



This is the accepted manuscript made available via CHORUS. The article has been published as:

Topological orders with global gauge anomalies

Yi-Zhuang You and Cenke Xu

Phys. Rev. B **92**, 054410 — Published 6 August 2015

DOI: [10.1103/PhysRevB.92.054410](https://doi.org/10.1103/PhysRevB.92.054410)

Topological Orders with Global Gauge Anomalies

Yi-Zhuang You¹ and Cenke Xu¹

¹*Department of physics, University of California, Santa Barbara, CA 93106, USA*

By definition, the physics of the d -dimensional (dim) boundary of a $(d+1)$ -dim symmetry protected topological (SPT) state cannot be realized as itself on a d -dim lattice. If the symmetry of the system is unitary, then a formal way to determine whether a d -dim theory must be a boundary or not, is to couple this theory to a gauge field (or to “gauge” its symmetry), and check if there is a gauge anomaly. In this paper we discuss the following question: can the boundary of a SPT state be driven into a fully gapped topological order which preserves all the symmetries? We argue (conjecture) that if the gauge anomaly of the boundary is “perturbative”, then the boundary must remain gapless; while if the boundary only has global gauge anomaly but no perturbative anomaly, then it is possible to gap out the boundary by driving it into a topological state, when $d \geq 2$. We will demonstrate this conjecture with two examples: (1) the $3d$ spin-1/2 chiral fermion with the well-known Witten’s global anomaly [1], which can be realized on the boundary of a $4d$ topological superconductor with $SU(2)$ or $U(1) \rtimes Z_2$ symmetry; and (2) the $4d$ boundary of a $5d$ topological superconductor with the same symmetry. We show that these boundary systems can be driven into a fully gapped Z_{2N} topological order with topological degeneracy, but this Z_{2N} topological order cannot be further driven into a trivial confined phase that preserves all the symmetries due to some special properties of its topological defects. Our study also leads to exotic states of matter in pure $3d$ space.

I. 1. INTRODUCTION

The contrast between bulk and boundary is the most general and important feature of all symmetry protected topological (SPT) states. A SPT state has a gapped and nondegenerate bulk state, but it must also have a nontrivial boundary. A nontrivial boundary must satisfy two criteria: (1) the boundary must be either gapless or degenerate, as long as the symmetry of the system is not explicitly broken; (2) the low energy physics of the boundary cannot be realized as a lower dimensional system as itself. For example, in the noninteracting case, the $2d$ boundary of the $3d$ topological insulator is a single (or odd number of) massless $2d$ Dirac fermion, which cannot exist in any $2d$ free fermion lattice model with time-reversal and charge $U(1)$ symmetry, and it will remain gapless as long as both symmetries are preserved [2–4].

The second criterion of SPT states is especially important, it implies that if we attempt to regularize the boundary of a SPT state as a lower dimensional system, some “anomaly” will occur. The most well-known example of anomaly is the $U(1)$ gauge anomaly of chiral fermions in odd spatial dimensions. For example, let us consider a $1d$ left-moving complex chiral fermion, and let us assume there is an exact $U(1)$ symmetry associated with the charge conservation (this exact $U(1)$ symmetry is an important assumption in the no-go theorem proved in Ref. [5, 6]). If this $U(1)$ symmetry exists in the fully regularized lattice model, then there should be no problem of enhancing this global $U(1)$ symmetry to a local $U(1)$ gauge symmetry, *i.e.* we should be able to couple this chiral fermion to a $U(1)$ gauge field. However, it is well-known that a chiral fermion coupled to $U(1)$ gauge field will have gauge anomaly: namely the gauge current is no longer conserved: $\partial_\mu j_\mu \sim F_{01}$, which causes inconsistency (anomaly). This anomaly implies that a $1d$ chi-

ral fermion can only exist at the boundary of a $2d$ system, and the physical interpretation of the chiral anomaly is merely the quantum Hall physics: charge is accumulated at the boundary when magnetic flux is adiabatically inserted in the $2d$ bulk. The anomaly of the boundary of $3d$ topological insulator was discussed in Ref. 7. More general relation between boundary anomaly and bulk SPT states has been studied systematically in Ref. 8–12.

Generally speaking, bulk states and boundary states do not have one-to-one correspondence, *i.e.* the bulk state does not uniquely determine its boundary, but the boundary state will determine the bulk state [13]. The boundary state of a SPT state depends on the Hamiltonian at the boundary, or in other words depends on how the bulk Hamiltonian “terminates” at the boundary. Thus different boundary states can belong to the same “universality class”, if they correspond to the same bulk state. Different boundary states belonging to the same universality class must share the same universal properties, and these universal properties are precisely the “anomalies”.

Based on the definition of SPT states, the boundary of all $1d$ SPT states must be degenerate; the boundary of all $2d$ SPT states must be either gapless or spontaneously break certain discrete symmetry which leads to ground state degeneracy; the boundary of SPT states on three and higher spatial dimensions has even richer possibilities: besides gapless spectrum and spontaneous symmetry breaking, the boundary can also have fully gapped topological order which preserves all the symmetries of the system. This last possibility is what we will study in this paper. The boundary topological order, although gapped, must still be anomalous, namely it cannot be realized as a lower dimensional system itself, and it cannot be further driven into a fully gapped and nondegenerate state without breaking symmetry. For example,

the “anomalous” boundary topological order of $3d$ topological insulator and topological superconductor $^3\text{He-B}$ phase has already been studied [14–18]. And one natural boundary state of a $3d$ bosonic SPT state [19, 20] is a \mathbb{Z}_2 topological orders whose e and m excitations carry fractional quantum numbers of the symmetry [21, 22] (some systems can also have a different boundary topological order with semions [23]), and this particular kind of fractionalization cannot exist in $2d$, even though it is consistent with all the fusion rules of e and m excitations.

In this paper we will focus on the SPT states whose symmetry group G is unitary. With unitary symmetries, there is a formal way to determine if a d –dimensional low energy theory is anomalous or not: we can couple the system to gauge field with gauge group G (or in other words we “gauge” the symmetry G), and check if there is any gauge anomaly. This procedure does not directly apply to the $2d$ boundary of many $3d$ TI/TSC, because these systems usually involve a nonunitary time-reversal symmetry which cannot be “gauged”.

Gauge anomaly is very well studied in high energy physics. It turns out that there is a *precise* correspondence between the free fermion topological insulator/superconductor with unitary symmetry G in $(d + 1)$ –dimensional space, and the gauge anomaly at its d –dimensional boundary space after gauging: if the bulk classification is \mathbb{Z} , its boundary must have perturbative gauge anomaly; if the bulk classification is \mathbb{Z}_2 , its boundary must have global gauge anomaly after gauging. Notice that for TSC with no symmetry at all, because its boundary modes can only couple to gravitational field, its boundary has a precise correspondence with gravitational anomalies computed in Ref. 24, see Tab. I [40].

Not all SPT states can have fully gapped symmetric boundary topological order, even in dimensions higher than 3. First of all, if after gauging, the boundary of a SPT state has perturbative anomaly (such as $U(1)$ chiral anomaly) which can be calculated using standard perturbation theory, it can never be gapped out into a boundary topological order, because this boundary must respond to weak background gauge field configurations by generating persistent gauge currents and charge accumulation, thus the boundary must remain gapless as long as its symmetry is preserved. The well-known “anomaly matching condition” was meant to deal with the perturbative anomaly only [25, 26], although the concept of topological order was not developed by then. This conclusion will be further demonstrated with concrete examples in the next section. However, if after gauging a boundary has global gauge anomaly, then gaplessness is no longer a necessity, which means that it is at least possible to drive the boundary into a fully gapped topological order which inherits the global anomaly. In this paper we will investigate the only two systems in Tab. I with global gauge anomalies: the Witten’s anomaly [1] in $(3 + 1)d$ (boundary of a $(4 + 1)d$ topological superconductor), and the analogue of Witten’s anomaly in $(4 + 1)d$ (boundary of a $(5 + 1)d$ system). We will demonstrate

that it is possible to drive these boundary systems into a topological order, and the topological order cannot be further driven into a gapped nondegenerate symmetric trivial confined phase, because of their anomalies. Although we use the boundary of $(4 + 1)d$ system as an example, the exotic topological order constructed in our work can also be realized in $3d$ bulk system as well, which we will discuss in section 5.

We also note that in Ref. 27, a “symmetry enforced gapless” state is proposed for the $2d$ boundary of the $3d$ fermionic SPT state with $SU(2)$ and time-reversal symmetry. The authors argued that this boundary cannot be gapped into a topological order with the full $SU(2)$ and time-reversal symmetry. We want to stress that in our paper we restrict our discussion to the cases with unitary symmetries, so that we can “gauge” all the symmetries, and make a precise comparison between the classification of TI/TSC with the well-known gauge anomalies. Please note that in this work we only “gauge” the global symmetry G to expose the anomaly of the boundary. When we construct the topological order, the global symmetry G is actually *not* gauged, so the topological order is a G –symmetry enriched topological order.

