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T. R. Kirkpatrick and D. Belitz

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Exponent relations at quantum phase transitions, with applications to metallic quantum ferromagnets

T. R. Kirkpatrick¹, D. Belitz^{2,3}

¹ *Institute for Physical Science and Technology,
and Department of Physics, University of Maryland,
College Park, MD 20742, USA*

² *Department of Physics and Institute of Theoretical Science,
University of Oregon, Eugene, OR 97403, USA*

³ *Materials Science Institute,
University of Oregon, Eugene, OR 97403, USA*

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Relations between critical exponents, or scaling laws, at both continuous and discontinuous quantum phase transitions are derived and discussed. In general there are multiple dynamical exponents at these transitions, which complicates the scaling description. Some rigorous inequalities are derived, and the conditions needed for these inequalities to be equalities are discussed. New scaling laws involving the specific-heat exponents are derived and contrasted with their counterparts at classical phase transitions. We also generalize the ideas of Fisher and Berker and others for applying (finite-size) scaling theory near a classical first-order transition to the quantum case. We then apply and illustrate all of these ideas by using the quantum ferromagnetic phase transition in metals as an explicit example. This transition is known to have multiple dynamical scaling exponents, and in general it is discontinuous in clean systems, but continuous in disordered ones. Furthermore, it displays many experimentally relevant crossover phenomena that can be described in terms of fixed points, originally discussed by Hertz, that ultimately become unstable asymptotically close to the transition and give way to the asymptotic fixed points. These fixed points provide a rich environment for illustrating the general scaling concepts and exponent relations. We also discuss the quantum-wing critical point at the tips of the tricritical wings associated with the discontinuous quantum ferromagnetic transition from a scaling point of view.

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I. INTRODUCTION

Classical or thermal phase transitions are special points in parameters space where the free energy is not analytic as a consequence of strong thermal fluctuations, see, e.g., Refs. 1–4. It is customary to distinguish between second-order or continuous phase transitions, where the order parameter (OP)⁵ goes to zero continuously at the transition, and first-order or discontinuous ones, where the order parameter displays a discontinuity. Classic examples for the former are the ferromagnetic transition at the Curie temperature in zero magnetic field, or the liquid-gas transition at the critical point; for the latter, a ferromagnet below the Curie temperature in a magnetic field, or the liquid-gas transition below the critical pressure. The nonanalytic free energy translates into singular behavior of observables. At a second-order transition, this usually takes the form of power laws, in which case the singularity is characterized by critical exponents. Let m be the order parameter (in the case of a ferromagnet, m is the magnetization, in the case of a liquid-gas transition, the density difference between the phases), h the external field conjugate to the order parameter, and r the dimensionless distance from the critical point for $h = 0$;⁶ at a thermal phase transition, r is usually chosen to be the distance from the critical temperature T_c , $r = T/T_c - 1$. Then

the specific heat C behaves as $C \propto |r|^{-\alpha}$, the order parameter m vanishes according to $m(r \rightarrow 0-) \propto (-r)^\beta$, the order-parameter susceptibility χ diverges according to $\chi(T) \propto |r|^{-\gamma}$, etc.⁷ α , β , and γ are examples of critical exponents. Underlying all of these singularities is a diverging length scale, the correlation length ξ . It measures the distance over which order-parameter fluctuations are correlated, and diverges as $\xi \propto |r|^{-\nu}$, which defines the exponent ν . Two other important critical exponents are δ , which describes the behavior of the order parameter as a function of its conjugate field h at criticality, $m(r = 0, h) \propto h^{1/\delta}$, and the exponent η , which describes the decay of the order-parameter correlation function at criticality, $G(r = 0, |\mathbf{x}| \rightarrow \infty) = \langle m(\mathbf{x}) m(0) \rangle \propto 1/|\mathbf{x}|^{d-2+\eta}$, where d is the spatial dimensionality.

The critical exponents are not all independent. At most classical transitions, it turns out that specifying two exponents determines all of the others. This was recognized early on and put in the form of exponent relations (also referred to as “scaling relations”, or “scaling laws”; the latter not to be confused with the homogeneity laws that are often called “scaling laws”). For instance, the six exponents α , β , γ , δ , η , and ν are related by the four relations listed in Eqs. (B1). These were initially derived from scaling assumptions, i.e., generalized homogeneity laws for various observables near a critical point, and later understood more deeply in terms of the renormal-

ization group (RG), which allowed for a derivation of the homogeneity laws. Some of the exponent relations are more robust than others. For instance, for a ϕ^4 -theory in $d > 4$ spatial dimensions, the Widom, Fisher, and Essam-Fisher scaling relations expressed in Eqs. (B1) remain valid, whereas the hyperscaling relation, Eq. (B1d), fails. This is related to the notion of dangerous irrelevant variables (DIVs), i.e., coupling constants that flow to zero under RG transformations but that some observables depend on in a singular way. As a result, “strong scaling”, i.e., a simple homogeneity law for the free energy, breaks down. More generally, there are constraints on the critical exponents that take the form of inequalities. They rely on much weaker assumptions than strong scaling and in some cases are rigorous. Examples are the Rushbrooke inequality, Eq. (B5), and the lower bound for the correlation-length exponent ν in disordered systems that is discussed in Appendix C.

A quantum phase transition (QPT) occurs by definition at zero temperature, $T = 0$, as a function of some non-thermal control parameter such as pressure, or composition, or an external magnetic field.⁸ The critical behavior at $T = 0$ is governed by quantum fluctuations rather than thermal ones. The role of temperature is thus different from that at a thermal transition, which necessitates the introduction of additional critical exponents. For instance, the order-parameter susceptibility will vary as $|r|^{-\gamma}$, with r the dimensionless distance at $T = h = 0$ from the critical point, but as $T^{-\gamma_T}$ as a function of temperature for $r \rightarrow 0$, with γ_T in general different from γ . Similarly, we need to define exponents β_T and ν_T that describe the behavior of the order-parameter and the correlation length, respectively, as functions of the temperature in addition to r .

The case of the specific heat c is less straightforward since c vanishes at $T = 0$ even away from any critical point. Related to this, in the thermodynamic identity that underlies the Rushbrooke inequality the specific-heat coefficient $\gamma = c/T$ appears. For the purpose of discussing quantum phase transitions it therefore is sensible to define critical exponents $\bar{\alpha}$ and $\bar{\alpha}_T$ that describe the critical behavior of the specific-heat coefficient according to $\gamma(r, T = 0) \propto |r|^{-\bar{\alpha}}$, and $\gamma(r = 0, T \rightarrow 0) \propto T^{-\bar{\alpha}_T}$. For thermal phase transitions, $\bar{\alpha}$ obviously coincides with the ordinary specific-heat exponent α . For QPTs, we define α and α_T in terms of the susceptibility χ_r given by the second derivative of an appropriate free energy with respect to r ; the physical meaning of this susceptibility depends on the nature of r . The definition of critical exponents discussed in this paper is summarized in Appendix A, and we will refer to the exponents α , $\bar{\alpha}$, β , etc. as the r -exponents, and to α_T , $\bar{\alpha}_T$, β_T , etc. as the T -exponents.

The concepts of scaling, and exponent relations, also carry over to dynamical critical phenomena.^{3,9} A dynamical critical exponent z is defined by how the critical time scale τ_ξ scales with the diverging correlation length ξ : $\tau_\xi \propto \xi^z$. In the case of thermal phase transitions, the dy-

namics are decoupled from the statics, and the static exponents can be determined independently of z . At quantum phase transitions this is not true since the statics and dynamics are intrinsically coupled. However, as we will show, at $T = 0$ there still are many exponent relations that involve the static exponents only. A further complication is the fact that multiple critical time scales, and hence multiple dynamical exponents z , which do occur at classical transitions¹⁰ but are not common, are the rule rather than the exception at quantum phase transitions.

Interestingly, the same concepts can also be applied to first-order phase transitions. Fisher and Berker¹¹ have developed a scaling description of classical first-order transitions and have shown that they can be understood as a limiting case of second-order transitions or critical points.

The purpose of this paper is to thoroughly discuss these concepts in the context of quantum phase transitions. While scaling concepts have been used since the notion of quantum phase transitions was invented, the issue of multiple dynamical exponents and its interplay with dangerous irrelevant variables has never been systematically discussed. Similarly, the importance of exponent relations has not been stressed in this context, and even the distinction between r -exponents and T -exponents is not always clearly made. In the first part of the paper we discuss scaling concepts for quantum phase transitions in general, with an emphasis on which aspects can be taken over from classical phase-transition theory and which ones requires modifications or new ideas. In particular, we show that the specific-heat coefficient shows scaling behavior that sets it apart from other observables and is not in direct analogy to the classical case, and we derive new exponent relations that involve the specific-heat exponents. Exponent relations have proven very useful in the classical context, since they provide stringent checks for whether experiments are actually in the asymptotic region. They are expected to be similarly useful for QPTs. We also generalized the scaling theory for thermal first-order transitions by Fisher and Berker¹¹ to the quantum case. In the second part of the paper we apply all of these concepts to the problem of the quantum ferromagnetic transition in metals. This transition is known to be generically first order in clean systems, and second order in disordered ones, and it has multiple dynamic critical exponents, which allows for an illustration of all of the ideas developed in the first part. It also displays many crossover phenomena the understanding of which is crucial for the interpretation of experiments. These crossovers can be phrased in terms of fixed points, originally discussed by Hertz,⁸ that become unstable asymptotically close to the transition. The experimental relevance of Hertz’s fixed point in disordered systems in particular is discussed here for the first time.

This paper is organized as follows. In Sec. II we derive exponent relations for the quantum case, both for the usual exponents and for those unique to QPTs, and we generalize the notion of scaling at a first-order transition

to the quantum case. In Sec. III we apply our results to the problem of the quantum ferromagnetic transition in metals, and in Sec. IV we conclude with a general discussion. Definitions of common critical exponents are given in Appendix A, and various classical exponent relations are recalled in Appendix B. Appendix C summarizes the Harris criterion and its generalization for the correlation-length exponent in systems with quenched disorder.

II. SCALING AT QUANTUM PHASE TRANSITIONS

A. General concepts for quantum phase transitions

In this section we discuss some very general points related to the role of dangerous irrelevant variables and multiple time scales in the context of quantum phase transitions.

