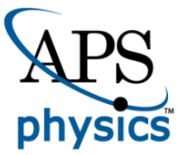

Equivalence of topological insulators and superconductors

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Equivalence of topological insulators and superconductors

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Systems of free fermions are classified by symmetry, space dimensionality, and topological properties described by K-homology. Those systems belonging to different classes are inequivalent. In contrast, we show that by taking a many-body/Fock space viewpoint it becomes possible to establish equivalences of topological insulators and superconductors in terms of duality transformations. These mappings connect topologically inequivalent systems of fermions, jumping across entries in existent classification tables, because of the phenomenon of symmetry transmutation by which a symmetry and its dual partner have identical algebraic properties but very different physical interpretations. To constrain our study to established classification tables, we define and characterize mathematically Gaussian dualities as dualities mapping free fermions to free fermions (and interacting to interacting). By introducing a large, flexible class of Gaussian dualities we show that any insulator is dual to a superconductor, and that fermionic edge modes are dual to Majorana edge modes, that is, the Gaussian dualities of this paper preserve the bulk-boundary correspondence. Transmutation of relevant symmetries, particle number, translation, and time reversal is also investigated in detail. As illustrative examples, we show the duality equivalence of the dimerized Peierls chain and the Majorana chain of Kitaev, and a two-dimensional Kekulé-type topological insulator, including graphene as a special instance in coupling space, dual to a p-wave superconductor. Since our analysis extends to interacting fermion systems we also briefly discuss some such applications.

I. INTRODUCTION

In this paper we establish equivalences of topologically non-trivial insulators[1–3] and superconductors.[3–5] By means of duality transformations, we show that any insulator has a dual superconducting partner, and the partners are either both topologically trivial or non-trivial. We will focus on non-interacting dual partners, since general classification schemes exist for free-fermion systems.[6–9] As it turns out, the duality transformations of this paper connect systems that are inequivalent from the point of view of these topological classifications. This is only possible because of the phenomenon of *symmetry transmutation*, by which a duality transformation maps a symmetry of one system with one physical interpretation, say particle number or time reversal, to a symmetry of the dual system with a different interpretation.

From an electromagnetic response viewpoint, insulating and superconducting phases of electron systems are dramatically different. While the bulk insulating phase is characterized by a vanishing steady current-carrying state at zero temperature, the superconducting phase supports a supercurrent and displays a perfect diamagnetic response, the Meissner effect. Geometrically, the states associated to these two phases can be distinguished by their distinctive localization properties.[10] Yet there is a basic sense in which both states of matter are equivalent, since many of their defining properties stem from a common factor, that is, the existence of a gap in the bulk energy spectrum of fermionic quasiparticles. The additional presence of gapless, symmetry-protected, extended surface excitations defines operationally their topologically non-trivial character. One of the objectives of topological band theory is to classify, based on a few preferred (discrete) symmetries and space dimensionality, topologically distinct non-interacting (single-particle) Hamiltonians and their concomitant gapless edge excitations.

For systems without gauge symmetries, duality transformations are implemented by unitary mappings,[11,12] and so they preserve symmetries; the symmetries of a system are in one-to-one correspondence with the symmetries of its dual partner. However, the physical interpretation of a symmetry and its dual image can be markedly different. Holographic symmetries[13] constitute a most extreme example. For some pairs of dual partners, one of the systems displays boundary symmetries, mapped to global symmetries of its dual partner. In this case we call the boundary symmetry holographic. This phenomenon is remarkable because a symmetry that is formally lost in the thermodynamic limit, the holographic symmetry, is mapped by duality onto a symmetry that may become spontaneously broken in that limit. Because of this, not uncommon, example of symmetry transmutation, it is conceivable that a duality may map a particle conserving system to a non-conserving one, simply by mapping the $U(1)$ symmetry of particle number to a dual $U(1)$ symmetry that does not have that interpretation.

While these arguments are encouraging in the search for equivalences of insulators and superconductors, there are at least two other obstacles besides particle (non)conservation. First, in general, dualities for fermions will not preserve the quadratic (or Gaussian) character of a model system, often mapping free-fermion systems to interacting ones. Second, for topologically non-trivial systems, even if one were to find dualities matching non-interacting dual partners, there is in general no reason to expect that these dualities should also preserve the locality properties of the quasiparticle modes. For example, at zero energy, modes localized at a boundary may also be interpreted as boundary sym-

metries. Extrapolating from the experience with holographic symmetries, one would expect these modes to become delocalized after a duality transformation.

As it turns out, both obstacles may be overcome, and, as a consequence, there is no fundamental obstruction to the construction of equivalences of topological insulators and superconductors in terms of dualities. Section II introduces the special class of duality maps that establishes those equivalences. The starting point is the characterization of duality transformations that preserve the quadratic fermionic nature of a given model system; we will call this transformation *Gaussian duality*. Next we will construct a large class of such dualities in any number of spatial dimensions, in order to create a toolkit for generating a topological superconductor from any given topological insulator in a systematic fashion. In other words, given a topological insulator one can always find at least one dual topological superconductor associated to it. This process is, of course, reversible since the duality transformation is an isometry. Hence our results strongly suggest that there may exist equivalences of topological insulators and superconductors across all entries of the topological classification table, at least for constant space dimension. Dimensional reduction by dualities is possible,[14] but we will not obtain any Gaussian instance of this phenomenon in this paper. Section II ends with the fundamental concept of symmetry transmutation as applied to fermion parity, translation and time-reversal symmetries. In particular, we will find a quantitative connection between changes in translation symmetry and breaking of particle conservation.

Particular and emblematic examples include the proof that the insulating dimerized Peierls[15,16] and superconducting Kitaev (at vanishing chemical potential)[17,18] chains are dual partners, and the equivalence of graphene to a popular example[19,20] of a weak[23] topological superconductor in two spatial dimensions. It is in Section III that we present these two prototypical equivalences. These dual partners do not simply *resemble* each other, but are isospectral from a many-body standpoint, for finite lattices and various boundary conditions. No doubt, this fact seems odd at first sight, since the spinfull Peierls chain for example partially breaks translation symmetry but not (the standard) time reversal or particle conservation, while its dual partner, Kitaev's Majorana chain, breaks time reversal and particle conservation but not translation symmetry. (In Appendix A we derive the superconducting dual of the m-merized Peierls chain, and discuss the differences between m odd and even.) Similarly, graphene displays the symmetry of the honeycomb lattice[21] while it dual superconducting partner sits on a square lattice. Remarkably, the Gaussian duality allows us to qualitatively understand the difference between zigzag and armchair terminations in graphene.[22] The explanation to all these seemingly paradoxical observations is symmetry transmutation. In Appendix B we describe a simple Gaussian duality mapping a (s-wave) BCS superconductor to an insulator in any number of dimensions.

Another important issue addressed in Section III is the locality character of our Gaussian dualities, i.e., the problem of showing that localized zero-energy modes are mapped to dual zero-modes that are also localized. It is remarkable to have the possibility to generate *localization-preserving* Gaussian dualities. In other words, there are no holographic symmetries associated to the Gaussian dualities of this paper: global symmetries map to global symmetries, and the localization properties of energy modes are also preserved, though edge modes may be shuffled among boundaries. In particular, the zig-zag boundary of graphene is exactly dual to the "Kitaev edge" of Refs. 19 and 20. We show how topological defects and edge states map, and also how the nature of those excitations transmutes from (canonical) fermionic to Majorana character by duality. Interestingly, we analytically construct *exact* (as opposed to asymptotic) zero-energy modes for any finite length Kitaev wire when the length is an odd number of lattice constants.

An interesting outcome of our investigation is further confirmation that non-trivial topological quantum order is a property of a manifold of states interpreted relative to a given language (a set of preferred observables[24]); and not a property of the energy spectrum alone[25] and some Hamiltonian singling out those states as energy eigenstates. We also investigate in Section III the interplay between dualities and topological invariants of the single-particle Hamiltonian. Indeed, our equivalences are duality mappings and hence necessarily isospectral.[12] However, some of these duality mappings connect systems with ground states characterized by different topological quantum numbers, thus belonging to different topological classes. For instance, Kitaev wire model belongs to the class D of the Dyson-Altland-Zirnbauer tenfold-way classification,[6] while its dual, the dimerized (spinless) Peierls chain, belongs to the class AIII. In the past, we have studied dualities mapping systems with topologically quantum-ordered ground-state manifolds to systems characterized by local (Landau) orders.[12,13]

In order to take advantage of duality transformations, it is crucial to recognize that the many-body, and not the single-particle, representation of the system is the relevant one. There is absolutely no doubt that, from a computational standpoint, the single-particle representation (e.g., the Bogoliubov-de Gennes equations) is the appropriate methodology to adopt in the non-interacting or mean-field case. Computationally, it reduces a problem of exponential complexity into one of polynomial complexity, thus allowing diagonalization of quite large system sizes. However, care must be exercised at the moment of analyzing properties like particle conservation that involve the whole many-body system. In particular, topological classification schemes and counting of many-body zero-modes relate directly to the many-body ground-state manifold. These cautionary remarks are entirely appropriate since Gaussian dualities connecting topologically non-trivial dual partners often seem at odds with one form or another of standard wisdom. However,

they are entirely natural if one adopts the many-body (Fock-space) language of second quantization, and not the vector bundle analysis of single-particle Bogoliubov-de Gennes Hamiltonian matrices.

Technically, Majorana operators, defined (up to normalization) as the real and imaginary parts of the canonical fermionic field, generate a complex Clifford algebra naturally represented in Fock space, and our Gaussian dualities are characterized most naturally as isomorphisms of these Clifford algebras. The effect in single-particle (mode) space, where a different, exponentially smaller Clifford algebra emerges[6,7] is induced *a posteriori*. Crucially, it follows that our Gaussian dualities can also be used for investigating interacting many-body systems (see for example Section III A 3).[26] Several different mathematical simplifications arise when Gaussian dualities are investigated in terms of Majorana operators. These simplify not only the search for equivalences, but also the analysis of symmetries, topological invariants and their transformation, and most importantly the mapping of boundary excitations.

At this point it becomes natural to ask about the extension of our work to bosons, since it is clear that the notion of Gaussian duality applies to canonical bosons just as well. However, the real and imaginary parts of the bosonic field satisfy the Heisenberg commutation relation, and so the theory of Gaussian dualities for bosons is bound to be markedly different from that of fermions. Due to this crucial technical difference, we defer the systematic study of bosonic Gaussian dualities to future research. Nonetheless, we would still like to illustrate explicitly the point that symmetry transmutation is also operative in bosonic systems. Hence, in Appendix C we describe a duality mapping of a bosonic Mott insulator to a quartet superconductor. We also comment briefly on the relevance of this example for cold atoms.

Section IV concludes with a summary and outlook.

II. GAUSSIAN DUALITIES

The duality mappings of statistical mechanics are, in the absence of gauge symmetries, unitary similarity transformations that respect the locality structure of particular many-body Hamiltonians or transfer matrices.[11,12] In what follows we will set up the foundations to establish equivalences via dualities. Particularly, we will characterize operator maps relating Hermitian quadratic forms of fermions, i.e., Gaussian dualities, including the connection between such many-body dualities and the associated transformation of the single-particle Hamiltonian. Next we will introduce general techniques to decompose a very large class of free fermion models into sums of commuting Hamiltonians, and finally we will use this technique to construct a general class of Gaussian dualities.

A. What is a Gaussian duality ?

We are interested in establishing the conditions under which a duality transformation may be classified as Gaussian. Let us focus for definitenes on systems of free fermions defined on a lattice. Then, the most general free fermion Hamiltonian is of the form

$$H = \sum_{i,j=1}^{L} \Big[K_{ij} c_i^\dagger c_j + \frac{1}{2}\Delta_{ij} c_i^\dagger c_j^\dagger + \frac{1}{2}\Delta_{ij}^* c_j c_i \Big], \quad (1)$$

with one-body and pairing interaction matrices

$$K^\dagger = K, \quad \Delta^T = -\Delta, \quad (2)$$

where † is the adjoint, ^{T} the transpose, and * complex conjugation of a matrix. The creation/annihilation operators of a fermion $c_j^\dagger/c_j$ in the single-particle orbital ϕ_j ($\{c_i, c_j^\dagger\} = \delta_{ij}$) are labelled by the generic subindex j encoding arbitrary quantum numbers like position, spin, orbital/band, angular momentum, etc. The total number of single-particle orbitals is L.