TABLE I: The correspondence between bulk classifications of noninteracting topological insulator (TI) and topological superconductor (TSC) in each spatial dimension d [28–30] and gauge anomalies at $d - 1$ dimensional boundary. This table is periodic with periodicity 8. The first row corresponds to the TSC without any symmetry, thus its boundary can only have gravitational anomaly. G and P stand for global and perturbative anomalies respectively.

Bulk TI/TSC classification										
d	1	2	3	4	5	6	7	8	9	10
none	\mathbb{Z}_2	\mathbb{Z}	0	0	0	\mathbb{Z}	0	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}
$U(1)$	0	\mathbb{Z}	0	\mathbb{Z}	0	\mathbb{Z}	0	\mathbb{Z}	0	\mathbb{Z}
$SU(2)$	0	\mathbb{Z}	0	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}	0	0	0	\mathbb{Z}
Boundary anomaly										
$d - 1$	1	2	3	4	5	6	7	8	9	
Grav.	P				P		G	G	P	
$U(1)$	P		P		P		P		P	
$SU(2)$	P		G	G	P				P	

II. 2. EXAMPLE WITH PERTURBATIVE ANOMALY

In this section we will discuss the most classic system with perturbative gauge anomaly: the $(3 + 1)d$ chiral fermion with a chiral $U(1)$ global symmetry. The Hamiltonian of this system reads

$$H = \int d^3x \psi^\dagger (i\boldsymbol{\sigma} \cdot \boldsymbol{\partial}) \psi, \quad (1)$$

with an exact U(1) symmetry $\psi \rightarrow e^{i\theta}\psi$, this Hamiltonian can never be regularized as a $3d$ system, it must be a boundary of a $4d$ integer quantum Hall state. Notice that this exact U(1) symmetry was an important assumption in the famous no-go theorem proved in Ref. 5, 6. If we couple Eq. (1) to a dynamical U(1) gauge field A_μ , then the gauge current would be anomalous:

$$\partial_\mu j_\mu \sim \epsilon_{\mu\nu\rho\tau} F_{\mu\nu} F_{\rho\tau}. \quad (2)$$

This anomaly means that we cannot view the boundary as an independent quantum system. Once we take the entire system into account, gauge anomalies from two opposite boundaries will cancel each other.

This gauge anomaly is “perturbative”, in the sense that it can be computed by standard perturbation theory. Based on the argument from the introduction, Eq. (1) cannot be gapped out without breaking the U(1) symmetry. We can try gapping out Eq. (1) following the same strategy as Ref. 15–18, 27, namely we first break the U(1) symmetry and gap out Eq. (1), then try to restore the U(1) symmetry by proliferating/condensing the topological defects of the U(1) order parameter. Ref. 15–18, 27 discussed how to drive the boundary of $3d$ topological insulator to a topological order, starting with the superconductor phase of the boundary which spontaneously breaks the U(1) symmetry. At the boundary of $3d$ topological insulator, the simplest vortex that can condense has winding number 4 (a strength 4 vortex), and after condensation the boundary is driven into a fully gapped topological state.

In our current case, the complex U(1) order parameter ϕ would couple to the fermions in the following way:

$$\phi \psi^\dagger i\sigma^y \psi + H.c. \quad (3)$$

In $3d$ space, the topological defect of a U(1) order parameter is its vortex loop, and in principle after condensing the vortex loops the U(1) global symmetry will be restored. However, the resultant state may or may not be a fully gapped state, depending on the spectrum of the vortex loop. If the vortex loop itself is gapless, then the condensate of the vortex loop cannot be a gapped state. In our current case, the vortex loop has a $1d$ chiral fermion, which cannot be gapped out at all by any interaction. Thus the way to gap out the boundary states in Ref. 15–18, 27 fails in this situation.

The same conclusion holds for arbitrary copies of Eq. (1), and for vortex loops with arbitrary strength (winding number). For example, for a single chiral fermion, a strength- N vortex loop has $1d$ chiral fermions with chiral central charge $c = N$, which still cannot be gapped out at all. Thus our conclusion is that Eq. (1) with U(1) anomaly can never be gapped out even by topological order.

III. 3d TOPOLOGICAL ORDER WITH WITTEN’S ANOMALY

A. Physical consequence of Witten’s anomaly

The most well-known example of global anomaly, is the SU(2) global anomaly of $(3+1)d$ chiral fermions discovered by Witten [1]. Let us consider a $(3+1)d$ chiral fermion which forms a fundamental representation of SU(2):

$$H = \int d^3x \sum_{a=1}^2 \psi_a^\dagger i\boldsymbol{\sigma} \cdot \boldsymbol{\partial} \psi_a + \dots, \quad (4)$$

under SU(2) transformation, $\psi_a \rightarrow \exp(i\boldsymbol{\tau} \cdot \boldsymbol{\theta}/2)_{ab} \psi_b$. In order to guarantee the chemical potential locates right at the Dirac point, we assume an extra inversion combined with particle-hole symmetry on the system:

$$\mathcal{IC} : \psi \rightarrow \sigma^y \tau^y \psi^\dagger, \quad \mathbf{r} \rightarrow -\mathbf{r}. \quad (5)$$

This symmetry commutes with the global SU(2). After we couple this system to a dynamical SU(2) gauge field, then there is a large gauge transformation that changes the sign of the partition function [1], which implies that the total partition function of Eq. (4) vanishes after considering all the gauge sectors. This anomaly comes from the mathematical fact that $\pi_4[S^3] = \mathbb{Z}_2$, and it implies that *even without the SU(2) gauge field*, odd copies of $(3+1)d$ chiral fermions with exact SU(2) global symmetry cannot be realized in $3d$ space, it must be the boundary of a $4d$ system.

A $4d$ topological superconductor with SU(2) symmetry has \mathbb{Z}_2 classification, namely for a single copy of Eq. (4), without breaking the SU(2) symmetry, the system cannot be gapped out at all; while two copies of Eq. (4) can be trivially gapped out without breaking SU(2) symmetry (see appendix A for details). Our goal is to study whether we can gap out one single copy of Eq. (4) by driving the system into a topological order, without breaking any symmetry. In order to do this, we should first make sure the system has no perturbative anomaly. Thus Eq. (4) should *not* have an extra U(1) symmetry $\psi_a \rightarrow e^{i\theta} \psi_a$. The apparent U(1) symmetry of Eq. (4) is merely a low energy emergent phenomenon, or in other words, the U(1) symmetry must be explicitly broken by the lattice model in the bulk, thus rigorously speaking the bulk state must be a topological superconductor rather than a topological insulator.

What is the physical meaning of the SU(2) Witten anomaly? If we view gauge transformation $U(x, \tau)$ as an evolution from $\tau = -\infty$ to $+\infty$, then the space-time configuration of the trouble-making large gauge transformation $U(x, \tau)$ corresponds to first creating a pair of SU(2) soliton and anti-soliton pair in space (the existence of SU(2) soliton is due to the fact that $\pi_3[SU(2)] = \mathbb{Z}$), then rotating the soliton by 2π , and eventually annihilating the pair. Now let us couple the fermion ψ to a SU(2)

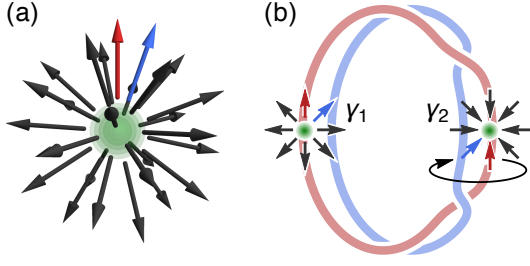


FIG. 1: (a) A hedgehog monopole traps a Majorana zero mode at its core. To keep track of the framing, the $\mathbf{n} = (0, 0, 1)$ vector and a nearby vector which is deviated along a tangent direction are colored in red and blue respectively. (b) The Hopf soliton is created by an event which corresponds to twisting a hedgehog monopole in the $3d$ space, i.e. the red and the blue vectors trace out the two edges of a self-twisted ribbon.

vector \mathbf{n} :

$$\mathbf{n} \cdot \text{Re}[\psi^\dagger \sigma^y \otimes \tau^y \psi], \quad (6)$$

The large gauge transformation $U(x, \tau)$ can be translated into a space-time configuration of $\mathbf{n}(x, \tau)$: the process that causes the partition function to change sign, corresponds to first creating a pair of Hopf soliton and anti-soliton pair of \mathbf{n} in space (the existence of Hopf soliton of \mathbf{n} is due to the fact $\pi_3[S^2] = \mathbb{Z}$), then rotating the Hopf soliton by 2π , and eventually annihilating the pair (more detail about this process is explained in the appendix B). This interpretation of $SU(2)$ anomaly using Hopf soliton of \mathbf{n} is equivalent to Witten's interpretation.

