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Phys. Rev. B **92**, 020504 — Published 13 July 2015

DOI: [10.1103/PhysRevB.92.020504](https://doi.org/10.1103/PhysRevB.92.020504)

Band-edge superconductivity

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We show that superconductivity can arise in semiconductors with a band in the shape of a Mexican hat when the chemical potential is tuned close to the band edge, but not intersecting the band, as long as interactions are sufficiently strong. Hence, this is an example where superconductivity can emerge from a band insulator when interactions exceed a threshold. Semiconductors with simple cubic symmetry point groups and with strong spin-orbit coupling provide an example of a system with such band dispersion.

The BCS theory of superconductivity is perhaps the most successful mean-field theory [1, 2]. It explains the phenomenology of many known superconductors, although, notably, it fails to describe the cuprate high T_c superconductors. BCS theory takes as starting point a good metal, with a sizable Fermi sea, and then explains the formation of the Cooper pairs at the Fermi surface, mediated by the electron-phonon interaction. Because the Cooper pairs occur only on a thin momentum shell, one may wonder if there may be a more “economical” way to form the pairs, without the sizable filled Fermi sea.

In this paper, we start with a system *without* a Fermi surface, a semiconductor where the chemical potential does not intercept the dispersing band. In the absence of interactions, this system has zero conductivity at zero temperature. We show, however, that for certain geometries of bands near the band edge, interactions can lead to superconductivity. The favorable dispersion (band geometry) is when the locus of the band edge in the Brillouin zone is not a single point (as in a parabolic band), but instead is a $d - 1$ dimensional momentum *shell* \mathcal{S}_0 (in the case of a d -dimensional semiconductor). The electrons in this shell are those responsible for superconductivity in the presence of interactions. Fig. (1) depicts the relevant situation, showing a Mexican hat dispersion and the chemical potential just missing the edge of the band. Near the extrema, the density of electronic states scales as in a one-dimensional system as long as the radius of the momentum shell \mathcal{S}_0 extremum is non-zero. As we shall see, the interactions (when sufficiently large) are responsible for a non-trivial occupation of the shell even when the chemical potential does not cross the bands. Hence, at non-zero temperature this system undergoes a superconducting phase transition as a function of the interaction strength. The superconducting transition temperature depends on the strength of the interactions and on the detuning of the chemical potential from the band edge. We find that such systems can have rather large transition temperatures, possibly on the order of room temperature for reasonable interaction strengths.

In [3], while discussing the possibility of inducing su-

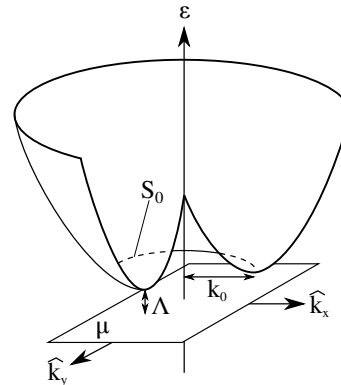


FIG. 1: Semiconductor with a rotationally symmetric band extremum (here minimum). The chemical potential is tuned right below the band edge, at distance Λ from the bottom.

perconductivity in large-gap semiconductors by shining an ac electric field, we had emphasized the importance of the Mexican hat geometry of the effective band which emerges in the rotating frame. Such a geometry also arises in driven Dirac systems [4]. But this geometry is not tied to a driven system and it can also be found in equilibrium situations such as non-centrosymmetric systems with spin-orbit interaction and cubic symmetry, such as $\text{Li}_2(\text{Pd}_{1-x}\text{Pt}_x)_3\text{B}$ [5–7]. In the latter case, for instance, the single particle Hamiltonian reads (we set $\hbar = 1$)

$$\mathcal{H}_{\mathbf{k}} = \frac{k^2}{2m^*} + \alpha \mathbf{k} \cdot \boldsymbol{\sigma}, \quad (1)$$

