Shear viscosity in a non-Fermi-liquid phase of a quadratic semimetal

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Shear Viscosity in a Non-Fermi Liquid Phase of a Quadratic Semimetal

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We study finite temperature transport in the Luttinger-Abrikosov-Beneslavskii phase – an interacting, scale invariant, non-Fermi liquid phase found in quadratic semimetals. We develop a kinetic equation formalism to describe the d.c. transport properties, which are dominated by collisions, and compute the shear viscosity η. The ratio of shear viscosity to entropy density η/s is a measure of the strength of interaction between the excitations of a quantum fluid. As a consequence of the quantum critical nature of the system, η/s is a universal number and we find it to be consistent with a bound proposed from gauge-gravity duality.

Transport coefficients such as the electrical and thermal conductivities or the viscosities play a central role in describing condensed matter systems. They are experimentally measurable and contain signatures which characterize the different phases of matter. Nonetheless, for strongly interacting systems at finite temperatures, as in the vicinity of a quantum critical point, there remain difficulties in calculating transport coefficients analytically due to the lack of a quasiparticle description [1]. Recently, insight has been gained from the relationship between strongly coupled field theories and classical gravitational theories in the context of gauge-gravity duality [2, 3]. In particular, Kovtun, Son and Starinets [4, 5] conjectured a lower bound on the ratio of the shear viscosity η and the entropy density s for a general class of finite temperature field theories

$$\frac{\eta}{s} \geq \frac{\hbar}{4\pi k_B}. \tag{1}$$

Since all scattering channels between excitations are saturated in the quantum critical regime, the mean free time of interaction approaches the thermal equilibration time $\tau_{eq} \sim \hbar/k_B T$. This gives rise to universal amplitude ratios which characterise the interacting field theory [1]. The bound (1) is then similar to the Mott-Ioffe-Regel limit for the minimal conductivity in disordered metals [6]. With the exception of particle-hole symmetric theories, however, the conductivity is determined by mechanisms which break translational symmetry. The ratio η/s is therefore singled out as a good indicator for studying the intrinsic strength of interaction of the underlying carriers. While there are theoretical counter-examples [7, 8] to the bound (1), most phases which have been studied satisfy it [9–13].

Motivated by these developments, we consider the shear viscosity in a model of a non-Fermi liquid phase where the quasiparticle description fails. Namely, we consider the zero-gap semiconductor with a quadratic band touching and Coulomb interactions in three spatial dimensions, as first developed in [14–16]. The low energy excitations of the system are described by an interacting scale-invariant field theory, which is manifestly non-relativistic – this fixed point has recently been dubbed the 'Luttinger-Abrikosov-Beneslavskii' (LAB) phase by [17]. It has been suggested [17] that this phase could arise in strongly correlated pyrochlore iridates and might explain some of the the interesting properties of $Pr_2Ir_2O_7$. More generally, the model has already garnered recent attention as part of a focus on emergent quantum phases in materials with strong spin-orbit coupling and in semi-metals – these include systems as different as graphene, the surface states of topological insulators, Weyl semimetals or nodal superconductors [18, 19]. Although there are still open question about the low energy properties of the model [20], the LAB phase might give more general insights into the nature of non-Fermi Liquid phases arising from electron-electron interactions. We briefly review the origin and properties of this model following [16]. We shall from now on work in units where $\hbar = k_B = 1$.

The bandstructure of this model is a version of the Luttinger Hamiltonian [21], in which the four $P_{3/2}$ bands form a quadratic zero-gap semiconductor (Figure 1). The classic examples of this band-structure are α-Sn and HgTe, the later has received much attention due to its importance in making topological insulators [22, 23]. The Hamiltonian for the effective model with the Fermi level at the degeneracy point and Coulomb interactions is

$$H = \int \mathrm{d}^3x \left[c(\partial_i \psi^\dagger) A_{ij} (\partial_j \psi) + \frac{1}{2} \int \mathrm{d}^3x' \frac{\rho(\mathbf{x})\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} \right] \tag{2}$$

where $\psi(\mathbf{x})$ are four component electron fields, $\rho(\mathbf{x}) = e\psi^\dagger(\mathbf{x})\psi(\mathbf{x})$ is the electric charge density, and A_{ij} are matrices characterizing the bands. We take