1. Dangerous irrelevant variables and dynamical scaling

The classical exponent relations recalled in Appendix B fall into two distinct classes: The Widom, Essam-Fisher, and Fisher relations hold quite generally, including the case of systems above an upper critical dimensionality. By contrast, the hyperscaling relation, Eq. (B1d), holds only below an upper critical dimension, when no dangerous irrelevant variables (DIVs) are present. At quantum critical points the role of DIVs is more complex than in the classical case. A DIV can affect the temperature scaling of an observable, and it can do so with or without also affecting the static scaling. We will refer to the assumptions underlying scaling that can or cannot be altered by DIVs as “strong” and “weak” scaling assumptions, respectively, and to the resulting scaling behavior as “weak scaling” and “strong scaling”. We will further distinguish between weak and strong static and dynamic scaling as appropriate. With this nomenclature, hyperscaling in the usual sense is a special case of strong static scaling. (Note that weak scaling is more robust than strong scaling.) The DIV concept is more important for quantum phase transitions than for classical ones, since the latter are more likely to be above their upper critical dimension and strong scaling is more often violated.

To illustrate these points, consider an observable \mathcal{O} . Let $[\mathcal{O}]$ be the scale dimension of \mathcal{O} as determined by power counting within the framework of an appropriate field theory, and define a critical exponent ω by the critical behavior of \mathcal{O} as a function of the dimensionless distance from criticality at $T = 0$, which we denote by r : $\mathcal{O}(r, T = 0) \propto |r|^\omega$. If strong static scaling holds, \mathcal{O} obeys a homogeneity law¹²

$$\mathcal{O}(r, T = 0) = b^{-[\mathcal{O}]} \Phi_{\mathcal{O}}(r b^{1/\nu}, T = 0) . \quad (2.1a)$$

Here ν is the correlation length exponent, and $\Phi_{\mathcal{O}}$ is a

scaling function. Strong scaling thus leads to

$$\omega = \nu[\mathcal{O}] . \quad (2.1b)$$

This does not remain true if there is an irrelevant variable that is dangerous with respect to the r -dependence of \mathcal{O} . This changes the value of ω , but usually it does not change the fact that \mathcal{O} obeys a homogeneity law. Weak static scaling thus still holds in general,¹³ as expressed by

$$\mathcal{O}(r, T = 0) = b^{-\omega/\nu} \Phi_{\mathcal{O}}(r b^{1/\nu}, T = 0) . \quad (2.2)$$

The value of ω is now affected by the pertinent DIV and can no longer be determined by scaling arguments alone, even if $[\mathcal{O}]$ and ν are known.

So far we have just stated the usual concept of a DIV,^{3,4} applied to the static scaling behavior at a quantum phase transition. Now consider the temperature dependence of \mathcal{O} and define a critical exponent ω_T by $\mathcal{O}(r = 0, T) \propto T^{\omega_T}$. Let z be the dynamical exponent as obtained from the underlying field theory by power counting. (For now we assume there is only one such z , we will generalize to the case of multiple dynamical exponents below.) Strong dynamical scaling then is expressed by a homogeneity law

$$\mathcal{O}(r, T) = b^{-\omega/\nu} \Phi_{\mathcal{O}}(r b^{1/\nu}, T b^z) , \quad (2.3a)$$

which yields

$$\omega_T = \omega/\nu z . \quad (2.3b)$$

If an irrelevant variable is dangerous with respect to the temperature dependence of \mathcal{O} , the relation (2.3b) no longer holds. However, as in the static case we still have a homogeneity law as long as weak scaling holds. We write

$$\mathcal{O}(r, T) = b^{-\omega/\nu} \Phi_{\mathcal{O}}(r b^{1/\nu}, T b^{z_{\mathcal{O}}}) , \quad (2.4a)$$

where

$$z_{\mathcal{O}} = \omega/\nu\omega_T \quad (2.4b)$$

with ω and ω_T the physical exponents that include the effects of the DIV. Equations (2.4) are valid as long as weak scaling holds. If in addition strong static and/or dynamic scaling holds one also has the relations (2.1b) and/or (2.3b) between exponents.

Now consider the free-energy density f as a function of r , T , and the field h conjugate to the order parameter. Assuming strong scaling, we have

$$f(r, h, T) = b^{-(d+z)} \Phi_f(r b^{1/\nu}, h b^{[h]}, T b^z) , \quad (2.5)$$

where $[h]$ is the power-counting scale dimension of h . We define a “control-parameter susceptibility” χ_r as $\chi_r = -\partial^2 f / \partial r^2$. The physical meaning of χ_r depends on the nature of the control parameter. For instance, if the QPT

is triggered by hydrostatic pressure, χ_r will be proportional to the compressibility of the system. We further define a critical exponent α by $\chi_r(r, T = 0) \propto |r|^{-\alpha}$. Note that the such-defined α at a QPT has nothing to do with the specific heat; however, at a thermal transition our definition makes χ_r the specific-heat coefficient, and α has its usual meaning. We can now incorporate the effect of any DIVs by writing

$$f(r, h, T) = b^{-(2-\alpha)/\nu} \Phi_f(r b^{1/\nu}, h b^{\beta\delta/\nu}, T b^{\beta/\nu\beta_T}), \quad (2.6)$$

which is valid as long as weak scaling holds.

2. Multiple time scales

So far we have assumed that there is a single underlying dynamical exponent z that may or may not be modified by a DIV. An additional complication is that often more than one dynamical exponent is present. This can happen at classical phase transitions,¹⁰ but it is much more common at quantum phase transitions, chiefly because there are more soft or massless modes at $T = 0$ than at $T > 0$. For instance, at any quantum phase transition in a metallic system the coupling of the order parameter to the conduction electrons introduces a second time scale; a ballistic one ($z = 1$) in clean systems, or a diffusive one ($z = 2$) in disordered ones.¹⁴ This issue is independent of whether or not DIVs invalidate strong scaling, so for simplicity we will assume strong scaling in this subsection.

For definiteness, let us assume that there are two critical time scales with dynamical exponents z_1 and $z_2 < z_1$. The generalization of the homogeneity law (2.1a) for \mathcal{O} then reads

$$\mathcal{O}(r, T) = b^{-[\mathcal{O}]} \Phi_{\mathcal{O}}(r b^{1/\nu}, T b^{z_1}, T b^{z_2}). \quad (2.7)$$

Let us assume $[\mathcal{O}] > 0$ (for $[\mathcal{O}] < 0$ obvious modifications of the following statements hold). Then the leading temperature dependence of \mathcal{O} at $r = 0$ will be given by z_1 , i.e., $\omega_T = [\mathcal{O}]/z_1$. However, there is no guarantee that $\Phi(0, x, y)$ depends on x . If it does not, then $\omega_T = [\mathcal{O}]/z_2$, and the singular T -dependence of \mathcal{O} at criticality is weaker than one would naively expect. We will see explicit examples of this in Sec. III.

For the free energy, multiple dynamical exponents result in multiple scaling parts. Let us again consider the case of two critical time scales with dynamical exponents $z_1 > z_2$. All of the following considerations can be trivially simplified to the case $z_1 = z_2$, and they are easily generalized to the case of more than two dynamical exponents. We generalize Eq. (2.5) to

$$f(r, h, T) = b^{-(d+z_1)} \Phi_f^{(1)}(r b^{1/\nu}, h b^{[h]}, T b^{z_1}, T b^{z_2}) + b^{-(d+z_2)} \Phi_f^{(2)}(r b^{1/\nu}, h b^{[h]}, T b^{z_2}). \quad (2.8)$$

Note that the scaling function $\Phi_f^{(2)}$ does not depend on the argument $T b^{z_1}$. This is because the entropy den-

sity, $s = \partial f / \partial T$, must vanish at least as fast as $1/\xi^d$ for $\xi \rightarrow \infty$.¹⁵ For f and its derivatives with respect to r and h the second term obviously yields the most singular contribution. We thus can obtain homogeneity laws for the order parameter $m = \partial f / \partial h$, the order-parameter susceptibility $\chi_m = \partial^2 f / \partial h^2$, and the control-parameter susceptibility $\chi_r = \partial^2 f / \partial r^2$ from a scaling part of the free-energy density

$$f_2(r, h, T) = b^{-(d+z_2)} \Phi_f^{(2)}(r b^{1/\nu}, h b^{[h]}, T b^{z_2}). \quad (2.9)$$

For the specific-heat coefficient $\gamma = c/T = -\partial^2 f / \partial T^2$, on the other hand, the largest z yields the most singular contribution and we have

$$\gamma(r, T) = b^{z_1-d} \Phi_\gamma(r b^{1/\nu}, T b^{z_1}, T b^{z_2}). \quad (2.10)$$

The leading temperature dependence at $r = 0$ is in general given by the first argument of the scaling function, i.e., $\gamma(r = 0, T) \propto T^{d/z_1-1}$.

We emphasize again that all of the above relations assume strong scaling. See below for a discussion of the weaker statements that remain valid if strong scaling is violated. We also stress that Eq. (2.8), with its two additive scaling parts, is an *ansatz*. In the case of the quantum ferromagnetic transition discussed as our prime example in Sec. III it is known explicitly that the scaling part of the free energy has this form.

B. Exponent relations at quantum critical points

We now use the concepts laid out above to derive various relations between critical exponents.

1. Exponent relations that rely on weak scaling

Let the observable in question be the order parameter m , let h be the field conjugate to the order parameter, and let $\chi = \partial m / \partial h$ be the order-parameter susceptibility. As long as weak scaling holds, Eqs. (2.4), generalized to allow for the dependence of m on h , will be valid with $\omega = \beta$ and $\omega_T = \beta_T$:

$$m(r, h, T) = b^{-\beta/\nu} \Phi_m(r b^{1/\nu}, h b^{\beta\delta/\nu}, T b^{z_m}), \quad (2.11)$$

where $z_m = \beta/\nu\beta_T$ is the dynamical exponent relevant for the order parameter. In the context of Eq. (2.8), z_m is z_2 , possibly modified by DIVs. Differentiating with respect to h we find a corresponding homogeneity law for the order-parameter susceptibility:

$$\chi_m(r, T) = b^{(\delta-1)\beta/\nu} \Phi_\chi(r b^{1/\nu}, T b^{z_m}). \quad (2.12)$$

With the definitions of the exponents γ and γ_T , Appendix A, this implies the

Widom equality:

$$\gamma = \beta(\delta - 1), \quad (2.13a)$$

$$\gamma_T = \beta_T(\delta - 1), \quad (2.13b)$$

which holds at a QPT for both the r -exponents and the T -exponents as a consequence of weak scaling only. It is not affected by the presence of multiple time scales.

