$$

In this regime the noise parameter decreases with decreasing temperature. All these conclusions are consistent with the temperature and field dependence of the noise parameter [Eq. (18)] shown in Fig. 3.

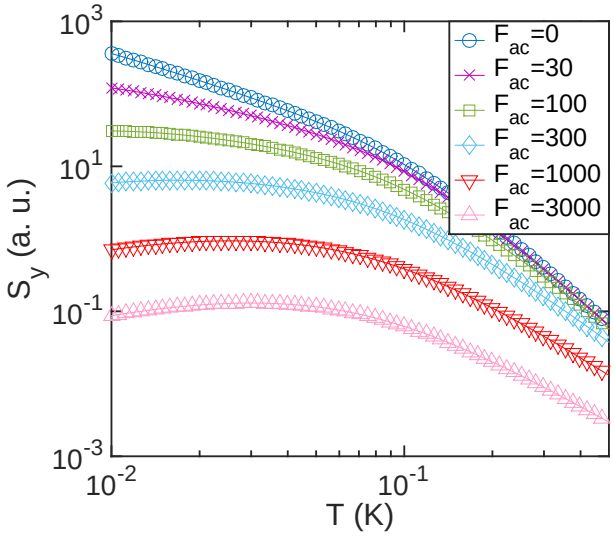

FIG. 4: (Color online) Temperature dependence of the noise power spectral density S_y in typical glasses ($\chi = 5\cdot 10^{-4}$) at frequency $f = 0.1$ Hz for various electric fields.

The temperature dependence of the noise power spectral density at frequency $f = 0.1$ Hz is shown in Fig. 4. At low temperatures, $T \ll \hbar\omega/k_{\rm B}$, one has $\tanh(\hbar\omega/2k_{\rm B}T) \approx 1$ in Eq. (19) and the power spectral density behaves as $S_y \propto \alpha_{\rm TLS}/T$ [see Eq. (18)]. In the limit of small fields it shows the power law dependence, $S_y \propto T^{-1-\eta}$, discussed in Refs. [16, 18]. The inverse temperature dependence is of similar origin and reflects the inverse dependence on the density of thermal TLSs. The additional exponent η is due to the logarithmic temperature dependence predicted within the STM. This exponent can vary depending on the specific system under consideration and future experiments can be compared to the proposed theory by evaluating Eq. (18) for the specific material under consideration. The noise decreases and its temperature dependence becomes weaker with increasing field F_{ac}. At large fields, $F_{ac} > 1000$ V/m, the noise power spectral density decreases with decreasing temperature.

However, the regime of low temperatures, $T \ll \hbar\omega/k_{\rm B}$, in which one expects $S_y \propto T^{-1-\eta}$, is not applicable to the experimental results of Ref. [16] which are reported at temperatures $T \sim \hbar\omega/k_{\rm B}$. In this regime one expects the temperature dependence of the factor $\tanh^2(\hbar\omega/2k_{\rm B}T)$ to be significant. Thus, it cannot be approximated as unity when comparing the data of Ref. [16] with theory. For instance, at high temperatures, $T \gg \hbar\omega/k_{\rm B}$, the relevant limit is $\tanh^2(\hbar\omega/2k_{\rm B}T) \propto T^{-2}$ for which Eq. (17) as well as Refs. [16, 18] predict a strong reduction of the noise with increasing temperature, $S_y \propto T^{-3-\eta}$. For intermediate temperatures, $T \sim \hbar\omega/k_{\rm B}$ as in Ref. [16], one should not use any of these approximations for the factor $\tanh^2(\hbar\omega/2k_{\rm B}T)$. To show the importance of this factor in the analysis of the data of Ref. [16], we plot in Fig. 5 the temperature dependence of the noise power spectral density obtained in this experiment divided by $\tanh^2(\hbar\omega/2k_{\rm B}T)$ [cf. Eq. (17)]. After this rescaling the temperature dependence almost disappears in contrast with the predictions of Ref. [18] and the present work for typical glasses (which predict that this rescaled power spectral density should behave as $T^{-1-\eta}$). As discussed above, we conjecture that this discrepancy is because the material studied in Ref. [16] cannot be treated as a typical glass. In Sec. IV B we suggest the interpretation of this experiment by assuming a small spectral diffusion width compared to the TLS relaxation rate which may be a consequence of a smaller density of TLSs compared to ordinary glasses.