Equivalently, one can re-write H in Nambu form

$$H = \frac{1}{2}\,\alpha^\dagger\, h_{\mathsf{BdG}}\, \alpha + \frac{1}{2}\mathsf{Tr}\, K, \quad (3)$$

where the column vector of fermion operators is given by

$$\alpha = \begin{pmatrix} c \\ c^\dagger \end{pmatrix}, \text{ with } \alpha_j = c_j \ , \ \alpha_{L+j} = c_j^\dagger \ , \ j = 1, \cdots, L, \quad (4)$$

and the Bogoliubov-de Gennes single-particle Hamiltonian ($2L \times 2L$ matrix)

$$h_{\mathsf{BdG}} = \begin{pmatrix} K & \Delta \\ -\Delta^* & -K^* \end{pmatrix} = \quad (5)$$
$$\mathsf{i}\mathbb{1} \otimes \Im(K) + \mathsf{i}\tau^x \otimes \Im(\Delta) + \mathsf{i}\tau^y \otimes \Re(\Delta) + \tau^z \otimes \Re(K),$$

where τ^ν, $\nu = x, y, z$, are Pauli matrices, and $\Re(\cdot)(\Im(\cdot))$ denotes the real (imaginary) part of the matrix. No matter what the specific matrices K and Δ are, the single-particle Hamiltonian h_{BdG} always anticommutes with the antiunitary (particle-hole or charge conjugation) operator

$$\mathcal{C} = \mathcal{K}\tau^x \otimes \mathbb{1}, \quad \mathcal{C}^2 = \mathbb{1}, \quad (6)$$

i.e., $\{h_{\mathsf{BdG}}, \mathcal{C}\} = 0$, where $\mathcal{K}$ denotes complex conjugation. That means that the single particle energy spectrum is antisymmetric with respect to its zero value, i.e., a particle-hole symmetric spectrum. By contrast, a chiral symmetry $\mathcal{U}_{\mathsf{chiral}}$ is a unitary transformation that anticommutes with h_{BdG}. For example, if $\Im(K) = 0 = \Im(\Delta)$, then $\mathcal{U}_{\mathsf{chiral}} = \tau^x \otimes \mathbb{1}$.

Suppose now that the unitary transformation $\mathcal{U}_{\mathsf{d}}$ implements a duality transformation,

$$H^D = \mathcal{U}_{\mathsf{d}}\, H\, \mathcal{U}_{\mathsf{d}}^\dagger, \quad (7)$$

meaning that it transforms a local non-interacting Hamiltonian H into another, dual H^D, that preserves the property of being also local. The map, however, could generate fermionic density-density interactions for instance.

What are the general conditions under which H^D is also an Hermitian quadratic form of fermions?

To answer this question we will recast Hamiltonian H of Eq. (1) in terms of Majorana operators

$$\gamma_{2j-1} = c_j + c_j^\dagger, \quad \mathsf{i}\gamma_{2j} = c_j - c_j^\dagger, \tag{8}$$

such that $\{\gamma_r, \gamma_{r'}\} = 2\delta_{r,r'}$ for $r, r' = 1, \cdots, 2L$. (The notation

$$a_j = c_j + c_j^\dagger, \quad \mathsf{i}b_j = c_j - c_j^\dagger. \tag{9}$$

will be favored in later sections). Then

$$H = \frac{\mathsf{i}}{2} \sum_{r,s=1}^{2L} h_{rs}\gamma_r\gamma_s + \frac{1}{2}\mathsf{Tr}\, K \tag{10}$$

becomes a quadratic form of Majorana fermions, with a $2L \times 2L$ matrix h that is real and antisymmetric.

We can now investigate the dual Hamiltonian. Remember that we want the duality map to be Gaussian, i.e., H^D should also be a quadratic form of Majorana fermions. A naive first, and trivial, attempt would be to keep the localization properties *identical*, i.e.

$$H^D = \frac{\mathsf{i}}{2} \sum_{r,s=1}^{2L} h_{rs}\gamma_r^D\gamma_s^D + \frac{1}{2}\mathsf{Tr}\, K \tag{11}$$

where the dual operators are related to the originals as

$$\gamma_r^D = \mathcal{U}_\mathsf{d}\, \gamma_r\, \mathcal{U}_\mathsf{d}^\dagger, \tag{12}$$

and γ_r^D is a Majorana fermion operator. This extreme local map, although Gaussian, is very restrictive and will not allow us to establish interesting equivalences between insulators and superconductors. We would like to relax the extreme locality constraint and allow for changes in the range of the dual matrix. In other words, we would like to realize a more general Gaussian duality

$$H^D = \frac{\mathsf{i}}{2} \sum_{r,s=1}^{2L} h_{rs}^D\, \tilde{\gamma}_r\tilde{\gamma}_s + \frac{1}{2}\mathsf{Tr}\, K, \tag{13}$$

for some new Majorana operator $\tilde{\gamma}_r$ significantly different from the dual Majorana γ_r^D and the original one γ_r.

The argument above suggests setting up the relation

$$\mathcal{U}_\mathsf{d}\, \gamma_r\, \mathcal{U}_\mathsf{d}^\dagger = \sum_{s=1}^{2L} {O_\mathsf{d}}_r^s\, \tilde{\gamma}_s, \tag{14}$$

so that the matrix O_d may be computed explicitly as

$${O_\mathsf{d}}_r^s = \frac{1}{2^L}\mathsf{tr}(\tilde{\gamma}_s\, \mathcal{U}_\mathsf{d}\, \gamma_r\, \mathcal{U}_\mathsf{d}^\dagger). \tag{15}$$

Therefore, the duality map $\mathcal{U}_\mathsf{d}$ is Gaussian if, and only if, the matrix O_d is invertible, in which case it is also orthogonal. In the absence of a more educated choice, one may always set $\tilde{\gamma}_s = \gamma_s$ in Eq. (14).

The association $\mathcal{U}_\mathsf{d} \mapsto O_\mathsf{d}$ shows that Gaussian dualities, though many-body in nature, also induce *a posteriori* a duality of the single-particle Hamiltonian h by the relation

$$h_{rs}^D = \sum_{r',s'=1}^{2L} h_{r's'}\, {O_\mathsf{d}}_{r'}^r {O_\mathsf{d}}_{s'}^s. \tag{16}$$

How much of the locality of the original system one preserves in the dual model will depend on the range of the matrix O_d.

Finally, let us contrast Gaussian dualities to other type of dualities that do not preserve the quadratic fermionic nature of the original theory. Consider the non-interacting Hamiltonian

$$H = \mathsf{i}\sum_{j=1}^{L-1}[t_x\, \gamma_{2j}\gamma_{2j+1} - t_y\, \gamma_{2j-1}\gamma_{2j+2}], \tag{17}$$

which can be also described as a spin-1/2 Hamiltonian,

$$H = -\sum_{j=1}^{L-1}[t_x\, \sigma_j^x\sigma_{j+1}^x + t_y\, \sigma_j^y\sigma_{j+1}^y], \tag{18}$$

after the Jordan-Wigner map of Majorana operators

$$\gamma_{2j-1} = \sigma_j^x \prod_{l=1}^{j-1}\sigma_l^z, \quad \gamma_{2j} = \sigma_j^y \prod_{l=1}^{j-1}\sigma_l^z, \tag{19}$$

in terms of Pauli matrices σ_j^ν, $\nu = x, y, z$. A simple local rotation around the spin y axis,

$$\sigma_j^x \mapsto \sigma_j^z, \quad \sigma_j^z \mapsto -\sigma_j^x \quad (j = 1, \cdots, L), \tag{20}$$

induces a non-trivial change in the dual fermionic Hamiltonian. Although local, it is an interacting Hamiltonian

$$H^D = \sum_{j=1}^{L-1}[t_x\, \gamma_{2j-1}\gamma_{2j}\gamma_{2j+1}\gamma_{2j+2} - \mathsf{i}t_y\, \gamma_{2j-1}\gamma_{2j+2}]. \tag{21}$$

Therefore, the matrix O_d of Eq. (14) should fail to be invertible, and one can check that this is indeed the case.

The dual Hamiltonian of Eq. (21) has an interesting physical interpretation. It describes the competition between a p-wave superconducting chain in its topological phase and a density-density interaction. In the limit in which t_y vanishes, its ground state is number conserving, a Mott insulating state, otherwise its ground state is superconducting. For $t_x = 0$, H^D has two exact zero-energy modes, γ_2 and γ_{2L-1}. The evolution of these modes with t_x may be computed exactly by exploiting the duality transformation connecting H^D to the free-fermion Hamiltonian H.

B. Decoupling transformations

The results of the previous section, especially the example at the end of the section, show that generic dualities do not preserve the non-interacting character of a

theory. Hence, to systematically establish equivalences of topological superconductors and insulators it is necessary to determine all possible Gaussian dualities. Recall that the key difficulty in searching for dualities is to identify unitary transformations that respect the locality structure of the Hamiltonian, meaning that Eq. (14) is not the full answer to our problem. We still need to address the issue of locality for the specific purpose of relating *topological insulators* to *topological superconductors*. The notion of locality necessitates a metric, i.e., a way of defining a distance between points in a manifold or, in the present case, a lattice with its associated notion of nearest neighbors (and next nearest, and so on).

Let us denote by $\mathbf{r}$ the sites of a lattice Λ, i.e., $\mathbf{r} \in \Lambda$, defined in arbitrary space dimensions. A generic (second-quantized) electron system where the number of electrons N is conserved is described by the Hamiltonian

$$H = -\sum_{\mathbf{r},\mathbf{r}',\sigma} \left[t_{\mathbf{r},\mathbf{r}'}\, c^\dagger_{\mathbf{r},\sigma} c_{\mathbf{r}',\sigma} + t_{\mathbf{r}',\mathbf{r}}\, c^\dagger_{\mathbf{r}',\sigma} c_{\mathbf{r},\sigma} \right], \quad (22)$$

where $c^\dagger_{\mathbf{r},\sigma}$ represents a canonical fermion creation operator at site $\mathbf{r}$ and spin $\sigma =\uparrow,\downarrow$. The hopping amplitude $t_{\mathbf{r},\mathbf{r}'}$ is related to $t_{\mathbf{r}',\mathbf{r}}$ by complex conjugation, $t_{\mathbf{r}',\mathbf{r}} = t^*_{\mathbf{r},\mathbf{r}'}$, and no spin-flip processes are included, i.e., terms such as $c^\dagger_{\mathbf{r},\uparrow} c_{\mathbf{r}',\downarrow}$.

Generically, the Hamiltonian above displays a broken time reversal symmetry unless the hopping amplitudes are purely real or purely imaginary. For purely imaginary amplitudes, an internal decoupling occurs in the system that splits the particle conserving Hamiltonian H into four identical, independent and decoupled superconductors. The proof of this assertion relies on rewriting the particle conserving Hamiltonian H above in terms of Majorana fermions $a_{\mathbf{r},\sigma}$ and $b_{\mathbf{r},\sigma}$, such that

$$a_{\mathbf{r},\sigma} = c_{\mathbf{r},\sigma} + c^\dagger_{\mathbf{r},\sigma}, \quad \mathsf{i} b_{\mathbf{r},\sigma} = c_{\mathbf{r},\sigma} - c^\dagger_{\mathbf{r},\sigma}, \quad (23)$$

with the result

$$H = -\sum_{\mathbf{r},\mathbf{r}',\sigma} \left[\left(\frac{t_{\mathbf{r},\mathbf{r}'} - t_{\mathbf{r}',\mathbf{r}}}{4} \right) (a_{\mathbf{r},\sigma} a_{\mathbf{r}',\sigma} + b_{\mathbf{r},\sigma} b_{\mathbf{r}',\sigma}) + \mathsf{i} \left(\frac{t_{\mathbf{r},\mathbf{r}'} + t_{\mathbf{r}',\mathbf{r}}}{4} \right) (a_{\mathbf{r},\sigma} b_{\mathbf{r}',\sigma} - b_{\mathbf{r},\sigma} a_{\mathbf{r}',\sigma}) \right]. \quad (24)$$

Hence, if $t_{\mathbf{r},\mathbf{r}'}$ is purely imaginary,

$$H = \sum_\sigma (\tilde{H}_{1,\sigma} + \tilde{H}_{2,\sigma}) \qquad (t^*_{\mathbf{r},\mathbf{r}'} = -t_{\mathbf{r},\mathbf{r}'}), \quad (25)$$

where the particle non-conserving Hamiltonians

$$\tilde{H}_{1,\sigma} = -\frac{1}{2} \sum_{\mathbf{r},\mathbf{r}'} t_{\mathbf{r},\mathbf{r}'}\, a_{\mathbf{r},\sigma} a_{\mathbf{r}',\sigma}, \quad (26)$$

$$\tilde{H}_{2,\sigma} = -\frac{1}{2} \sum_{\mathbf{r},\mathbf{r}'} t_{\mathbf{r},\mathbf{r}'}\, b_{\mathbf{r},\sigma} b_{\mathbf{r}',\sigma}, \quad (27)$$

are all independent,

$$[\tilde{H}_{1,\sigma}, \tilde{H}_{2,\sigma'}] = 0. \quad (28)$$

The four decoupled superconductors $\tilde{H}_{1,\uparrow}, \tilde{H}_{2,\downarrow}, \tilde{H}_{1,\uparrow}, \tilde{H}_{2,\downarrow}$ are isospectral: Any one of them can be mapped into any other one by a local unitary transformation. On one hand, $\tilde{H}_{1,\uparrow}$ ($\tilde{H}_{2,\uparrow}$) is mapped to $\tilde{H}_{1,\downarrow}$ ($\tilde{H}_{2,\downarrow}$) by a rotation in spin space. On the other hand, the unitary transformation $\mathcal{U}_\sigma = \prod_{\mathbf{r}} \left(\frac{\mathbb{1} + a_{\mathbf{r},\sigma} b_{\mathbf{r},\sigma}}{\sqrt{2}} \right)$ maps $\tilde{H}_{1,\sigma}$ to $\tilde{H}_{2,\sigma}$. Hence it is possible to unequivocally associate H with a new Hamiltonian H_{reduced}

$$H \mapsto H_{\mathsf{reduced}} = -\frac{1}{2} \sum_{\mathbf{r},\mathbf{r}'} t_{\mathbf{r},\mathbf{r}'} \gamma_{\mathbf{r}} \gamma_{\mathbf{r}'}, \quad (29)$$

of spinless Majorana fermions $\gamma_{\mathbf{r}}$ that includes only one fourth of the original number of fermionic degrees of freedom, and represents any of the four Hamiltonians obtained by decoupling H.