where $k \equiv |\mathbf{k}|$, α is the spin-orbit coupling, $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ are the Pauli matrices acting on the spin degrees of freedom, and m^* is an effective band mass (positive or negative, depending on the band). The dispersion relation $\epsilon(\mathbf{k}) = \frac{1}{2m^*}k^2 \pm \alpha k$ has extrema located on \mathcal{S}_0 , the $d - 1$ -sphere of radius

$$k_0 = |m^*| |\alpha|, \quad (2)$$

see Fig. (1). The near-extrema dispersion relation is quadratic, $\epsilon(\mathbf{k}) \approx \frac{1}{2m^*}(k - k_0)^2 + \epsilon_0$, resulting in a density of states which diverges in the same fashion as

in a 1D system, as noted in Ref. [4, 6, 7], even for 2D or 3D crystals: $\rho_{3D}(\epsilon) = 4\pi k_0^2 \sqrt{2|m^*|/\sqrt{|\epsilon - \epsilon_0|}}$ or $\rho_{2D}(\epsilon) = 2\pi k_0 \sqrt{2|m^*|/\sqrt{|\epsilon - \epsilon_0|}}$. If the Fermi energy is set near ϵ_0 so as to just cross the bands slightly, the Fermi surface consists of two concentric spherical shells (in 3D) or two concentric circles (in 2D) at momenta $k_0 \pm \delta k_F$. However, we shall concentrate instead in the case where the chemical potential does not cross the band, and therefore there is no Fermi surface and consequently either no occupation or complete occupation of the bands (at zero temperature) in the absence of interactions. All the phenomena discussed below emerge because of interactions.

The rest of the paper is organized as follows. First, we use a mean-field analysis to derive the conditions for p -wave superconductivity in the case of systems with large spin-orbit coupling. Then, we address the case without spin-orbit coupling, which is pertinent to a situation where the Mexican hat potential may arise by some other mechanism. In this case, s -wave pairing is allowed. Finally we go beyond weak-coupling mean-field theory and compute the critical temperature T_c by means of a calculation *à la* Migdal-Eliashberg [8], *i.e.* by considering the interplay between Coulombic repulsive interactions and phonon-mediated attractive interactions.

Mean-field analysis. For the sake of generality, let us depart from the Hamiltonian in Eq. (1) and simply assume that the semiconductor has one relevant band with dispersion $\epsilon(\mathbf{k})$ and a shallow extremum (either a band maximum or band minimum) located on a surface \mathcal{S}_0 . For all wave vectors $\mathbf{k}_0 \in \mathcal{S}_0$, $\epsilon(\mathbf{k}_0) \approx \text{cte}$, see Fig. (1). Additionally, let us assume that $\epsilon(\mathbf{k}) = \epsilon(-\mathbf{k})$. Note that this does not necessarily require inversion symmetry. The spin structures at \mathbf{k} and $-\mathbf{k}$ are locked to the respective momenta due to the spin-orbit coupling, and hence we drop any reference to spins, which do not play any role in what follows. Also, since the cases with a band maximum or a band minimum are alike, let us consider only the case of a band minimum ($m^* > 0$) with the following Hamiltonian

$$H = \int (d\mathbf{k}) \epsilon(\mathbf{k}) c_{\mathbf{k}}^\dagger c_{\mathbf{k}} + \int (d\mathbf{k})(d\mathbf{k}') V_{\mathbf{k},\mathbf{k}'} c_{\mathbf{k}}^\dagger c_{-\mathbf{k}}^\dagger c_{\mathbf{k}'} c_{-\mathbf{k}'}, \quad (3)$$

in which $c_{\mathbf{k}}$ is the annihilation operator of an electron with momentum \mathbf{k} and we use the shorthand notation $(d\mathbf{k}) \equiv d^d \mathbf{k} / (2\pi)^d$. To further simplify the discussion, let us assume that close to \mathcal{S}_0 the band structure $\epsilon(\mathbf{k})$ and the interaction term $V_{\mathbf{k},\mathbf{k}'}$ are isotropic to leading order so that $\epsilon(\mathbf{k}) = \epsilon(k)$ and $V_{\mathbf{k},\mathbf{k}'} = V(\hat{\mathbf{k}} \cdot \hat{\mathbf{k}}')$ can be decomposed into spherical harmonics