$$A_{ij} = \frac{1}{2}\{J_i, J_j\} - \frac{1}{3} J^2 \delta_{ij} \tag{3}$$

where the J_i are the usual 3/2 angular momentum matrices. The energy spectrum of the free states has the simple form

$$\epsilon(\mathbf{k}) = \pm c k^2 \tag{4}$$

with each band being two-fold degenerate. The O_h symmetry in a crystal allows a more general form of the kinetic energy, which breaks both the isotropy and particle-hole symmetry of the dispersion [24], such as adding the term $\propto (\partial_i \psi^\dagger)(\partial_i \psi)$. However, it is found that these terms are irrelevant in the renormalization-group (RG) sense [16, 17]. We shall consider the low energy regime, where these terms are asymptotically small and can be neglected.

The Coulomb interaction between electrons is RG relevant in three spatial dimensions, but marginal if we consider the model generalized to four dimensions. We can therefore use an epsilon expansion ($\varepsilon = 4 - d \ll 1$) with non-integer dimension d to control the perturbative RG calculation and access the zero-temperature fixed point. Within the ε-expansion, the dimensionless coupling constant $\alpha = e^2/c\Lambda^\varepsilon$ evolves under the renormalization group and one finds that the system flows towards a stable Wilson-Fisher fixed point [16, 17]; here Λ is the energy scale where the Hamiltonian (2) is defined. In particular, in the scaling limit close to the fixed point the propagator of the auxiliary Coulomb field remains unscreened and takes the form

$$V(\omega, \mathbf{q}) = \frac{1}{q^{2-\eta_\varphi}} S\left(\frac{\omega}{q^z}\right). \tag{5}$$

Here z is the anomalous exponent describing the scaling between energy and momentum $\epsilon(\mathbf{k}) = ck^z$ and η_φ is the anomalous scaling dimension of the Coulomb field. The scaling function $S(x)$ is defined so that in the asymptotic limit $S \to 1$ as $x \to 0$. Using momentum shell integration we find the leading corrections $z = 2 - (9/56)\varepsilon$ and $\eta_\varphi = \alpha^*/2\pi$, where $\alpha^* = (32\pi/21)\varepsilon$ is the fixed point coupling. We note that these values depend on the way we perform the analytic continuation of the $J = 3/2$ spin structure to non-integer dimensions as discussed in detail in the Supplementary Material [25]. One can also develop a controlled RG calculation in $d = 3$ in the limit of a large number N of electronic fields [16], which gives a consistent picture with the fixed point found above [26].

We focus on studying the shear viscosity η at finite temperatures. The shear viscosity is the transport coefficient which characterizes the relaxation of a transverse momentum gradient back to local equilibrium. Considering slow variations of the local momentum $\mathbf{P}(\mathbf{x})$ on a lengthscale much larger than the mean free path, the leading dissipative contribution to the stress tensor T_{ij} defines the viscosities. For a pure shear flow

$$\Delta T_{ij} = -\eta \left[\frac{\partial V_i}{\partial x_j} + \frac{\partial V_j}{\partial x_i} - \frac{2}{d}\delta_{ij}(\partial_k V_k) \right], \tag{6}$$

where $\mathbf{V} = 2c\mathbf{P}$. Being linear response coefficients, the viscosities can be written as Kubo formulae; for the real part of the shear viscosity [27]

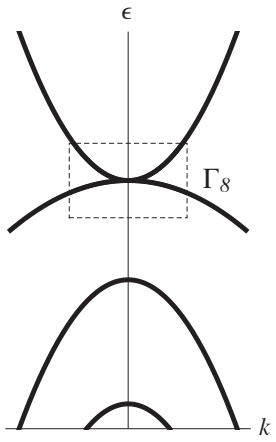

FIG. 1. Schematic of band-structure in a spin-orbit coupled s-p semiconductor with a band inversion. The quadratic band touching forms at the Γ_8 point; the region of the effective model giving rise to the LAB phase is shown in dashed box.

$$\eta(\omega) = \frac{1}{\omega L^d} \cdot \mathrm{Re} \int_0^\infty \mathrm{d}t \; e^{i\omega^+ t} \, \langle [T_{xy}(t), T_{xy}(0)] \rangle \tag{7}$$

where $\omega^+ = \omega + i0^+$ and L^d the volume of the system.

