Entanglement spectrum of a random partition: Connection with the localization transition

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Entanglement Spectrum of a Random Partition: Connection with the Localization Transition

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We study the entanglement spectrum of a translationally-invariant lattice system under a random partition, implemented by choosing each site to be in one subsystem with probability $p \in [0, 1]$. We apply this random partitioning to a translationally-invariant (i.e., clean) topological state, and argue on general grounds that the corresponding entanglement spectrum captures the universal behavior about its *disorder-driven* transition to a trivial localized phase. Specifically, as a function of the partitioning probability p, the entanglement Hamiltonian H_A must go through a topological phase transition driven by the percolation of a random network of edge-states. As an example, we analytically derive the entanglement Hamiltonian for a one-dimensional topological superconductor under a random partition, and demonstrate that its phase diagram includes transitions between Griffiths phases. We discuss potential advantages of studying disorder-driven topological phase transitions via the entanglement spectra of random partitions.

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In recent years, systematic studies of quantum entanglement have greatly advanced our understanding of topological states of matter that cannot be adiabatically connected to a trivial product state. For example, topological entanglement entropy is directly related to the total quantum dimension of fractional quasi-particles [1, 2]. More recently, there has been a growing interest in utilizing the full entanglement spectrum to extract other universal properties [3], especially in chiral topological phases (e.g., quantum Hall states) or symmetry-protected topological phases (e.g., topological insulators). As a common feature, these phases have topologically-protected gapless excitations on a physical boundary. When the ground state is spatially cut into left and right halves, the low-lying part of the entanglement spectrum shares the same universal characteristics as the energy spectrum of these boundary excitations [3–9].

A recent work studied the entanglement spectrum obtained from an extensive partition that divides a system into two *extensive* subsystems [10]. For topological states that support gapless edge states, the corresponding entanglement spectrum was found to encode a wealth of information about the universal quantum critical behavior that would arise at its phase transition to a trivial direct-product state, despite the fact that the system under study itself is non-critical. It has been further shown that the entanglement spectra of extensive partitions can be directly computed from the matrix product state or tensor network representation of ground-state wavefunctions [11], which may offer insights into topological phase transitions [12–15].

In this Letter, we study the entanglement spectum generated from a random partition that spatially bipartitions a system in a *probabilistic* manner: each physical site is chosen to be in subsystem B (or A) with a probability $p \in [0, 1]$ (or $1 - p$). We apply this random partition to a translationally-invariant (i.e., clean) topological state, and argue on general grounds that the corresponding entanglement spectrum reproduces the universal behavior about its *disorder-driven* transition to a trivial localized phase. As an example, we analytically derive the form of the entanglement Hamiltonian for a clean one-dimensional topological superconductor under random partition, and establish the entanglement phase diagram as a function of probability p, finding agreement with the physical phase diagram of a disordered superconductor [16].

We begin by considering a translationally-invariant topological state, which can be either a topological insulator/superconductor or a bosonic symmetry-protected topological phase. It has been shown [10] that upon varying the geometry of A and B subsystems in an extensive partition, the corresponding entanglement Hamiltonian undergoes a gap-closing transition that lies in the same universality class as the transition to a topologically trivial state realized by tuning the *physical* Hamiltonian. This intriguing connection is absent when applying the extensive partitioning to a topologically trivial wavefunction, and follows from the nature of topological phase transitions, which are driven by the percolation of gapless edge-states. For example, the transition from a quantum Hall insulator to a trivial insulator is described by the quantum percolation of chiral edge-states in a Chalker-Coddington network model [17]. Extensive partitioning of a quantum Hall insulator precisely creates, in the low-lying part of the entanglement spectrum, a network of chiral edge states moving along the percolating borders between A and B. This mapping explains why topological phase transitions and entanglement spectra of extensive partitions are intimately related. It further motivates us to study the random partitioning of a topological ground-state, for which the entanglement spectrum is expected to mimic the network model with randomness and thus connect with the localization transition.