$$V_{\mathbf{k},\mathbf{k}'} = \sum_{l=0}^{\infty} (2l+1) V_l P_l(\hat{\mathbf{k}} \cdot \hat{\mathbf{k}}'), \quad (4)$$

where $\hat{\mathbf{k}} \equiv \mathbf{k}/k$, P_l is the Legendre polynomial of degree l and $V_l \equiv \int_0^1 d\cos\theta V(\cos\theta) P_l(\cos\theta)$.

Since there is only one band only p -wave superconductivity is allowed. For p -wave superconductivity, the symmetry of the order parameter, $\Delta(\mathbf{k}) = -\Delta(-\mathbf{k})$, implies that only odd l are appropriate for pairing. The dominant pairing mechanism is given by the $l = 1$ channel and we may approximate the pairing potential as

$$V_{\mathbf{k},\mathbf{k}'} \approx -\frac{3g}{\mathcal{V}} \hat{\mathbf{k}} \cdot \hat{\mathbf{k}}', \quad (5)$$

where \mathcal{V} is the volume of the sample and g is a coupling constant. For $l = 1$ pairing, the most general order parameter is of the form [9] $\Delta(\mathbf{k}) = \mathbf{\Delta} \cdot \hat{\mathbf{k}}$ where $\mathbf{\Delta}$ is a constant vector. Choosing the order parameter with the fewest nodes, we consider $\Delta(\mathbf{k}) = \Delta(\hat{k}_x + i\hat{k}_y)$ where Δ is a scalar to be determined self-consistently. Carrying out a Hubbard-Stratonovich transformation we obtain the mean field Hamiltonian

$$H = \int (d\mathbf{k}) \epsilon(\mathbf{k}) c_{\mathbf{k}}^\dagger c_{\mathbf{k}} + \sum_{\mathbf{k} \in \frac{1}{2} \text{ B.Z.}} \Delta(\mathbf{k}) c_{\mathbf{k}}^\dagger c_{-\mathbf{k}}^\dagger + \text{h.c.} + \frac{2}{3g} |\Delta|^2. \quad (6)$$

The integration over the fermions (keeping Δ fixed) yields the free energy

$$F = -T \int (d\mathbf{k}) \ln \left(\cosh \left(\frac{\beta E(\mathbf{k})}{2} \right) \right) + \frac{2}{3g} |\Delta|^2, \quad (7)$$

where $E(\mathbf{k}) \equiv \sqrt{\epsilon(\mathbf{k})^2 + |\Delta(\mathbf{k})|^2}$ and $T = 1/\beta$ is the temperature (we set $k_B = 1$). The saddle-point equation, obtained by taking the variation of F with respect to Δ^* [9], reads

$$\frac{2}{3g} = \frac{1}{4} \int (d\mathbf{k}) \tanh \left(\frac{\beta E(\mathbf{k})}{2} \right) \frac{\hat{k}_x^2 + \hat{k}_y^2}{E(\mathbf{k})}. \quad (8)$$

Notice the extra factor of $1/2$ compared to the regular BCS theory which stems from the fact that only one “spin species” is considered. To estimate the integral in Eq. (8), we use the fact that close to the surface \mathcal{S}_0 , the dispersion relation can be Taylor-expanded as $\epsilon = \Lambda + \kappa k_\perp^2 + \dots$, where k_\perp is the momentum perpendicular to \mathcal{S}_0 and $\kappa = 1/2m^* > 0$. $\Lambda > 0$, the distance of the band edge to the chemical potential, will play a key role in what follows. Again, notice that $\epsilon > 0$, so the chemical potential never intercepts the band.