The fact that Hopf soliton changes sign under 2π rotation, implies that Hopf soliton is a fermion. How do we understand the fermion carried by the Hopf soliton? This was answered in Ref. 31, 32. To create a Hopf soliton from vacuum, we can first create a pair of hedgehog monopole anti-monopole pair of \mathbf{n} , then rotate the monopole by 2π , and annihilate the pair, as illustrated in Fig. 1. The final configuration of \mathbf{n} at $\tau = +\infty$ compared with the initial configuration at $\tau = -\infty$ has one extra Hopf soliton. It is well-known that a hedgehog monopole of \mathbf{n} has a Majorana fermion zero mode γ localized at the core of the monopole. A pair of well-separated monopole anti-monopole defines two different quantum states with opposite fermion parity: $(-1)^{N_f} = 2i\gamma_1\gamma_2 = \pm 1$. If the monopole anti-monopole pair has a finite distance, then after rotating the monopole by 2π , there will be a level crossing in the fermion spectrum, which causes change of fermion parity of the ground state of the system [32]. This analysis explains why the Hopf soliton carries a fermion, and also explains the physical meaning of the Witten's anomaly.

The above physical interpretation of Witten's anomaly implies that this global anomaly also exists in systems whose symmetry is a subgroup of $SU(2)$, as long as the system still has hedgehog monopole defect, and the de-

fect carries a Majorana fermion zero mode. For the convenience of later analysis, let us consider Eq. (4) with $U(1) \rtimes Z_2$ symmetry (a rotation around \hat{z} axis and π -rotation around \hat{x} axis), which is a subgroup of $SO(3)$:

$$U(1) : \psi \rightarrow e^{i\tau^z \theta/2} \psi, \quad (n_1 + in_2) \rightarrow e^{i\theta} (n_1 + in_2)$$

$$R_{x,\pi} : \psi \rightarrow i\tau^x \psi, \quad (n_1, n_2, n_3) \rightarrow (n_1, -n_2, -n_3). \quad (7)$$

With this $U(1) \rtimes Z_2$ symmetry, the classification of the $4d$ bulk topological superconductor is unchanged, namely Eq. (4) with this reduced $U(1) \rtimes Z_2$ symmetry still cannot exist in $3d$ space. But with this $U(1) \rtimes Z_2$ symmetry, a hedgehog monopole of \mathbf{n} becomes a domain wall of Ising order parameter n_3 inside a vortex line of $U(1)$ order parameter $n_1 + in_2$, and it indeed still carries a Majorana zero mode. The Hopf soliton, though also deformed compared with the $SU(2)$ invariant case (see Fig. 2), must still be a fermion.

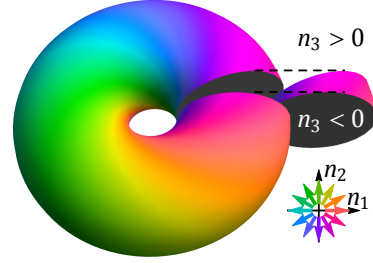


FIG. 2: Hopf soliton in the Z_2 anisotropic limit can be considered as a torus shape domain wall of n_3 , on which $(n_1 + in_2)$ (color coded) winds by 2π around both the meridian and the longitudinal circles of the torus. A slice of the torus is cut out to show the sign change of n_3 across the interior and the exterior of the torus. This configuration can also be viewed as a link of two meron loops with opposite sign of n_3 respectively.

In Ref. 31, 32, because there is no such Z_2 symmetry which transforms $n_3 > 0$ to $n_3 < 0$ (n_3 in our case corresponds to the mass term of bulk Dirac fermion in Ref. 31, 32, and the system always polarizes n_3 to be either $n_3 > 0$ or $n_3 < 0$, except for the bulk quantum critical point between TI and trivial insulator), the system discussed therein can exist in $3d$, and it is precisely the ordinary $3d$ topological insulator. In $3d$ TI, the hedgehog monopole and Hopf soliton are always confined because n_3 is always polarized; while in our case, these defects can be deconfined, and this is a key difference between our $3d$ boundary system and the $3d$ bulk system in Ref. 31, 32.

B. \mathbb{Z}_{2N} topological order

Now let us first gap out Eq. (4) by condensing a superfluid order parameter $n_1 + in_2$, then try to restore the $U(1)$ symmetry by condensing the vortex loops. Inside the vortex loop, if $n_3 = 0$, there will be a $1d$ counter

propagating nonchiral gapless Majorana fermion localized in the vortex loop. A nonzero n_3 will open up a gap for this localized modes, and lower the energy of the vortex loop. Thus energetically the system favors n_3 to be nonzero in the vortex loop, hence the vortex loop actually becomes a “meron” loop, and unlike an ordinary vortex loop, the vector order parameter \mathbf{n} is always smooth around the meron loop. Because n_3 can take either positive or negative expectation values, thus there are two flavors of meron loops, whose domain wall is the hedgehog monopole of \mathbf{n} . With nonzero n_3 inside the meron loop, the fermion spectrum remains fully gapped. The only potential low energy fermion excitations are localized inside the hedgehog monopole (domain of n_3 in a meron loop), but in this work we will always keep the hedgehog monopole either gapped or confined. The Hopf soliton now becomes a *link* between the two flavors of meron loops (see Fig. 2). Because of nonzero n_3 in the meron loop, this meron link is a nonsingular smooth configuration of vector \mathbf{n} .

Let us tentatively ignore the background fermions and the Witten’s anomaly. Using the standard dual description of superfluid in $(3+1)d$, we can describe these two meron loops by two gauge fields $b_{1,\mu}$ and $b_{2,\mu}$. The effective $4d$ Euclidean space-time theory for meron loops read [33]:

$$\mathcal{S} = \sum_{\mathbf{x}} \sum_{\mu\nu} \sum_{c=1,2} -t \cos(\nabla_\mu b_{c,\nu} - \nabla_\nu b_{c,\mu} - 2\pi B_{\mu\nu}) + \frac{1}{K} (\epsilon_{\nu\rho\tau} \nabla_\nu B_{\rho\tau})^2. \quad (8)$$

The sum is taken over all space-time position \mathbf{x} and plaquettes. $B_{\mu\nu}$ is a rank-2 antisymmetric tensor field, which is the dual of the Goldstone mode of the order parameter $(n_1 + in_2)$. $\Psi_{c,\mu}^\dagger \sim \exp(ib_{c,\mu})$ creates a segment of meron loop with flavor c along the direction μ , and $\exp(i\nabla \times b)$ creates a small meron loop.

The pure bosonic theory Eq. (8) can have the following different phases:

(1) Ordinary superfluid phase. This phase corresponds to the case when all loops are small and gapped. Then the system has only one gapless mode described by $B_{\mu\nu}$.

(2) U(1) liquid phase with gapless photon excitation, which was discussed in Ref. 33. This phase corresponds to the case when all monopoles are gapped, while both flavors of meron loops $b_{1,\mu}$ and $b_{2,\mu}$ condense. In this phase, the linear combination $b_{1,\mu} + b_{2,\mu}$ (which corresponds to the bound state between the two flavors of vortices $\Psi_{1,\mu}^\dagger \Psi_{2,\mu}^\dagger$) will “Higgs” and gap out $B_{\mu\nu}$, and the combination $b_{1,\mu} - b_{2,\mu}$ becomes the gapless photon mode of the U(1) liquid.

(3) A *fully gapped* \mathbb{Z}_{2N} topological order which preserves all the symmetries. This is a phase where individual loop $b_{1,\mu}$ and $b_{2,\mu}$ does *not* condense, but meron bound state $(\Psi_{1,\mu}^\dagger \Psi_{2,\mu}^\dagger)^N \sim \exp(iNb_{1,\mu} + iNb_{2,\mu}) = \exp(ib_\mu)$ condense and gap out $B_{\mu\nu}$ through Higgs mechanism. Because now b_μ is a bound state of $2N$ meron

loops, this phase is a \mathbb{Z}_{2N} topological order. It is well-known that condensation of double vortex/meron loops will lead to a \mathbb{Z}_2 topological order with fractionalization, for instance see Ref. 34–36. A condensate of $2N$ vortex/meron loop bound state can be effectively described by the following action:

$$\mathcal{S} = \sum_{\mathbf{x}} \sum_{\mu\nu} -t \cos(\nabla_\mu b_\nu - \nabla_\nu b_\mu - 2\pi(2NB_{\mu\nu})) + \frac{1}{K} (\epsilon_{\nu\rho\tau} \nabla_\nu B_{\rho\tau})^2. \quad (9)$$

It is clear that when b_μ condenses, $B_{\mu\nu}$ takes only $2N$ discrete values $0, \frac{1}{2N}, \dots, \frac{2N-1}{2N}$, hence the condensate is a \mathbb{Z}_{2N} topological order. Also, under the \mathbb{Z}_2 symmetry transformation, $\Psi_{1,2} \rightarrow \Psi_{2,1}^\dagger$, *i.e.* $b \rightarrow -b$, the vortex/meron loop condensate explicitly preserves the \mathbb{Z}_2 symmetry as long as we take $b = 0$ in the condensate.

The topological order (3) is what we will focus on in this paper. In our case, because of the background fermions and the Witten’s anomaly, there is one subtlety that we need to be careful with. With odd N , say $N = 1$, a bound state $\Psi_1 \Psi_2$ could be a fermion if Ψ_1 and Ψ_2 has odd number of links in the space, due to the Witten’s anomaly. Thus we should condense only the configurations of vortex/meron loop bound state in which Ψ_1 and Ψ_2 are always parallel and properly separately so that they are not linked at all. We assume this can be achieved by turning on local interactions between the loops, although we do not prove this. For even integer N this subtlety does not arise at all, because the link between $(\Psi_1)^N$ and $(\Psi_2)^N$ is always a boson, thus their bound state is free to condense. In the following we will take $N = 1$ as an example (\mathbb{Z}_2 topological order), but our discussion can be generalized to arbitrary integer N .