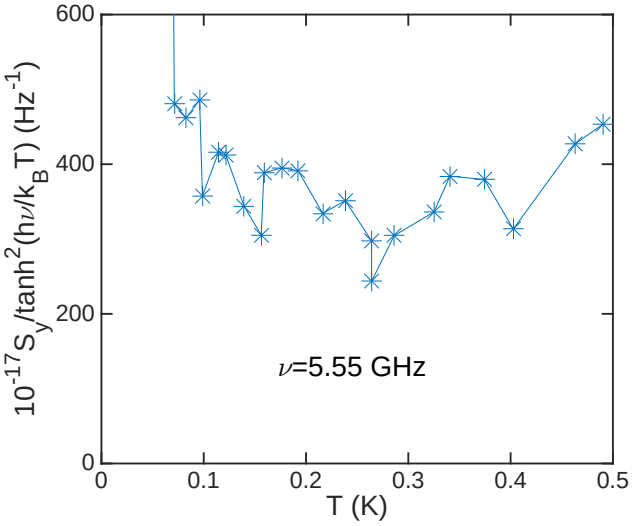

FIG. 5: (Color online) Temperature dependence of $S_y/\tanh^2(\hbar\omega_0/2k_{\rm B}T)$ for the noise power spectral density S_y obtained in a Pt capped Nb resonator at resonance frequency $\nu_0 = 5.55$ GHz and at the smallest measuring field [16].

B. 1/f noise in Pt capped Nb resonator

In this section we attempt to fit the experimental data of Ref. [16]. This experiment was performed using superconducting Nb resonators. Following Refs. [34, 35] we assume that TLSs cannot exchange energy with Nb superconducting electrons because the superconductor has a gap in energy spectrum which is much larger than the thermal energy. We also use the model developed above for $1/f$ noise due to interacting TLSs forming a three-dimensional system. The assumption of a three-dimensional geometry is valid until the distance between thermal TLSs is smaller compared to the thickness of the oxide film at the Nb surface where TLSs are located. At low temperature this assumption fails and two-dimensional consideration is needed. Based on the qualitative consistency between theory and experiment we assume that our consideration is qualitatively valid at $T > 0.1$K. At low temperatures the generalization of our theory to two-dimensional geometry may be required, which is beyond the scope of the present paper.

In order to fit the experimental data we assume that the relaxation rate of TLSs, $1/T_1$ [Eqs. (6) and (7)], exceeds the spectral diffusion width $W_0(t_f)$ [Eq. (11)]. This could be a result of the small amount of TLSs in the sample, as reflected by the high quality factor $Q \sim 10^6$ [16], leading to a smaller value of $W_0(t_f)$. In addition, the relaxation rate in the experimental setup of Ref. [16] may be larger compared to ordinary glasses, due to interaction of TLSs with conduction electrons in the Pt capping layer. Assuming $W_0(t_f)T_1 \ll 1$, one can ignore the contributions of spectral diffusion and decoherence in the denominator of Eq. (17). The noise power spectral density then takes the form

$$S_y(f) \propto \frac{k_{\rm B}T}{2f}\tanh^2\left(\frac{\hbar\omega}{2k_{\rm B}T}\right)\int_0^1 x^3 dx \times\left\langle \frac{p_z^4}{x^4\coth^2\left(\frac{\hbar\omega}{2k_{\rm B}T}\right)\frac{1}{\tau_{\rm min}^2}+x^2\left(\frac{p_z F_{ac}}{\hbar}\right)^2}\right\rangle. \quad (27)$$

Here the time $\tau_{\rm min}$ stands for the minimum TLS relaxation time in the zero temperature limit. This time is defined by $\tau_{\rm min}^{-1} = A\left(\hbar\omega/k_{\rm B}\right)^3$ for the relaxation induced by TLS-phonon interaction [Eq. (6)], or $\tau_{\rm min}^{-1} = A_e\hbar\omega/k_{\rm B}$ for the relaxation induced by the interaction of TLSs with conduction electrons [Eq. (7)].