To understand the transport of the LAB phase at finite temperatures, we emphasize the similarity to a system in the vicinity of a quantum critical point [1]. The long distance behaviour of the system is described by a scale invariant, interacting field theory, whose transport properties are described by universal functions. At finite frequencies there are two regimes for the transport coefficients. The collision-less limit $\omega/T \gg 1$ is dominated by particle-hole production; the effect of interactions is small and the form of η may be found from the RG flow. In contrast, the limit $\omega/T \ll 1$ is dominated by collisions between thermally excited particles and holes. In a non-interacting system, the excited particles move ballistically and give a delta-function peak; this delta function is broadened by the Coulomb interactions. Phenomenologically, we can understand the dc transport properties from a simple mean free time argument. The temperature is the only characteristic energy scale in the problem; the characteristic length scale is $\sim (c/T)^{1/z}$ and the mean free path is $\ell \sim \alpha^{-2}(c/T)^{1/z}$, which includes the dimensionless scattering rate. A simple Drude theory would predict a dc shear viscosity $\eta \sim n\bar{v}\ell/c \sim \alpha^{-2}(T/c)^{d/z}$. The scaling of the temperature is exactly that of the entropy density $s \sim (T/c)^{d/z}$ as expected.

We now develop a kinetic equation approach to calculate the shear viscosity to leading order in the ε-expansion, which will recover the above considerations and also give a numerical estimate for the value of η/s. Formally, the kinetic equation makes use of a semi-classical expansion and assumes quasiparticles which interact locally. It was shown that in the weakly coupled

finite-temperature theory of a scalar field [28, 29], the diagrammatic summation in a Kubo calculation using (7) matches the results from a high temperature effective kinetic equation. The LAB phase does not contain quasi-particles, but for small ε the quasi-particles become well defined and this approach is justified during the calculation. The long range Coulomb interaction will be screened at finite temperature on the Debye lengthscale $\ell_D \sim \sqrt{\alpha T/c}$, which is much less then the mean free path, justifying the local collision term.

We write the fields ψ in terms of particle and hole eigenstates:

$$\psi(\mathbf{x}) = \int_{\mathbf{k}} \left[u_\sigma(\mathbf{k}) c_\sigma(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}} + v_\sigma(\mathbf{k}) h^\dagger_\sigma(\mathbf{k}) e^{-i\mathbf{k}\cdot\mathbf{x}} \right] \quad (8)$$

where $\sigma = \pm$ is the helicity of the state, $u_\sigma(\mathbf{k}), v_\sigma(\mathbf{k})$ are spinor factors and $\int_{\mathbf{k}} = \int (\mathrm{d}^d k)/(2\pi)^d$. Since we have a four-band model, the most general semi-classical kinetic equation is for a matrix of distribution of functions including both diagonal distribution functions $\sim \langle c^\dagger_\sigma c_\sigma \rangle, \langle h^\dagger_\sigma h_\sigma \rangle$ and distribution function that describe particle-hole pairs $\sim \langle c_\sigma h_{\sigma'} \rangle, \langle c^\dagger_\sigma h^\dagger_{\sigma'} \rangle$. Here, we neglect the latter type of distribution function, since we are interested in the dc limit of the viscosity. Additionally, we shall drop spin-orbit coupling in matrix elements in the equation and neglect scattering in the Mandelstam s-channel, which involves electron-hole annihilation and production. This is justified in a 'leading q' approximation, where the dominant contribution is from particles with small momentum transfer; however, this would tend to overestimate the viscosity. In this limit, the scattering matrix elements are not affected by the spin structure of the theory. We note that at leading order in ε, the details of the RG scheme do not enter the form of the Coulomb interaction and we use the lowest order expression $4\pi\alpha^* c/q^2$ as the interaction between excitations. At this order, the thermal screening of the interaction is also negligible. Unlike the bare Coulomb interaction in $d = 3$, there is no low-momentum divergence of the collision integral and thermal screening is not needed as a regulator [26].