We define the probabilistic partitioning of a clean, topological state $|\Psi\rangle$ as follows. We independently choose each physical site in the full system to be in the B subsystem with probability p; the remaining sites are defined to be in subsystem A. A partial trace of the density matrix over sites in subsystem B yields a reduced density matrix for the A subsystem $\rho_A \equiv \mathrm{Tr}_B|\Psi\rangle\langle\Psi|$, which can be interpreted as the *thermal* density matrix at temperature $T = 1$ for an entanglement Hamiltonian H_A: $\rho_A \equiv e^{-H_A}$.

Our goal is to determine the phase diagram of H_A as a function of the partitioning probability p. First, when $p \to 0$, the probabilistic partitioning yields a vanishingly small B subsystem, with most sites belonging to the A subsystem. In this limit, the ground-state of the entanglement Hamiltonian, denoted by $|\psi_A\rangle$, must share the same topological index as the original ground state $|\Psi\rangle$. As $p \to 1$, however, most sites become part of the B subsystem, so that $|\psi_A\rangle$ becomes a trivial product state over the disjoint regions of the A subsystem, and hence must be topologically trivial. Since the topological character of $|\psi_A\rangle$ changes as we tune the partitioning probability p, we conclude that the entanglement Hamiltonian H_A must go through a *phase transition* at some critical partitioning probability $p = p_c$. Physically, the transition is driven by the percolation of a random network of gapless edge-states propagating around traced-out regions of the B subsystem, as in the case of the aforementioned checkerboard-type extensive partition. Even though the original state $|\Psi\rangle$ is translationally-invariant, the probabilistic partitioning procedure introduces randomness into the entanglement Hamiltonian H_A, with the probability p effectively tuning disorder strength.

The phase diagram of the entanglement Hamiltonian as a function of p satisfies additional constraints. For a given bipartition, the eigenvalue spectra of the reduced density matrices ρ_A and ρ_B are identical, though their Hilbert spaces are distinct. By definition, for a fixed partitioning probability p, the B subsystem is, on average, equivalent to the A subsystem obtained with a partitioning probability $1 - p$. Therefore, the ensemble-averaged spectra of the entanglement Hamiltonians $H_A(p)$ and $H_A(1 - p)$ must be identical. As a result, the presence of a phase transition in the entanglement Hamiltonian with partition probability p implies another transition at probability $1-p$. In the case where the topological index of $|\Psi\rangle$ cannot be evenly divided between two subsystems, as is the case for topological insulators with a $\mathbb{Z}_2$ index or quantum Hall insulators with an odd Chern number, we further expect that H_A exhibits at least a topological phase transition at partitioning probability $p = 1/2$, when the two subsystems are equivalent on average.

For the remainder of the paper, we apply our random partitioning procedure to Kitaev's model [18] for a clean one-dimensional p-wave superconductor, extract the phase diagram of the entanglement Hamiltonian as a function of partitioning probability, and demonstrate its

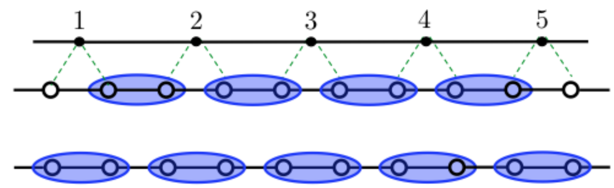

FIG. 1: The two ground-states of the Kitaev p-wave superconductor with dimerization of Majorana fermions *across* (top) and *within* (bottom) lattice sites, corresponding to a TSC and a trivial p-wave superconductor, respectively.