Similarly to the previous section and to other works [7, 25], we assume a single value p for the magnitude of TLS dipole moment, as well as for the relaxation parameter A (A_e). Other assumptions (e.g., Gaussian distribution) lead to similar results and the present uncertainty of available experimental data [16] (see Figs. 6 and 7) does not permit us to make any specific choice. For the direct comparison of our theory with the experimental data of Ref. [16], expressed in terms of the average number of photons $\langle n\rangle$ within the cavity, we write the external field F_{ac} in terms of the vacuum field F_{vac} as [37]

$$F_{ac} = F_{vac}\sqrt{\langle n\rangle}, \qquad F_{vac} = \sqrt{\frac{4\pi\hbar\omega}{\epsilon V}}. \quad (28)$$

The field is treated classically ($\langle n\rangle > 1$), so that the difference between the factors $\sqrt{\langle n\rangle}$ and $\sqrt{\langle n\rangle + 1}$ is neglected, which is approximately satisfied for the average number of photons $\langle n\rangle \geq 3$ used in the experiment [16].

Then one can evaluate the integral in Eq. (27) and obtain the noise power spectral density in terms of the average number of photons $\langle n\rangle$ within the cavity as

$$S_y(f) \propto \frac{k_{\rm B}T}{f}\tanh^4\left(\frac{\hbar\omega}{2k_{\rm B}T}\right)\int_0^1 x_1^4 dx_1 \times \ln\left[\frac{1+c^2\langle n\rangle x_1^2\tanh^2\left(\frac{\hbar\omega}{2k_{\rm B}T}\right)}{c^2\langle n\rangle x_1^2\tanh^2\left(\frac{\hbar\omega}{2k_{\rm B}T}\right)}\right]. \quad (29)$$

Here the integration is over the polar angle θ ($x_1 = \cos\theta$ and $p_z = px_1$) and the dimensionless parameter c is

$$c = \frac{pF_{vac}\tau_{\rm min}}{\hbar}. \quad (30)$$

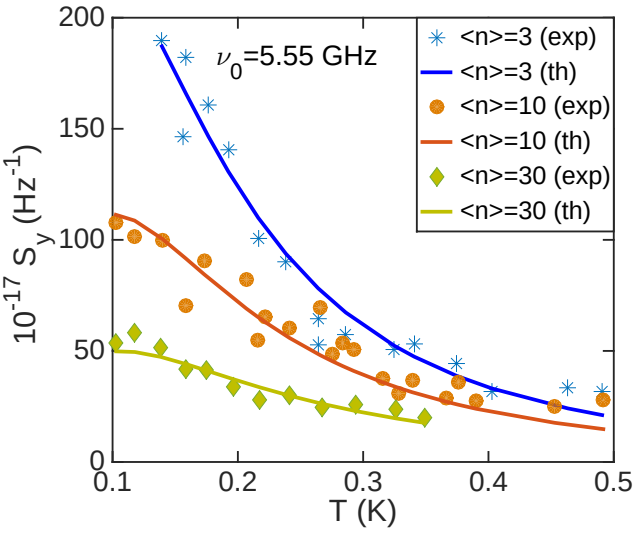

FIG. 6: (Color online) Fit of the experimental data by Burnett *et al.* [16] at resonance frequency $\nu_0 = 5.55$ GHz (markers) to Eq. (29) (solid lines).

Using the constant c as an adjustable parameter we attempted to fit the experimental data of Burnett *et al.* [16]. Only data points at temperatures higher than 0.1 K have been considered because of the strong variation of the noise spectral density at lower temperatures. According to Ref. [36], this is not due to an experimental error; we believe that it can be caused by the reduction of the number of thermal TLSs and the narrowing of the resonance linewidth with decreasing temperature, giving

rise to strong noise fluctuations if either the number of thermal TLSs or the number of TLSs within the resonant linewidth approaches unity (cf. Ref. [37]).

The optimum data fits were obtained with the fitting parameters $c_{5.55} = 0.384$ for $\nu_0 = 5.55\,\mathrm{GHz}$ (Fig. 6) and $c_{6.68} = 0.348$ for $\nu_0 = 6.68\,\mathrm{GHz}$ (Fig. 7). The theory fits the experimental data reasonably well at temperatures between 0.1 K and 0.4 K. At higher temperatures some excess contribution is seen especially for the high frequency sample. It is possibly associated with the effect of non-resonant TLSs [10] or the elevation of the noise floor with decreasing microwave driving-field, generally observed across many measurements [36].