In 3D, the self-consistency equation (8) becomes

$$\begin{aligned} \frac{2}{3g} &\approx \frac{k_0^2}{4(2\pi)^2} \int_0^\pi d\theta \sin^3 \theta \\ &\times \int dk_\perp \frac{\tanh \left(\frac{\beta}{2} \sqrt{(\Lambda + \kappa k_\perp^2)^2 + |\Delta|^2 \sin^2 \theta} \right)}{\sqrt{(\Lambda + \kappa k_\perp^2)^2 + |\Delta|^2 \sin^2 \theta}}. \end{aligned} \quad (9)$$

In a standard BCS approach, the cut-off is typically set by the Debye frequency. Here, the divergence of the density of states at the band edge makes this scale irrelevant and we can simply afford to extend the k_\perp integrals to infinity since they are convergent.

The superconducting phase transition is located at $\Delta = 0$, *i.e.*,

$$1 = \frac{g k_0^2}{2(2\pi)^2} \int_{-\infty}^{\infty} dk_\perp \frac{\tanh\left(\frac{\beta}{2}(\Lambda + \kappa k_\perp^2)\right)}{\Lambda + \kappa k_\perp^2} \quad (10)$$

$$\gtrsim \frac{g k_0^2}{2(2\pi)^2} \frac{\pi}{\sqrt{\Lambda\kappa}} \tanh(\beta\Lambda). \quad (11)$$

The existence of a superconducting phase is therefore conditioned by

$$g > g_c = \frac{8\pi\sqrt{\Lambda\kappa}}{k_0^2}. \quad (12)$$

Notice that the non-zero value of k_0 for this type of the band geometry is *essential* to give a finite value for the critical coupling g_c . Recall that in the case where the band geometry derives from the spin-orbit interaction, the value of k_0 is proportional to $|\alpha|$ as given by Eq. (2). Also notice that the threshold condition can be satisfied even with rather pessimistic estimates: for $\Lambda \approx 0.1$ eV, $\kappa \approx (10^{-6} \text{ eV})^{-1} c^2$, and $gk_0^2 \approx 10^{-2}c$, the threshold condition is satisfied with $1 > 0.79$. We note that in the optimistic limit where $\Lambda \rightarrow 0$ we have $g_c \rightarrow 0$ and the critical temperature is given by $\beta_c = 859\kappa \frac{1}{g^2 k_0^4}$. Furthermore the superconducting order parameter at zero temperature is given by $\Delta_0 = \frac{g^2 k_0^4}{\kappa} \left(\frac{3\sqrt{\pi}\Gamma^2(\frac{5}{4})}{2(2\pi)^2} \right)^2$, this leads to $\beta_c \Delta_0 = 2.63$.

For a more realistic limit with a sizable Λ the transition temperature grows as the coupling exceeds the threshold,

$$T_c \approx \frac{\Lambda}{\tanh^{-1}(g/g_c)}, \quad \text{for } g > g_c. \quad (13)$$

The magnitude of the superconducting gap can be estimated by focusing on the low-temperature regime $T \ll \Lambda$ for which the gap equation Eq. (9) yields

$$1 \approx \frac{g}{g_c} \frac{\sqrt{2}}{5} \frac{\frac{9}{2}x^2 + \sqrt{1+x^2} - 1}{x^2 \sqrt{\sqrt{1+x^2} + 1}}, \quad (14)$$

with $x \equiv \Delta/\Lambda$, from which it follows that a superconducting gap on the order of $\Delta \sim \Lambda$ can be achieved when the ratio g/g_c starts to increase away from unity. Near the threshold, the superconducting order parameter scales as

$$\Delta \approx \Lambda 2\sqrt{5/3} \sqrt{1 - \frac{g_c}{g}}, \quad \text{for } g > g_c. \quad (15)$$

Singlet case. We now extend the previous analysis to cases in which the Mexican hat band geometry is not a consequence of spin-orbit coupling but finds its origin in another mechanism. In this case, the bands are spin-degenerate and an *s*-wave pairing, $\Delta(\mathbf{k}) = \Delta(-\mathbf{k})$, is allowed by symmetry. Thus, the dominant pairing channel sees a potential $V(\mathbf{q}) = V(-\mathbf{q})$, and the pairing now occurs between the two spin bands.