Next we consider, in addition to m , the order-parameter susceptibility χ_m and the control-parameter susceptibility χ_r , which all are obtained from the scaling part f_2 of the free-energy density, Eq. (2.9). Incorporating the effects of DIVs, if any, as in Sec. II A 1, we can write

$$m(r, h, T) = b^{(\beta\delta + \alpha - 2)/\nu} \Phi_m(r b^{1/\nu}, h b^{\beta\delta/\nu}, T b^{z_m}) , \quad (2.14a)$$

$$\chi_m(r, T) = b^{(2\beta\delta + \alpha - 2)/\nu} \Phi_\chi(r b^{1/\nu}, T b^{z_m}) , \quad (2.14b)$$

$$\chi_r(r, T) = b^{\alpha/\nu} \Phi_r(r b^{1/\nu}, T b^{z_m}) . \quad (2.14c)$$

From the definitions of the exponents β , γ , β_T , γ_T , and α_T , and using Eq. (2.13a) we then obtain the

Essam-Fisher equality:

$$\alpha + 2\beta + \gamma = 2 , \quad (2.15a)$$

for the r -exponents, and its analog

$$\alpha_T + 2\beta_T + \gamma_T = 2/\nu z_m \quad (2.15b)$$

for the T -exponents. Equation (2.15a) depends on weak scaling only. Note that α is *not* the specific-heat exponent, but rather the control-parameter-susceptibility exponent defined in Sec. II A 1 and Appendix A. Equation (2.15a) is the natural extension of the classical Essam-Fisher equality to quantum phase transitions, even though it does not involve a specific-heat exponent. The simple relation between Eq. (2.15a) and (2.15b), which is consistent with Eq. (2.4b) with $z_O = z_m$, is due to the fact that the dominant dynamical exponent is the same for all three observables m , χ_m , and χ_r . The latter can be guaranteed only in the presence of strong scaling, see the remarks after Eq. (2.8). However, while strong scaling is sufficient for Eq. (2.15b), it is not necessary.

Equation (2.14a) also implies $-\beta = \beta\delta + \alpha - 2$. Together with the Essam-Fisher equality this allows to express δ in terms of α and γ :

$$\delta = \frac{2 - \alpha + \gamma}{2 - \alpha - \gamma} . \quad (2.16)$$

This extends to QPTs another relation that is well known for thermal phase transitions.¹ Note that Eq. (2.16) is not independent; it follows from a combination of the Widom and Essam-Fisher equalities. We also stress again that in the current context α is *not* the specific-heat coefficient.

Now consider the order-parameter two-point correlation function $G(\mathbf{x}) = \int_0^\infty d\tau \langle m(\mathbf{x}, \tau) m(\mathbf{0}, 0) \rangle$, with τ the imaginary-time variable. The spatial integral of $G(\mathbf{x})$ yields the order-parameter susceptibility, $\chi_m = \int d\mathbf{x} G(\mathbf{x})$. It has the form

$$G(\mathbf{x}) = \frac{e^{-|\mathbf{x}|/\xi}}{|\mathbf{x}|^{d-2+\eta}} , \quad (2.17)$$

which defines both the correlation length ξ and the critical exponent η . One thus has

$$\chi_m \propto \xi^{2-\eta} . \quad (2.18)$$

An equivalent argument is to generalize the homogeneity equation for the order-parameter susceptibility to include the wave-number dependence:

$$\chi_m(r, T; k) = b^{\gamma/\nu} \Phi_\chi(r b^{1/\nu}, T b^{z_m}; k b) . \quad (2.19)$$

With the definitions of the exponents ν and ν_T , Appendix A, we obtain from either Eq. (2.18) or (2.19) the

Fisher equality:

$$\gamma = (2 - \eta)\nu , \quad (2.20a)$$

$$\gamma_T = (2 - \eta)\nu_T \quad (2.20b)$$

for both the r -exponents and the T -exponents. It depends on weak scaling only.

In summary, we have found three independent weak-scaling exponent relations for the r -exponents, viz. the Widom, Fisher, and Essam-Fisher equalities (2.13a), (2.15a), and (2.20a), and three corresponding relations for the T -exponents, Eqs. (2.13b), (2.15b), and (2.20b).

2. A rigorous inequality

Exponent relations that involve the specific-heat exponent $\bar{\alpha}$ (see the Introduction and Appendix A for a definition and discussion of $\bar{\alpha}$) fall into a different class, since the critical behavior of the specific-heat coefficient is in general governed by a scaling part of the free energy that is different from the one that determines the exponents discussed so far, see Eq. (2.10) and the related discussion. We start with the thermodynamic identity that underlies the classical Rushbrooke inequality^{1,16}

$$1 - \gamma_m/\gamma_h = [(\partial m/\partial T)_h^2] / \chi_T \gamma_h . \quad (2.21)$$

Here γ_m and γ_h denote the specific-heat coefficient at fixed order parameter and fixed conjugate field, respectively, and χ_T is the isothermal OP susceptibility. Note that, in order to apply this relation to QPTs, it is crucial to formulate it in terms of the specific-heat coefficients; the usual formulation in terms of the specific heats leads to a factor of T on the right-hand side that makes the $T \rightarrow 0$ limit ill defined.¹⁷ With the T -exponents as defined in Appendix A this yields an exponent inequality at a QPT in the form

$$\bar{\alpha}_T + 2\beta_T + \gamma_T \geq 2 . \quad (2.22a)$$

This is rigorous, since it depends only on thermodynamic stability arguments. If in addition we use weak scaling of the order parameter, Eq. (2.11), we find a corresponding inequality for the r -exponents,

$$\bar{\alpha} + 2\beta + \gamma \geq 2\beta/\beta_T . \quad (2.22b)$$

This follows since weak scaling of the OP implies $(\partial m / \partial T)_{T=0, h=0} \propto (-r)^{\beta(1-1/\beta_T)}$. Note that this is different from the classical Rushbrooke inequality, Eq. (B5). As for the latter, Eqs. (2.22) hold as equalities if $\gamma_m / \gamma_h \rightarrow 1$ for $T \rightarrow 0$ at $r = 0$, and for $r \rightarrow 0$ at $T = 0$, respectively, and as inequalities otherwise. In the classical limit, both of the Eqs. (2.22) turn into the the classical Rushbrooke inequality: The T -exponents coincide with the r -exponents, and $\bar{\alpha}$ coincides with the classical specific-heat exponent α , see the discussion in Appendix A.

3. Hyperscaling relations

The exponent relations we derived and discussed so far depended at most on weak scaling (with the exception of Eq. (2.15b), which relies to some extent on a strong-scaling assumption). If strong scaling is valid we can derive additional constraints on the exponents. For this purpose, we return to Eq. (2.8). As we discussed in this context, $z_2 = z_m$ is the dynamical exponent for the order parameter, the order-parameter susceptibility, and the control-parameter susceptibility, while $z_1 = z_c \geq z_m$ is the dynamical exponent for the specific-heat coefficient. In general z_c could be z_1 modified by DIVs, but the strong-scaling assumption means $z_c = z_1$. Strong scaling thus implies the quantum versions of the usual hyperscaling relation (B1d),

$$\alpha = 2 - \nu(d + z_m), \quad (2.23a)$$

$$\alpha_T = 2/\nu z_m - d/z_m - 1, \quad (2.23b)$$

the latter in agreement with Eq. (2.3b). We will refer to these as the α -hyperscaling relations. Analogously, we find from the first term in Eq. (2.8) hyperscaling relations for the specific-heat exponents

$$\bar{\alpha} = \nu(z_c - d), \quad (2.24a)$$

$$\bar{\alpha}_T = 1 - d/z_c, \quad (2.24b)$$

which we will refer to as the $\bar{\alpha}$ -hyperscaling relations.

Equations (2.23) and (2.24) are two independent strong-scaling relations. In conjunction with the weak-scaling relations from Sec. II B 1 we can use them to derive additional relations that are not independent. For instance, Eqs. (2.23) and (2.24) imply a relation between the control-parameter exponent α and the specific-heat exponent $\bar{\alpha}$,

$$\bar{\alpha} = \alpha - 2 + \nu(z_m + z_c), \quad (2.25a)$$

$$\bar{\alpha}_T = \alpha_T - 2/\nu z_m + 2 + d(1/z_m - 1/z_c). \quad (2.25b)$$

Combining these with the Essam-Fisher relations (2.15) we find

$$\bar{\alpha} + 2\beta + \gamma = \nu(z_m + z_c), \quad (2.26a)$$

$$\bar{\alpha}_T + 2\beta_T + \gamma_T = 2 + d(1/z_m - 1/z_c), \quad (2.26b)$$

Both of these equalities are consistent with the Eqs. (2.22), since $z_c \geq z_m$ always. Note that they are physically different from the Essam-Fisher equalities (2.15), which depend on weak scaling only.

We can use Eq. (2.25a) together with Eq. (2.16) to express δ in terms of $\bar{\alpha}$, γ , and the dynamical exponents,

$$\delta = \frac{\nu(z_m + z_c) - \bar{\alpha} + \gamma}{\nu(z_m + z_c) - \bar{\alpha} - \gamma}. \quad (2.27)$$

We note that all of the above relations rely on strong scaling, even if they do not explicitly involve the dimensionality.

Finally, the electrical conductivity is dimensionally an inverse length to the power $(d - 2)$. A strong-scaling hypothesis thus implies that the scaling part $\Delta\sigma$ of the conductivity obeys a homogeneity law

$$\Delta\sigma(r, T) = b^{-(d-2)} F_\sigma(r b^{1/\nu}, T b^{z_c}). \quad (2.28)$$

For the exponents s and s_T defined in Eq. (A6) this implies the

Wegner equality:

$$s = \nu(d - 2), \quad (2.29a)$$

$$s_T = (d - 2)/z_c. \quad (2.29b)$$

These relations play an important role in the theory of electron localization,^{18,19} but are much more generally applicable. They do, however, depend on strong scaling and are not valid if DIVs affect the conductivity. Note that in general, with DIVs taken into account, s and s_T can be either positive or negative in any dimension, with negative values applying to either the clean limit or regimes where the residual resistivity is small compared to the temperature-dependent part.²⁰

C. Scaling at quantum first-order transitions

Fisher and Berker have shown how a classical first-order transition can understood within a standard scaling and RG framework.¹¹ In a RG context, at a first-order transition all gradient-free operators in a Landau-Ginzburg-Wilson (LGW) theory are relevant and have the largest scale dimension that is thermodynamically allowed, viz., d . This just follows from the fact that the OP itself is dimensionless at a first-order transition, so the operator dimensions must make up for the scale dimension of the spatial integration measure.²¹ In particular, $\nu = 1/d$. This is reconciled with the usual notion of a finite correlation length at a first-order transition by means of finite-size scaling considerations. A dimensionless order parameter requires $[h] = d$, which is equivalent to $\beta = 0$, and the remaining classical exponent values follow readily: $\eta = 2 - d$, $\gamma = 1$, $\delta = \infty$, and $\alpha = 1$. All scaling relations, Appendix B, are fulfilled, including the hyperscaling relation. (Note that DIVs are not an issue in this case, since the suspect operators are relevant.)