With the above conditions, the full quantum kinetic equation encapsulating the non-equilibrium time-evolution of the system, reduces to the usual semi-classical case

$$\left(\partial_t + \mathbf{v}\cdot\partial_{\mathbf{x}} + \mathbf{F}_{ext}\cdot\partial_{\mathbf{k}}\right) f^a(t,\mathbf{x},\mathbf{k}) = -C[f] \quad (9)$$

where a labels both the species (electrons or holes) and helicity, $\mathbf{v} = 2c\mathbf{k}$ is the particle velocity and $C[f]$ is the collision term describing two-particle scattering. The method of extracting the viscosity is standard – one considers the slowly varying local momentum as an external disturbance $X_{ij} = \left[\partial_j P_i + \partial_i P_j - (2\delta_{ij}/d)(\partial_k P_k)\right]/2$,

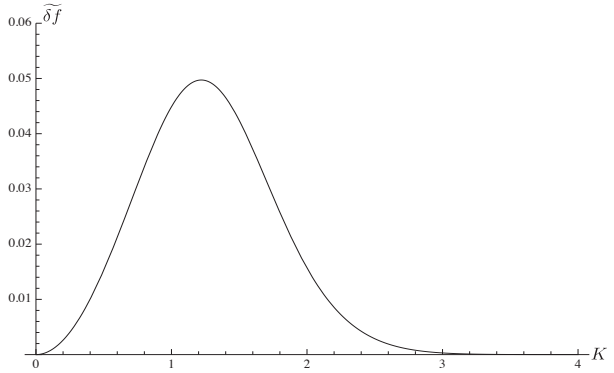

FIG. 2. Plot of optimized dimensionless function $\widetilde{\delta f}(K) = -(T(\alpha^*)^2 c)\cdot\chi(K)\,K^2[1-f_0(K)]f_0(K)$ as a function of $K = k\sqrt{T/c}$.

assumes this weakly modifies the equilibrium distribution function $f_0(\mathbf{k}) = 1/\left[\exp(ck^2/T)+1\right]$ and uses this modified distribution function to find the stress tensor T_{ij} and hence η. We linearize the distribution function $f^a(\mathbf{k}) = f_0(\mathbf{k}) + \delta f^a(\mathbf{k})$, where

$$\delta f^a(\mathbf{k}) = \beta[1-f_0(\mathbf{k})]f_0(\mathbf{k})\chi^a_{ij}(\mathbf{k})X_{ij} \quad (10)$$

and $\chi^a_{ij}(\mathbf{k}) = \sqrt{d/(d-1)}\left(k_i k_j - \delta_{ij}k^2/d\right)\chi^a(k)$ has the appropriate symmetry for the shear flow factored out, so that $\chi^a(k)$ is a function of the magnitude k only.
For the stress tensor, we consider the contributions from the diagonal fermion distribution functions

$$T_{ij} = \sum_a \int_{\mathbf{k}} 2ck_i k_j f^a_\sigma(\mathbf{k}) \quad (11)$$

In addition to neglecting the distribution function describing particle-hole creation, we neglect the contribution from the coulomb interaction between carriers to the stress tensor; we find this to be sub-leading in ε in our approach (see Supplementary Material [26]).

We solve the kinetic equation using a variational approach [30–32]. The kinetic equation, linearized for χ, can be viewed as an operator equation on function space $|S_{ij}\rangle = C|\chi_{ij}\rangle$. Here S_{ij} corresponds to the streaming term in (9); C is an operator encoding the collision term and is Hermitian with respect to an inner product $\langle f, g\rangle = \sum_a \sum_{i,j} \int_{\mathbf{k}} f^a_{ij}(\mathbf{k}) g^a_{ij}(\mathbf{k})$. Finding the solution $\chi(k)$ of is equivalent to maximizing the functional $Q[\chi] = \langle \chi_{ij}, S_{ij}\rangle - \langle \chi_{ij}, C\chi_{ij}\rangle/2$ with respect to variations in χ. Physically, this means that the realized solution is that which maximizes entropy production subject to the constraints imposed by the external disturbance and subsequent time evolution of the system. We project χ onto a set of 12 basis functions and optimise the coefficients numerically to find the variational approximation

(see Supplementary Material [26]); the optimal function is shown in Figure 2.