The self-consistency equation is now given by

$$\frac{1}{g} = \frac{1}{2} \int (d\mathbf{k}) \frac{1}{E(\mathbf{k})} \tanh\left(\frac{\beta E(\mathbf{k})}{2}\right), \quad (16)$$

where $E(\mathbf{k}) \equiv \sqrt{\epsilon(\mathbf{k})^2 + |\Delta|^2}$. The phase transition ($\Delta = 0$) occurs at

$$1 = \frac{g k_0^2}{(2\pi)^2} \int_{-\infty}^{\infty} dk_\perp \frac{\tanh\left(\frac{\beta}{2}(\Lambda + \kappa k_\perp^2)\right)}{\Lambda + \kappa k_\perp^2} \quad (17)$$

$$\gtrsim \frac{g k_0^2}{(2\pi)^2} \frac{\pi}{\sqrt{\Lambda\kappa}} \tanh(\beta\Lambda). \quad (18)$$

The existence of a superconducting phase is now conditioned by

$$g > g_c = \frac{4\pi\sqrt{\Lambda\kappa}}{k_0^2}. \quad (19)$$

The analysis of this case is analogous to the previous: once g exceeds the threshold g_c , a superconducting phase appears. The critical temperature can be of order Λ once the threshold starts to be exceeded. We note that in the optimistic limit where $\Lambda \rightarrow 0$ we have $g_c \rightarrow 0$ and the critical temperature is given by $\beta_c = 215\kappa \frac{1}{g^2 k_0^4}$. Furthermore the superconducting order parameter at zero temperature is given by $\Delta_0 = \frac{g^2 k_0^4}{\kappa} \left(\frac{8\Gamma^2(\frac{5}{4})}{(2\pi)^2 \sqrt{\pi}} \right)^2$, this leads to $\beta_c \Delta_0 = 1.90$.

For the more realistic limit where Λ is sizable the magnitude of the gap can be estimated again by focusing on the low-temperature regime when $T \ll \Lambda$ and using the gap equation, yielding

$$1 = \frac{g}{g_c} \frac{2}{\pi} E_K \left(\frac{1}{\sqrt{2}} \sqrt{1 - 1/\sqrt{1+x^2}} \right) / (1+x^2)^{1/4}, \quad (20)$$

with $x \equiv |\Delta|/\Lambda$ and E_K is the complete elliptic integral of the first kind. Again, it follows that a superconducting gap on the order of $\Delta \sim \Lambda$ can be achieved when the ratio g/g_c starts to increase away from unity, and near the threshold we obtain

$$\Delta \approx \Lambda \frac{4}{\sqrt{3}} \sqrt{1 - \frac{g_c}{g}}, \quad \text{for } g > g_c. \quad (21)$$

Strong-coupling approach. Let us now give a description beyond weak-coupling mean-field theory by means of a Migdal-Eliashberg approach [8]. This consists in

including explicitly the screened electron-phonon interaction and the screened Coulomb interaction, and establishing a self-consistent equation on the resulting self-energy, yielding an estimate of the critical temperature. If the phonon frequency is much smaller than the electronic energy scale, Migdal's theorem states that the phononic vertex corrections can be neglected, even if the electron-phonon coupling constant is large [10–12]. We derive a version of this theorem applicable to our case in [13].

To simplify, we consider the second scenario described above, *i.e.*, the *s*-wave pairing case in the absence of spin-orbit coupling. The *p*-wave case with spin-orbit coupling is conceptually identical and, in the strong-coupling limit, can be shown to have the same critical temperature up to numerical factors of order unity. However, as the precise form of the spin-orbit coupling is not available, these factors cannot be reliably computed.