The equivalence of Eq. (29) states that the spectrum of H can be reconstructed from that of H_{reduced}. Let $|E, d_i\rangle$ denote an eigenstate of H_{reduced}, of energy E and degeneracy d_E labelled by $d_i = 1, \cdots, d_E$, and let

$$|E, d_1, d_2, d_3, d_4\rangle = |E, d_1\rangle |E, d_2\rangle |E, d_3\rangle |E, d_4\rangle \quad (30)$$

denote the product state associated to the four independent copies of H_{reduced}. Then, because of Eq. (25),

$$H|E, d_1, d_2, d_3, d_4\rangle = E|E, d_1, d_2, d_3, d_4\rangle \quad (31)$$

with degeneracy d_E^4. In particular, the zero-energy modes of the particle conserving Hamiltonian H are explained by the zero-energy modes of the spinless superconductor H_{reduced}.

It is possible to add non-diagonal spin terms and retain some level of decoupling. For example, spin-orbit terms of the Rashba, Dimmock, or Dresselhaus type are linear in momentum and hence purely imaginary. Thus they couple $\tilde{H}_{1,\uparrow}$ to $\tilde{H}_{1,\downarrow}$ (and $\tilde{H}_{2,\uparrow}$ to $\tilde{H}_{2,\downarrow}$), but they do not couple a to b Majoranas. In the presence of these types of spin terms, the decoupling transformation decomposes the particle conserving system into two rotationally-invariant superconductors with non-trivial spin dynamics.

A less general but more often useful version of the decoupling transformation exists for purely real hopping amplitudes on bipartite lattices $\Lambda = A \cup B$, with generic lattice sites $\mathbf{r} \in \Lambda$. Let us denote by $\mathbf{x} \in A$ and $\mathbf{y} \in B$ the sites of each sublattice. The generic bipartite Hamiltonian

$$H = -\sum_{\mathbf{x},\mathbf{y},\sigma} \left[t_{\mathbf{x},\mathbf{y}}\, c^\dagger_{\mathbf{x},\sigma} c_{\mathbf{y},\sigma} + t_{\mathbf{y},\mathbf{x}}\, c^\dagger_{\mathbf{y},\sigma} c_{\mathbf{x},\sigma} \right], \quad (32)$$

allows only for hopping from sublattice A to B or vice versa. If $t_{\mathbf{x},\mathbf{y}}$ is purely real-valued, that is, $t_{\mathbf{x},\mathbf{y}} = t_{\mathbf{y},\mathbf{x}}$, then

$$H = \sum_\sigma \left[H_{1,\sigma} - H_{2,\sigma} \right], \quad (33)$$

where the superconducting Hamiltonians

$$H_{1,\sigma} = -\frac{\mathsf{i}}{2}\sum_{\mathbf{x},\mathbf{y}} t_{\mathbf{x},\mathbf{y}}\, a_{\mathbf{x},\sigma} b_{\mathbf{y},\sigma}, \tag{34}$$

$$H_{2,\sigma} = -\frac{\mathsf{i}}{2}\sum_{\mathbf{x},\mathbf{y}} t_{\mathbf{x},\mathbf{y}}\, b_{\mathbf{x},\sigma} a_{\mathbf{y},\sigma}, \tag{35}$$

commute, $[H_{1,\sigma}, H_{2,\sigma'}] = 0$. The spectral equivalence of $H_{1,\sigma}$ and $H_{2,\sigma}$ is established by the unitary transformation

$$\mathcal{U}_\sigma = \prod_{\mathbf{x},\mathbf{y}} \left(\frac{\mathbb{1} + b_{\mathbf{x},\sigma} a_{\mathbf{x},\sigma}}{\sqrt{2}}\right)\left(\frac{\mathbb{1} + a_{\mathbf{y},\sigma} b_{\mathbf{y},\sigma}}{\sqrt{2}}\right), \tag{36}$$

that maps $H_{1,\sigma} \leftrightarrow H_{2,\sigma}$. Just as before it is possible to associate a reduced Hamiltonian to H.

An interesting corollary to Eqs. (33) and (36) is that the unitary transformation $\mathcal{C} = \mathcal{U}_\uparrow \mathcal{U}_\downarrow$ anticommutes with the Hamiltonian, $\mathcal{C}H = -H\mathcal{C}$, and so it defines a chiral symmetry: for each positive eigenvalue E_α there exists a negative $-E_\alpha$.

Consider, as an example, the chain of spinless fermions

$$H = -\sum_{j=1}^{2M}\Big[t_j\,(c_j^\dagger c_{j+1} + c_{j+1}^\dagger c_j) + \epsilon_j(n_j - 1/2)\Big], \tag{37}$$

with quenched disorder in the real-valued hopping amplitudes t_j and on-site atomic energy ϵ_j. The chain has $L = 2M$ lattice sites and periodic boundary conditions are assumed ($c_{L+1}^\dagger = c_1^\dagger$). Since the lattice is bipartite, H is the difference of two identical, independent superconductors, coupled by the on-site atomic energies. Let us associate a pair of Majorana fermions a_j, b_j to each site j, as in Eq. (23), and rewrite the Hamiltonian above in terms of Majorana degrees of freedom

$$\begin{aligned} H &= -\frac{\mathsf{i}}{2}\sum_{j=1}^{2M}[t_j\,(a_{j+1}b_j - b_{j+1}a_j) + \epsilon_j\, a_j b_j] \\ &= H_1 - H_2 + H_\epsilon, \end{aligned} \tag{38}$$

with commuting Hamiltonians

$$H_1 = -\frac{\mathsf{i}}{2}\sum_{j=1}^{M}\left[t_{2j-1}\, a_{2j} b_{2j-1} - t_{2j}\, b_{2j+1} a_{2j}\right],$$

$$H_2 = -\frac{\mathsf{i}}{2}\sum_{j=1}^{M}\left[t_{2j-1}\, b_{2j} a_{2j-1} - t_{2j}\, a_{2j+1} b_{2j}\right], \tag{39}$$

and

$$H_\epsilon = -(\mathsf{i}/2)\sum_{j=1}^{M}[\epsilon_{2j-1} a_{2j-1} b_{2j-1} + \epsilon_{2j} a_{2j} b_{2j}]. \tag{40}$$

The Hamiltonian H_1, or equivalently (isospectrally) H_2, obtained from decoupling the number conserving chain is precisely the Majorana chain of Kitaev (at vanishing chemical potential). A closely related observation was made in Ref. 27. Many of the Majorana lattice models investigated in the literature can be obtained as an H_{reduced} associated to a particle-conserving Hamiltonian. Consider for example the Hamiltonian

$$H_{\mathsf{honeycomb}} = -\sum_{\mathbf{x}\in A}[c_{\mathbf{x}}^\dagger(t_x c_{\mathbf{y}_x} + t_y c_{\mathbf{y}_y} + t_z c_{\mathbf{y}_z}) + \text{H.c.}] \tag{41}$$

on the honeycomb lattice with sublattices A and B as usual. The sites $\mathbf{y}_\nu \in B$ ($\nu = x, y, z$) are nearest neighbors to $\mathbf{x} \in A$ in the so called x, y, or z directions of the honeycomb lattice. Since this lattice is bipartite, our decoupling transformation applies for real-valued t_ν. The associated reduced Hamiltonian is precisely Kitaev's honeycomb model, projected onto its gauge invariant sector.[29] Other sectors are obtained by modulating the sign of the hopping amplitudes t_ν. From this perspective, it is interesting to notice that the number conserving $H_{\mathsf{honeycomb}}$ is precisely the simplest model of graphene[30] for $t_x = t_y = t_z$. Finally, let us notice in passing that decoupling on the square lattice obtains variations of the Majorana arrays of Ref. 19.

C. A class of Gaussian duality transformations

We are now ready to introduce a large class of Gaussian duality transformations. Consider the cases for which the sublattices A and B of previous section are equivalent, meaning that there is a shortest, typically non-unique translation $\boldsymbol{\delta}_1$ ($\boldsymbol{\delta}_2$) mapping sublattice A (B) to sublattice B (A). In set notation,

$$A + \boldsymbol{\delta}_1 = B, \quad B + \boldsymbol{\delta}_2 = A. \tag{42}$$

For the purpose of the duality transformation that we are about to introduce it is often convenient to choose $\boldsymbol{\delta}_1, \boldsymbol{\delta}_2$ to be as parallel and short as possible. This condition guarantees that the range of the hoppings in the dual Hamiltonian deviates as little as possible from that of the original Hamiltonian. A hypercubic lattice is simplest in that one may choose $\boldsymbol{\delta}_1 = \boldsymbol{\delta}_2$.

The mapping

$$\begin{aligned} a_{\mathbf{y},\sigma} &\to b_{\mathbf{y}+\boldsymbol{\delta}_2,\sigma}, & a_{\mathbf{x},\sigma} &\to a_{\mathbf{x},\sigma}, \\ b_{\mathbf{y},\sigma} &\to b_{\mathbf{y},\sigma}, & b_{\mathbf{x},\sigma} &\to -a_{\mathbf{x}+\boldsymbol{\delta}_1,\sigma}, \end{aligned} \tag{43}$$

induces a unitary transformation that leaves $H_{1,\sigma}$ unchanged and transforms $H_{2,\sigma}$ as

$$H_{2,\sigma} \to H_{2,\sigma}^D = -\frac{\mathsf{i}}{2}\sum_{\mathbf{x},\mathbf{y}} t_{\mathbf{y}-\boldsymbol{\delta}_1,\mathbf{x}-\boldsymbol{\delta}_2}\, b_{\mathbf{x},\sigma} a_{\mathbf{y},\sigma}. \tag{44}$$

In rearranging the sum over sites, we have assumed periodic boundary conditions or that the system is infinite. The dual superconducting Hamiltonian,

$$\begin{aligned} H^D &= \sum_\sigma \left[H_{1,\sigma} - H_{2,\sigma}^D\right] = \\ &\quad -\frac{\mathsf{i}}{2}\sum_{\mathbf{x},\mathbf{y},\sigma}\left[t_{\mathbf{x},\mathbf{y}}\, a_{\mathbf{x},\sigma} b_{\mathbf{y},\sigma} - t_{\mathbf{y}-\boldsymbol{\delta}_1,\mathbf{x}-\boldsymbol{\delta}_2}\, b_{\mathbf{x},\sigma} a_{\mathbf{y},\sigma}\right], \end{aligned} \tag{45}$$

can be rewritten in terms of creation and annihilation operators,

$$H^D = -\sum_{\mathbf{x},\mathbf{y},\sigma}\Big[t^{\mathsf{av}}_{\mathbf{x},\mathbf{y}}(c^\dagger_{\mathbf{x},\sigma}c_{\mathbf{y},\sigma}+c^\dagger_{\mathbf{y},\sigma}c_{\mathbf{x},\sigma})+ \Delta_{\mathbf{x},\mathbf{y}}(c^\dagger_{\mathbf{y},\sigma}c^\dagger_{\mathbf{x},\sigma}+c_{\mathbf{x},\sigma}c_{\mathbf{y},\sigma})\Big], \quad (46)$$

where

$$t^{\mathsf{av}}_{\mathbf{x},\mathbf{y}}=\frac{t_{\mathbf{x},\mathbf{y}}+t_{\mathbf{y}-\boldsymbol{\delta}_1,\mathbf{x}-\boldsymbol{\delta}_2}}{2}, \quad \Delta_{\mathbf{x},\mathbf{y}}=\frac{t_{\mathbf{x},\mathbf{y}}-t_{\mathbf{y}-\boldsymbol{\delta}_1,\mathbf{x}-\boldsymbol{\delta}_2}}{2}.$$

Even though $H+H_\epsilon$ is roughly as general as possible for a band electronic system of independent fermions, $H^D+H^D_\epsilon$ remains a superconductor at vanishing chemical potential μ. It is possible to include spin terms in H and still obtain a dual superconductor featuring only local interactions. Just as the duality breaks particle conservation in general, we expect it to modify rotational properties since it has a highly non-trivial action on the operators of total spin. Since, and when, spin does not play any decisive role in the studied physical phenomenon we will drop it from the discussion in order to avoid confusing notation and obscure explanations.