An ordinary \mathbb{Z}_2 topological order can be driven into a trivial gapped confined phase by proliferating/condensing the “vison loops”. A vison loop in our case is bound with a single meron loop of order parameter $n_1 + in_2$. In the following we will argue that our \mathbb{Z}_2 topological order is a special one, it *cannot* be further driven into a trivial confined phase.

First of all, we still have two flavors of vison loops with fully gapped fermion spectrum, which corresponds to $n_3 > 0$ or $n_3 < 0$ at the vortex/meron core. We will primarily consider the vison loops with uniform n_3 , or in other words the vison loops in which the \mathbb{Z}_2 symmetry and the \mathcal{IC} symmetry are spontaneously broken. This is because if inside the vison loop there is a domain wall of n_3 , at the domain wall there will be a Majorana fermion zero mode, and it is unclear whether these vison loops with Majorana zero modes can condense at all due to the non-Abelian statistics introduced by the Majorana zero modes. And if a vison loop has $\langle n_3 \rangle = 0$, then the fermions will be gapless along the vison loop, and condensing these vison loops will not lead to a trivial gapped confined phase.

If we do not want to break any symmetry, the two flavors of fully gapped vison loops must condense simultane-

ously. However, there is a clear obstacle for condensing both vison loops like an ordinary \mathbb{Z}_2 topological order. This is because when these two different vison loops are linked, the \mathbf{n} configuration around the vison link is a Hopf soliton, and hence it must be a fermion, which is a consequence of the Witten's anomaly. By contrast, if the \mathbb{Z}_2 symmetry is explicitly broken (for instance in the 3d TI), then we can condense just one flavor of vison loops while preserving all the symmetries, then it is possible to get a fully symmetric confined phase.

The hedgehog monopole of the superfluid phase becomes the end point of the loop $b_{1,\mu} - b_{2,\mu}$. While because this loop does not condense in the \mathbb{Z}_2 topological order, this loop still has a finite loop tension, hence the hedgehog monopole which carries Majorana fermion zero mode is still *confined* in the \mathbb{Z}_2 topological order. In Ref. 37 the authors discussed a gapless phase where the hedgehog monopole is deconfined. Whether there is a fully gapped topological phase with deconfined hedgehog monopole which carries Majorana fermion zero mode is an open question.

C. CP^1 formalism

All we have discussed so far can be equivalently formulated in the standard CP^1 formalism, which was also used in Ref. 37 to study the gapless photon phase. In the CP^1 formalism, the order parameter \mathbf{n} is fractionalized into the bosonic spinon $z = (z_1, z_2)^\top$ via $\mathbf{n} = z^\dagger \boldsymbol{\tau} z$ under the constraint of $z^\dagger z = 1$. The constraint can be implemented by the emergent $\text{U}(1)$ gauge field a_μ between the spinons. The symmetry acts on the spinon as $\text{U}(1) : z \rightarrow \exp(i\tau^z \theta/2)z$ and $\text{R}_{x,\pi} : z \rightarrow \tau^x z$. The field theory for both the CP^1 spinon and the $\text{SU}(2)$ chiral fermion on the 3d boundary reads[37]

$$\begin{aligned} \mathcal{S} = & \frac{1}{2g} |(\partial - a)_\mu z|^2 + \mu(z^\dagger z - 1) \\ & + \psi^\dagger (i\partial_0 + i\boldsymbol{\sigma} \cdot \boldsymbol{\partial})\psi + z^\dagger \boldsymbol{\tau} z \cdot \text{Re}[\psi^\dagger \boldsymbol{\sigma}^y \boldsymbol{\tau}^y \psi]. \end{aligned} \quad (10)$$

The spinons z_1 and z_2 carry $\pm 1/2$ $\text{U}(1)$ symmetry charges respectively, and both carry one $\text{U}(1)$ gauge charge.

Let us start with the ordered phase of $\mathbf{n} = z^\dagger \boldsymbol{\tau} z$, *i.e.* the spinon condensed phase $\langle z \rangle \neq 0$. In this phase, the chiral fermion ψ is fully gapped, the gauge $\text{U}(1)$ fluctuation a_μ is Higgsed out by the z condensate, and the symmetry $\text{U}(1)$ is spontaneously broken leading to one gapless Goldstone mode of $n_1 + in_2 \simeq z_1^* z_2$.

We consider the Hopf soliton configuration of \mathbf{n} , which is also a pair of $n_1 + in_2$ vortices (2π symmetry fluxes) linked together. Each vortex must be bound to a π gauge flux of a_μ to reduce the kinetic energy of the spinon, thus the Hopf soliton corresponds to a linking of π gauge fluxes whose linking number is counted by the Chern-Simon term as $\frac{1}{\pi^2} \int a \wedge da = 1$ (see Appendix B the correspondence of Hopf soliton and gauge flux link). Suppose the typical length scale of the Hopf soliton is R , to preserve the linking number given by the

Chern-Simon term, a must scale with R as $a \sim 1/R$, so the Maxwell term of the $\text{U}(1)$ gauge field will contribute energy $E \simeq \int \frac{\kappa}{2} (da)^2 \sim \kappa/R$. In the ordered phase, the condensate of z_1 and z_2 will generate a mass term a^2 to the effective action, then the soliton energy is given by $E \simeq \int \frac{\kappa}{2} (da)^2 + \frac{\rho}{2} a^2$, which scales with R as $E \sim \kappa/R + \rho R$ and is minimized at finite length scale $R_0 \sim (\kappa/\rho)^{1/2}$ with a finite energy $E_0 \sim (\kappa\rho)^{1/2}$. In the \mathbb{Z}_2 topological order we discussed in the last section, although z_1 and z_2 are not individually condensed, the boundary state $z_1 z_2$ is still condensed which breaks the $\text{U}(1)$ gauge field down to \mathbb{Z}_2 , and a mass term a^2 still exists for the gauge field. Thus the Hopf soliton becomes a local object and can be fully gapped out. Since the spinon pair $z_1 z_2$ carries two units of gauge charge and no symmetry charge, the $\text{U}(1)$ gauge structure is broken down to \mathbb{Z}_2 without breaking any physical symmetry, therefore we obtain a fully gapped symmetric \mathbb{Z}_2 topological order on the 3d boundary.

To make connection to the loop theory in Eq. (8), we evoke the duality transformation. To start, we rewrite the CP^1 field $z_c \sim e^{i\theta_c}$ ($c = 1, 2$) in terms of the phase angles θ_c . We can neglect the amplitude fluctuation of each z_c component, as long as we take the easy-plane limit of the system, *i.e.* n_1 and n_2 are energetically more favorable than n_3 . In this limit, the effective action in the Euclidean space-time reads

$$\mathcal{S} = \sum_{c=1,2} -K \cos(d\theta_c - a) \quad (11)$$

We can take the standard Villain form of the action, by expanding the cosine function at its minimum, and introducing the 1-form fields $l_c \in \mathbb{Z}$ and $k_c \in \mathbb{R}$ ($c = 1, 2$):

$$\begin{aligned} \mathcal{Z} = & \text{Tr} \exp \left[\sum_{c=1,2} -\frac{K}{2} (d\theta_c - a - 2\pi l_c)^2 \right] \\ \sim & \text{Tr} \exp \left[\sum_{c=1,2} \frac{1}{2K} k_c^2 + k_c \cdot (d\theta_c - a - 2\pi l_c) \right] \\ \sim & \text{Tr} \exp \left[\sum_{c=1,2} \frac{1}{2K} k_c^2 - k_c \cdot (a + 2\pi l_c) \right] \delta[\partial k_c] \\ \sim & \text{Tr} \exp \left[\sum_{c=1,2} \frac{1}{2K} (dB_c)^2 + (a + 2\pi l_c) \wedge dB_c \right]. \end{aligned} \quad (12)$$

In the last line, we introduce the 2-form fields B_c ($c = 1, 2$) on the dual space-time manifold, such that $k_c = \star dB_c$ resolves the constraint $\partial k_c = 0$. Summing over l_c will require B_c to take only integer values, which could be imposed by adding a $\cos(2\pi B_c)$ term, and the theory now becomes

$$\mathcal{Z} \sim \text{Tr} \exp \left[\sum_{c=1,2} \frac{1}{2K} (dB_c)^2 + a \wedge dB_c - t \cos(2\pi B_c) \right] \quad (13)$$

Integrating out the gauge field a will impose the constraint $d(B_1 + B_2) = 0$, which can be resolved by $B_1 = B - db_1/(2\pi)$, $B_2 = -B + db_2/(2\pi)$. Therefore the

final action takes the form of $\mathcal{S} \sim \sum_{c=1,2} -t \cos(db_c - 2\pi B) + K^{-1}(dB)^2$ which is identical to Eq. (8).