The frequency dependence of the fitting parameter c can be used to determine the dominant mechanism of TLS relaxation, if one assumes the sample volumes to be identical. For the phonon-induced relaxation [Eq. (6)] the characteristic time $\tau_{\min}$ depends on the frequency as ν^{-3}, leading to $c \propto \nu^{-2.5}$, whereas for the relaxation due to the interaction of TLSs with conduction electrons [Eq. (7)] one has $\tau_{\min} \propto \nu^{-1}$, corresponding to $c \propto \nu^{-0.5}$. The ratio of the two fitting parameters $c_{5.55}/c_{6.68} = 1.101$ scales almost exactly as the inverse square root of the ratio of the cavity resonance frequencies $\sqrt{6.68/5.55} \approx 1.097$. Thus, provided that the volume of both resonators is the same, our analysis suggests that the TLS relaxation in the experimental setup of Ref. [16] is mainly due to interaction of TLSs with conduction electrons. Such electrons may reside in the Pt capping layer used in the experimental setup of Ref. [16].

The fit of the experimental data is made with the accuracy to the unknown sample-dependent, but field independent proportionality constant which cannot be estimated based on the present information since the TLS density, dipole moment, and electron-TLS coupling constant are unknown. Loss tangent measurements with a time-varying bias similar to Ref. [25] can help to extract these parameters separately. The non-linear threshold at the number of photons $\langle n \rangle \sim c^{-2} \approx 10$ is quite typical for the microwave cavities under consideration (cf. Ref. [37]).

V. CONCLUSION

In the present paper we investigated $1/f$ noise in the microwave dielectric constant produced by TLSs in amorphous solids. The noise power spectral density has been calculated analytically within the standard tunneling model involving the long-range elastic or dipolar TLS-TLS interactions. For amorphous solids characterized by "typical" parameters we predict the increase of the noise with decreasing temperature according to the power-law behavior $S_y(T) \propto T^{-1-\eta}$ at low temperatures, $T \ll \hbar\omega_0/k_\mathrm{B}$, and vanishing field $F_{ac} \to 0$. The additional exponent η originates in the logarithmic time and temperature dependence of the spectral diffusion width and its specific value may vary among different materi-

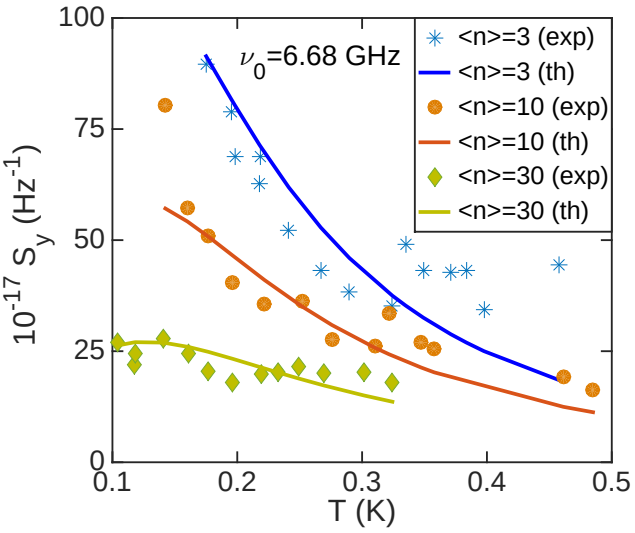

FIG. 7: (Color online) Fit of the experimental data by Burnett *et al.* [16] at resonance frequency $\nu_0 = 6.68\,\mathrm{GHz}$ (markers) to Eq. (29) (solid lines).

als. This result should be verified in experiments to be performed in the limit $T \ll \hbar\omega_0/k_\mathrm{B}$ and $F_{ac} \to 0$.

The experimental data by Burnett *et al.* [16], performed at $T \sim \hbar\omega_0/k_\mathrm{B}$, cannot be explained in terms of our theoretical results for typical amorphous solids. In this regime of temperatures the factor $\tanh^2(\hbar\omega/2k_\mathrm{B}T)$ in Eq. (17) cannot be neglected and one expects a temperature dependence much stronger than $T^{-1-\eta}$ with $\eta \approx 0.3$. Instead, we show that these data may be explained by assuming the opposite limit of small spectral diffusion width compared to the TLS relaxation rate. The general expression for the noise power spectral density then reduces to a form which fits the experimental data above 0.1 K reasonably well. This expression contains two fitting parameters for each sample, which are independent of the field intensity. The extracted frequency dependence of the TLS relaxation rate suggests that the TLS relaxation is caused by TLS interaction with conduction electrons. Such electrons may be available in the experimental setup of Ref. [16] due to the Pt capping layer used in this setup. It should be noted, however, that this conclusion assumes equal volumes [as well as other parameters in Eqs. (28) and (30)] for the two resonators studied in Ref. [16]. Either way, the main results of this paper for the noise in the dielectric constant do not depend on the specific relaxation mechanism.