It is now straightforward to apply the general duality transformation of bipartite models to the disordered chain of the previous section. For this one-dimensional system, the mapping defined in Eqs. (43) reduces to

$$\begin{aligned} a_{2j-1}&\to b_{2j}, & a_{2j}&\to a_{2j},\\ b_{2j-1}&\to b_{2j-1}, & b_{2j}&\to -a_{2j+1}, \end{aligned} \quad (47)$$

always identifying the index $L+1$ with 1, and 0 with $L-1$. Thus, while $H_1=H_1^D$ remains invariant, H_2 transforms as

$$H_2^D=-\frac{\mathsf{i}}{2}\sum_{j=1}^{M}\left[t_{2j-2}\,b_{2j}a_{2j-1}-t_{2j-1}\,a_{2j+1}b_{2j}\right] \quad (48)$$

(the case $M=1$ is special in that the duality map keeps H_2^D also invariant). The on-site atomic energy term transforms like

$$H^D_\epsilon=-\frac{\mathsf{i}}{2}\sum_{j=1}^{M}[\epsilon_{2j-1}b_{2j}b_{2j-1}-\epsilon_{2j}a_{2j}a_{2j+1}] \quad (49)$$

$$=\frac{\mathsf{i}}{2}\sum_{j=1}^{2M}\Big[\epsilon_j(c^\dagger_jc_{j+1}-c^\dagger_{j+1}c_j)+(-1)^j\epsilon_j(c^\dagger_jc^\dagger_{j+1}-c_{j+1}c_j)\Big].$$

Combining all of these results, we obtain the dual superconductor $H^D=H_1-H_2^D+H_\epsilon^D$

$$H^D=-\sum_{j=1}^{2M}\Big[\left(\frac{t_j+t_{j-1}-\mathsf{i}\epsilon_j}{2}\right)c^\dagger_jc_{j+1}+ (-1)^j\left(\frac{t_{j-1}-t_j-\mathsf{i}\epsilon_j}{2}\right)c^\dagger_jc^\dagger_{j+1}+\text{H.c.}\Big]. \quad (50)$$

1. *Symmetry transmutation: particle number and fermionic parity*

The duality transformation Eqs. (43), breaks particle conservation in general because the particle number (charge) operator

$$\hat{N}=\sum_{\mathbf{r},\sigma}[n_{\mathbf{r},\sigma}-1/2]=\frac{\mathsf{i}}{2}\sum_{\mathbf{r},\sigma}a_{\mathbf{r},\sigma}b_{\mathbf{r},\sigma}. \quad (51)$$

associated to, and a symmetry of, H is drastically modified by the duality. Since

$$\begin{aligned} a_{\mathbf{x},\sigma}b_{\mathbf{x},\sigma}&\to -a_{\mathbf{x},\sigma}a_{\mathbf{x}+\boldsymbol{\delta}_1,\sigma},\\ a_{\mathbf{y},\sigma}b_{\mathbf{y},\sigma}&\to -b_{\mathbf{y},\sigma}b_{\mathbf{y}+\boldsymbol{\delta}_2,\sigma}, \end{aligned} \quad (52)$$

the duality transformation maps $\hat{N}$ to a symmetry $\hat{N}^D$ of H^D that does not have the interpretation of a charge operator,

$$\hat{N}^D=-\frac{\mathsf{i}}{2}\sum_\sigma\left[\sum_{\mathbf{x}}a_{\mathbf{x},\sigma}a_{\mathbf{x}+\boldsymbol{\delta}_1,\sigma}+\sum_{\mathbf{y}}b_{\mathbf{y},\sigma}b_{\mathbf{y}+\boldsymbol{\delta}_2,\sigma}\right], \quad (53)$$

while it is still true that $[\hat{N}^D,H^D]=0$.

There is, however, a quantum number important from the point of view of superconductivity that is *almost* preserved by duality: fermionic parity. The operator of fermionic parity

$$(-1)^F=e^{\mathsf{i}\pi\sum_{\mathbf{r},\sigma}n_{\mathbf{r},\sigma}}=\prod_{\mathbf{r},\sigma}(-\mathsf{i}a_{\mathbf{r},\sigma}b_{\mathbf{r},\sigma}) \quad (54)$$

measures the parity of the total number of fermions. The BCS mean field approximation breaks the symmetry of particle conservation down to conservation of fermionic parity. The duality transformation maps

$$(-1)^F\to \prod_{\mathbf{x},\sigma}(\mathsf{i}a_{\mathbf{x},\sigma}a_{\mathbf{x}+\boldsymbol{\delta}_1,\sigma})\prod_{\mathbf{y},\sigma}(\mathsf{i}b_{\mathbf{y},\sigma}b_{\mathbf{y}+\boldsymbol{\delta}_2,\sigma})=(-1)^{\pi_\sigma}(-1)^F, \quad (55)$$

where $(-1)^{\pi_\sigma}$ is the sign accumulated after permutations of the Majorana fermions to establish the original order $(-1)^F$.

Incidentally, Eq. (52) shows that the duality $H\to H^D$ is more general in scope than the decoupling transformation that motivated it. For example, adding an on-site energy term

$$H_\epsilon=-\sum_{\mathbf{r},\sigma}\epsilon_{\mathbf{r}}(n_{\mathbf{r},\sigma}-1/2) \quad (56)$$

($\mathbf{r}\in A\cup B$) to H, couples the reduced superconductors $H_{1,\sigma}$ and $H_{2,\sigma}$. It transforms as

$$H^D_\epsilon=\frac{\mathsf{i}}{2}\sum_{\mathbf{x},\sigma}\epsilon_{\mathbf{x}}a_{\mathbf{x},\sigma}a_{\mathbf{x}+\boldsymbol{\delta}_1,\sigma}+\frac{\mathsf{i}}{2}\sum_{\mathbf{y},\sigma}\epsilon_{\mathbf{y}}b_{\mathbf{y},\sigma}b_{\mathbf{y}+\boldsymbol{\delta}_2,\sigma}. \quad (57)$$

Hence the effect of the on-site atomic energy term $\epsilon_{\mathbf{r}}$ is to renormalize (by a purely imaginary amount) the hopping and pairing amplitudes of the dual superconductor in the directions $\boldsymbol{\delta}_1$, and $\boldsymbol{\delta}_2$.

2. Translation Symmetry

For the Gaussian dualities of Eq. (43), the non-conservation of particle number for the dual partner H^D is precisely related to (partial) breaking of translation symmetry for H, since the pairing potential $\Delta_{\mathbf{x},\mathbf{y}}$ vanishes if $t_{\mathbf{y}-\boldsymbol{\delta}_1,\mathbf{x}-\boldsymbol{\delta}_2} = t_{\mathbf{x},\mathbf{y}}$ (and $H^D = H$ in this case). What is less obvious is that the translation symmetry of H^D may be be higher than that of H, in which case we are enlarging one group of symmetries (translations) at the expense of breaking another symmetry, particle conservation. This is explained by the transmutation of the translation operation under duality.

Let us focus for simplicity on a closed chain of spinless fermions. The extension to more general settings is straightforward but notationally cumbersome. The map of Eq. (47) leads to

$$c_{2j-1} \mapsto c^D_{2j-1} = \frac{1}{2}(b_{2j} + \mathsf{i}b_{2j-1}),$$
$$c_{2j} \mapsto c^D_{2j} = \frac{1}{2}(a_{2j} - \mathsf{i}a_{2j+1}), \tag{58}$$

or, more explicitly,

$$c^D_{2j-1} = \frac{1}{2}(c_{2j-1} - \mathsf{i}c_{2j}) - \frac{1}{2}(c^\dagger_{2j-1} - \mathsf{i}c^\dagger_{2j}),$$
$$c^D_{2j} = \frac{1}{2}(c_{2j} - \mathsf{i}c_{2j+1}) + \frac{1}{2}(c^\dagger_{2j} - \mathsf{i}c^\dagger_{2j+1}). \tag{59}$$

These expressions show already transmutation of the translation operation, let us call it $\hat{T}$. On one hand,

$$\hat{T} c_j \hat{T}^\dagger = c_{j+1} \quad (j = j + L), \tag{60}$$

and consequently ($\hat{T}^D = \mathcal{U}_\mathsf{d}\,\hat{T}\,\mathcal{U}_\mathsf{d}^\dagger$),

$$\hat{T}^D c^D_j (\hat{T}^D)^\dagger = c^D_{j+1} \quad (j = j + L). \tag{61}$$

On the other hand,

$$\hat{T} c^D_j \hat{T}^\dagger \neq c^D_{j+1}, \tag{62}$$

and so

$$\hat{T} \neq \hat{T}^D. \tag{63}$$

This is the point to notice. If $\hat{T}$ happens to be a symmetry of H, then $\hat{T}^D$ is necessarily a symmetry of H^D. However, $\hat{T}^D$ cannot possibly have the interpretation of a translation by one site, since that physical interpretation continues to be attached to $\hat{T}$! (The action of $\hat{T}^D$ on the c_j may be computed by inverting Eqs. (59).) Notice, however, by the same reasoning that

$$\hat{T}^2 = e^{\mathsf{i}\alpha_\mathsf{d}} (\hat{T}^D)^2, \tag{64}$$

where the (possibly trivial) phase on the right-hand side is determined by the actual duality transformation.

The concrete significance of this result will become apparent in the next section when we investigate the equivalence of the dimerized Peierls chain and the Majorana chain of Kitaev. What happens in that case actually is the following. The dimerized Peirls chain commutes with $\hat{T}^2$ and a very non-evident symmetry $\mathcal{U}_\mathsf{d}^\dagger\,\hat{T}\,\mathcal{U}_\mathsf{d}$ (not to be confused with $\hat{T}^D = \mathcal{U}_\mathsf{d}\,\hat{T}\,\mathcal{U}_\mathsf{d}^\dagger$). As a consequence, its dual partner (the Majorana chain) commutes with $\hat{T}$. This illustrates how Gaussian dualities may increase translation symmetry at the expense of breaking (transmuting) other symmetries, e.g., particle conservation.

It is revealing to rewrite the Gaussian duality of Eq. (47) as a map of fermions in crystal momentum space. The even-odd structure of the duality mapping evinced by Eqs. (59) for example suggests that we should take a unit cell with two sites. To keep the notation simple, we will assume that $L = 2M$, with M odd. Then

$$c_{2j-1} = \sum_{l=-\frac{M-1}{2}}^{\frac{M-1}{2}} \frac{e^{-\mathsf{i}kj}}{\sqrt{M}}\,\hat{c}_{1,k}, \quad c_{2j} = \sum_{l=-\frac{M-1}{2}}^{\frac{M-1}{2}} \frac{e^{-\mathsf{i}kj}}{\sqrt{M}}\,\hat{c}_{0,k}, \tag{65}$$

where $k = 2\pi l/M$. Let us emphasize that we are not making any assumption about the symmetries of any particular Hamiltonian. We are just going to recast our duality transformation in a new light. With these definitions, the dual fermions in momentum space are

$$\hat{c}^D_{1,k} = \frac{1}{2}(\hat{c}_{1,k} - \mathsf{i}\hat{c}_{0,k}) - \frac{1}{2}(\hat{c}^\dagger_{1,-k} - \mathsf{i}\hat{c}^\dagger_{0,-k}), \tag{66}$$
$$\hat{c}^D_{0,k} = \frac{1}{2}(\hat{c}_{0,k} - \mathsf{i}e^{-\mathsf{i}k}\hat{c}_{1,k}) + \frac{1}{2}(\hat{c}^\dagger_{0,-k} - \mathsf{i}e^{-\mathsf{i}k}\hat{c}^\dagger_{1,-k}).$$

The key point is that the dual fermions of momentum k are combinations of the original fermions of momentum k and $-k$. It follows that the induced single-particle duality O_d is not block diagonal with respect to momentum.

3. Time-reversal Symmetry

The standard antiunitary operation $\mathcal{T}$ of motion reversal may be specified by its action on the creation and annihilation operators,

$$\mathcal{T}\; c_{j,\uparrow}\; \mathcal{T}^{-1} = c_{j,\downarrow}, \quad \mathcal{T}\; c^\dagger_{j,\uparrow}\; \mathcal{T}^{-1} = c^\dagger_{j,\downarrow},$$
$$\mathcal{T}\; c_{j,\downarrow}\; \mathcal{T}^{-1} = -c_{j,\uparrow}, \quad \mathcal{T}\; c^\dagger_{j,\downarrow}\; \mathcal{T}^{-1} = -c^\dagger_{j,\uparrow}. \tag{67}$$

Consequently, $\mathcal{T}^2$ is a unitary transformation that anticommutes with all the creation and annihilation operators. Fermionic parity, that in this paper is written in Eq. (54) as $(-1)^F$, is also a unitary map that anticommutes with all the creation and annihilation operators. It follows that the two operators must be equal up to a phase and, in particular,

$$\mathcal{T}^2 = e^{\mathsf{i}\theta}(-1)^{\hat{N}}. \tag{68}$$

The duality transformation Eq. (43) is spin diagonal. Hence the dual fermions

$$c^D_{2j-1,\sigma} = \frac{1}{2}(c_{2j-1,\sigma} - \mathsf{i}c_{2j,\sigma}) - \frac{1}{2}(c^\dagger_{2j-1,\sigma} - \mathsf{i}c^\dagger_{2j,\sigma}),$$
$$c^D_{2j,\sigma} = \frac{1}{2}(c_{2j,\sigma} - \mathsf{i}c_{2j+1,\sigma}) + \frac{1}{2}(c^\dagger_{2j,\sigma} - \mathsf{i}c^\dagger_{2j+1,\sigma}), \tag{69}$$

are just as before, except for the additional spin label. By construction, the dual antiunitary operation $\mathcal{T}^D = \mathcal{U}_{\rm d}\,\mathcal{T}\,\mathcal{U}_{\rm d}^\dagger$, with

$$(\mathcal{T}^D)^2 = e^{{\rm i}\theta}(-1)^{\hat{N}^D}, \tag{70}$$

acts as standard time-reversal on the dual fermions (for a discussion of $\hat{N}^D$, see Section II C 1). One may check that

$$\mathcal{T} \neq \mathcal{T}^D, \tag{71}$$

and so there is transmutation of time-reversal symmetry. To put this result in perspective, suppose that both H and H^D commute with $\mathcal{T}$. As we will see, this is the case for example in polyacetylene and its dual superconducting partner (class DIII). Then, since $[H^D, \mathcal{T}^D] = 0$, we have uncovered a (unitary) symmetry $\mathcal{T}^D\mathcal{T}$ of H^D.