$b_{1,\mu}$ and $b_{2,\mu}$ introduced in Eq. (8) correspond to vortex of z_1 and anti-vortex of z_2 respectively, which both correspond to a vortex of the original order parameter $n_1 + in_2 \sim z_1^* z_2$. condensation of the vortex bound state $b_{1,\mu} + b_{2,\mu}$ in Eq. (8) will disorder the physical order parameter $z_1^* z_2$, but not disorder the condensate $z_1 z_2$, thus the \mathbb{Z}_2 topological state after condensing $b_{1,\mu} + b_{2,\mu}$ is precisely the same \mathbb{Z}_2 topological state after condensation of bound state $z_1 z_2$ in the CP^1 formalism. And the loop excitations $b_{1,\mu}$ and $b_{2,\mu}$ of this \mathbb{Z}_2 topological state both correspond to the π -flux lines of a_μ .

Since our \mathbb{Z}_2 topological order is obtained by condensing pair of $z_1 z_2$ from the $\text{U}(1)$ photon phase, the hedgehog monopole of \mathbf{n} , which in the $\text{U}(1)$ photon phase becomes the Dirac monopole of a_μ , will be *confined* in our \mathbb{Z}_2 topological order. The remnant of the Witten's anomaly is completely encoded in the fact that the link of two flavors of vison loops must be a fermion, thus the \mathbb{Z}_2 topological order cannot be driven into a trivial confined phase that preserves all the symmetries.

As we discussed in the last section, if we start with the superfluid phase and condense vortex/meron bound state $(\Psi_1 \Psi_2)^N$, the system will enter a \mathbb{Z}_{2N} topological order. Starting with the CP^1 formalism, this \mathbb{Z}_{2N} topological order can be understood as following: the condensate $z_1 z_2$ implies that $z_1 \sim z_2^*$. The \mathbb{Z}_{2N} gauge field is introduced by fractionalizing the CP^1 field as $z_1 \sim z_2^* \sim w^N$ with bosonic parton field w . Equivalently we can write $n_1 - in_2 \sim z_1 z_2^* \sim w^{2N}$. Because $z_1 \sim z_2^*$ carries $\text{U}(1)$ charge $1/2$, the parton w carries global $\text{U}(1)$ charge $1/(2N)$, which is consistent with the $2N$ flux condensate. The parton w is coupled to the \mathbb{Z}_{2N} gauge field.

IV. 4. 4d TOPOLOGICAL ORDER WITH GLOBAL ANOMALY

Analogue of Witten's anomaly can be found in higher dimensions. The simplest generalization is in one higher dimension: one single copy of $(4+1)d$ Dirac fermion with $\text{SU}(2)$ or $\text{U}(1) \rtimes \mathbb{Z}_2$ symmetry cannot exist in $(4+1)d$ itself, it must be a boundary of a $5d$ topological superconductor:

$$H = \int d^4x \psi^\dagger (i\mathbf{\Gamma} \cdot \partial) \psi \quad (14)$$

where $\mathbf{\Gamma} = (\Gamma^1, \Gamma^2, \Gamma^3, \Gamma^4)$ and we choose $\Gamma^{1,2,3} = \sigma^3 \otimes \sigma^{1,2,3}$, $\Gamma^{4,5} = \sigma^{1,2} \otimes \sigma^0$. The fermions transform under the symmetry as $\text{U}(1)$: $\psi \rightarrow e^{i\theta/2} \psi$ and $\text{R}_{x,\pi} : \psi \rightarrow i\Gamma^5 \Gamma^2 \psi^\dagger$. Now we couple the fermion ψ to a vector \mathbf{n} as

$$(n_1 - in_2) \psi^\dagger \Gamma^2 \psi + n_3 \psi^\dagger \Gamma^5 \psi + H.c. \quad (15)$$

such that the vector \mathbf{n} transforms as $\text{U}(1)$: $(n_1 + in_2) \rightarrow e^{i\theta} (n_1 + in_2)$ and $\text{R}_{x,\pi} : n_{2,3} \rightarrow -n_{2,3}$.

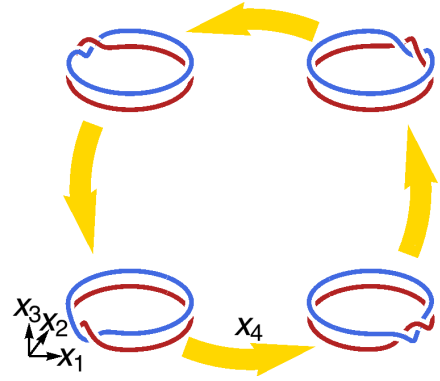


FIG. 3: $\pi_4[S^2]$ soliton in the $4d$ space. As one circles around the 4th dimension x_4 , the Hopf soliton in each $3d$ slice rotates around. The Hopf soliton is represented as linked preimages of $n_3 = \pm 1$ points on S^2 . The configuration also corresponds to two $2d$ gauge flux membranes linked in the $4d$ space, s.t. in each $3d$ slice, the sections of flux membranes are linked flux loops (red and blue loops).

In $4d$ space vector \mathbf{n} also has a nontrivial soliton. The $\pi_4[S^2]$ soliton configuration is given by the non-trivial map $f : S^4 \rightarrow S^2$, which can be considered as the composition of two non-trivial maps $g : S^4 \rightarrow S^3$ and $h : S^3 \rightarrow S^2$ as $f = h \circ g$. The first map g is such that the preimage of each point in S^3 is a circle in S^4 , along which the $3d$ framing twists around once. The second map h is just the standard Hopf map. So the $\pi_4[S^2]$ soliton can be understood by considering $3d$ slices embedded in the $4d$ space, with each slice hosting a Hopf soliton, and the Hopf soliton rotates by 2π as the slice evolves along the 4th dimension. Also, while mapping $4d$ space to S^2 , every preimage of S^2 is a $2d$ manifold (for instance T^2 or S^2). And two disconnected m -dimensional manifolds can have nontrivial linking in $(m+2)$ -dimensional space (knot with codimension-2). A nontrivial $\pi_4[S^2]$ soliton corresponds to the case when the preimages of two arbitrary points on S^2 will be two $2d$ manifolds linked in the $4d$ space.

Now we argue that the $\pi_4[S^2]$ soliton on the $4d$ boundary of the $5d$ topological superconductor is also fermionic. We first consider the $5d$ bulk as a $\mathcal{M}_4 \times S^1$ manifold (see Fig. 4) where \mathcal{M}_4 is a $4d$ manifold, and then compactify the S^1 dimension. Depending on the flux Φ threaded through the S^1 , the compactified effective $4d$ system can either be a trivial superconductor ($\Phi = 0$) or a topological superconductor ($\Phi = \pi$). This can be shown explicitly by the cut-and-glue strategy: first cut the $5d$ bulk along the \mathcal{M}_4 to expose the upper and the lower $4d$ boundaries (green boundaries in Fig. 4), described by $H_{\text{cut}} = \int d^4x \psi_1^\dagger (i\mathbf{\Gamma} \cdot \partial) \psi_1 - \psi_2^\dagger (i\mathbf{\Gamma} \cdot \partial) \psi_2$, and then glue the boundaries together by a coupling term $H_{\text{glue}} = u \int d^4x i\psi_1^\dagger \psi_2 + H.c.$ with the coupling coefficient $u \sim e^{i\Phi}$ depending on the flux Φ through the S^1 . $H_{\text{cut}} + H_{\text{glue}}$ together describes an effective $4d$

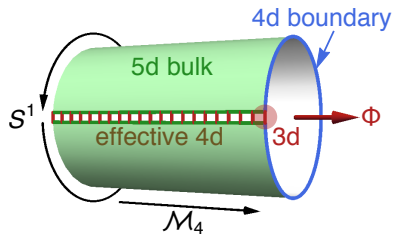


FIG. 4: Compactify the 5d bulk with a flux Φ through the compactified dimension S^1 .

superconductor with the $U(1) \rtimes Z_2$ symmetry that $U(1): \psi_a \rightarrow e^{i\theta/2} \psi_a$ and $R_{x,\pi}: \psi_a \rightarrow i\Gamma^5 \Gamma^2 \psi_a$ ($a = 1, 2$). As the flux $\Phi = 0, \pi$: $u = \pm 1$ plays the role of the topological mass that tunes the 4d effective bulk state between the trivial and the topological phases. In the presence of the π flux ($\Phi = \pi$), $H_{\text{cut}} + H_{\text{glue}}$ together describes an effective 4d topological superconductor protected by $U(1) \rtimes Z_2$ whose 3d boundary has the Witten anomaly, namely the Hopf soliton is fermionic on the compactified 3d boundary. If we revert the compactification, the original 4d boundary (blue boundary in Fig. 4) of the 5d bulk is the S^1 extension of the 3d boundary of the effective 4d bulk. The π flux of the fermions corresponds to the 2π vortex of the order parameter $n_1 + in_2$. So the configuration of \mathbf{n} on the 4d boundary is indeed a 3d Hopf soliton rotated by 2π as it translated around in the S^1 dimension, which corresponds to a $\pi_4[S^2]$ soliton. Thus the fermionic nature of the Hopf soliton on the 3d boundary of the 4d topological superconductor implies that the $\pi_4[S^2]$ soliton on the 4d boundary of the 5d topological superconductor is also fermionic.