correspondence with a disordered superconductor. The Kitaev model is described by the Hamiltonian

$$H = -w\sum_n \left(c^\dagger_{n+1}c_n + \text{h.c.}\right) + \sum_n \left(\Delta c^\dagger_{n+1}c^\dagger_n + \text{h.c.}\right) - \mu\sum_n \left(c^\dagger_n c_n - \frac{1}{2}\right) \quad (1)$$

with fermion operators c_n, $c^\dagger_n$ satisfying canonical anti-commutation relations. To simplify the calculation below, we restrict ourselves to the case where $w = \Delta \in \mathbb{R}$. Introducing two species of Majorana fermions at each lattice site $\gamma_n \equiv c_n + c^\dagger_n$ and $\chi_n \equiv (c^\dagger_n - c_n)/i$, we may write the Hamiltonian as

$$H = \frac{iw}{2}\sum_n \left[\eta\gamma_n\chi_n + \chi_n\gamma_{n+1}\right] \quad (2)$$

where the dimensionless parameter $\eta \equiv \mu/2w$ distinguishes the topologically trivial strong pairing phase ($|\eta| > 1$) and topologically non-trivial weak pairing phase ($|\eta| < 1$). As shown in Figure 1, in the topological superconductor (TSC) phase, Majorana fermions couple more strongly across two adjacent lattice sites than within a lattice site, leading to two unpaired Majorana fermions at the ends of the chain. In the extreme limit when $\eta = 0$ in the topological phase, pairs of Majorana fermions within lattice sites completely decouple. We will refer to this special point $\eta = 0$ as the *Kitaev limit* of the Hamiltonian (2), which proves to be a useful starting point for our analysis below.

Let $|\Psi(\eta)\rangle$ be the ground-state of the Kitaev Hamiltonian with parameter $|\eta| < 1$ in the topological regime. Applying the random partitioning procedure to this ground-state, we trace over *physical* lattice sites (each of which contains two Majorana fermions) with probability p, and obtain an entanglement Hamiltonian $H_A(\eta; p)$. Clearly, $H_A(\eta; p)$ contains random couplings between sites in the A subsystem, and mimics the physical Hamiltonian of a disordered superconductor, with the partitioning probability p playing the role of disorder strength.

To derive $H_A(\eta; p)$, we note that the entanglement Hamiltonian of a free fermion system such as the Kitaev model can only contain fermion bilinear terms [19].

The spectrum of the entanglement Hamiltonian over an N-site subsystem A can be determined from the $N \times N$ correlation matrices in the original ground-state $C_{nm} \equiv \langle \Psi \,|\, c_n^\dagger c_m \,|\, \Psi \rangle$ and $F_{nm} \equiv \langle \Psi \,|\, c_n^\dagger c_m^\dagger \,|\, \Psi \rangle$, by solving the eigenvalue problem [19]:

$$(2\hat{C} - 2\hat{F} - 1)(2\hat{C} + 2\hat{F} - 1)\phi_\ell = \tanh^2\left(\frac{\epsilon_\ell}{2}\right)\phi_\ell \quad (3)$$

where ϵ_ℓ is an eigenvalue of the entanglement Hamiltonian, with eigenvector ϕ_ℓ. Rewriting the complex fermions in terms of Majorana operators, we may define the $2N \times 2N$ skew-symmetric correlation matrix $\hat{\Gamma}$ for the Majorana fermions which has eigenvalues $\pm \tanh(\epsilon_\ell/2)$ [20, 27]. Since we are interested in the low-lying part of the entanglement spectrum $\epsilon_\ell \to 0$, $\tanh(\epsilon_\ell/2) \to \epsilon_\ell/2$ and hence $\hat{\Gamma}$ satisfies:

$$2\hat{\Gamma}\phi_\ell \approx \pm\epsilon_\ell\phi_\ell \quad (4)$$

Therefore, the correlation matrix for Majorana fermions in the original ground-state is *equivalent* to the entanglement Hamiltonian acting on low-lying states $\epsilon_\ell \to 0$ in the entanglement spectrum. By building the correlation matrix for Majorana fermions in the ground-state $|\Psi(\eta)\rangle$, we may now construct the entanglement Hamiltonian for the Kitaev model after a random partition.