An example of transmutation of time-reversal for canonical bosons (phonons) can be found in Ref. 12, page 730.

III. EQUIVALENCES OF TOPOLOGICAL INSULATORS AND SUPERCONDUCTORS

Can Gaussian dualities in general, and in particular, the ones of this paper, establish equivalences between topological insulators and topological superconductors? The discussion of the previous section shows that there is no obstruction for this to be the case: many-body dualities can jump across entries in single-particle classification schemes simply by transmuting key symmetries. But symmetry transmutation is a necessary, not a sufficient condition. For example, the duality of Eq. (47) maps the clearly trivial insulator,

$$H = -\epsilon \sum_{j=1}^{2M} (n_j - 1/2) \tag{72}$$

to the equally trivial superconductor,

$$H^D = \frac{{\rm i}\epsilon}{2} \sum_{j=1}^{2M} \Big[(c_j^\dagger c_{j+1} - c_{j+1}^\dagger c_j) + (-1)^j (c_j^\dagger c_{j+1}^\dagger - c_{j+1} c_j) \Big], \tag{73}$$

in spite of the symmetry rearrangements it causes.

For the Gaussian dualities of the previous section in particular, it is not hard to convince oneself that it must be the case that they map topologically (non)trivial systems to equally (non)trivial dual partners. In this section we will study some paradigmatic examples of non-trivial partners. In one dimension, we find that the dimerized Peirls chain and the Majorana chain of Kitaev are dual partners, and we also investigate the mapping of topological defects under duality. The more general $m-$merized Peierls chain is investigated in the Appendix A. In two dimensions, we study a topological insulator based on a Kekulé-like pattern of hopping matrix elements that includes graphene as a special case. Its dual partner is a p-wave superconductor. In the limit where the insulator becomes graphene (a semi-metal), the dual superconductor reduces to a stack of Kitaev chains interconnected by pure kinetic hopping in the direction perpendicular to the chains. This superconducting realization of Dirac cones seems to be new in the literature. Appendix B shows an equivalence of a trivial BCS superconductor to a trivial insulator in any number of space dimensions.

A. The Peierls chain is dual to the Kitaev chain

In one dimension, the dimerized Peierls chain at half-filling, proposed by Su, Schrieffer and Heeger (SSH) to model polyacetylene,[16] is the prototype of a topologically non-trivial band insulator, while the Kitaev chain is the prototype of a topologically non-trivial superconductor. In spite of their physical differences, the Kitaev and Peierls chains are isospectral *in second quantization*, that is, as many-fermion systems. The reason is that there exists a Gaussian duality connecting both models. As we will see below, the mapping is different in detail for periodic and open boundary conditions. The Gaussian duality for open boundary conditions is crucial to understand the way boundary excitations are related, i.e., how the Majorana charge-neutral zero-energy edge modes of the Kitaev chain map into charged (canonical fermion) zero-modes of the Peierls chain.

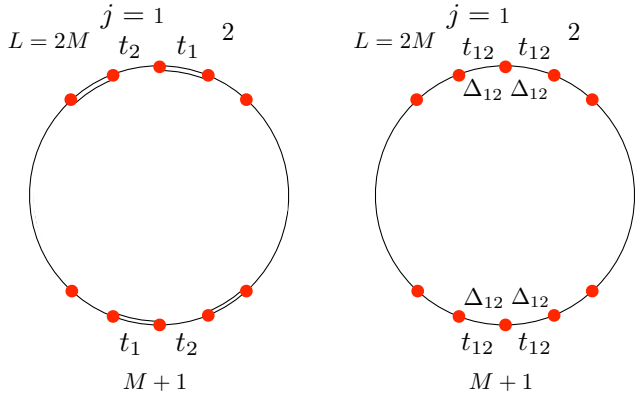

FIG. 1. Peierls chain with periodic boundary conditions on the left and its dual Kitaev chain superconductor on the right. Here, $L = 2M$.

For simplicity, and pedagogical reasons, in the following we will consider the spinless case. (The original model for polyacetylene[16] involves spin-1/2 electrons and this fact is relevant for topological classification purposes but it is not from the standpoint of Gaussian dualities that preserve spin, such as the ones defined in this paper.) Since the unit cell of the dimerized Peierls chain consists of two sites, the length (number of sites) L of the closed chain must be even, $L = 2M$. There are only two in-

dependent, periodically repeated hopping terms t_1 and t_2, see Fig. 1. It follows that the Peierls chain is a special case of the generic one-dimensional Hamiltonian of Eq. (37), with

$$t_{2j-1} = t_1, \quad t_{2j} = t_2, \quad \epsilon_j = 0, \; j = 1, \cdots, L. \quad (74)$$

The dual of the Peirls chain,

$$H^D = -\frac{\mathrm{i}}{2} \sum_{j=1}^{L} \Big[t_1 \, a_{j+1} b_j - t_2 \, b_{j+1} a_j \Big] \quad (75)$$
$$= -\sum_{j=1}^{L} \Big[t_{12} \, c_j^\dagger c_{j+1} + \Delta_{12} \, c_j^\dagger c_{j+1}^\dagger + \text{H.c.} \Big],$$

is obtained by the same specialization of Eq. (50), with

$$t_{12} = \frac{t_1 + t_2}{2}, \quad \Delta_{12} = \frac{t_1 - t_2}{2}. \quad (76)$$

The Hamiltonian H^D is precisely the Majorana chain of Kitaev, at vanishing chemical potential μ. The interplay of symmetries is noteworthy. While the dimerized Peierls chain shows reduced translation symmetry and conservation of particle number, its dual the Kitaev chain has full translation symmetry and particle-number conservation is broken.

Next, we would like to comment on polyacetylene (class AII), that is the case with real electrons (spinfull fermions). In this case, we obtain two copies of a Peierls chain, one for each spin component. Time-reversal symmetry is preserved and our Gaussian duality maps the SSH model to two copies of Kitaev's chain (class DIII), one for each spin component.

In Appendix A we analyze the superconducting equivalent of an m-merized Peierls chain with $m \geq 3$. There is a clear distinction between distortions with m even and odd. In the latter case, the periodicity of the dual superconductor is doubled, and there are no zero-energy modes. For $m \geq 4$ even, the periodicities of the insulator and its dual superconductor are the same, in contrast to the dimerized case discussed above. It is well-known that for m even there are zero-energy modes.

1. *Mapping of topological invariants*

A main goal of any topological classification of matter is to divide all possible quantum states of matter into equivalence classes. There is some degree of arbitrariness in the criteria used to define those classes. Once the criteria is established, e.g., by symmetry/dimension and reduced K-homology of bundles,[7] two states in the same class are connected through a continuous map whose inverse is also continuous, i.e., a homeomorphism. To establish a characterization of the class one uses topological invariants, i.e., quantities that are preserved under the homeomorphism. That a particle-conserving system such as a Peierls insulator may be dual to a superconductor raises some conceptual issues for the topological classification of systems of free fermions. What is the relation between the topological invariants characterizing these different but dual states of matter? The fermionic parity of the ground state of a superconductor such as Kitaev's, with $L = 2M$ even, constitutes a good quantum number whose value depends on the boundary conditions. For periodic boundary conditions, the topologically nontrivial ground state of the Kitaev chain is non-degenerate and fermionic parity is odd, while it is even in the trivial phase.[26] These facts hold independently of $M = L/2$ (the relevance of this remark will become clear below).

Fermionic parity may be computed in several equivalent ways. The topological character of this quantum number is revealed by its connection to the Majorana number,[17]

$$\mathcal{M} = \text{sgn}(\mu + 2t_{12}) \; \text{sgn}(\mu - 2t_{12}), \quad (77)$$

defined as the product of signs of Pfaffians of an antisymmetric matrix at momenta 0 and -π. This quantity identifies the topologically non-trivial phase as the one with $\mathcal{M} = -1$. On the other hand, we have seen that the dimerized Peierls chain maps into a Kitaev's chain at $\mu = 0$. The Peierls chain is always in a topologically nontrivial insulating phase, as long as $t_1 \neq t_2$. However, the fermion parity of the Peierls' insulating non-degenerate many-body ground state is given by $(-1)^M$, and therefore it is defined by the parity of M, i.e., it can be odd or even depending on M. It is instructive to express the Peierls' chain in terms of Majorana fermions and compute the Majorana number, now with a doubled unit cell, to realize that indeed $\mathcal{M} = (-1)^M$. The point is that fermion parity *is not* the good topological quantum number to characterize the Peierls' insulating phase despite the fact that it is exactly dual to Kitaev's chain model.

From the point of view of the many-body duality transformation, the mismatch is explained by the (very mild) transmutation of fermionic parity, Eq. (55).

2. *Mapping of topological defects and boundary modes*

Consider for simplicity a dimerized, spinless, chain with periodic boundary conditions, $L = 2M$ with $M \in$ odd, and two defects symmetrically located at positions $j = 1$ and $j = M$. This corresponds to the Hamiltonian

$$H = -\sum_{j=2}^{M-1} t_{j-1(\text{mod}\,2)} (c_j^\dagger c_{j+1} + c_{j+1}^\dagger c_j) \quad (78)$$
$$-\sum_{j=M+2}^{L-1} t_{j-(M+1)(\text{mod}\,2)} (c_j^\dagger c_{j+1} + c_{j+1}^\dagger c_j)$$
$$-t_2 (c_1^\dagger c_2 + c_L^\dagger c_1 + c_M^\dagger c_{M+1} + c_{M+1}^\dagger c_{M+2} + \text{H.c.}),$$

where the first two terms represent two identical dimerized Peierls chains each of length $M - 1$, and the last term represents the pair of defects, see Fig. 2.

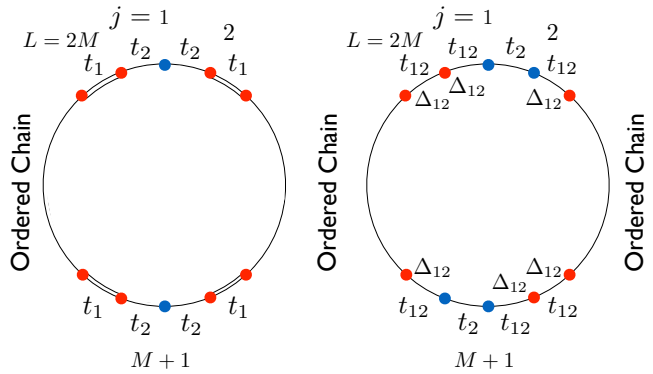

FIG. 2. Peierls chain (periodic boundary conditions) with a couple of defects on the left and its dual Josephson junctions of Kitaev chain superconductors on the right. Here, $L = 2M$.

Applied to the Hamiltonian (78), the duality transformation of the periodic chain, Eq. (47), produces two dual superconductors coupled by two particle-conserving segments,

$$H^D = -\sum_{j=2}^{M-1} (t_{12} c_j^\dagger c_{j+1} + \Delta_{12} c_j^\dagger c_{j+1}^\dagger + \text{H.c.}) \tag{79}$$

$$-\sum_{j=M+2}^{L-1} (t_{12} c_j^\dagger c_{j+1} - \Delta_{12} c_j^\dagger c_{j+1}^\dagger + \text{H.c.})$$

$$-t_2 (c_1^\dagger c_2 + c_{M+1}^\dagger c_{M+2} + \text{H.c.}). \tag{80}$$

Notice the change in sign of the superconducting order parameter across the links. Because of this phase difference, a Majorana zero mode is trapped at each weak link.

For an infinite chain with a single defect located at the origin, one may use the same duality map, Eq. (47) in order to obtain a dual Majorana zero-mode localized at the origin. The defect that famously traps fractional charge $\pm e/2$, per spin direction, in the Peierls chain is dual to a defect that traps a Majorana zero-mode!