Let us start with the following Hamiltonian

$$H = \int (d\mathbf{k}) \left(\epsilon(\mathbf{k}) c_{\mathbf{k}}^\dagger c_{\mathbf{k}} + V_C(\mathbf{k}) \rho_{\mathbf{k}\downarrow}^\dagger \rho_{\mathbf{k}\uparrow} + \Omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} \right) + \sum_{\sigma} \int (d\mathbf{k}) (d\mathbf{q}) g_{\mathbf{q}} c_{\mathbf{k}+\mathbf{q},\sigma}^\dagger c_{\mathbf{k},\sigma} (b_{\mathbf{q}} + b_{-\mathbf{q}}^\dagger). \quad (22)$$

Here, $\rho_{\mathbf{k}\uparrow} = \int (d\mathbf{q}) c_{\mathbf{q}\uparrow}^\dagger c_{\mathbf{q}+\mathbf{k}\uparrow}$ is the electron density, $V_C(\mathbf{k})$ is the screened Coulomb interaction, and $g_{\mathbf{q}}$ is the electron phonon-coupling matrix (which is also screened). Let us introduce the phonon propagator in Matsubara time [10]

$$D(\mathbf{q}, \omega_n) = - \int_0^\beta d\tau e^{i\omega_n \tau} \left\langle T_\tau \left(b_{\mathbf{q}}(\tau) + b_{-\mathbf{q}}^\dagger(\tau) \right) (b_{-\mathbf{q}}(0) + b_{\mathbf{q}}^\dagger(0)) \right\rangle = - \frac{2\Omega_{\mathbf{q}}}{\omega_n^2 + \Omega_{\mathbf{q}}^2} \approx - \frac{2}{\Omega_{\mathbf{q}}} \delta_{n,0}. \quad (23)$$

Here, $\omega_n = n 2\pi/\beta$ and in the last equality we have taken the high temperature limit $T \gg \Omega_{\mathbf{q}}$.

Below, we derive a self-consistency equation for the pairing amplitude. It is convenient to introduce the standard Nambu Green's functions [14]:

$$G(\mathbf{k}, \tau) = - \begin{pmatrix} \left\langle T_\tau c_{\mathbf{k}\uparrow}(\tau) c_{\mathbf{k}\uparrow}^\dagger(0) \right\rangle & \left\langle T_\tau c_{\mathbf{k}\uparrow}(\tau) c_{-\mathbf{k}\downarrow}(0) \right\rangle \\ \left\langle T_\tau c_{-\mathbf{k}\downarrow}^\dagger(\tau) c_{\mathbf{k}\uparrow}^\dagger(0) \right\rangle & \left\langle T_\tau c_{-\mathbf{k}\downarrow}^\dagger(\tau) c_{-\mathbf{k}\downarrow}(0) \right\rangle \end{pmatrix}. \quad (24)$$

The corresponding self-energy obeys the matrix equation

$$\Sigma(\mathbf{k}, \omega_n) = G_0^{-1}(\mathbf{k}, \omega_n) - G^{-1}(\mathbf{k}, \omega_n), \quad (25)$$

with the non-interacting Green's function $G_0(\mathbf{k}, \omega_n) = [i\omega_n \tau_0 - \epsilon(\mathbf{k}) \tau_3]^{-1}$. τ_0 and τ_1, τ_2, τ_3 denote the identity and Pauli matrices in the Nambu space. To leading order,

the self-energy is given by

$$\Sigma(\mathbf{k}, \omega_n) = -T \sum_{\omega_{n'}} \int (d\mathbf{q}) \tau_3 G(\mathbf{k} - \mathbf{q}, \omega_{n'}) \tau_3 \times (|g_{\mathbf{q}}|^2 D(\mathbf{q}, \omega_n - \omega_{n'}) + V_C(\mathbf{q})). \quad (26)$$

Next, we note that as long as our system is isotropic, *e.g.* $\epsilon(\mathbf{k}) = \epsilon(k)$, $g_{\mathbf{q}} = g_q$, $\Omega_{\mathbf{q}} = \Omega_q$, $V_C(\mathbf{q}) = V_C(q)$, and the quantities $V_C(q)$ and $|g_{\mathbf{q}}|^2/\Omega_q$ do not have strong dependence on q for $q \sim \sqrt{\Lambda/\kappa}$ then the self-energy does not depend on the wavevector, *e.g.* $\Sigma(k, \omega_n) \approx \Sigma(\omega_n)$. On symmetry grounds, the self-energy can be decomposed as [14]

$$\Sigma(\omega_n) = (1 - Z) i\omega_n \tau_0 + \Delta(\omega_n) \tau_1 + \chi \tau_3. \quad (27)$$