Using this solution, we find the viscosity

$$\eta \simeq \frac{3.1}{(\alpha^*)^2}\left(\frac{T}{c}\right)^2, \tag{12}$$

Since we are calculating η/s to leading order in ε, we can use the entropy density for the unrenormalized band-structure in $d = 4$

$$s = \frac{9\zeta(3)}{16\pi^2}\left(\frac{T}{c}\right)^2, \tag{13}$$

along with the fixed point value $\alpha^* = (32\pi/21)\varepsilon$ to finally obtain

$$\frac{\eta}{s} = \frac{0.63}{\varepsilon^2}. \tag{14}$$

Setting $\varepsilon = 1$ gives $4\pi\eta/s = 8.0$; consistent with the original bound (1). A priori, there was perhaps reason to believe that this model might strongly violate the original bound, given the unusual properties of the LAB phase – strong interactions, both particle and hole carriers, no Galilean invariance and an anomalous scaling z with softer excitations than in a relativistic system. The value of $\eta/s \simeq 0.63$ is, however, similar to the values found in the unitary Fermi gas ($\eta/s \leq 0.5$) or in the quark-gluon plasma ($\eta/s \leq 0.4$) [9].

The LAB phase only emerged with the chemical potential at the Γ_8 degeneracy point, so the system lacked a Fermi surface. Changing the chemical potential away sufficiently, we expect to recover the usual Fermi liquid behaviour. For a Fermi liquid $\eta/s \sim (\varepsilon_F/T)^3$ is temperature dependant [33]; in this limit the quasi-particles are well defined and weakly interacting. By comparison, graphene with Coulomb interactions is described by a field theory with marginally irrelevant interactions, but stays in a scaling regime for a large range of energies [34, 35]. The ratio $\eta/s \sim 1/\alpha(T)^2$ has a form similar to the quantum critical case (14), except that the effective coupling constant $\alpha(T) \sim 1/\log(T_\Lambda/T)$ ultimately renormalizes to zero as $T \to 0$ [13]. In both of these cases, η/s diverges as $T \to 0$ in contrast to the universal ratio obtained for the LAB phase.

Throughout our analysis, we have worked in the ε-expansion to retain analytic control, both for the description of the RG flow and the kinetic equation calculation. The use of the ε-expansion is well established [36, 37]. Determining its accuracy for finite temperature transport coefficients in quantum critical systems by comparison to numerical simulations is an active research area for which we refer to the literature [38]. The extrapolation to the physical dimension is, however, always a sensitive matter and relies on the fixed point evolving smoothly as $\varepsilon \to 1$. This may fail if there is interference from another fixed point. While there is no other fixed point at small ε, it was suggested in [20] that such a fixed point might arise at strong coupling and destabilize LAB phase for the physical dimension $d = 3$; this is still an open question [39].

Finally, we want to briefly comment on the experimental implications of the above discussion. In order to define the viscosity, the mean free path of effects which violate momentum conservation, such as disorder or umklapp processes, must be much larger than the mean free path of electron-electron interaction. This hydrodynamic regime is generally difficult to access in a solid state system, although, for example, recent experiments in $PdCoO_2$ observe very low resistivity and a temperature behaviour consistent with phonon drag [40]. In this regime, we might expect the viscosity to dominate the damping of acoustic waves and also to observe viscous drag effects, which may be a route to accessing η. Alternatively, the possibility of more dramatic signatures due to electronic turbulence has been suggested in low viscosity phases [13]; the LAB phase would be a prime candidate for searching for such effects. Overall, we anticipate that the hydrodynamic regime will become increasingly important in the study of solid state systems.

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[1] S. Sachdev, *Quantum Phase Transitions*, 2nd ed. (Cambridge University Press, Cambridge, 2011).
[2] S. Sachdev, Annu. Rev. Con. Mat. Phys. **3**, 9 (2012).
[3] S. A. Hartnoll, Classical and Quantum Gravity **26**, 224002 (2009).
[4] P. K. Kovtun, D. T. Son, and A. O. Starinets, Phys. Rev. Lett. **94**, 111601 (2005).
[5] D. T. Son and A. O. Starinets, Ann. Rev. Nucl. Part. Sci. **57**, 95 (2007).
[6] O. Gunnarsson, M. Calandra, and J. E. Han, Rev. Mod. Phys. **75**, 1085 (2003).
[7] M. Brigante, H. Liu, R. C. Myers, S. Shenker, and S. Yaida, Phys. Rev. Lett. **100**, 191601 (2008).
[8] Y. Kats and P. Petrov, J. High Energy Phys. **2009**, 044 (2009).
[9] T. Schäfer and D. Teaney, Rep. Prog. Phys. **72**, 126001 (2009).
[10] T. Schäfer, Phys. Rev. A **76**, 063618 (2007).
[11] G. M. Bruun and H. Smith, Phys. Rev. A **75**, 043612 (2007).
[12] T. Enss, R. Haussmann, and W. Zwerger, Ann. Phys. (NY) **326**, 770 (2011).
[13] M. Müller, J. Schmalian, and L. Fritz, Phys. Rev. Lett.