For open boundary conditions, it is necessary to take the number of sites $L = 2M + 1$ to be odd, see Fig. 3. The duality transformation

$$\begin{aligned} a_{2j-1} &\to a_{L-2(j-1)} \ , \ a_{2j} \to a_{2j}, \\ b_{2j-1} &\to b_{2j-1} \quad , \ b_{2j} \to b_{L+1-2j}, \end{aligned} \tag{81}$$

leaves $H_1 = H_1^D$ invariant, and transforms H_2 as follows

$$H_2^D = -\frac{\mathrm{i}}{2} \sum_{j=1}^{M} (t_{2j}\, a_{L-2j} b_{L+1-2j} - t_{2j-1}\, b_{L+1-2j} a_{L-2(j-1)})$$

$$= -\frac{\mathrm{i}}{2} \sum_{j=1}^{M} (t_{L+1-2j}\, a_{2j-1} b_{2j} - t_{L-2j}\, b_{2j} a_{2j+1}). \tag{82}$$

In the dimerized case there are only two different alternating hopping terms which satisfy

$$t_{2j-1} = t_{L-2j} = t_1 \ , \ t_{2j} = t_{L+1-2j} = t_2, \tag{83}$$

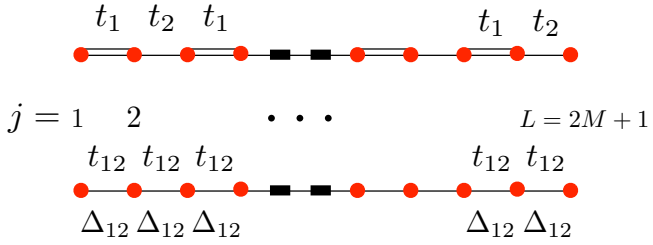

FIG. 3. Peierls chain with open boundary conditions on the top and its dual Kitaev chain superconductor on the bottom. Here, $L = 2M + 1$.

with the end result that the dual total Hamiltonian represents a spinless superconductor

$$H^D = -\frac{\mathrm{i}}{2} \sum_{j=1}^{L-1} (t_1\, a_{j+1} b_j - t_2\, b_{j+1} a_j) \tag{84}$$

$$= -\sum_{j=1}^{L-1} \left(t_{12}\, c_j^\dagger c_{j+1} + \Delta_{12}\, c_j^\dagger c_{j+1}^\dagger + \text{H.c.} \right),$$

that is again the Kitaev chain Hamiltonian at vanishing, $\mu = 0$, chemical potential.

The Kitaev chain of this section (open boundary conditions, odd length, and vanishing chemical potential) has two *exact* zero-energy modes, one per boundary point. The chain is reflection symmetric with respect to the central site $j = M + 1$, and the many-body ground state is two-fold degenerate. The zero-energy mode associated to the left boundary can be computed from the set of commutators

$$\begin{aligned} [-\mathrm{i} H^D, a_1] &= t_2 b_2, \\ [-\mathrm{i} H^D, a_3] &= t_1 b_2 + t_2 b_4, \\ &\vdots \\ [-\mathrm{i} H^D, a_{L-2}] &= t_1 b_{L-3} + t_2 b_{L-1}, \\ [-\mathrm{i} H^D, a_L] &= t_1 b_{L-1}. \end{aligned} \tag{85}$$

Let $\eta = t_2/t_1$, and assume without loss of generality $|\eta| < 1$. From these commutators it is possible to show that the combination

$$\gamma^D_{\text{left}} = \frac{1}{\mathcal{N}} \sum_{j=0}^{M} (-\eta)^j \ a_{2j+1} \tag{86}$$

of a_j fermions at odd sites j commutes with the Hamiltonian H^D. It constitutes an *exact symmetry* for any finite M. The normalization factor is

$$\mathcal{N}(\eta) = \sqrt{\frac{1 - \eta^{2(M+1)}}{1 - \eta^2}}. \tag{87}$$

A similar calculation establishes the right symmetry

$$\gamma^D_{\text{right}} = \frac{1}{\mathcal{N}} \sum_{j=0}^{M} (-\eta)^j \ b_{L-2j}. \tag{88}$$

These exact Majorana zero-energy modes are exponentially localized. If $|\eta| > 1$, the corresponding localized symmetries are obtained by rescaling $\gamma_\alpha^D \to (-1/\eta)^M \gamma_\alpha^D$, and changing the normalization factor to $\mathcal{N}(1/\eta)$. At $|\eta| = 1$ the mass gap vanishes in the thermodynamic limit $L \to \infty$.

Notice the fundamental difference between chains of even or odd lengths L. While it is possible to determine *exact* zero modes when L is odd, this is not the case for L even where the Majorana character of the edge modes is only *asymptotically exact* in the thermodynamic limit. The reason is simple. For *any* finite L, H^D commutes with the global symmetries

$$U_z = \prod_{j=1}^{L}(1-2n_j)\ ,\ U_x = \prod_{j=1}^{L} \mathrm{i}(b_j^\dagger + b_j), \qquad (89)$$

where $b_j^\dagger = c_j^\dagger \prod_{l=1}^{j-1}(1-2n_l)$ is a hard-core boson. However, it is *only when L is odd* that $\{U_z, U_x\} = 0$. In turn, this implies that the whole many-body spectrum of H^D is, at least, exactly two-fold degenerate. This symmetry analysis applies just as well to the more general Hamiltonian of Eq. (37) with open boundary conditions, provided the on-site potential ϵ_j vanishes.

The duality transformation of Eq. (81) maps the boundary Majorana zero-modes of the Kitaev chain into corresponding boundary symmetries of the Peierls chain,

$$\gamma_{\mathsf{right},1} = \frac{1}{\mathcal{N}} \sum_{j=0}^{M} (-\eta)^j\ a_{L-2j},$$
$$\gamma_{\mathsf{right},2} = \frac{1}{\mathcal{N}} \sum_{j=0}^{M} (-\eta)^j\ b_{L-2j}. \qquad (90)$$

Unlike for the Kitaev chain, these two zero modes reside on one and the same edge. Hence, it is natural to recombine them into one exponentially localized fermionic mode,

$$c_{\mathsf{right}}^\dagger = \frac{\gamma_{\mathsf{right},1} - \mathrm{i}\gamma_{\mathsf{right},2}}{2}. \qquad (91)$$

It is not surprising that both boundary symmetries of the Peierls chain appear on one boundary point, and not both as in the Kitaev case. The reason is the lack of reflection symmetry about $j = M+1$ in the mapped Peierls chain. Nonetheless, its many-body ground state is two-fold degenerate, just as its dual Kitaev superconductor, indicating that the $\mathbb{Z}_2$ symmetry of fermionic parity is odd in the thermodynamic limit also for the Peierls chain. This statement is of course confirmed by the exact solution of the Peierls Hamiltonian.

3. Density-density interactions

As mentioned above Gaussian dualities can be used to establish equivalences in interacting many-body systems. Here, we illustrate this fact in another paradigmatic example. Consider the case of a dimerized Peierls chain (of length $L = 2M$) at half-filling where electrons interact through a density-density Coulomb repulsion V

$$H = \sum_{j=1}^{2M} \Big[-t_j\,(c_j^\dagger c_{j+1} + c_{j+1}^\dagger c_j) + \frac{V}{4}(1-2n_j)(1-2n_{j+1})\Big]. \qquad (92)$$

The duality transformation of Eq. (47) maps the density operators as follows

$$n_{2j-1} \to \frac{1}{2}(1 - \mathrm{i}b_{2j-1}b_{2j}),$$
$$n_{2j} \to \frac{1}{2}(1 - \mathrm{i}a_{2j}a_{2j+1}), \qquad (93)$$

and the resulting dual superconducting equivalent is given by

$$H^D = -\sum_{j=1}^{L}\Big[t_{12}\,c_j^\dagger c_{j+1} + \Delta_{12}\,c_j^\dagger c_{j+1}^\dagger + \mathrm{H.c.}\Big] + \frac{V}{4}\sum_{j=1}^{L} e^{\mathrm{i}\pi n_{j+1}}\Big[-c_j^\dagger c_{j+2} + e^{\mathrm{i}\pi j}\,c_j^\dagger c_{j+2}^\dagger + \mathrm{H.c.}\Big], \qquad (94)$$

which clearly shows the competition and interplay between the band and Mott gaps.

A natural question that emerges is how robust is the Peierls phase to the presence of Coulomb interactions? The second, related, question is can interactions alone generate a topological Mott phase in the case where the non-interacting phase is metallic, i.e., $\Delta_{12} = 0$? Since in the latter case the model is exactly (Bethe ansatz) solvable, we know that for sufficiently large repulsion V, there exists a Mott phase.

In Appendix C we analyze the phenomenon of symmetry transmutation in interacting boson systems.

B. Graphene is dual to a "weak" Topological Superconductor

In this section we establish non-trivial equivalences in two space dimensions. For conciseness, we investigate a topological insulator on a square lattice, characterized by a Kekulé-like pattern of hopping parameters t_1, t_1', t_2, t_2' and an on-site potential $\epsilon_{\mathbf{x}}, \epsilon_{\mathbf{y}}$ that is constant on each sublattice, see Figure 4. In this way we manage to address several interesting models in a unified fashion. The semimetal graphene for example is realized on the line $t_1 = t_1' = t_2'$, $t_2 = 0 = \epsilon_{\mathbf{x}}$. (A few other lines obtain graphene as well. Notice that the honeycomb lattice is represented as a brick wall lattice in Figure 4). As a consequence,[31] our model realizes a condensed matter analog of the (2+1)-dimensional parity anomaly,[32] and our duality transformation provides a superconducting dual representation of this phenomenon. As was the

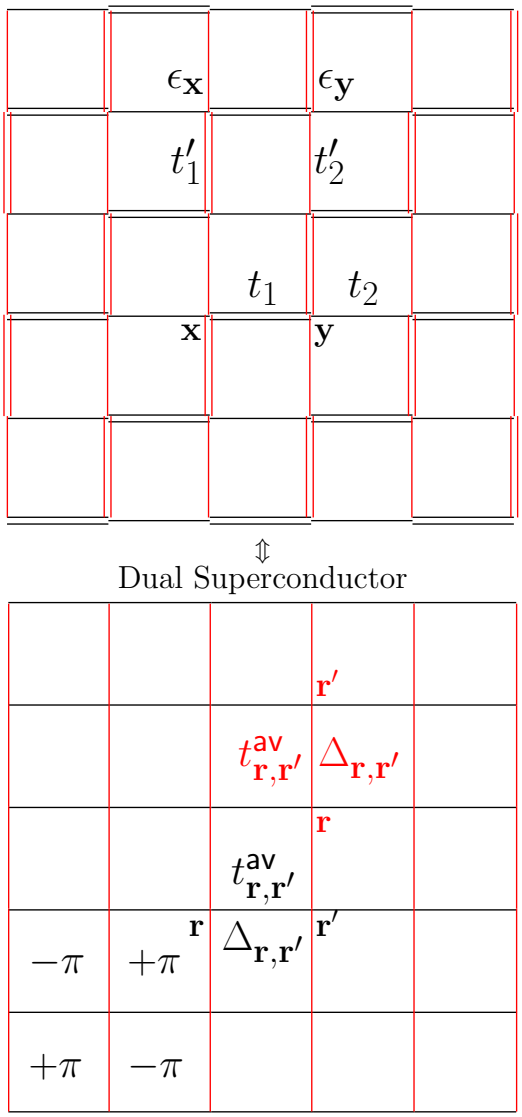

FIG. 4. (Top) Topological insulator characterized by a particular Kekulé-type pattern of hopping matrix elements as depicted in the figure. Double bonds along the horizontal (vertical) direction represent the hopping matrix element t_1 (t_1'), while single bonds along the horizontal (vertical) direction represent t_2 (t_2'). The $\epsilon_{\mathbf{x}(\mathbf{y})}$ are on-site atomic energies, constant on each sublattice. (Bottom) Dual chiral topological superconductor with π fluxes per square plaquette distributed antiferromagnetically, and a $p_x + \mathrm{i}p_y$ superconducting order parameter. The dual relation between parameters is explained in the main text.

case in one dimension, the Gaussian duality mapping the Kekulé-type insulating model to a topological p-wave superconductor has the key property of preserving the locality of the edge mode excitations.

The two-dimensional underlying lattice Λ considered in the following is bipartite with lattice points $\mathbf{x} = (x_1, x_2)$ and $\mathbf{y} = (y_1, y_2)$, such that $x_1 + x_2 \in$ even and $y_1 + y_2 \in$ odd integers. The total number of lattice points along the horizontal direction is L_x, and L_y along the vertical direction, such that $L_x \times L_y$ defines the size of the lattice and where, for simplicity, L_x and L_y represent even integers. Figure 4 (Top) is an example of a lattice Λ. Consider in particular the lattice shown in Fig. 4 (Top) where, for a given point $\mathbf{x} = (x_1, x_2)$, the corresponding hopping amplitudes of Hamiltonian (32) are given by:

$$t_{\mathbf{x},\mathbf{y}} = \begin{cases} t_1 \ , \ \text{for } \mathbf{y} = (x_1 + 1, x_2) \\ t_2 \ , \ \text{for } \mathbf{y} = (x_1 - 1, x_2) \\ t_1' \ , \ \text{for } \mathbf{y} = (x_1, x_2 - 1) \\ t_2' \ , \ \text{for } \mathbf{y} = (x_1, x_2 + 1) \end{cases} \tag{95}$$

and with on-site energies $\epsilon_{\mathbf{x}} = \epsilon = -\epsilon_{\mathbf{y}}$ (see Eq. (56)).