The reasoning of Fisher and Berker can readily be extended to QPTs.⁷ For the case of making the logic clear, we first do so for a QPT with only one dynamical exponent, and then generalize to the case of two dynamical exponents. Consider again the homogeneity law for the free energy, Eq. (2.5). In order for the order parameter $m = \partial f / \partial h$ to be dimensionless we must have

$$[h] = d + z. \quad (2.30)$$

A dimensionless OP in turn implies

$$1/\nu = d + z, \quad (2.31a)$$

which generalizes the classical $1/\nu = d$. It further implies

$$\eta = 2 - d - z, \quad \delta = \infty, \quad \beta = 0, \quad \beta_T = 0. \quad (2.31b)$$

Considering the OP susceptibility $\chi_m = \partial^2 f / \partial h^2$, and using Eqs. (2.30) and (2.31a), we have

$$\gamma = 1, \quad \gamma_T = (d + z)/z. \quad (2.31c)$$

Similarly, we find from the homogeneity law for the control-parameter susceptibility $\chi_r = \partial^2 f / \partial r^2$ the exponents

$$\alpha = 1, \quad \alpha_T = (d + z)/z. \quad (2.31d)$$

The value $\alpha = 1$ implies that the first derivative of the free energy with respect to r is discontinuous, and the second derivative contains a delta-function. This is analogous to the entropy being discontinuous, and the existence of a latent heat, at a thermal phase transition. In the quantum case the physical interpretation of $\alpha = 1$ depends on the nature of the control parameter; see the discussion in Sec. IV B 1. From the homogeneity law for the specific-heat coefficient $\gamma = \partial^2 f / \partial T^2$ we find

$$\bar{\alpha} = \frac{z - d}{z + d}, \quad \bar{\alpha}_T = \frac{z - d}{z}. \quad (2.31e)$$

Finally, Eq. (2.2) holds in particular for the correlation-length exponent, which yields

$$\nu_T = 1/z. \quad (2.31f)$$

We see that all scaling relations discussed in Sec. II A hold, in analogy to the classical case.

In the above discussion we have assumed what is called a *block* geometry in the finite-size scaling theory of classical first order phase transitions. Physically it is realized in a finite system whose linear sizes in all dimensions, including the imaginary-time or inverse temperature one, are comparable. Another choice is a *cylinder* geometry in which the linear size in one dimension is large compared to the others. In the quantum case this is of special interest for a system with fixed finite volume in the zero-temperature limit, where the size in the imaginary-time direction becomes infinitely large. In this case arguments identical to those used in classical finite-size scaling theory²² lead to a time scale that scales as

$$\tau \sim e^{\sigma L^d}, \quad (2.32)$$

with σ a positive constant. In the classical case the analogous result has led to a number of remarkable conclusions that have been confirmed experimentally.²³ The physical meaning of τ in the quantum case is the time it takes for a droplet of volume L^d containing a certain state to transform into a different state via a tunneling process.

Now we turn to the case of two different dynamical exponents for the OP and the specific heat, which we again denote by z_m and z_c , respectively. The imaginary-time integral in the mass term and the Zeeman term in a LGW functional will then have a scale dimension of $-z_m$ (taking the scale dimension of a length to be -1). Making the OP dimensionless thus requires

$$[h] = d + z_m, \quad (2.33a)$$

and

$$1/\nu = d + z_m. \quad (2.33b)$$

The OP is now dimensionless by construction, which implies

$$\eta = 2 - d - z_m, \quad \delta = \infty, \quad \beta = 0, \quad \beta_T = 0. \quad (2.33c)$$

From the OP susceptibility, $\chi_m = \partial m / \partial h$, we obtain

$$\gamma = 1, \quad \gamma_T = (d + z_m)/z_m, \quad (2.33d)$$

and from the control-parameter susceptibility $\chi_r = \partial^2 f / \partial r^2$

$$\alpha = 1, \quad \alpha_T = (d + z_m)/z_m. \quad (2.33e)$$

Since the correlation length is entirely a property of the OP-OP correlation function, its temperature dependence is also governed by z_m and Eq. (2.4b) holds for $\omega = \nu$ with $z = z_m$:

$$\nu_T = 1/z_m. \quad (2.33f)$$

The specific-heat coefficient is governed by the dynamical exponent z_c , and we thus have

$$\bar{\alpha} = \nu(z_c - d) = \frac{z_c - d}{z_m + d}, \quad \bar{\alpha}_T = 1 - d/z_c. \quad (2.33g)$$

All of the scaling relations discussed in Sec. II A still hold.

We will see explicit examples for the general relations developed here in Sec. III, and we will further discuss the underlying concepts and their consequences in Sec. IV.

III. APPLICATION TO QUANTUM FERROMAGNETS

The scaling theory developed in Section II is in general quite complex because of the occurrence of multiple time scales at many quantum phase transitions. As an explicit illustration we now apply the theory to the quantum ferromagnetic phase transitions in metallic systems. For a review of the ferromagnetic quantum phase transition problem, see Ref. 24.

A. Clean systems

The ferromagnetic transition in metals was one of the earliest quantum phase transitions considered, and it was used as an example by Hertz in his seminal paper on a renormalization-group approach to quantum phase transitions.⁸ Hertz considered a ferromagnetic order parameter \mathbf{m} coupled to conduction electrons. The latter lead to Landau damping of the magnetic fluctuations, and on general grounds, using symmetry arguments as well as the known structure of the Fermi-liquid dynamics, one can write down an action

$$\mathcal{A}_{\text{Hertz}} = - \sum_{\mathbf{k}, n} [r + a\mathbf{k}^2 + c|\Omega_n|/|\mathbf{k}|] \times \mathbf{m}(\mathbf{k}, \Omega_n) \cdot \mathbf{m}(\mathbf{k}, -\Omega_n) + O(\mathbf{m}^4), \quad (3.1)$$

where we have written only the term quadratic in the order parameter explicitly. Here Ω_n is a bosonic Matsubara frequency, and r , a , and c are parameters of the LGW functional. The term $|\Omega_n|/|\mathbf{k}|$ reflects the dynamics of the conduction electrons. This is the action considered by Hertz, who explicitly derived it from a specific model. From this he concluded that the dynamical critical exponent is $z = 3$ (since $\Omega \sim |\mathbf{k}|^3$ for $r = 0$), and that the upper critical dimension is $d_c^+ = 4 - z = 1$. This in turn led to the conclusion that the quantum phase transition is second order and described by a simple Gaussian fixed point, and that the static critical behavior is mean-field-like for all $d \geq 1$ with logarithmic corrections to scaling for $d = 1$.²⁵ The behavior at finite temperature was considered in detail by Millis,²⁶ and the results of the RG treatment for $d = 3$ confirmed results obtained earlier by Moriya and co-workers by means of what is often called self-consistent spin-fluctuation theory.²⁷ This combined body of work is often referred to as Hertz-Millis-Moriya theory.

It was later shown that the above conclusions do not hold due to properties of the conduction electrons that are not captured in Hertz's action. A careful analysis of the soft fermionic modes coupling to the magnetization shows that, in three-dimensional systems, the leading wave-vector dependence in the Gaussian action is a term proportional to $\mathbf{k}^2 \ln(1/|\mathbf{k}|)$ with a *negative* prefactor,²⁸ and such a term is indeed generated under renormalization if one starts with Hertz's action. An equivalent statement is that there is a $m^4 \ln(1/m)$ term in a generalized Landau theory for the ferromagnetic quantum phase transition.²⁴ As a result, Hertz's fixed point is not stable, and the ferromagnetic quantum phase transition in clean metals is generically first order.^{29,30} However, depending on quantitative details involving the strength of the electron correlations and the inevitable weak disorder, there can be a sizable regime where the critical behavior associated with Hertz's fixed point is observable, even though asymptotically close to the transition it crosses over to a first-order transition.²⁴ In this subsection we therefore discuss, and include in Table I, the critical exponents

at Hertz's fixed point, together with the exponents associated with the ultimate first-order transition, and the critical behavior at the termination point of the tricritical wings that result from the first-order transition.³¹

1. Hertz's fixed point

Table I lists the critical exponents associated with Hertz's fixed point. The values for the static exponents are straightforward generalizations of the results obtained in Refs. 8,26. The value for the resistivity exponent s_T is Mathon's result³³ generalized to d dimensions, and $z_c = 3$ is Hertz's value for the dynamical critical exponent.⁸ There was no notion of a separate dynamical exponent z_m in the original work. However, this concept naturally arises if we take the point of view expressed in Eqs. (2.4) and incorporate the effects of the DIVs in the homogeneity laws. Equation (2.4b) applied to the order parameter then implies $z_m = \beta/\nu\beta_T$, which yields $z_m = 1/\beta_T$ as listed in Table I. Alternatively, one can work with only one dynamical exponent, $z = 3$, and consider the DIV explicitly, as was done by Millis,²⁶ see also Ref. 24. This reasoning effectively leads to $z_m = z/(1 + \nu(d - 1))$, which is the same result as above. At the upper critical dimension $d = 1$ the effects of the DIVs disappear, and there is only one dynamical exponent $z_m = z_c = 3$. Note that there is only one fundamental time scale in Hertz's theory, and the appearance of a second dynamical exponent is solely due to the fact that the fundamental z is modified by a DIV in some contexts, but not in others. This is very different from the theories discussed in Secs. III A 2, III A 3, and III B 2, which intrinsically contain two time scales of different physical origin. This is an indication that important physics related to the conduction electrons is missing in Hertz's theory; see Sec. IV B 2 for additional comments on this issue.