103, 025301 (2009).
[14] A. A. Abrikosov and S. D. Beneslavskii, Sov. Phys. JETP **32**, 699 (1971).
[15] A. A. Abrikosov and S. D. Beneslavskii, J. Low. Temp. Phys. **5**, 141 (1971).
[16] A. A. Abrikosov, Sov. Phys. JETP **66**, 1443 (1974).
[17] E.-G. Moon, C. Xu, Y. B. Kim, and L. Balents, Phys. Rev. Lett. **111**, 206401 (2013).
[18] O. Vafek and A. Vishwanath, Annu. Rev. Con. Mat. Phys. **5**, 83 (2014).
[19] W. Witczak-Krempa, G. Chen, Y. B. Kim, and L. Balents, Annu. Rev. Con. Mat. Phys. **5**, 57 (2014).
[20] I. F. Herbut and L. Janssen, Phys. Rev. Lett. **113**, 106401 (2014).
[21] J. M. Luttinger, Phys. Rev. **102**, 1030 (1956).
[22] B. A. Bernevig, T. L. Hughes, and S.-C. Zhang, Science **314**, 1757 (2006).
[23] M. König, S. Wiedmann, C. Brüne, A. Roth, H. Buhmann, L. W. Molenkamp, X.-L. Qi, and S.-C. Zhang, Science **318**, 766 (2007).
[24] We note that there is no true particle-hole symmetry due to the spin structure of the system (see Supplementary Material [26]).
[25] See Supplementary Material at [URL] for details about the renormalization-group analysis of the LAB model and technical details of the kinetic equation calculation.
[26] This restriction is due to technical feature of the kinetic equation calculation. A consistent large-N theory needs to includes the full frequency dependent 1-loop renormalized interaction; see [1].
[27] This form assumes the hydrodynamic limit, so the viscosity is well defined and certain $\delta(\omega)$ terms in the general Kubo formula cancel. In general, the Kubo formula for the complex viscosity also contains additional contact terms. See the in-depth discussion in B. Bradlyn, M. Goldstein, and N. Read, Phys. Rev. B **86**, 245309 (2012).
[28] S. Jeon, Phys. Rev. D **52**, 3591 (1995).
[29] S. Jeon and L. G. Yaffe, Phys. Rev. D **53**, 5799 (1996).
[30] E. M. Lifshitz and L. P. Pitaevskii, *Physical Kinetics*, Course of Theoretical Physics, Vol. 10 (Pergamon Press, Oxford, 1981).
[31] J. M. Ziman, *Electrons and Phonons* (Oxford University Press, Oxford, 1960).
[32] P. Arnold, G. D. Moore, and L. G. Yaffe, J. High Energy Phys. **2000**, 001 (2000).
[33] A. A. Abrikosov and I. M. Khalatnikov, Reports on Progress in Physics **22**, 329 (1959).
[34] D. T. Son, Phys. Rev. B **75**, 235423 (2007).
[35] D. E. Sheehy and J. Schmalian, Phys. Rev. Lett. **99**, 226803 (2007).
[36] K. G. Wilson and J. Kogut, Physics Reports **12**, 75 (1974).
[37] J. Zinn-Justin, *Quantum field theory and critical phenomena*, 4th ed., International series of monographs on physics, Vol. 113 (Clarendon Press, Oxford, 2002).
[38] See e.g. [1, 41, 42] and references therein.
[39] L. Janssen and I. F. Herbut, ArXiv e-prints (2015), arXiv:1503.04242 [cond-mat.str-el].
[40] C. W. Hicks, A. S. Gibbs, A. P. Mackenzie, H. Takatsu, Y. Maeno, and E. A. Yelland, Phys. Rev. Lett. **109**, 116401 (2012).
[41] S. Gazit, D. Podolsky, A. Auerbach, and D. P. Arovas, Phys. Rev. B **88**, 235108 (2013).
[42] W. Witczak-Krempa, E. S. Sorensen, and S. Sachdev, Nat Phys **10**, 361 (2014).