This model, endowed with periodic (toroidal) boundary conditions, has a single-particle energy spectrum (bulk bands) given by

$$\begin{aligned} E_{1,\mathbf{k}\sigma} &= -\sqrt{\epsilon_{\mathbf{x}}^2 + A_{\mathbf{k},+}^2 + B_{\mathbf{k},-}^2} \ , \ E_{4,\mathbf{k}\sigma} = -E_{1,\mathbf{k}\sigma}, \\ E_{2,\mathbf{k}\sigma} &= -\sqrt{\epsilon_{\mathbf{x}}^2 + A_{\mathbf{k},-}^2 + B_{\mathbf{k},+}^2} \ , \ E_{3,\mathbf{k}\sigma} = -E_{2,\mathbf{k}\sigma}, \end{aligned} \tag{96}$$

where the wavevectors $\mathbf{k} = (k_x, k_y)$ are defined in the Brillouin zone ($k_x = \frac{4\pi}{L_x} n_x$, $k_y = \frac{4\pi}{L_y} n_y$) with $n_x = 0, 1, \cdots, \frac{L_x}{2} - 1$ and $n_y = 0, 1, \cdots, \frac{L_y}{2} - 1$, and

$$\begin{aligned} A_{\mathbf{k},\pm} &= (t_1 + t_2) \cos\left(\frac{k_x}{2}\right) \pm (t_1' + t_2') \cos\left(\frac{k_y}{2}\right), \\ B_{\mathbf{k},\pm} &= (t_1 - t_2) \sin\left(\frac{k_x}{2}\right) \pm (t_1' - t_2') \sin\left(\frac{k_y}{2}\right). \end{aligned} \tag{97}$$

There is a chiral symmetry at work, since the energy levels are symmetrically distributed around zero energy and time reversal is not broken.

The Gaussian duality of Eq. (43) with $\boldsymbol{\delta}_1 = \boldsymbol{\delta}_2 = (0, 1)$ maps our Kekulé-type insulator into a p-wave superconductor with an antiferromagnetic distribution of π-fluxes per square plaquette (or more precisely, two spin copies of this system). The dual (chiral) superconductor is shown in Fig. 4 (Bottom) and corresponds to (see Eqs. (46) and (57))

$$\begin{aligned} H^D = -\sum_{\langle \mathbf{r},\mathbf{r}' \rangle, \sigma} \Big[& t^{\mathsf{av}}_{\mathbf{r},\mathbf{r}'} \, c^\dagger_{\mathbf{r},\sigma} c_{\mathbf{r}',\sigma} + (t^{\mathsf{av}}_{\mathbf{r},\mathbf{r}'})^* \, c^\dagger_{\mathbf{r}',\sigma} c_{\mathbf{r},\sigma} + \\ & \Delta_{\mathbf{r},\mathbf{r}'} \, c^\dagger_{\mathbf{r}',\sigma} c^\dagger_{\mathbf{r},\sigma} + (\Delta_{\mathbf{r},\mathbf{r}'})^* \, c_{\mathbf{r},\sigma} c_{\mathbf{r}',\sigma} \Big], \end{aligned} \tag{98}$$

where $\langle \mathbf{r}, \mathbf{r}' \rangle$ represents nearest-neighbor links of a rectangular lattice with lattice points $\mathbf{r} = (r_1, r_2)$ and

$$t^{\mathsf{av}}_{\mathbf{r},\mathbf{r}'} = \begin{cases} \frac{t_1 + t_2}{2} & , \ \text{for } \mathbf{r}' = (r_1 + 1, r_2) \\ \frac{t_1' + t_2'}{2} - \frac{\mathrm{i}(-1)^{r_1 + r_2}}{2} \epsilon & , \ \text{for } \mathbf{r}' = (r_1, r_2 + 1), \end{cases}$$

$$\Delta_{\mathbf{r},\mathbf{r}'} = \begin{cases} -\frac{t_1 - t_2}{2} & , \ \text{for } \mathbf{r}' = (r_1 + 1, r_2) \\ -\frac{t_1' - t_2'}{2} - \frac{\mathrm{i}}{2} \epsilon & , \ \text{for } \mathbf{r}' = (r_1, r_2 + 1) \end{cases}. \tag{99}$$

1. Mapping of topological boundary modes

Our Kekulé-type insulator may display zero energy modes if the on-site potential vanishes. It is instructive to consider explicitly the case of graphene to highlight the differences between zig-zag and armchair edge terminations[33] from the point of view of the dual superconductor. So let us take open boundary conditions along the r_1-direction and periodic along the r_2-direction, i.e., and open cylinder, and the parameter set $t_1 = t_1' = t_2'$, $t_2 = 0 = \epsilon_{\mathbf{x}} = \epsilon_{\mathbf{y}}$. Then, the Gaussian duality map used above for the toroidal boundary conditions (bulk) also works for the cylinder, since $\boldsymbol{\delta}_1 = \boldsymbol{\delta}_2 = (0,1)$ describes a translation along the periodic direction. One can see from Fig. 4 that this situation corresponds to a zig-zag edge, while the parameter set $t_1 = t_2 = t_1'$, $t_2' = 0 = \epsilon_{\mathbf{x}} = \epsilon_{\mathbf{y}}$ would correspond to an armchair termination. The corresponding dual superconductors represent (two copies of) a stack of horizontal or vertical Kitaev chains respectively, in the topologically non-trivial regime. When the chains are horizontal, they obtain the topological superconducting Majorana edge modes dual to zig-zag terminated graphene. When the chains are vertical, the superconductor does not display edge modes, and its dual corresponds to the armchair terminated graphene.

The general duality transformation of Section II C breaks down for open boundary conditions in both r_1 and r_2 directions. Nonetheless, all of our conclusions hold just as well in this case were zig-zag and armchair terminations coexist. In order to illustrate this point explicitly it becomes necessary to introduce a different Gaussian duality, showcasing once more the fact that exact dualities are very sensitive to boundary conditions.

Let us focus for simplicity on a particular, spinless, case of our Kekulé-type insulator with open boundary conditions in both directions. The Hamiltonian of interest is given by

$$H = H_1 + H_2 + H_v, \tag{100}$$

where

$$H_1 = -\sum_{r_2=1}^{M_y}\sum_{r_1=1}^{M_x}\Big[t_1\, c^\dagger_{2r_1-1,2r_2-1}c_{2r_1,2r_2-1} + t_2\, c^\dagger_{2r_1,2r_2-1}c_{2r_1+1,2r_2-1} + \text{H.c.}\Big], \tag{101}$$

$$H_2 = -\sum_{r_2=1}^{M_y}\sum_{r_1=1}^{M_x}\Big[t_2\,(c^\dagger_{2r_1-1,2r_2}c_{2r_1,2r_2} + t_1\, c^\dagger_{2r_1,2r_2}c_{2r_1+1,2r_2} + \text{H.c.}\Big], \tag{102}$$

$[H_1, H_2] = 0$, and

$$H_v = -t_1\sum_{r_2=1}^{L_y-1}\sum_{r_1=1}^{L_x}\Big[c^\dagger_{r_1,r_2}c_{r_1,r_2+1} + \text{H.c.}\Big], \tag{103}$$

with $L_x = 2M_x + 1$ and $L_y = 2M_y$, has an appealing interpretation as a stack of dimerized Peierls chains. Because the Peirls chain are alternating, the bulk translation symmetry in either direction is generated by translations by two sites. As a function of t_2, the stack interpolates between a trivial metal, $t_2 = t_1$, and spinless graphene, $t_2 = 0$, with zig-zag vertical and armchair horizontal boundaries.

The description of H as a stack of Peierls chains suggests a natural way to map the system to a superconductor. Let us apply, to each horizontal chain, the duality of Section III A 2, Eq. (81). Unfortunately, this simplest alternative would obtain a non-local dual representation of H_v. There is, however, a way to fix this problem. Recall that the Gaussian dualities of this paper are motivated by the observation that some systems may be split into independent subsystems, and then it is possible to rearrange one subsystem relative to the other. So, given the splitting of the open Peierls chain into two subsystems, we could rearrange one subsystem for, say, the chains at odd height $2r_2 - 1$, and the other subsystem for the chains at even height $2r_2$. This idea is implemented by the Gaussian duality

$$\begin{aligned} a_{2r_1-1,2r_2-1} &\to a_{L_x-2(r_1-1),2r_2-1},\\ b_{2r_1,2r_2-1} &\to b_{L_x+1-2r_1,2r_2-1},\\ a_{2r_1,2r_2} &\to a_{L_x+1-2r_1,2r_2},\\ b_{2r_1-1,2r_2} &\to b_{L_x-2(r_1-1),2r_2}. \end{aligned} \tag{104}$$

The Majorana operators that are not explicitly listed remain unchanged.

Now, the alternating structure of the duality mapping in the vertical direction obtains the trivial transformation

$$H_v^D = H_v. \tag{105}$$

The effect of the first half of the transformation on H_1 follows immediately from the work in Section III A 2. In H_2, the hoppings t_1 and t_2 are exchanged relative to H_1. However, the second half of the transformation is also modified relative to the first half in such a way as to precisely compensate for this exchange. Explicitly,

$$H^D = H_1^D + H_2^D + H_v, \tag{106}$$

with

$$H_1^D + H_2^D = -\sum_{r_2=1}^{L_y}\sum_{r_1=1}^{L_x-1}\Big[t_{12}c^\dagger_{r_1,r_2}c_{r_1+1,r_2} + \Delta_{12}c^\dagger_{r_1,r_2}c^\dagger_{r_1+1,r_2} + \text{H.c.}\Big]. \tag{107}$$

As before, the dual system may be described as a stack of Kitaev wires, but now with open boundary conditions in both directions. It realizes on its vertical boundaries the Kitaev edge,[19] a one-dimensional p-wave superconductor robust against statistical translation invariant disorder and/or interactions.[20] We see that the zig-zag boundary of spinless graphene is dual to the Kitaev edge.

IV. SUMMARY AND OUTLOOK

In this paper we have developed the general theory of Gaussian duality transformations for fermions, defined as maps that preserve a quadratic Hermitian form. The dual partner of a system of free fermions is also a system of free fermions if the duality transformation is Gaussian, and both systems are equally local in space. The theory and practice of fermionic Gaussian dualities benefits from "Majorana fermions," the complex Clifford algebra canonically represented in Fock space. As a consequence, the theory of Gaussian dualities for canonical bosons is markedly different and left for a future publication.

As transformations of statistical mechanics, dualities are valued for mapping strongly-coupled systems to weakly coupled ones. Gaussian dualities seem unconventional from this point of view, since systems of free fermions are, by definition, all weakly coupled. To some extent, the conceptual mismatch is just a matter of choice of language, that is, of physical representation of the systems under consideration.[24] Take for example the simplest model of magnetic ordering, the transverse-field Ising chain. The self-duality of this model, a non-local transformation of spins, is the prototype of a strong coupling/weak coupling duality transformation. However, the Jordan-Wigner mapping transforms the Ising model into the Majorana chain, and the self-duality of the Ising model into a Gaussian self-duality of the Majorana chain.

The key property shared by all duality transformations is symmetry transmutation in face of the locality constraint. This phenomenon occurs when a symmetry and its dual partner, necessarily a symmetry of the dual Hamiltonian, have different physical interpretations. For Gaussian dualities in particular we demonstrated transmutation of particle number, fermionic parity, translation, spin rotation, and time-reversal symmetry in various space dimensions. Transmutation of particle number is most conspicuous, since it allows for insulators (or semimetals) and superconductors (with zeroes of the gap function) to appear as dual partners.

Because of symmetry transmutation, Gaussian dualities can establish equivalences of topological insulators and superconductors, relating systems classified as inequivalent from a single-particle viewpoint. Here we investigated in detail two paradigmatic examples of such pairs of dual partners: the dimerized Peierls/Majorana chain (at vanishing chemical potential), and graphene which happens to be dual to a weak topological superconductor. Transmutation of lattice symmetry is manifest in both examples since, for example, the superconductor dual to graphene is naturally defined on a square lattice.

While our list of equivalences is far from exhaustive, our approach constitutes a general framework. Hence, future research should be focused on searching systematically for other classes of Gaussian dualities besides the ones presented here, aiming at obtaining all possible equivalences across entries in established classification tables of noninteracting electronic matter. We can offer two comments as to what "possible" should entail. First, we do not necessarilly expect equivalences of systems of different space dimensionality, since dimensional reduction by duality is possible but uncommon.[14] Second, to obtain equivalences between topologically trivial and non-trivial systems, it will become necessary to obtain the Gaussian analog of a holographic symmetry.[13] The Gaussian dualities of this paper do not realize holographic symmetries: boundary symmetries (zero-energy modes) of a topologically non-trivial system are mapped to equally localized boundary symmetries of its dual partner, and topologically trivial systems are mapped to equally trivial dual partners. In short, our dualities preserve the bulk-boundary correspondence (for a useful discussion of this correspondence, see for example Ref. [34] and references therein).