For arbitrary η, performing a random partition will generally produce an entanglement Hamiltonian with highly non-local couplings, due to the non-vanishing correlations between distant Majorana fermions in the ground state. However, for a sufficiently small $|\eta|$, i.e., when the system is close to the Kitaev limit, we may derive the form of $H_A(\eta;p)$ *analytically*. Let us first consider the case $\eta = 0$, when pairs of Majorana fermions decouple. A single cut between two adjacent lattice sites then produces, in the entanglement spectrum of the A subsystem, an unpaired Majorana fermion at the end of A. Aside from this, the entanglement spectrum at $\eta = 0$ is identical to the energy spectrum (properly normalized) of decoupled Majorana pairs in the Kitaev Hamiltonian. Therefore, performing a random partition with several cuts will yield an A subsystem that consists of disjoint segments, each of which hosts unpaired Majorana fermions at the two ends.

We now explicitly construct the entanglement Hamiltonian $H_A(\eta;p)$ near the Kitaev limit by perturbing away from $\eta = 0$. As one may expect, a small η induces a small coupling between the unpaired Majorana fermions at ends of disjoint segments with the rest of the A subsystem. By an analytical calculation [27], we find that couplings between two Majorana fermions in $H_A(\eta;p)$ decrease exponentially with their separation in the original lattice. Therefore, it suffices to include *nearest-neighbor* couplings within subsystem A only in $H_A(\eta;p)$.

Two types of nearest-neighbor couplings appear in $H_A(\eta;p)$. First, to leading order in η, couplings belonging to a connected sequence of sites in the A subsystem

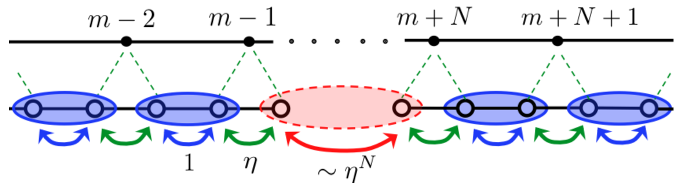

FIG. 2: The entanglement Hamiltonian $H_A(\eta;p)$, derived in the Kitaev limit $|\eta| << 1$, consists of several couplings between adjacent Majorana fermions in the subsystem A. Blue and green hoppings between nearest-neighbor Majorana fermions appear with dimensionless coupling 1 and η, respectively. Tracing over N lattice sites induces a coupling $O(\eta^N)$ between the dangling Majorana modes on the adjacent chains in the A subsystem.

are identical to those appearing in the original Kitaev Hamiltonian, after a proper normalization. Second, couplings between Majorana fermions belonging to different segments in the A subsystem are computed from their two-point correlation function. If a series of N consecutive lattice sites − between sites $m-1$ and $m+N$ − are determined to be within the B subsystem and traced over in the random partition, a coupling will be induced between the Majorana fermions at the right and left edges of the two lattice sites, which is found to be proportional to $C(N) \equiv \langle \Psi(\eta) \,|\, i\chi_{m-1}\gamma_{m+N} \,|\, \Psi(\eta)\rangle$. An explicit calculation [27] yields the result that at long distances, i.e. large N,

$$C(N) \sim \eta^N/\sqrt{N} + O(\eta^{N+1}) \quad (5)$$

We then conclude that the entanglement Hamiltonian takes the form:

$$H_A(\eta;p) = \frac{iw}{2}\sum_{n\in A}\eta\gamma_n\chi_n + \frac{iw}{2}\sum_{m>n\in A} f_{nm}\chi_n\gamma_m \quad (6)$$

where f_{nm} has non-zero elements $f_{n,n+1} = 1$ and $f_{nm} = C(m-n-1)$ if n and m label lattice sites at the right and left edges of two adjacent segments in the A subsystem. The couplings in the entanglement Hamiltonian are illustrated in Fig. 2.