We acknowledge Kevin Osborn and Georg Weiss for stimulating discussions and John Burnett and Tobias Lindström for many critical remarks related to the interpretation of their work. A.L.B. acknowledges support by the Louisiana EPSCORE LA Sigma Program (EPS-1003897) and LINK Program (NSF(2014)-LINK-90). M.S. and S.M. acknowledge support by the German-Israeli Foundation (GIF) and by the Israeli Science Foun-

dation (ISF).

[1] P. Dutta and P. M. Horn, Rev. Mod. Phys. **53**, 497 (1981).
[2] M. B. Weissman, Rev. Mod. Phys. **60**, 537 (1988).
[3] Sh. M. Kogan, *Electronic Noise and Fluctuations in Solids* (Cambridge University Press, Cambridge, 1996).
[4] W. Schottky, Ann. Phys. (Berlin) **362**, 541 (1918).
[5] S-I. Han, R. Almy, E. Apodaca, W. Bergmann, S. Deiker, A. Lesser, D. McCammon, K. Rawlins, R. L. Kelley, S. H. Moseley, F. S. Porter, C. K. Stahle, and A. E. Szymkowiak, in *EUV, X-ray, and Gamma-Ray Instrumentation for Astronomy IX*, edited by O. H. Seigmund and M. A. Gummin, Proc. of SPIE **3445**, 660 (1998); D. McCammon, in *Cryogenic Particle Detection*, edited by C. Enss, Top. Appl. Phys. **99**, 35 (Springer, Berlin, 2005).
[6] D. McCammon, M. Galeazzi, D. Liu, W. T. Sanders, B. Smith, P. Tan, K. R. Boyce, R. Brekosky, J. D. Gygax, R. Kelley, D. B. Mott, F. S. Porter, C. K. Stahle, C. M. Stahle, and A. E. Szymkowiak, Phys. Stat. Sol. b **230**, 197 (2002).
[7] J. M. Martinis, K. B. Cooper, R. McDermott, M. Steffen, M. Ansmann, K. D. Osborn, K. Cicak, S. Oh, D. P. Pappas, R. W. Simmonds, and C. C. Yu, Phys. Rev. Lett. **95**, 210503 (2005).
[8] A. L. Burin, B. I. Shklovskii, V. I. Kozub, Y. M. Galperin, and V. Vinokur, Phys. Rev. B **74**, 075205 (2006).
[9] Y. Lemonik and I. Aleiner, Phys. Rev. B **90**, 184203 (2014).
[10] C. C. Yu, J. Low Temp. Phys. **137**, 251 (2004); M. Constantin, C. C. Yu, and J. M. Martinis Phys. Rev. B **79**, 094520 (2009).
[11] P. W. Anderson, B. I. Halperin, and C. M. Varma, Philos. Mag. **25**, 1 (1972); W. A. Phillips, J. Low-Temp. Phys. **7**, 351 (1972).
[12] J. Q. You and F. Nori, Phys. Today **58**, 42 (2005); J. Clarke and F. K. Wilhelm, Nature **453**, 1031 (2008).
[13] C. Neill, A. Megrant, R. Barends, Y. Chen, B. Chiaro, J. Kelly, J. Y. Mutus, P. J. J. O'Malley, D. Sank, J. Wenner, T. C. White, Y. Yin, A. N. Cleland, and J. M. Martinis, Appl. Phys. Lett. **103**, 072601 (2013).
[14] K. W. Murch, S. J. Weber, E. M. Levenson-Falk, R. Vijay, and I. Siddiqi, Appl. Phys. Lett. **100**, 142601 (2012).
[15] J. Burnett, T. Lindström, M. Oxborrow, Y. Harada, Y. Sekine, P. Meeson, and A. Ya. Tzalenchuk, Phys. Rev. B **87**, 140501(R) (2013).
[16] J. Burnett, L. Faoro, I. Wisby, V. L. Gurtovoi, A. V. Chernykh, G. M. Mikhailov, V. A. Tulin, R. Shaikhaidarov, V. Antonov, P. J. Meeson, A. Ya. Tzalenchuk, T. Lindström, Nat. Commun. **5**, 4119 (2014).
[17] A. N. Ramanayaka, B. Sarabi, K. D. Osborn, arXiv:1507.06043 [cond-mat.supr-con].
[18] L. Faoro and L. B. Ioffe, Phys. Rev. B **91**, 014201 (2015).
[19] S. Hunklinger and A. K. Raychaudchary, Progr. Low Temp. Phys. **9**, 267 (1986).