Let us now discuss some possible applications of our results. It is interesting to think of dual partner systems, with at least one open boundary, as scattering regions and ask what happens at the level of dualities if we attach leads to the system. The remarkable answer is that it is often possible to extend the Gaussian duality to apply to the whole system (scattering region plus leads), so that its dual partner also has an unambigous interpretation as a scattering region with leads attached. Depending on the nature of the leads and Gaussian duality, the dual leads may be rearranged in space and/or become superconducting. In any case it is now possible to obtain quantitative dual mappings of transport properties to further extend the equivalence of insulators and superconductors. Notice that it is no problem to add disorder, but symmetry transmutation of the disorder ensemble should be expected. That is, if the disorder ensemble of one system displays some statistical symmetry, the dual system will be distributed according to a dual ensemble with a dual statistical symmetry of possibly very different nature. These brief comments set the ground for investigating equivalences of statistical topological insulators and superconductors.[35]

Topologically non-trivial insulators and superconductors display zero-energy boundary modes associated to the degeneracy of their many-body ground-state energy level. In general, it is possible to trace the superconductor ground degeneracy back to some discrete symmetry left over from the breaking of particle conservation. This picture does not apply to insulators and so the origin of their topological degeneracy is harder to unveil. But our equivalence can help here. In terms of its superconducting equivalent, the ground degeneracy of a topological insulator is explained by spontaneous symmetry breaking. Moreover, it can be detected and characterized numerically by investigating the system in terms of Josephson-like physics.

Finally, we would like to point out a very important application of our duality approach to strongly correlated systems. Since our duality transformations are not restricted to free fermion/boson systems (as shown in a couple of examples of this paper), it is apparent that they

may become a powerful tool for a potential topological classification of interacting fermion/boson systems. One can always split a generic many-body Hamiltonian into non-interacting and interacting components. A Gaussian duality preserves this decomposition by mapping the non-interacting part into another non-interacting Hamiltonian, and the interactions into a dual interaction term. Imagine now, since one would like to focus on nonperturbative effects, that one approaches these systems armed with tools borrowed from renormalization group techniques, for example (similar arguments apply to numerical methods as well). Because of the phenomenon of symmetry transmutation, it is natural to expect that the renormalization flow will look very different for the original and dual Hamiltonians, and it is in principle possible to achieve considerable simplifications and insight by dualizing renormalization group flows. These ideas seem quite reasonable if, as is usually done, one approaches the problem by first taking the continuum limit. The reason is that Gaussian dualities are naturally suited to induce, starting from the lattice, highly non-trivial dualities of effective quantum field theories.

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Appendix A: The m-merized Peierls chain

Let us describe briefly the superconducting model associated to the general m-merized Peierls case with $m > 2$. Consider the $m = 3$ (trimerized) Peierls chain with hoppings t_1, t_2, and t_3 and periodic boundary conditions ($L = 2M$ is divisible by $m = 3$). Then, the dual superconducting Hamiltonian is given by

$$H^D = -\sum_{j=1}^{L} \left(w_j \, c_j^\dagger c_{j+1} + \Delta_j \, c_j^\dagger c_{j+1}^\dagger + \text{H.c.} \right), \quad \text{(A1)}$$

with coupling constants

$$w_1 = \frac{t_1 + t_3}{2} \ , \ w_2 = \frac{t_1 + t_2}{2} \ , \ w_3 = \frac{t_3 + t_2}{2}, \quad \text{(A2)}$$
$$\Delta_{1,4} = \pm\frac{t_1 - t_3}{2} \ , \ \Delta_{2,5} = \pm\frac{t_1 - t_2}{2} \ , \ \Delta_{3,6} = \pm\frac{t_3 - t_2}{2},$$

and the rest of the couplings periodically repeat. Repeating the same line of reasoning, the case $m = 4$ leads to a dual Hamiltonian such as Eq. (A1), where the coupling constants are given by $w_1 = \frac{t_1+t_4}{2}, w_2 = \frac{t_1+t_2}{2}, w_3 = \frac{t_3+t_2}{2}, w_4 = \frac{t_3+t_4}{2}$, $\Delta_1 = \frac{t_1-t_4}{2}, \Delta_2 = \frac{t_1-t_2}{2}, \Delta_3 = \frac{t_3-t_2}{2}, \Delta_4 = \frac{t_3-t_4}{2}$. This clearly shows the difference between the cases where m is odd from those where m is even. In the latter case, the periodicity is always m lattice constants, except for the particular dimerized $m = 2$ case where the periodicity is one lattice constant, while the periodicity becomes $2m$ when m is odd.

Appendix B: The BCS-insulator map

There is an elementary Gaussian duality of an s-wave BCS superconductor to a trivial insulator. Let us denote by $\hat{c}_{\mathbf{k},\sigma}$ the annihilation fermionic field in momentum space $\mathbf{k}$, and assume $\epsilon_{\mathbf{k}} = \epsilon_{-\mathbf{k}}$. Then we may define the Gaussian duality

$$\hat{c}_{\mathbf{k},\uparrow} \mapsto \hat{c}_{\mathbf{k},\uparrow}, \quad \hat{c}_{\mathbf{k},\downarrow} \mapsto \hat{c}^\dagger_{-\mathbf{k},\downarrow}, \quad \text{(B1)}$$

acting non-trivially only on the spin-down fermions. The mean-field BCS Hamiltonian is

$$H = \sum_{\mathbf{k},\sigma} (\epsilon_{\mathbf{k}} - \mu) \, \hat{c}^\dagger_{\mathbf{k},\sigma} \hat{c}_{\mathbf{k},\sigma} - \sum_{\mathbf{k}} \Delta \, (\hat{c}^\dagger_{\mathbf{k},\uparrow} \hat{c}^\dagger_{-\mathbf{k},\downarrow} + \text{H.c.}), \quad \text{(B2)}$$

and gets mapped to

$$H^D = \sum_{\mathbf{k}} \left[\epsilon_{\mathbf{k}} \hat{c}^\dagger_{\mathbf{k},\alpha} \sigma^z_{\alpha,\beta} \hat{c}_{\mathbf{k},\beta} - h_\nu \hat{c}^\dagger_{\mathbf{k},\alpha} \sigma^\nu_{\alpha,\beta} \hat{c}_{\mathbf{k},\beta} \right] + C, \quad \text{(B3)}$$

with $C = \sum_{\mathbf{k}} (\epsilon_{\mathbf{k}} - \mu)$, $\nu = x, y, z$, and

$$h_x = \Delta, \quad h_y = 0, \quad h_z = \mu. \quad \text{(B4)}$$

Particle conservation is restored in H^D at the expense of broken (by symmetry transmutation) time-reversal and spin-rotation symmetry.

Appendix C: A Bosonic insulator and its dual superfluid

In recent years, lattice models of bosons have acquired new relevance thanks to the spectacular experimental development of ultracold atom physics. In this section we will focus on a case of practical importance in which only three states $\{|\bar{n}-1\rangle, |\bar{n}\rangle, |\bar{n}+1\rangle\}$ per lattice site j are physically active. Then the particle operators $g_j, g_j^\dagger, n_j$ act on these states as

$$\begin{aligned} &g_j|\bar{n}-1\rangle = 0 \ , && g_j^\dagger|\bar{n}-1\rangle = \sqrt{\bar{n}} \ |\bar{n}\rangle, \\ &g_j|\bar{n}\rangle = \sqrt{\bar{n}} \ |\bar{n}-1\rangle, && g_j^\dagger|\bar{n}\rangle = \sqrt{\bar{n}} \ |\bar{n}+1\rangle, \\ &g_j|\bar{n}+1\rangle = \sqrt{\bar{n}} \ |\bar{n}\rangle, && g_j^\dagger|\bar{n}+1\rangle = 0, \end{aligned} \quad \text{(C1)}$$

and

$$\begin{aligned} &n_j|\bar{n}-1\rangle = (\bar{n}-1)|\bar{n}-1\rangle, \\ &n_j|\bar{n}\rangle = \bar{n}|\bar{n}\rangle, \\ &n_j|\bar{n}+1\rangle = (\bar{n}+1)|\bar{n}+1\rangle. \end{aligned} \quad \text{(C2)}$$

These are bosonic operators restricted to a finite(three)-dimensional Hilbert space per site. Their algebra and application to optical lattices[36] has been extensively investigated in Refs. 37 and 24. Here it will suffice to notice that there is a connection to spin $S = 1$ operators that, in the present paper and because of the duality we will apply later on, we take to be

$$\sqrt{\frac{2}{\bar{n}}}\, g_j^\dagger = S_j^z + \mathsf{i} S_j^x = S_j^+, \quad n_j = S_j^y + \bar{n}. \tag{C3}$$

Here we will focus on a system of bosonic atoms distributed with (integer) average particle density $\bar{n} \geq 1$ on a one-dimensional optical lattice. We assume that the dynamics of the system is described by the Hamiltonian

$$H = \sum_j \Big[-\frac{t}{2}(g_{j+1}^\dagger g_j + \text{H.c.}) - \mu\, n_j + V\, n_j n_{j+1} \Big]. \tag{C4}$$

The chemical potential μ fixes $\bar{n}$, the interaction $V > 0$ is repulsive, and the total number of particles $\hat{N} = \sum_j n_j$ is conserved. This bosonic t-V model displays a Mott insulating phase.

As a result of Eq. (C3), H has an interesting interpretation as an XXZ spin $S = 1$ model in a magnetic field,

$$H = \sum_j \Big[-\frac{\bar{n}\, t}{2}(S_j^x S_{j+1}^x + S_j^z S_{j+1}^z) - (\mu - 2\bar{n}V) S_j^y + V S_j^y S_{j+1}^y \Big] \tag{C5}$$

(up to an additive constant). In the following the chemical potential will be kept fixed at $\mu = 2\bar{n}V$ in order to eliminate the magnetic field term.

The quantum phase diagram of the $S = 1$ XXZ is well known.[38] For $t = 0$, the ground state is antiferromagnetically (Ising) ordered in the spin language and in a Mott insulating state in the boson language. There is a mass gap in the quasiparticle spectrum. According to the Haldane gap conjecture, the mass gap remains as the Heisenberg antiferromagnetic line $t = -2V/\bar{n}$ is reached. For the bosonic t-V model this means that the atomic system remains in a Mott insulating state. Next we will show that the bosonic t-V model is dual to a particle non-conserving Hamiltonian with an unconventional superfluid ground state.

To establish the equivalence of the bosonic t-V model to a superfluid, we will exploit a duality transformation first investigated in the context of spin $S = 1$ models, see Ref. 39 and references therein. A compact expression for the unitary transformation associated to this duality is[40]

$$\mathcal{U}_\mathsf{d} = \prod_{j<k} e^{\mathsf{i}\pi S_j^z S_k^x}. \tag{C6}$$

It follows that

$$\begin{aligned} \mathcal{U}_\mathsf{d} S_j^x S_{j+1}^x \mathcal{U}_\mathsf{d}^\dagger &= -S_j^x S_{j+1}^x, \\ \mathcal{U}_\mathsf{d} S_j^y S_{j+1}^y \mathcal{U}_\mathsf{d}^\dagger &= S_j^y e^{\mathsf{i}\pi(S_j^z + S_{j+1}^x)} S_{j+1}^y, \\ \mathcal{U}_\mathsf{d} S_j^z S_{j+1}^z \mathcal{U}_\mathsf{d}^\dagger &= -S_j^z S_{j+1}^z. \end{aligned} \tag{C7}$$

Therefore, the dual spin model $H^D = \mathcal{U}_\mathsf{d}\, H\, \mathcal{U}_\mathsf{d}^\dagger$ is

$$H^D = \sum_j \Big[\frac{\bar{n}t}{2}(S_j^x S_{j+1}^x + S_j^z S_{j+1}^z) + V\, S_j^y e^{\mathsf{i}\pi(S_j^z + S_{j+1}^x)} S_{j+1}^y \Big]. \tag{C8}$$

The dual model of bosonic atoms is obtained by rewriting the spin Hamiltonian H^D in terms of particle operators. This task is easy once the identity

$$S_j^y e^{\mathsf{i}\pi(S_j^z + S_{j+1}^x)} S_{j+1}^y = -\frac{1}{4} S_j^y ((S_j^+)^2 + (S_j^-)^2)((S_{j+1}^+)^2 + (S_{j+1}^-)^2) S_{j+1}^y \tag{C9}$$

is established. Then it follows that the dual bosonic t-V model is described by the Hamiltonian

$$H^D = \sum_j \Big[\frac{\bar{n}t}{2}(g_j^\dagger g_{j+1} + g_{j+1}^\dagger g_j) - \frac{V}{\bar{n}^2} \bar{n}_j (g_j^2 + g_j^{\dagger\,2})(g_{j+1}^2 + g_{j+1}^{\dagger\,2}) \bar{n}_{j+1} \Big], \tag{C10}$$

with $\bar{n}_j = n_j - \bar{n}$.

The terms

$$-\frac{V}{\bar{n}^2} \sum_j \bar{n}_j (g_j^2 g_{j+1}^2 + g_j^{\dagger\,2} g_{j+1}^{\dagger\,2}) \bar{n}_{j+1}. \tag{C11}$$

break particle conservation. Bosons are created and annihilated in quartets. Hence the U(1) symmetry of particle conservation of the t-V model is explicitly broken down to a discrete $\mathbb{Z}_4$ symmetry in the dual Hamiltonian. The operator $\mathsf{i}^{\hat{N}} = e^{\mathsf{i}\frac{\pi}{2}\sum_j n_j}$ commutes with H^D, and has the interpretation of counting the total number of bosons modulo four.

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