We can now drive the 4d boundary to a Z_2 (or Z_{2N}) topological order, following the same strategy of the previous section. We can first condense (n_1, n_2) and spontaneously break the $U(1)$ symmetry. In 4d space, the topological defects of a superfluid phase are 2d vortex membranes, and there are still two flavors of vortex/meron membranes $\Psi_{1,\mu\nu} \sim \exp(ib_{1,\mu\nu})$ and $\Psi_{2,\mu\nu} \sim \exp(ib_{2,\mu\nu})$ depending on the sign of n_3 in the vortex core. Without the n_3 component in the vortex core, the vortex membrane will host a single 2d gapless Majorana cone. In this case, the vortex condensation will lead to gapless boundary which is not what we are after. However, once we introduce the n_3 component in the vortex core, the Majorana cone is gapped out in both b_1 membrane and b_2 membranes. Then by condensing vortex/meron membrane bound state $(\Psi_1 \Psi_2)^N$, this system is driven into a Z_{2N} topological order. Again, in this topological order, Ψ_1 and Ψ_2 become two flavors of 2d unit gauge flux membranes, and when they “link” in 4d space, the configuration of \mathbf{n} around this link will be a $\pi_4[S^2]$ soliton, and hence it must be a fermion. Thus this Z_{2N} topological order can not be further driven into a trivial confined phase, unless we explicitly break the Z_2 symmetry.

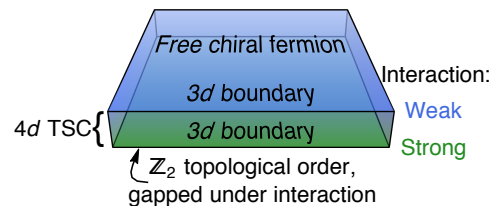


FIG. 5: A slab of 4d topological superconductor with $U(1) \rtimes Z_2$ symmetry. The interaction strength changes from weak on the top boundary (in blue) to strong on the bottom boundary (in green), such that the bottom boundary is gapped out by the Z_2 topological order.

V. 5. IMPLICATION AND SUMMARY

Our analysis in this work implies that we can realize some exotic states in 3d systems. For example, let us consider a slab of 4d system, with a thin fourth dimension, as shown in Fig. 5. Because the fourth dimension is finite, the entire system is three dimensional, but we can still realize two different 3d boundary states on two opposite boundaries: the top boundary is a free chiral fermion Eq. (4) with exact $U(1) \rtimes Z_2$ symmetry, the bottom boundary is the fully gapped Z_2 topological order in which the link of two vison loops is a fermion. This state is possible as long as we make the interaction stronger on the bottom boundary, but weaker on the top boundary. Because a short range interaction on Eq. (4) is irrelevant, Eq. (4) will survive at low energy for weak interaction. But because the bottom boundary is fully gapped, any low energy experiment can only probe the top surface, which may lead to the conclusion that this system is “anomalous”. But the entire system, including both the top and bottom boundary, is anomaly free.

Another way to realize such exotic 3d state, is to separate the left and right chiral fermions in the 3d momentum space, like the situation in the Weyl semimetal 38, and the left (right) Weyl fermions will form a doublet under $SU(2)$ or $U(1) \rtimes Z_2$. The left and right Weyl fermions will each carry the Witten’s anomaly, but the entire 3d system is anomaly free. Then in principle by introducing proper interactions we can drive the left Weyl fermions into the topological order discussed in this work, while keeping the right fermions gapless. One point we need to stress is that, in this realization the global charge $U(1)$ symmetry (which has nothing to do with the $U(1) \rtimes Z_2$ symmetry we want to preserve) needs to be broken to get rid of the perturbative anomaly of the left (or right) Weyl fermions, so rigorously speaking this system is different from a Weyl semimetal.

Bosonic SPT states also have anomalous and nontrivial boundaries [8–12, 19, 20]. For those bosonic SPT states with the same perturbative anomalies as fermionic systems, for instance bosonic integer quantum Hall states with $U(1)$ symmetry in all even dimensions, their bound-

ary cannot be gapped into a topological order at all, for the same reason as the fermionic systems discussed in section 2. But for those bosonic SPT states whose boundary anomalies cannot be represented as response to weak background gauge field (for example a discrete gauge field can not have arbitrarily weak background field), then we believe their boundaries can be driven into a gapped topological order. Examples of these boundary topological orders of bosonic SPT states were discussed in Ref. [21, 22].

In the following we will list a few open questions that we were not able to address in this paper:

i. In this paper we have understood the topological order for systems with $U(1) \rtimes Z_2$ symmetry, which still has Witten's anomaly. However, our formalism in terms of vortex loop condensation does not directly apply to systems with $SU(2)$ symmetry, because a precise duality formalism has not been developed for systems with $SU(2)$ symmetry, thus we have not proved that our topological order can survive in the $SU(2)$ limit, although we do not see a fundamental obstacle for that.

ii. As we explained in this paper, in our topological order, the nonabelian topological defect which carries the

Majorana fermion zero mode, *i.e.* the hedgehog monopole of \mathbf{n} , is still confined. Whether there is a fully gapped topological order with deconfined $3d$ nonabelian defect is still an open question.

iii. In this paper we studied the boundary of two topological superconductors whose boundary states have global gauge anomaly after “gauging”, and we demonstrated that these two systems can both be driven into a boundary topological orders. But if a system involves nonunitary symmetries that cannot be “gauged”, the situation seems to be more complicated. As we mentioned in the introduction, Ref. 27 has given us an example of $3d$ topological superconductor whose boundary can never be driven into a gapped topological order. Thus a more refined classification of “gappable” and “ungappable” anomalous systems is demanded for systems that involve time-reversal symmetry.

The authors are supported by the the David and Lucile Packard Foundation and NSF Grant No. DMR-1151208. The authors thank Xiao-Gang Wen, Chetan Nayak, Xiao-Liang Qi and Sung-Sik Lee for very helpful discussions.

-
- [1] E. Witten, Phys. Lett. **117B**, 324 (1982).
 - [2] L. Fu, C. L. Kane, and E. J. Mele, Phys. Rev. Lett. **98**, 106803 (2008).
 - [3] J. E. Moore and L. Balents, Phys. Rev. B **75**, 121306(R) (2007).
 - [4] R. Roy, Phys. Rev. B **79**, 195322 (2009).
 - [5] H. B. Nielsen and M. Ninomiya, Nucl. Phys. B **185**, 20 (1981).
 - [6] H. B. Nielsen and M. Ninomiya, Nucl. Phys. B **193**, 173 (1981).
 - [7] M. A. Metlitski, C. L. Kane, and M. P. A. Fisher, Phys. Rev. B **88**, 035131 (2013).
 - [8] X.-G. Wen, Phys. Rev. D **88**, 045013 (2013).
 - [9] A. Kapustin, arXiv:1404.6659 (2014).
 - [10] A. Kapustin, arXiv:1403.1467 (2014).
 - [11] J. Wang, Z.-C. Gu, and X.-G. Wen, Phys. Rev. Lett. **114**, 031601 (2015).
 - [12] A. Kapustin and R. Thorngren, Phys. Rev. Lett. **112**, 231602 (2014).
 - [13] L. Kong and X.-G. Wen, arXiv:1405.5858 (2014).
 - [14] L. Fidkowski, X. Chen, and A. Vishwanath, Phys. Rev. X **3**, 041016 (2013).
 - [15] X. Chen, L. Fidkowski, and A. Vishwanath, Phys. Rev. B **89**, 165132 (2014).
 - [16] P. Bonderson, C. Nayak, and X.-L. Qi, J. Stat. Mech. p. P09016 (2013).
 - [17] C. Wang, A. C. Potter, and T. Senthil, Phys. Rev. B **88**, 115137 (2013).
 - [18] M. A. Metlitski, C. L. Kane, and M. P. A. Fisher, arXiv:1306.3286 (2013).
 - [19] X. Chen, Z.-C. Gu, Z.-X. Liu, and X.-G. Wen, Phys. Rev. B **87**, 155114 (2013).
 - [20] X. Chen, Z.-C. Gu, Z.-X. Liu, and X.-G. Wen, Science **338**, 1604 (2012).
 - [21] A. Vishwanath and T. Senthil, Phys. Rev. X **3**, 011016 (2013).
 - [22] Z. Bi, A. Rasmussen, and C. Xu, Phys. Rev. B **91**, 134404 (2015).
 - [23] X. Chen, F. J. Burnell, A. Vishwanath, and L. Fidkowski, arXiv:1403.6491 (2014).
 - [24] L. Alvarez-Gaume and E. Witten, Nucl. Phys. B **234**, 269 (1983).
 - [25] G. 't Hooft, *Recent Developments in Gauge Theories*, (Plenum Press, New York; reprinted in “Unity of Forces in the Universe”, edited by A. Zee (World Scientific, Singapore, 1982), Vol. II, p. 1004., 1980).
 - [26] A. Zee, Physics Letters B **95**, 290 (1980).
 - [27] C. Wang and T. Senthil, Phys. Rev. B **89**, 195124 (2014).
 - [28] A. P. Schnyder, S. Ryu, A. Furusaki, and A. W. W. Ludwig, AIP Conf. Proc. **1134**, 10 (2009).
 - [29] S. Ryu, A. Schnyder, A. Furusaki, and A. Ludwig, New J. Phys. **12**, 065010 (2010).
 - [30] A. Kitaev, AIP Conf. Proc. **1134**, 22 (2009).
 - [31] J. C. Y. Teo and C. L. Kane, Phys. Rev. Lett. **104**, 046401 (2010).
 - [32] Y. Ran, P. Hosur, and A. Vishwanath, Phys. Rev. B **84**, 184501 (2011).
 - [33] O. I. Motrunich and T. Senthil, Phys. Rev. B **71**, 125102 (2005).
 - [34] L. Balents, M. P. A. Fisher, and C. Nayak, Phys. Rev. B **60**, 1654 (1999).
 - [35] T. Senthil and M. P. A. Fisher, Phys. Rev. B **63**, 134521 (2001).
 - [36] L. Balents, M. P. A. Fisher, and S. M. Girvin, Phys. Rev. B **65**, 224412 (2002).
 - [37] M. Freedman, M. B. Hastings, C. Nayak, X.-L. Qi, K. Walker, and Z. Wang, Phys. Rev. B **83**, 115132 (2011).
 - [38] X. Wan, A. M. Turner, A. Vishwanath, and S. Y.