[20] M. Von Haumeder, U. Strom, and S. Hunklinger, Phys. Rev. Lett. **44**, 84 (1980).
[21] J. L. Black and B. I. Halperin, Phys. Rev. B **16**, 2879 (1977).
[22] In a recent work [S. T. Skacel *et al.*, Appl. Phys. Lett. **106**, 022603 (2015)], an anomalously strong energy dependence of the DOS has been reported in *a*-SiO film. This dependence looks much stronger than the one assumed in Ref. [18] and is probably related to the lack of oxygen compared to the standard SiO_2 stoichiometry. The material can have a large number of dangling bonds possibly responsible for a "non-glassy" behavior.
[23] A. L. Burin, J. Low Temp. Phys. **100**, 309 (1995).
[24] A. L. Burin, D. Natelson, D. D. Osheroff, Yu. Kagan, in *Tunneling Systems in Amorphous and Crystalline Solids*, edited by P. Esquinazi, p. 243 (Springer, 1998).
[25] M. S. Khalil, S. Gladchenko, M. J. A. Stoutimore, F. C. Wellstood, A. L. Burin, and K. D. Osborn, Phys. Rev. B **90**, 100201(R) (2014); A. L. Burin, M. S. Khalil, K. D. Osborn, Phys. Rev. Lett. **110**, 157002 (2013).
[26] A. L. Burin, J. M. Leveritt, G. Ruyters, C. Schötz, M. Bazrafshan, P. Fassl, M. von Schickfus, A. Fleischmann, and C. Enss, Euro Phys. Lett. **104**, 57006 (2013).
[27] H. M. Carruzzo, E. R. Grannan, and C. C. Yu, Phys. Rev. B **50**, 6685 (1994).
[28] M. Von Schickfus and S. Hunklinger, Phys. Lett. A **64**, 144, (1977).
[29] C. C. Yu and A. J. Leggett, Comments Condensed Matter Phys **14**, 231 (1988).
[30] J. Lisenfeld et. al., unpublished.
[31] At very high temperatures, $T \gg \hbar\omega_0/k_\mathrm{B}$, the resonant contribution will not be dominating any more and the noise will be determined by TLSs with energies comparable to the thermal energy as suggested in Ref. [18] (see estimates of the integral in Eq. (39) there, given by Eq. (41) for low temperature and Eq. (42) for high temperature limits, respectively). Indeed, this contribution scales as $k_\mathrm{B}T \cdot AT^3/\hbar\omega_0^2$ at $k_\mathrm{B}T > \hbar\omega$ compared to the resonant contribution [18]. Consequently, this high temperature contribution can be neglected at $T < \left(\hbar\omega_0^2/k_\mathrm{B}A\right)^{1/4} \sim 3\,\mathrm{K}$ which is well above the temperatures explored in Ref. [16]. Accordingly, the squared population difference factors $\tanh^2\left(\frac{E_i}{2k_\mathrm{B}T}\right)$ containing individual TLS energies E_i are replaced in Eq. (14) with the common factor $\tanh^2\left(\frac{\hbar\omega}{2k_\mathrm{B}T}\right)$. The inclusion of this factor changes the results of the model of Ref. [18] so that it no longer agrees with the experiment.
[32] F. N. Hooge, Physica **60**, 130 (1972); F. N. Hooge, T. G. M. Kleinpenning, and L. K. J. Vandamme, Rep. Prog. Phys. **44**, 479 (1981).
[33] R. O. Pohl, X. Liu, and E. Thompson, Rev. Mod. Phys. **74**, 991 (2002).
[34] J. L. Black and P. Fulde, Phys. Rev. Lett. **43**, 453 (1979).
[35] C. C. Yu and A. V. Granato, Phys. Rev. B **32**, 4793 (1985).
[36] J. Burnett, T. Lindström, private communication.
[37] B. Sarabi, A. N. Ramanayaka, A. L. Burin, F. C. Wellstood and K. D. Osborn, Appl. Phys. Lett. **106**, 172601 (2015).