We now demonstrate that the topological character of the entanglement Hamiltonian (6) changes at a critical partitioning probability $p = 1/2$. Specifically, we demonstrate that when $p < 1/2$ the entanglement Hamiltonian in the A subsystem supports an exponentially localized edge-state $|\psi\rangle$ at zero energy and corresponds to a topological superconductor phase, but that such a state does not exist for $p > 1/2$.

To see this, we first note that a random partitioning of a chain with probability p will produce an A subsystem that consists of clusters of lattice sites. Let us now introduce a boundary in the A subsystem and explicitly construct a zero-energy state of the entanglement spectrum under a random partition. Recall that an exact zero-energy left-boundary eigenstate of the translationally-invariant Kitaev Hamiltonian (2) takes the form $|\psi\rangle = (1, 0, \eta, 0, \eta^2, 0, \ldots)$, with $|\eta| < 1$, in the basis of Majorana sites on a semi-infinite chain [18]. We find a similar zero-energy state can be obtained for the entanglement Hamiltonian (6) of the A subsystem, which consists of consecutive clusters of sites of lengths $\{\ell_k\}$, each separated by distance $\{d_k\}$. An edge-state of the entanglement Hamiltonian now takes the form:

$$|\psi_A\rangle \propto \Bigg(\underbrace{1, 0, \eta, 0, \ldots, \eta^{\ell_1}, 0,}_{\text{1st cluster of sites}} \frac{\eta^{\ell_1}}{C(d_1)}, 0, \ldots, \frac{\eta^{\ell_1+\ell_2}}{C(d_1)}, 0, \ldots \underbrace{\prod_{k=1}^{N-1} \frac{\eta^{\ell_k}}{C(d_k)}, 0, \ldots, \eta^{\ell_N} \prod_{k=1}^{N-1} \frac{\eta^{\ell_k}}{C(d_k)}, 0, \ldots}_{N^{\text{th}} \text{ cluster of sites}} \Bigg) \tag{7}$$

in a basis of Majorana sites in A. From the form of the two-point function computed previously we see that amplitude for the edge-state on the first Majorana site in the Nth cluster of the A subsystem is given by:

$$\psi_N = \prod_{k=1}^{N-1} \frac{\eta^{\ell_k}}{C(d_k)} \sim \prod_{k=1}^{N-1} \eta^{\ell_k - d_k} = \eta^{\sum_k \ell_k - \sum_k d_k} \tag{8}$$

Now, if we consider the amplitude at the end of the chain, we see that $\psi_N \to \eta^{L_A - L_B}$ where L_A and L_B are the sizes of the A and B subsystems, respectively. For a system of size L, regardless of the probability distributions for the lengths $\{\ell_k\}$ and $\{d_k\}$, the sizes of the two subsystems are determined from the partitioning probability to be $L_A = (1-p)L$ and $L_B = pL$ in the thermodynamic limit $L \to +\infty$ so that

$$\psi_N \to \left[\eta^{1-2p}\right]^L \tag{9}$$

When $p < 1/2$ we observe that the state (7) is an exact zero-energy eigenstate of the entanglement Hamiltonian. When $p > 1/2$, however, the amplitude at the end of the chain diverges and the above state becomes non-normalizable for an infinite set of clusters.