Savrasov, Phys. Rev. B **83**, 205101 (2011).

[39] Y.-Z. You and M. Cheng, ArXiv e-prints (2015), 1502.03192.

[40] This table is for illustration purposes, and only the well-understood free fermion topological insulators/superconductors are listed here. So far it is not

completely clear whether strong interaction can give us new topological insulators with the symmetries in Table Tab.I. If future studies confirm that such new topological insulators do exist, this table may need updates.

Appendix A: A. Topological superconductors with SU(2) symmetry

The topological superconductors/insulators with SU(2) symmetry belongs to the symmetry class C in the classification table[28, 29] of fermion SPT phases. In $(4+1)d$ and $(5+1)d$, the SU(2) fermion SPT phases are \mathbb{Z}_2 classified, and the classification remains the same under the interaction. In the following we will give a brief introduction to these topological superconductors/insulators with explicit model Hamiltonian and symmetry actions.

1. A1. $(4+1)d$ bulk with $(3+1)d$ boundary

The $4d$ topological superconductor with SU(2) symmetry can be described by the following lattice model

$$H = \sum_{\mathbf{k}} \sum_{a=1,2} \psi_a^\dagger \left(\sum_{i=1}^4 \sin k_i \Gamma^i + \left(\sum_{i=1}^4 \cos k_i - 4 + m \right) \Gamma^5 \right) \psi_a, \quad (\text{A1})$$

where the fermion $\psi = (\psi_1, \psi_2)$ forms a fundamental representation of the SU(2) symmetry: $\psi \rightarrow \exp(i\boldsymbol{\tau} \cdot \boldsymbol{\theta}/2)\psi$. The Γ^i matrices are defined as $\Gamma^{1,2,3} = \sigma^3 \otimes \sigma^{1,2,3}$, $\Gamma^4 = \sigma^1 \otimes \sigma^0$, $\Gamma^5 = \sigma^2 \otimes \sigma^0$ with $\sigma^{0,1,2,3}$ being the Pauli matrices. The model has an emergent U(1) symmetry $\psi \rightarrow e^{i\theta}\psi$, which is not required. It is possible to turn on a weak SU(2)-singlet pairing ($\Delta\psi^\dagger\tau^y\psi + H.c.$) to break the U(1) symmetry explicitly in the bulk, while still retaining the SU(2) symmetry, hence the system is a superconductor in general. When $m > 0$ ($m < 0$), the model is in its topological (trivial) phase.

The $3d$ boundary of the $4d$ topological superconductor will host the gapless chiral fermion which carries the SU(2) fundamental representation. The boundary effective Hamiltonian is

$$H_\partial = \int d^3x \sum_{a=1,2} \psi_a^\dagger (i\boldsymbol{\partial} \cdot \boldsymbol{\sigma}) \psi_a, \quad (\text{A2})$$

where $\psi = (\psi_1, \psi_2)^\dagger$ forms the SU(2) doublet (as inherited from the bulk fermion). The boundary fermion mode can not be gapped out (on the free fermion level) due to the SU(2) anomaly. The only possible fermion mass term that can be added to the boundary theory is the pairing term $\Delta_{ab}\psi_a^\dagger i\sigma^2\psi_b$, however such term necessarily breaks the SU(2) symmetry. Because the fermion statistics requires $\Delta_{ab} = \Delta_{ba}$ to be symmetric, so the pairing term must be an SU(2) triplet, and thus breaks the symmetry. However if we double the system, then we can gap out the boundary by introducing the pairing term $\psi^\dagger i\sigma^2\tau^2\mu^2\psi$ where τ^i and μ^i are the Pauli matrices that act in the spaces of the SU(2) spinor and the two copies of the fermions respectively. Therefore the SU(2) topological superconductor in $4d$ is \mathbb{Z}_2 classified. This classification will not be further reduced by the fermion interaction.

2. A2. $(5+1)d$ bulk with $(4+1)d$ boundary

The $5d$ topological insulator with SU(2) symmetry can be described by the following lattice model

$$H = \sum_{\mathbf{k}} \sum_{i=1}^5 \sin k_i (\psi_{1,\mathbf{k}}^\dagger \Gamma^i \psi_{1,\mathbf{k}} - \psi_{2,\mathbf{k}}^\dagger \Gamma^i \psi_{2,\mathbf{k}}) + \left(\sum_{i=1}^5 \cos k_i - 5 + m \right) (i\psi_{1,\mathbf{k}}^\dagger \psi_{2,\mathbf{k}} + H.c.), \quad (\text{A3})$$

where $\psi = (\psi_1, \psi_2)^\dagger$ is an 8-component complex fermion field (ψ_1 and ψ_2 are both 4-component), and the five gamma matrices Γ^i ($i = 1, \dots, 5$) are defined below Eq. (A1). The model has an SU(2) symmetry, which transforms $(\psi_{\mathbf{k}}, \Gamma^2\Gamma^5\psi_{-\mathbf{k}}^\dagger)^\dagger$ as an SU(2) doublet (fundamental representation). The obvious U(1) symmetry $\psi \rightarrow e^{i\theta}\psi$ is a subgroup of the SU(2) symmetry. When $m > 0$ ($m < 0$), the model is in its topological (trivial) phase.

The $4d$ boundary of the $5d$ topological insulator will host the gapless fermion

$$H_\partial = \int d^4x \psi^\dagger (i\boldsymbol{\partial} \cdot \boldsymbol{\Gamma}) \psi, \quad (\text{A4})$$

where $(\psi, \Gamma^2 \Gamma^5 \psi^\dagger)^\top$ forms the SU(2) doublet (as inherited from the bulk fermion). The boundary fermion mode can not be gapped out (on the free fermion level) due to the SU(2) anomaly. The only possible fermion mass terms that can be added to gap out the boundary are $\psi^\top \Gamma^2 \psi$ and $\psi^\dagger \Gamma^5 \psi$. It can be verified that all these mass terms break the SU(2) symmetry. However if we double the system, then we can gap out the boundary by introducing the SU(2) symmetric mass term $\psi_A^\dagger i \Gamma^5 \psi_B + H.c.$ where A, B labels the two copies of the fermions. Therefore the SU(2) topological superconductor in $5d$ is \mathbb{Z}_2 classified. This classification will not be further reduced by the fermion interaction.

Appendix B: B. Topological defects

1. B1. SU(2) soliton and Hopf soliton

Consider an SU(2) field $U(\mathbf{r})$ in the $3d$ space with U being a 2×2 unitary matrix, which can be parameterized by an O(4) vector $\vec{u} = (u_0, u_1, u_2, u_3) \in S^3$ as

$$U = u_0 \sigma^0 + i u_1 \sigma^1 + i u_2 \sigma^2 + i u_3 \sigma^3. \quad (B1)$$

An SU(2) soliton can be given by the following configuration (in Cartesian coordinate)

$$\vec{u}(\mathbf{r}) = \frac{1}{1+r^2} (1-r^2, 2x, 2y, 2z), \quad (B2)$$

where $\mathbf{r} = (x, y, z)$ and $r = |\mathbf{r}|$. It is straight forward to verify that $|\vec{u}(\mathbf{r})| = 1$ through out the space. As shown in Fig. 6(a), the configuration of \vec{u} is a hedgehog monopole of $\mathbf{u} = (u_1, u_2, u_3) \sim \mathbf{r}$ around $|\mathbf{r}| \sim 1$, with its interior filled by $u_0 \rightarrow +1$ and its exterior filled by $u_0 \rightarrow -1$, which is exactly an $\pi_3[S^3]$ soliton of unit strength. The corresponding configuration of U will be an SU(2) soliton of unit strength. The energy density $(U^\dagger \nabla U)^2$ of the soliton is localized around the origin, verifying that the soliton is a local excitation (in the \vec{u} -ordered limit).