Since for $d > 1$ the system is above its upper critical dimension, DIVs in general invalidate any scaling relations that rely on strong scaling as defined in Sec. II A, while those that rely on weak scaling only will remain valid. Indeed, considering the values in the table, we see that the Widom equalities (2.13), the Essam-Fisher equalities (2.15), the Fisher equalities (2.20), and the Rushbrooke inequalities (2.22) are satisfied for all $d \geq 1$. The hyperscaling relations (2.23), on the other hand, do not hold. The hyperscaling relations (2.24) are a more complicated case. The values of $\bar{\alpha}$ and $\bar{\alpha}_T$ listed in Table I reflect the leading fluctuation contribution to the specific-heat coefficient, which do obey strong scaling. For $1 \leq d < 3$ they dominate the constant mean-field contribution which violates strong scaling, and as a result Eqs. (2.24) hold even though the system is above its upper critical dimension. For $d > 3$ they are subleading compared to the mean-field contribution, the specific-heat exponents lock into their mean-field values $\bar{\alpha} = \bar{\alpha}_T = 0$, and hyperscaling breaks down.

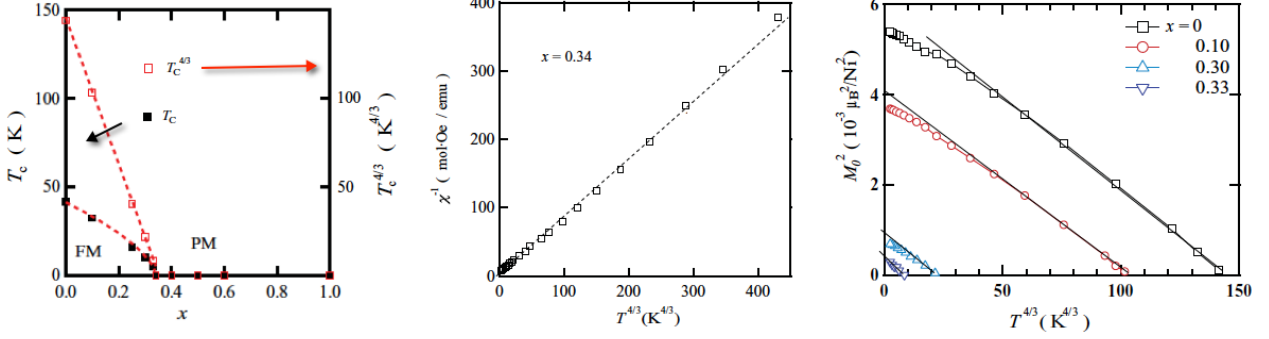


Figure 1: Hertz-type scaling behavior as observed in $\text{Ni}_3\text{Al}_{1-x}\text{Ga}_2$. From left to right: T_c vs x phase diagram, inverse magnetic susceptibility vs. $T^{4/3}$ for the critical sample, magnetization squared vs. $T^{4/3}$ for various concentrations. Figure adapted from Ref. 32.

The contribution to the resistivity ρ due to scattering of electrons by critical fluctuations was calculated by Mathon,³³ who found $\Delta\rho(r=0, T) \propto T^{5/3}$ in $d=3$. A simple generalization to general d yields $\Delta\rho(r=0, T) \propto T^{(d+2)/3}$, or $s_T = -(d+2)/3$ for the conductivity exponent s_T defined in Appendix A. From a scaling point of view, this result can be made plausible as follows. Consider the strong-scaling homogeneity law (2.28) for the conductivity contribution $\Delta\sigma$. In a clean system, the backscattering factor in the Boltzmann equation provides an additional factor of the hydrodynamic momentum squared, which is not captured by power counting. This leads to a scale dimension of $\Delta\sigma$ that is effectively equal to $d-4$ rather than $d-2$. In addition, the DIV u with scale dimension $[u] = -(d-1)$ affects $\Delta\sigma$, and from Fermi's golden rule it is plausible that $\Delta\sigma \propto 1/u^2$. Combining these arguments, we have

$$\begin{aligned} \Delta\sigma(r, T) &= b^{-(d-4)} b^{2(d-1)} F_\sigma(r b^{1/\nu}, T b^{z_c}) \\ &= b^{(d+2)} F_\sigma(r b^{1/\nu}, T b^{z_c}), \end{aligned} \quad (3.2)$$

which yields $s_T = -(d+2)/z_c$ as quoted above and listed in Table I. Note that the Wegner equality (2.29b) is violated for two reasons: Firstly, the conductivity, or the underlying relaxation rate, does not have its power-counting scale dimension due to the backscattering factor in the Boltzmann equation. Secondly, the DIV further modifies the scale dimension and actually changes its sign for all dimensions. Finally, note that the exponent s as defined in Eq. (A6) does not exist since $\sigma(T=0) = \infty$. Rather, for $T \rightarrow 0$ at fixed $r \neq 0$ the conductivity will cross over to the Fermi-liquid result, $\Delta\sigma(T) \propto T^{-2}$, and Eq. (3.2) yields the r -dependence of the prefactor:

$$\Delta\sigma(r \neq 0, T \rightarrow 0) \propto |r|^{(4-d)/2} T^{-2}. \quad (3.3)$$

As an illustration, we discuss an experiment on the ferromagnet $\text{Ni}_3\text{Al}_{1-x}\text{Ga}_x$, which displays a QPT at $x = x_c \approx 0.34$.³² This system is only moderately disordered, and critical behavior associated with the clean Hertz fixed point is expected to be observable in a sizable transient regime.²⁴ There are three experimental observations, from which four exponents can be deduced:

- (1) The critical temperature scales as $T_c \sim r^{3/4}$, see the first panel in Fig. 1. This implies $\nu z_m = \nu z / (1 + 2\nu) = 3/4$, as can be seen, for instance, from Eq. (2.12): The critical temperature is determined by χ_m diverging, which must happen for a particular value of the argument x in $\Phi_\chi(1, x)$. This in turn implies $T_c/r^{\nu z_m} = \text{const.}$
- (2) At the critical concentration, the magnetic susceptibility scales as $\chi_m \sim T^{-4/3}$, see the second panel in Fig. 1. This implies $\gamma_T = 4/3$. Together with the first observation, it also implies $\gamma = 1$, since $r \sim T^{4/3} \sim 1/\chi$.
- (3) The magnetization vanishes as $m^2 \propto T_c^{4/3} - T^{4/3}$, see the third panel in Fig. 1. In addition to confirming the product $\nu z_m = 3/4$ this yields $\beta = 1/2$, see the discussion after Eq. (4.2). All of these results are in agreement with the theoretical results summarized in Table I, specialized to $d=3$. The conductivity exponent $s_T = -5/3$ has also been observed in various materials, see, e.g., Ref. 27.

2. The first-order transition

There are strong theoretical arguments for the quantum phase transition from a paramagnet to a homogeneous ferromagnet to be first order,^{29,30} and this is indeed the prevalent experimental observation.²⁴ Here we discuss the exponent values, and the scaling relations, at this particular first-order quantum phase transition as an example of the general scaling theory in Sec. II C.

Let us first discuss the dynamical exponents. The soft fermionic fluctuations that drive the transition first order are of a ballistic nature with $z = z_2 = 1$. Their coupling to the order-parameter fluctuations lead to $z_m = 1$. The static fermionic susceptibility in clean metals at $T = 0$ has a wave-number dependence $\chi(\mathbf{k} \rightarrow 0) \propto \text{const.} + |\mathbf{k}|^{d-1}$.²⁸ This plays against the $|\Omega|/|\mathbf{k}|$ Landau-damping term (see Eq. (3.1) with $a\mathbf{k}^2$ replaced by $|\mathbf{k}|^{d-1}$), which produces another dynamical exponent $z_1 = d$. For $d > 1$ we thus have $z_c = z_1 = d$. $\beta = \beta_T = 0$ and $\delta = \infty$ by the discontinuous nature of the transition. Also, the fact that the order parameter is dimensionless enforces $\nu = 1/(d + z_m) = 1/(d + 1)$ and $\eta = 2 - d - z_m = 1 - d$,

Table I: Static, dynamic, and transport critical exponents as defined in the text, and validity of exponent relations, at various fixed points for the quantum phase transition in metallic ferromagnets. The first two columns represent true asymptotic exponents (QWCP = quantum wing-critical point). The third and fourth column refer to critical behavior that is asymptotically not represented by pure power laws, and the fifth and sixth column represent power-law behavior that is observable only in a transient non-asymptotic regime. Strong-scaling relations for the T -exponents are not shown, they hold if and only if the corresponding relations for the r -exponents hold. See the text for additional explanation. N/A = not applicable.

			Ferromagnetic Fixed Point					
			True Critical Fixed Points				Unstable Fixed Points	
			clean (1 st order) ^a	clean (QWCP) ^{b,c}	dirty ^d	dirty (pre- asymptotic) ^e	clean (Hertz) ^c	dirty (Hertz) ^f
Exponents	static	ν	$\frac{1}{d+1}$	$\frac{1}{2}$	$\frac{1}{d-2}$	$\frac{1}{d-2+\lambda}$	$\frac{1}{2}$	$\frac{1}{2}$
		ν_T	1	$\frac{d+1}{6}$	$\frac{1}{2}$	$\frac{1}{2+\lambda}$	$\frac{d+1}{6}$	$\frac{d+2}{8}$
		β	0	$\frac{1}{2}$	$\frac{2}{d-2}$	$\frac{2}{d-2+\lambda}$	$\frac{1}{2}$	$\frac{1}{2}$
		β_T	0	$\frac{d+1}{6}$	1	$\frac{2}{2+\lambda}$	$\frac{d+1}{6}$	$\frac{d+2}{8}$
		δ	∞	3	$\frac{d}{2}$	$\frac{d+\lambda}{2}$	3	3
		γ	1	1	1	1	1	1
		γ_T	$d+1$	$\frac{d+1}{3}$	$\frac{d-2}{2}$	$\frac{d-2+\lambda}{2+\lambda}$	$\frac{d+1}{3}$	$\frac{d+2}{4}$
		η	$1-d$	0	$4-d$	$4-d-\lambda$	0	0
		α	1	0	$\frac{d-6}{d-2}$	$\frac{d-6+\lambda}{d-2+\lambda}$	0	0
		α_T	$d+1$	0	$\frac{d-6}{2}$	$\frac{d-6+\lambda}{2+\lambda}$	0	0
		$\bar{\alpha}$	0	$\frac{3-d}{2}$ or 0^g	0	$\frac{\lambda}{d-2+\lambda}$	$\frac{3-d}{2}$ or 0^g	$\frac{4-d}{2}$ or 0^h
		$\bar{\alpha}_T$	0	$\frac{3-d}{3}$ or 0^g	0	$\frac{\lambda}{d+\lambda}$	$\frac{3-d}{3}$ or 0^g	$\frac{4-d}{4}$ or 0^h
	dynamic	z_m	1	$\frac{6}{d+1}$	2	$2+\lambda$	$\frac{6}{d+1}$	$\frac{8}{d+2}$
		z_c	d	3	d	$d+\lambda$	3	4
	transport	s	N/A	N/A	1	$\frac{d-2}{d-2+\lambda}$	N/A	N/A
		s_T	N/A	$-\frac{d+2}{3}$	$\frac{d-2}{d}$	$\frac{d-2}{d+\lambda}$	$-\frac{d+2}{3}$	$(-\frac{d+4}{4})^i$
Exponent Relations	weak scaling	Widom	(yes)	yes	yes	yes	yes	yes
		Essam-Fisher	yes	yes	yes	yes	yes	yes
		Fisher	yes	yes	yes	yes	yes	yes
	strong	α -hyperscaling	yes	no ^j	yes	yes	no ^j	no ^k
		$\bar{\alpha}$ -hyperscaling	yes	(yes) ^l	yes	yes	(yes) ^l	(yes) ^m
		$\bar{\alpha} + 2\beta + \gamma = \nu(z_m + z_c)^n$	yes	no ^j	yes	yes	no ^j	no ^k
		Wegner	N/A	no	yes	yes	no	no
		Chayes et al	N/A	N/A	yes	no ^o	N/A	no

^a For $d > 1$.