Using Eq. (25), we obtain

$$G(k, \omega_n) = - \frac{Z i\omega_n \tau_0 + \Delta(\omega_n) \tau_1 + (\epsilon(k) + \chi) \tau_3}{Z^2 \omega_n^2 + (\epsilon(k) + \chi)^2 + \Delta(\omega_n)^2}. \quad (28)$$

For simplicity, we assume that $Z = 1$ and $\chi = 0$ (the former renormalizes Λ and can always be set to zero). Using Eq. (26) in Eq. (27), we obtain the self-consistency equation

$$\Delta(\omega_n) = \lambda T \int dk \frac{\Delta(\omega_n)}{\omega_n^2 + \epsilon(k)^2 + \Delta(\omega_n)^2} - \mu_C T \sum_{\omega_{n'}} \int dk \frac{\Delta(\omega_{n'})}{\omega_{n'}^2 + \epsilon(k)^2 + \Delta(\omega_{n'})^2}, \quad (29)$$

where we introduced

$$\mu_C \equiv \frac{1}{(2\pi)^3} \int_{S_0} V_C(\mathbf{k}) \quad \text{and} \quad \lambda \equiv \frac{1}{(2\pi)^3} \int_{S_0} |g_{\mathbf{q}}|^2 \frac{2}{\Omega_{\mathbf{q}}}, \quad (30)$$

respectively the Coulomb and phonon interactions integrated over the surface S_0 . Close to the critical temperature, we may drop the $\Delta(\omega_n)^2$ terms in the denominators of in Eq. (29) which, after integration over k using $\epsilon(k) = \Lambda + \kappa k^2 + \dots$, can be recast as

$$\Delta(\omega_n) = A_n \Delta(\omega_n) - \frac{\mu_C}{\lambda} \sum_{n'} A_{n'} \Delta(\omega_{n'}), \quad (31)$$

with $A_n \equiv \pi \frac{\lambda T}{\sqrt{\kappa}} [\sqrt{\Lambda^2 + \omega_n^2} (\sqrt{\Lambda + i\omega_n} + \sqrt{\Lambda - i\omega_n})]^{-1}$.

$$1 = \frac{\mu_C}{\lambda} \sum_n \frac{A_n}{A_n - 1}. \quad (32)$$

This equation admits a non-trivial solution whenever $\lambda \geq \lambda_c(\mu_C/\lambda)$, with $\lambda_c(0) < \lambda_c(\mu_C/\lambda) < \lambda_c(\infty)$. Computing explicitly $\lambda_c(0) = 2^{3/2} \sqrt{\kappa \Lambda}$ and $\lambda_c(\infty) = 3 \cdot 2^{3/2} \sqrt{\kappa \Lambda}$, this implies that the critical temperature always satisfies $T_c > \Lambda/\sqrt{3\pi^2}$.

Conclusions. We have studied the emergence of superconductivity for those semiconductors with a band in the shape of a Mexican hat, where the energy reaches an extremum on a band-edge surface S_0 . We have set the chemical potential for the semiconductor close to the band edge, but not intercepting the band. Therefore, in the absence of interactions, the system would have vanishing conductivity at zero temperature. We have shown, both through a mean field and through a strong coupling calculation, that phonon-mediated superconductivity arises and is robust to high temperatures. The mechanism benefits from a quasi one-dimensional divergent density of states at the band edge, making an “economical” use of the energy levels on a shell near the band extrema.

This work has been supported by the Rutgers CMT fellowship (G.G.), the NSF grant DMR-115181 (C.A.), and the DOE Grant DEF-06ER46316 (C.C.).

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