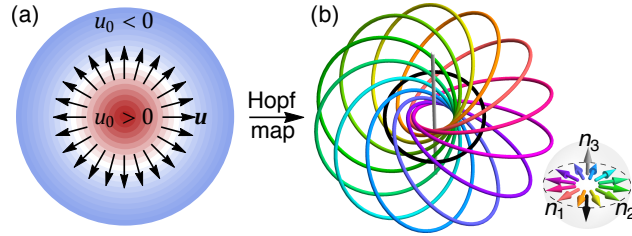


FIG. 6: (a) SU(2) soliton and (b) Hopf soliton.

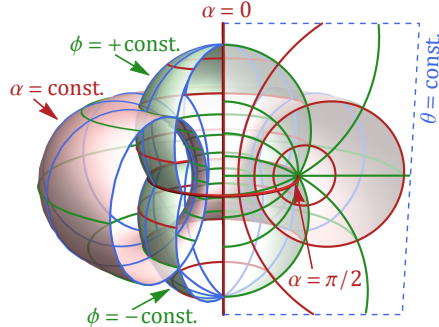


FIG. 7: Toroidal coordinate.

Consider an O(3) vector $\mathbf{n} = (n_1, n_2, n_3) \in S^2$ which transforms as a spin-1 representation of the SU(2) group. Then the SU(2) gauge transformation that creates an SU(2) soliton will correspondingly create a Hopf soliton in the

\mathbf{n} field. To see this, let us start from a trivial configuration of \mathbf{n} with all the vectors polarized to $\mathbf{n}(\mathbf{r}) = (0, 0, 1)$. After the SU(2) gauge transformation induced by the field $U(\mathbf{r})$, the configuration of $\mathbf{n}(\mathbf{r})$ will become

$$\mathbf{n} = \frac{1}{2} \text{Tr } U^\dagger \boldsymbol{\sigma} U \sigma^3. \quad (\text{B3})$$

This is a Hopf map from the SU(2) manifold to S^2 , under which the SU(2) soliton is mapped to a Hopf soliton. However it is difficult to visualize the Hopf soliton in the Cartesian coordinate, thus we switch to the toroidal coordinate (α, ϕ, θ) , which is defined by

$$\mathbf{r} = (x, y, z) = \frac{1}{\sec \alpha + \cos \phi} (\tan \alpha \cos \theta, \tan \alpha \sin \theta, \sin \phi), \quad (\text{B4})$$

where $\alpha \in [0, \pi/2]$ and $\phi, \theta \in [-\pi, \pi)$ are the . The geometric meaning of the toroidal coordinate is illustrated in Fig. 7. In the new coordinate system, Eq. (B2) is reduced to $\vec{u} = (\cos \phi \cos \alpha, \cos \theta \sin \alpha, \sin \theta \sin \alpha, \sin \phi \cos \alpha)$. Plugging into Eq. (B1) and Eq. (B3) yields

$$\mathbf{n} = (\sin(\phi - \theta) \sin 2\alpha, \cos(\phi - \theta) \sin 2\alpha, \cos 2\alpha). \quad (\text{B5})$$

The configuration can be described as follows: on the torus specified by $\alpha = \pi/4$, $n_3 = 0$ and (n_1, n_2) has a full winding along both the meridian ϕ and the longitude θ directions; while the exterior (interior) of the torus is gradually polarized to $n_3 = +1$ ($n_3 = -1$). In this configuration, the preimages of \mathbf{n} are mutually linked circles in the $3d$ space, as shown in Fig. 6(b), so it is exactly a Hopf soliton. Although the figure does not seem to be rotational invariant but in fact the energy density $|\nabla \mathbf{n}|^2$ of the Hopf soliton is spherical symmetric and localized around the origin.

When the \mathbf{n} field is coupled to the SU(2) Dirac fermion ψ in $3d$,

$$H = \int d^3x \psi^\dagger i \boldsymbol{\sigma} \cdot \boldsymbol{\partial} \psi + \mathbf{n} \cdot \text{Re}[\psi^\dagger \boldsymbol{\sigma}^y \boldsymbol{\tau}^y \boldsymbol{\tau} \psi], \quad (\text{B6})$$

the Hopf soliton will carries a fermion. More precisely, the creation of a Hopf soliton will change the fermion parity locally. To see this, we can first reduce the theory to the domain wall of n_3 on the torus of $\alpha = \pi/4$, on which we further reduce the theory to the domain wall of n_1 along the circle of $\phi - \theta = 0$, which reads $H = \int d\xi \frac{1}{2} \chi^\dagger [\sigma^1 (i\partial_\xi + \omega_\xi) + n_2 \sigma^2] \chi$ in terms of the Majorana fermion χ , where ξ parameterize the circle and ω_ξ is the spin connection along the circle. It turns out that the spin Berry phases along both the meridian and the longitude directions are both π on the $\alpha = \pi/4$ torus, so the total Berry phase along the circle is 2π , meaning that the spin connection can be gauged away. Thus the lowest momentum is quantized to $k = 0$. On the circle, the Hopf soliton is differed from a trivial configuration by the sign of n_2 : flipping n_2 from $n_2 < 0$ to $n_2 > 0$ corresponds to the creation of the Hopf soliton, which, according to the effective theory on the circle, will lead to a level crossing at $k = 0$, and hence change the fermion parity.

Due to the self-statistic of the fermion, the 2π rotation of the Hopf soliton is expected to produce a minus sign in the many-body wave function. In fact the Berry phase can be explicitly calculated. To simplify, we can reduce the problem to the n_3 domain wall on the torus of $\alpha = \pi/4$, and then compute the Berry phase accumulated over the S modular transformation of the torus, which is a $\pi/2$ rotation. It is found[39] that the S transformation will give a Berry phase of $\pi/4$, so the full 2π rotation (four times of S transformation) will produce a minus sign in the wave function.

2. B2. Vison loop and vison link

In the \mathbb{Z}_{2N} topological order phase, a vison line is a π gauge flux seen by the CP¹ spinon $z = (z_1, z_2)^\top$, meaning that the spinon going around the vison line will acquire a minus sign as $z \rightarrow -z$. The vison line will be bound with either a U(1) (gauge) half-vortex or an SU(2) (symmetry) half-vortex. Assuming the vison line is along the axis $\rho = 0$ in a cylindrical coordinate (ρ, φ, h) , then the U(1) and the SU(2) half-vortices are described respectively by

$$\text{U}(1): z = e^{i\varphi/2} z_{\text{ref}}, \quad \text{SU}(2): z = e^{i\varphi\sigma^3/2} z_{\text{ref}}, \quad (\text{B7})$$

where $\varphi \in [0, 2\pi)$ is the azimuthal angle around the vison line and $z_{\text{ref}} = (z_1, z_2)_{\text{ref}}^\top$ is a spinon reference state. As φ goes from 0 to 2π , z will get a minus sign under both vortex configurations. However in the \mathbb{Z}_{2N} topological order phase, the U(1) gauge vortex is gapped by the Higgs mechanism (because it corresponds to a vortex in the $z_1 z_2$ field, which is condensed in the \mathbb{Z}_2 topological order phase). So the SU(2) vortex is energetically favored around the vison line. In the following, we will focus on the case that the vison line is always bound with the SU(2) vortex.

To investigate the vortex link, it will be convenient to switch to the toroidal coordinate defined in Eq. (B4). Let the two vison loops (or lines) be the vertical axis $\alpha = 0$ and the horizontal ring $\alpha = \pi/2$ in the toroidal coordinate. The link of the SU(2) half-vortices of z can be described by

$$z = e^{i(\theta-\phi)\sigma^3/2} e^{-i\alpha\sigma^1} z_{\text{ref}}. \quad (\text{B8})$$

The operator $e^{i(\theta-\phi)\sigma^3/2}$ impose the SU(2) rotation by π in both the meridian and the longitude directions. As either θ or ϕ going from 0 to 2π , the spinon z will get a minus sign as required by the vison loops. Then in terms of the order parameter $\mathbf{n} = z^\dagger \boldsymbol{\sigma} z$, the configuration will be a pair of SU(2) vortices linked together

$$\mathbf{n} = e^{i(\theta-\phi)J_3} e^{-2i\alpha J_1} \mathbf{n}_{\text{ref}}, \quad (\text{B9})$$

where $(J_i)_{jk} = i\epsilon_{ijk}$ ($i, j, k = 1, 2, 3$) are the generators of SO(3). To avoid singularity in the configuration of \mathbf{n} (otherwise there will be gapless fermion modes), we must have $\mathbf{n}_{\text{ref}} = (0, 0, \pm 1)$. Suppose we choose $z_{\text{ref}} = (1, 0)^\top$ and $\mathbf{n}_{\text{ref}} = (0, 0, 1)$, then $\mathbf{n} = (\sin(\phi - \theta) \sin 2\alpha, \cos(\phi - \theta) \sin 2\alpha, \cos 2\alpha)$ will exactly be the Hopf soliton configuration given in Eq. (B5). Thus we conclude that the SU(2) vortex link (bound to the vison link) is equivalent to a Hopf soliton, which, after coupling the order parameters to the fermions, will also carry a fermion as required by the Witten anomaly.