^b This physical fixed point maps onto the unphysical (describing pre-asymptotic behavior only) clean Hertz fixed point. See the text for a discussion of what is observed if the critical point is approached along generic paths in the phase diagram.

^c For $d \geq 1$. At the upper critical dimension $d = 1$ there are logarithmic corrections to scaling.

^d For $2 < d < 4$. Values in this column equal values in the next column for $\lambda = 0$. The critical behavior consists of log-normal terms multiplying power laws with the exponents shown.

^e For $2 < d < 4$. λ depends on the distance from criticality. In $d = 3$, $\lambda \approx 2/3$ in a large region.

^f For $d \geq 0$. At the upper critical dimension $d = 0$ there are logarithmic corrections to scaling.

^g For $1 \leq d \leq 3$ and $d > 3$, respectively. ^h For $0 < d \leq 4$ and $d > 4$, respectively.

ⁱ See Sec. IIIB1 for the sense in which this result is valid. ^j Except for $d = 1$. ^k Except for $d = 0$.

^l For $1 \leq d \leq 3$ only. ^m For $0 \leq d \leq 4$ only. ⁿ Not an independent relation. ^o See the text for a discussion.

as explained in Sec. II C, while $\nu_T = \nu/\nu_{z_m} = 1$. It further implies that the free-energy density scales linearly with the control parameter, which implies $\alpha = 1$ and $\alpha_T = \alpha/\nu_{z_m} = d + 1$. Finally, in a fermionic system the specific-heat coefficient must display a discontinuity at a discontinuous phase transition, which implies $\bar{\alpha} = \bar{\alpha}_T = 0$. Note that, combined with the scaling result (2.33g), this implies that the dynamical exponent z_c must be equal to d .

All of these exponent values are displayed in Table I. We see that all exponent relations, including those that rely on strong scaling, are valid as expected. The Widom equation is obvious fulfilled only in the sense that $\beta(\delta - 1) \rightarrow 1$ as $\beta \rightarrow 0$, $\delta \rightarrow \infty$.

3. The quantum wing-critical point

Since the quantum ferromagnetic transition is first order, while the corresponding thermal transition is generically second order, there necessarily is a tricritical point in the phase diagram if the Curie temperature is continuously suppressed by means of some control parameter. This in turn leads to the existence of tricritical wings, i.e., a pair of surfaces of first-order transitions that emanate from the coexistence curve in the $h = 0$ plane.³¹ These wings are indeed commonly observed, see Fig. 2 for an example. They end in a pair of quantum critical points (QWCPs; in the literature they are often erroneously referred to as quantum critical end points) correspond to a fixed point that maps onto Hertz's fixed point; they thus are an example of a true quantum critical point that is correctly described by Hertz theory, and the critical behavior is known exactly for all $d > 1$.³¹ The critical exponents are thus the same as those discussed in Sec. III A 1, and the entries in Table I reflect this. However, the application of these results for the prediction of experimental observations must be handled with care, as we will now discuss.

A crucial point to remember is that at the QWCP, in contrast to Hertz's fixed point, the field h conjugate to the order parameter is *not* the physical magnetic field, and $h = 0$ only on special paths in the phase diagram that are not natural paths to choose for an experiment. Let H be the physical magnetic field, and for definiteness let us assume that the control parameter is hydrostatic pressure p . In the three-dimensional T - p - H phase diagram, let the QWCP be located at $(0, H_c, p_c)$, and let the magnetization at this point have the value m_c . Then the conjugate field is given by $h = 2m_c\delta p - \delta H$, where $\delta p = p - p_c$, and $\delta H = H - H_c$.^{24,31} In order to observe the exponent β , for instance, one therefore needs to approach the QWCP on a curve given asymptotically by $\delta H = 2m_c\delta p$, see path (1) in Fig. 2. This is in exact analogy to the case of a classical liquid-gas critical point in the p - T plane, where an observation of β requires that the critical point be approached on the critical isochore.^{4,34}

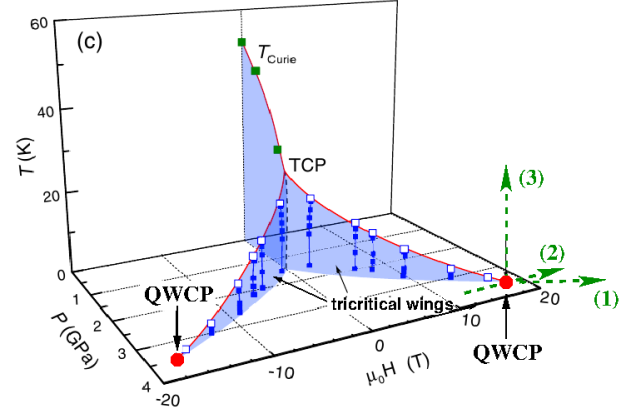


Figure 2: Tricritical wings observed in the temperature-pressure-magnetic field phase diagram of UGe_2 by Kotegawa et al., Ref. 35. Data points are represented by squares, blue planes are planes of first-order transitions, red solid lines are lines of second-order transitions, red points are the quantum wing-critical points (QWCP). Also shown is the tricritical point (TCP). Dotted green arrows labeled (1), (2), (3) are three different paths through a QWCP; the critical behavior predicted for each path is discussed in the text. Figure adapted from Ref. 35.

Probing the QWCP along a generic path in the $T = 0$ plane measures the exponent δ instead, e.g.,

$$m(p = p_c, H, T = 0) \propto |H - H_c|^{1/\delta} = |H - H_c|^{1/3}. \quad (3.4)$$

This is the behavior predicted for the magnetization along path (2) in Fig. 2. A related complication occurs for the temperature dependence of observables at the QWCP. Consider again the order parameter. The exponent β_T is defined via the T -dependence of m at criticality in zero conjugate field, see Appendix A. Since p_c is temperature dependent, $|p_c(T) - p_c(T = 0)| \propto T^{(d+1)/3}$, see Sec. III A 1, this means that β_T can be observed only on a particular surface in the three-dimensional parameter space spanned by T , p , and H . Along a generic path through the QWCP the temperature dependence of m is given by the exponent combination $2\nu_T/\delta$. In particular, just raising the temperature from zero at the critical point (path (3) in Fig. 2) yields

$$m(p = p_c, H = H_c, T) \propto |h|^{1/\delta} = -T^{2\nu_T/\delta} = -T^{(d+1)/9}. \quad (3.5a)$$

This is the result that was derived in Ref. 31, see also Ref. 24. An observable that is easier to measure is the magnetic susceptibility χ_m , which along the same path behaves as

$$\chi_m(p = p_c, H = H_c, T) \propto |h|^{-\gamma/\beta\delta} = T^{-2(d+1)/9}. \quad (3.5b)$$

B. Disordered systems

Quenched disorder in metallic ferromagnets introduces many different effects. Some, such as the change of the conduction-electron dynamics from ballistic to diffusive, directly affect the behavior at the quantum phase transition. Others, such as rare-region effects that can lead to the appearance of a quantum Griffiths region in the paramagnetic phase,^{24,36} are superimposed on critical singularities and may easily be confused with the latter. Disentangling these various effects is challenging from both a theoretical and an experimental point of view, and many open questions remain. Here we ignore rare-region effects and discuss the effects of quenched disorder on the phase transition itself.

This problem was also considered by Hertz,⁸ who argued that the only salient change compared to the clean case is in the Landau-damping term in Eq. (3.1), which now is $|\Omega_n|/k^2$ due to the diffusive dynamics of the conduction electrons. This obviously leads to a dynamical critical exponent $z = 4$, and to an upper critical dimension $d_c^+ = 0$. The finite-temperature behavior can be discussed in exact analogy to Millis's treatment of the clean case in Ref. 26.

Hertz's fixed point in the disordered case is again unstable. The physics behind this instability is the same as in the clean case, viz. the coupling of the magnetization to fermionic soft modes, which now are diffusive in nature at asymptotically small wave numbers. At larger wave numbers they cross over to the clean soft modes. For weak disorder, this crossover occurs at a very small wave number, and one expects the effects to be small. This expectation is indeed borne out in practical terms: Even though strictly speaking there cannot be a first-order transition with any amount of disorder,³⁷ the smearing of the transition is very small and the observable effect is just a suppression of the tricritical temperature, with the quantum phase transition remaining first order.³⁸ However, for a threshold value of the disorder strength the tricritical temperature reaches zero, the quantum phase transition becomes second order, and above this threshold it remains second order, albeit with an unusual critical behavior. The evolution of the phase diagram is shown in Fig. 3. A renormalized mean-field theory leads to unusual critical exponents, and a renormalization-group analysis of the fluctuations reveals that there are marginal operators in all dimensions $2 < d < 4$, which leads to log-normal terms that multiply the usual critical power laws.^{39–41} This modification of scaling mimics, in a sizable pre-asymptotic region, power laws with effective exponents that are quite different from the asymptotic ones.⁴² Furthermore, the behavior associated with Hertz's unstable fixed point can be observable in substantial regimes in parameter space, as we will show below. We therefore list in Table I, and discuss below, the critical behavior at Hertz's fixed point as well as that associated with the physical critical fixed point and its pre-asymptotic region.