The above calculation of an edge-state immediately implies that the entanglement Hamiltonian H_A changes from being topologically non-trivial at partitioning probability $p < 1/2$ to trivial when $p > 1/2$, and hence must be critical at the point $p = 1/2$. This transition can also be understood by integrating out the Majorana fermions in the interior of the clusters in the A subsystem and constructing an *effective* entanglement Hamiltonian H_A^{eff} acting exclusively on the dangling Majorana modes at the ends of each cluster. In this case, H_A^{eff} will describe a dimerized Majorana fermion chain, in which two adjacent Majorana fermions correspond to sites separated by lengths $\{...\ell_k, d_k, \ell_{k+1}, d_{k+1}...\}$ in the original lattice. The nearest-neighbor hopping in A, which is proportional to the corresponding correlation function in the ground-state, is determined by the lengths $\{d_k\}$ for intra-cluster hoppings or $\{\ell_{k+1}\}$ for inter-cluster hoppings. At $p = 1/2$, the A and B subsystems are equivalent on average, so that the length distributions $\{\ell_k\}$ and $\{d_k\}$ are identical, and the ensemble of H_A^{eff} is translationally-invariant, instead of dimerized. The corresponding ground state of a one-dimensional Majorana fermion chain is well-known to be critical [21].

We now demonstrate that in the vicinity of $p = 1/2$, the entanglement Hamiltonian is in Griffiths phases, characterized by a singularity in the density of states at zero energy due to the proliferation of segments of the topologically-ordered or trivial phase, respectively. Recall that when $p < 1/2$, near $p = 1/2$, the characteristic size of clusters in the A subsystem is larger than that of the B subsystem. Then, the dangling Majorana modes on adjacent clusters in the A subsystem, separated by distance x will mix to form localized bound-states with finite energy $\epsilon \sim \exp[c\,x \ln|\eta|]$, with $c > 0$ a constant. Since the probability of such a configuration of sites in the A subsystem is $p^x(1-p)^2$, the contribution of these low-energy modes to the density of states in the entanglement ground-state is [16]:

$$\rho(\epsilon) = \int_0^\infty dx\,(1-p)^2 p^x\, \delta(\epsilon - e^{c\,x\ln|\eta|}) \propto \frac{1}{\epsilon^{1-\beta(p)}} \tag{10}$$

with the non-universal exponent $\beta(p) \equiv \ln(p)/c\ln|\eta|$. The power-law singularity in the density of states signals the presence of a Griffiths region for an entanglement ground-state with p near $1/2$, due to the proliferation of low-energy configurations of Majorana edge-modes dimerizing across lattice sites. $p > 1/2$ also correspond to a Griffiths phase, with exponent $\beta(1-p)$ due to Majorana modes at the ends of the *same* chain forming bound-states with exponentially small energy. The two Griffiths phases at $p < 1/2$ and $p > 1/2$ are both characterized by a power-law singularity in the density of states at zero energy, but are topologically distinct, as shown by the presence and absence of zero-energy Majorana fermion at the boundary.

To summarize, we have introduced a random partitioning scheme to study the disorder-driven quantum critical behavior of a topological phase; applying this procedure to the one-dimensional p-wave superconductor yields an interesting phase diagram, consisting of two topologically distinct Griffiths phases separated by a critical point. In addition to its theoretical novelty, studying a disorder-driven topological phase transition via quantum entanglement poses distinct advantages over conventional numerics, especially when dealing with interacting systems. First, our approach only requires knowledge of a single translationally invariant ground state in the absence of disorder, while a numerical study of a disordered quantum critical point requires knowledge of the full low-lying spectrum, for every disorder realization. Second, the entanglement spectrum obtained from a random partition exhibits a duality between the partitioning probability p

and $1-p$. This guarantees a topological phase transition takes place at $p=1/2$ in topological states with an *irreducible* topological index [10], as shown by the above example. In contrast, identifying the location of the phase transition point is highly non-trivial in direct numerical studies of disordered Hamiltonians.

The random-partitioning scheme can be used straightforwardly to study spin-chains [22, 23], as well as higher-dimensional systems to numerically extract critical exponents of disorder-driven phase transitions, such as localization transitions in all Altland-Zirnbauer symmetry classes of non-interacting topological phases [24, 25]. It will be interesting to see whether our method applies to topological crystalline insulators, whose entanglement spectra show nontrivial features [26]. It might also be interesting to study the entanglement spectrum of fractional topological phases under a random partition, although the connection with topological phase transitions appears to be less direct.

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