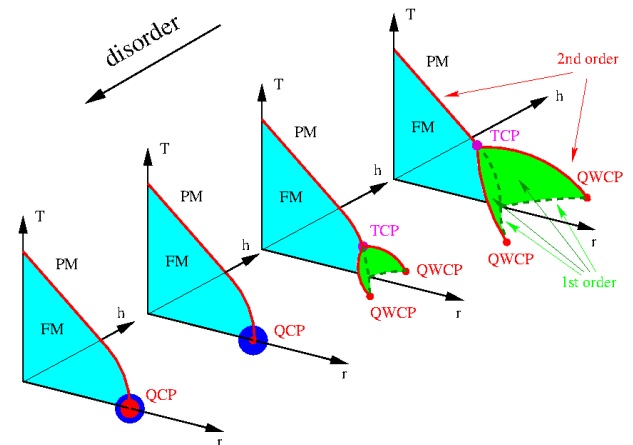


Figure 3: Evolution of the phase diagram in temperature-control parameter-field (T - r - h) space with increasing disorder. For weak disorder the phase diagram is a schematic version of the experimentally observed one shown in Fig. 2. Increasing disorder suppresses the tricritical point (TCP). At a critical disorder the TCP merges with the quantum wing-critical points (QWCP), and a quantum critical point (QCP) in zero field emerges. For disorder strengths close to the threshold the asymptotic quantum critical behavior is confined to a very small region (red circle), and the observable critical behavior (blue circle) is controlled by Hertz's fixed point. With further increasing disorder the instability of Hertz's fixed point becomes apparent farther away from the transition, and the region controlled by the true critical fixed point grows.

1. Hertz's fixed point

As mentioned above, Hertz's fixed point in the disordered case is never a physical fixed point; it is unstable with respect to the same physical processes as in the clean case, viz., coupling of the magnetization to soft fermionic excitations. For disordered systems, the latter are diffusive, and their effects are small in the limit $k_F \ell \gg 1$, with k_F the Fermi wave number and ℓ the elastic mean-free path. In addition, it is destabilized by a random-mass term in the LGW theory for the magnetic order parameter.^{39,40} This effect is small as long as the fluctuations of the distance from criticality, δr , are small compared to r itself: $\delta r \propto (\Delta/\xi^d)^{1/2} \propto (\Delta r^{d\nu})^{1/2} \ll r$. Here Δ is the disorder strength, and we assume that the fluctuation is proportional to the inverse square root of a correlation volume ξ^d , with ξ the correlation length. With mean-field exponents in $d = 3$, this condition becomes $\Delta^2 \ll r$. With $\Delta = 1/k_F \ell$ this is a very weak constraint for small disorder, $k_F \ell \gg 1$, and Hertz's mean-field description remains valid in a large parameter range. In $d = 3$, the condition becomes $\xi \ll \ell$. In Fig. 3, the region controlled by Hertz's fixed point is schematically indicated by the blue region around the quantum critical point, and the asymptotic critical region by the red circle inside the blue region. We conclude that in many systems one expects sizable regions in the phase diagram

where Hertz's fixed point yields the observable behavior, and only very close to the quantum critical point does the behavior cross over to the asymptotic one. This experimental relevance of Hertz's fixed point in disordered systems has not been appreciated before, and we therefore discuss it here.

Table I lists the critical exponents associated with Hertz's fixed point in the presence of disorder. The static exponents have their mean-field values for all $d > 0$, and the dynamical exponent $z_c = 4$ results from the diffusive dynamics of the conduction electrons, see above. The temperature dependence of the observables, expressed by the T -exponents, is obtained by repeating Millis's analysis with obvious modifications; the most important one being that the scale dimension of the DIV is now $[u] = -d$, rather than $[u] = -(d-1)$ in the clean case.

The system is above its upper critical dimensionality for all $d > 0$, and the exponent relations that depend on strong scaling therefore break down, while those that rely only on weak scaling hold, in perfect analogy to the clean case in $d > 1$. The discussion of the exponents $\bar{\alpha}$ and $\bar{\alpha}_T$ in Sec. III A 1 also carries over with obvious modifications. At the upper critical dimension $d_c^+ = 0$ hyperscaling holds as expected, and there is only one dynamical exponent, $z_m = z_c = 4$, as is the case in clean systems in $d = 1$. An important exponent relation that has no clean analog is the rigorous inequality $\nu > 2/d$.⁴³ This is violated for all $d < 4$, which by itself implies that the fixed point cannot be stable.

We next consider the temperature dependence of the conductivity. The arguments that led to Eq. (3.2) are easily modified to apply to the disordered case. Since the scale dimension of the DIV is now $[u] = -d$, the effective scale dimension of $\Delta\sigma$ is $[\Delta\sigma] = -(d+4)$. In addition, the dynamical exponent is now $z_c = 4$, which yields

$$\Delta\sigma(r, T) = b^{d+4} F_\sigma(r b^{1/\nu}, T b^4). \quad (3.6)$$

This immediately yields $s_T = -(d+4)/4$ as shown in Table I. Here we have assumed that the backscattering factor is still present, which requires that the residual resistivity is small compared to the temperature-dependent part; see the discussion below. The exponent s does not exist in this regime for the same reason as in the clean case.

The disorder dependence of the temperature-dependent conductivity can also be obtained from scaling. Let us start from Eq. (3.2) with $z_c = 3$. Dropping the meaningless r -dependence, and adding the dependence of $\Delta\sigma$ on the elastic mean-free path ℓ , we have

$$\Delta\sigma(T, \ell) = b^{d+2} F_\sigma(T b^3, \ell b^{-1}). \quad (3.7)$$

The scaling function F_σ now must have the following properties: (1) $F_\sigma(x, y \rightarrow \infty) = F_\sigma^{\text{clean}}(x)$, so we recover Eq. (3.2). (2) For $\ell/b \rightarrow 0$ the prefactor must change from b^{d+2} to b^{d+4} , and the temperature scale factor must

change from b^3 to b^4 in order to recover Eq. (3.6). This is achieved by $F_\sigma(x, y \rightarrow 0) = F_\sigma^{\text{disordered}}(x/y)/y^2$. We thus find the following generalization of Eq. (3.6):

$$\Delta\sigma(T) = b^{d+4} \ell^{-2} F_\sigma((T/\ell) b^4). \quad (3.8)$$

This yields

$$\Delta\sigma(T) \propto \ell^{-(4-d)/4} T^{-(d+4)/4}, \quad (3.9)$$

which yields the disorder dependence of the prefactor in addition to the exponent s_T . We have checked this result by means of an explicit calculation based on the Kubo formula for the conductivity, and have found agreement.

The above results for the conductivity depend on various assumptions that limit their regime of validity and require a discussion. Firstly, we have assumed diffusive electron dynamics, which requires $T\tau < 1$, with $\tau = \ell/v_F$ the elastic mean-free time. Secondly, we have assumed that the contribution from Eq. (3.9), $\Delta\rho \propto \ell^{(4-d)/4} T^{(d+4)/4}$, which is the contribution from small wave numbers in any transport theory, is larger than the clean contribution, $\Delta\rho \propto T^{(d+2)/3}$. Thirdly, we have assumed that the backscattering factor is still present. This requires that the residual resistivity ρ_0 is small compared to the temperature-dependent contribution $\rho(T)$. The last two assumptions both imply that the temperature must be large compared to a disorder-dependent energy scale, and the result (3.9) will thus be valid only in a temperature window. In a simple model for a metallic ferromagnet, where the Fermi energy is the only microscopic energy scale, this window does not exist, since the last requirement leads to an unrealistically large lower bound for the temperature. This is misleading, however. In any real ferromagnetic material a complicated band structure leads to multiple microscopic energy scales, some of which are strongly renormalized downward from the Fermi energy of a nearly-free-electron model. This is especially true for the low-Curie-temperature ferromagnets that are good candidates for observing a quantum ferromagnetic transition. Different factors of the temperature in the above arguments will be normalized by different microscopic scales, and generically one expects the relevant temperature window to exist. This is certainly true empirically, as the temperature-dependent resistivity is observed to dominate the residual resistivity for all but the lowest temperatures in many materials.⁴⁴

In the regime discussed above the exponent s as defined in Eq. (A6) does not exist for the same reason as in the clean case, and the low-temperature behavior of the conductivity away from criticality is again given by Eq. (3.3). Remarkably, the r -dependence of the prefactor of the T^{-2} dependence is the same in the clean and dirty cases.

2. The physical fixed point

Asymptotically close to the quantum critical point the behavior is described by a Gaussian fixed point with

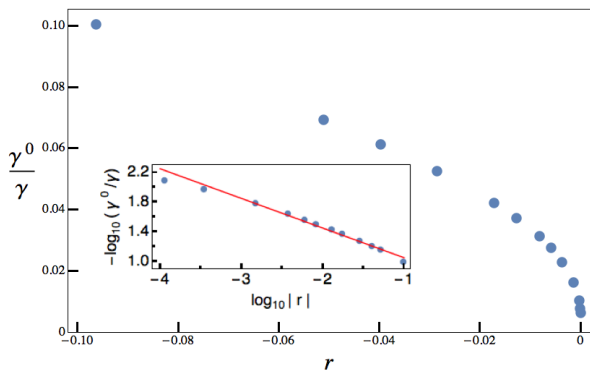


Figure 4: The inverse of the specific-heat coefficient γ normalized by the Fermi-liquid value γ^0 as a function of r in the ordered phase for $d = 3$. The inset demonstrates the effective power law $\gamma \propto |r|^{-\bar{\alpha}}$ with $\bar{\alpha} = \lambda/(1 + \lambda) \approx 0.4$. Figure adapted from Ref. 42.

marginal operators in all dimensions $2 < d < 4$,^{40,41} and with increasing disorder the region in parameter space that is controlled by this fixed point grows. The marginal operators result in critical behavior that is not given by pure power laws. For example, the magnetization at $T = 0$ in $d = 3$ asymptotically behaves as

$$m(r, T = 0) \propto (-r)^2 [g(\ln(-1/r))]^2. \quad (3.10a)$$

Here the function g is asymptotically log-normal,

$$g(x \rightarrow \infty) \propto e^{c x^2}, \quad (3.10b)$$

with c a constant; see Ref. 41 for details. In a large pre-asymptotic region these multiplicative corrections to scaling mimic power laws that span several decades in r or T .⁴² For instance, the magnetization obeys an effective homogeneity law

$$m(r, T, h) = b^{-2} F_m(r b^{d-2+\lambda}, T b^{2+\lambda}, h b^{d+\lambda}). \quad (3.11)$$

λ is an effective exponent that depends on d and r and goes to zero for $r \rightarrow 0$; however, it is approximately constant over a large r -range. For $d = 3$, $\lambda \approx 2/3$ for $0.001 \lesssim |r| \lesssim 0.1$. As an example, we show in Fig. 4 the divergence of the specific-heat coefficient, $1/\gamma(r, T = 0) \propto |r|^{\bar{\alpha}}$. The asymptotic value $\lambda = 0$ needs to be interpreted as signaling the presence of the log-normal terms. The resulting power laws are listed in the two middle columns in Table I. Since the system is below an upper critical dimension for $2 < d < 4$, all exponent relations hold, including the hyperscaling relations. This holds even for the pre-asymptotic effective exponents, with the exception of the generalized Harris criterion, Eq. (C1), which holds only asymptotically.

IV. SUMMARY, AND DISCUSSION

A. Summary

In summary, we have presented two conceptual advances for quantum phase transitions: Firstly, we have analyzed relations between critical exponents, also known as scaling relations. Compared to classical transitions, there are additional exponents to consider that describe the scaling behavior of observables with respect to temperature, which is independent from the scaling behavior with respect to the control parameter. The exponents describing the latter (“ r -exponents”) are related to the ones describing the former (“ T -exponents”) by means of a dynamical critical exponent; however, care must be taken since at many quantum phase transitions there is more than one dynamical exponent. We have shown that the Widom and Fisher equalities hold for both the r -exponents and the T -exponents under weak assumptions that are fulfilled at most quantum (and classical) phase transitions. The same is true for the Essam-Fisher equality relating the r -exponents. Additional relations, which are akin to the classical hyperscaling relation between the correlation-length exponent ν and the specific-heat exponent α , require stronger assumptions that break down if the system is above an upper critical dimension. We have also generalized the rigorous Rushbrooke inequality to the quantum case, and we have shown that at quantum phase transitions it is crucial to distinguish between exponents describing the critical behavior of the specific-heat coefficient, which we denote by $\bar{\alpha}$ and $\bar{\alpha}_T$, and the exponents α and α_T that govern the control-parameter susceptibility; at a classical transition, $\bar{\alpha}$ and α are the same. We have also discussed Wegner’s equality that relates critical exponents for the electrical conductivity and the correlation-length exponent ν , and have discussed the conditions under which it is valid.

Secondly, we have generalized the concepts of Fisher and Berker related to scaling at classical first-order transitions to quantum phase transitions. The scaling concepts all carry over, with the temperature playing the role of additional dimensions. Again, the presence of multiple time scales lead to complications that need to be dealt with carefully. With the proper choice of dynamical exponents all scaling relations, including the hyperscaling ones, hold under very weak assumptions. This reflects the fact that at a first-order transition there are no dangerous irrelevant variables.

We then have applied these concepts to the case of the ferromagnetic quantum phase transition as a specific example. This transition is well suited for this purpose, since it shows very different behavior in the presence or absence of quenched disorder, respectively, and in either case there are interesting and experimentally relevant crossover phenomena that are governed by different fixed points, some of which are above and some of which are at their upper critical dimension in the physically most interesting dimensions $d = 2, 3$.

B. Discussion

We conclude with some discussion points that augment remarks already made in the body of the paper.

1. General points

Number of independent exponents: There are seven r -exponents for thermodynamic quantities: α , $\bar{\alpha}$, β , γ , δ , η , and ν . These are related by three independent weak-scaling exponent relations (Widom, Eq. (2.13a), Essam-Fisher, Eq. (2.15a), and Fisher, Eq. (2.20a)), and two independent strong-scaling ones (α -hyperscaling, Eq. (2.23a), and $\bar{\alpha}$ hyperscaling, Eq. (2.24a)). In the presence of strong scaling one thus has two independent static r -exponents. In addition, there are five T -exponents: α_T , $\bar{\alpha}_T$, β_T , and γ_T that are constrained by two independent weak-scaling exponent relations (Eqs. (2.13b, 2.20b) plus three independent strong-scaling ones (Eqs. (2.15b, 2.23b, 2.24b)). In the presence of strong scaling, we thus have two independent static exponents, plus the dynamic ones. In the absence of strong scaling, there are either four or five independent static exponents plus the dynamic ones, depending on whether the Essam-Fisher equality for the T -exponents holds.

Observability of the exponent β_T The order parameter is nonzero only in the ordered phase, i.e., for $-r > T^{1/\nu z_m}$ (ignoring constant proportionality factors). For $-r \gg T^{1/\nu z_m}$ one observes static scaling with small temperature corrections, and for $-r < T^{1/\nu z_m}$ the scaling function vanishes identically. The exponent β_T therefore cannot be observed via the T -dependence of m at $r = 0$. Instead, consider the general weak-scaling homogeneity law for m

$$m(r, T) = b^{-\beta/\nu} \Phi_m(r b^{1/\nu}, T b^{\beta/\nu \beta_T}), \quad (4.1a)$$

which follows, e.g., by differentiating Eq. (2.6) with respect to h and using Eqs. (2.13a) and (2.15a). This can be written

$$\begin{aligned} m(r, T) &= T^{\beta_T} \Phi_m(r/T^{\beta_T/\beta}, 1) \\ &= r^\beta \Phi_m(1, T/r^{\beta/\beta_T}), \end{aligned} \quad (4.1b)$$

which defines the exponent β_T . Note that $\Phi_m(x, 1)$ has a zero for some value $x = x_c < 0$ that determines the phase boundary, and that $\Phi_m(x > x_c, 1) \equiv 0$. For the temperature derivative of the order parameter this implies

$$\frac{dm}{dT}(r, t) = T^{\beta_T-1} \tilde{\Phi}_m(r/T^{\beta_T/\beta}, 1), \quad (4.2)$$

where $\tilde{\Phi}_m(x, y) = \partial_y \Phi_m(x, y)$. This has been used to derive Eq. (2.22a) from (2.21).

As an illustration, consider the third panel of Fig. 1 again. The data imply $m^2 \propto \text{const.} - T^{4/3}$, where the constant depends on r , but the prefactor of the $T^{4/3}$ does

not. When interpreted by means of Eq. (4.1b) this implies $\beta = 1/2$ and $\beta_T/\beta = 4/3$. The latter ratio is equal to $1/\nu z_m$, which is consistent with the observed shape of the phase boundary in the first panel of the figure. All of this is consistent with the discussion in Sec. III A 1. Another way to measure β_T is to observe the T -dependence of the susceptibility at criticality, which yields γ_T , see the second panel in the figure, and use Eq. (2.13b) in conjunction with an independent determination of the exponent δ . We finally mention that for asymptotically small m at any nonzero temperature one crosses over to the classical critical behavior characterized by the classical value of the exponent β .

Coupling of statics and dynamics: Even though the statics and the dynamics are coupled in quantum statistical mechanics, the Widom, Essam-Fisher, and Fisher exponent equalities do not involve the dynamical exponents. Even at $T = 0$ there thus are exponent relations that do not reflect this coupling.

Relevance of quenched disorder: Since the generalized Harris criterion by Chayes et al., Eq. (C1), is rigorous it provides a necessary condition for the stability of a critical fixed point in the presence of quenched disorder. For instance, it immediately tells that Hertz's fixed point in disordered systems with its mean-field value $\nu = 1/2$ cannot be stable for $d < 4$, although it does not provide any hints as to what the fixed point is unstable against, or what replaces it asymptotically. Similarly, it shows that a first-order transition is strictly speaking impossible in a disordered system, since it would result in a correlation-length exponent $\nu = 1/(d + z_m) < 2/d$.

Significance of $\alpha = 1$ at a first-order QPT: A first-order classical phase transition is characterized by the appearance of a latent heat, i.e., a discontinuity of the entropy as a function of the temperature, and a δ -function contribution to the specific heat. The corresponding physical phenomenon at a first-order QPT is a discontinuity of the derivative of the free energy with respect to the control parameter, and a δ -function contribution to the second derivative. As an example, consider a QPT where the control parameter is the hydrostatic pressure p . With g the Gibbs free energy density, the compressibility $\kappa = -\partial^2 g / \partial p^2$ will thus have a δ -function contribution. That is, at any pressure-driven QPT the system will display a “latent-volume” effect, i.e., the system volume will change spontaneously and discontinuously at the critical pressure.

Choice of the control parameter: While deriving a LGW theory, the easiest choice for the control parameter or mass term r may be a linear combination of some non-thermal parameter and the temperature, or some power of the temperature. This is not a good choice for r for at least two reasons: (1) In order to measure, e.g., the exponent β it is necessary to keep $T = 0$ in addition to keeping the conjugate field equal to zero. Moving on a path that is not in the $T = 0$ plane is akin to deviating from the critical isochore at a classical liquid-gas transition. (2) The such-obtained temperature dependence

of the phase diagram may not be the leading one due to DIVs. This is what happens, for instance, in Hertz theory.

Applicability of scaling theory: We have restricted ourselves to phase transitions for which a local order parameter exists. We note, however, that the existence or otherwise of a local order parameter can be a matter of how the theory is formulated. In the case of the ferromagnetic quantum phase transition in metals that we have used as an example in Sec. III it is crucial that one treats the fermionic soft modes that couple to the order parameter explicitly, as in Ref. 40; if one integrates them out to formulate a theory entirely in terms of the order parameter a local description is not possible.³⁹ We also note that some theories of “exotic” quantum phase transitions that cannot be cast into the language of a simple Landau theory are structurally very similar to the ferromagnetic quantum phase transition problem in that they couple an order-parameter field to a gauge field,⁴⁵ or to fermions.⁴⁶

More generally, we note that the scaling arguments we have employed are extremely general and hinge only on the existence of a phase transition with power-law critical behavior. Given this, the free energy will obey a generalized homogeneity law irrespective of whether or not the transition allows for a traditional Landau description or is of a more exotic nature.

2. Points related to Hertz’s fixed point

Multiple time scales in Hertz theory: We come back to the identification of the dynamical critical exponent z_m at the beginning of Sec. III A 1. It is important to distinguish between theories that truly have multiple time scales that belong to different types of excitations, and theories where a single time scale may or may not get modified by the effects of DIVs, depending on the context, which effectively leads to multiple time scales. Hertz’s theory, both for the clean and the disordered case, belongs to the latter category. The time scale with $z_m = 1$ in the clean case, or $z_m = 2$ in the disordered one, at the physical fixed point (see Table I) is missing in Hertz theory, and the effective z_m listed for Hertz’s fixed point in Table I is due to a DIV modifying the sole critical $z = 3$ (clean) or $z = 4$ (disordered). Note that if a $z = 1$ were present in the clean theory, it would be smaller than the $z_m = 6/(d + 1)$ in Hertz-Millis theory, and thus would play the role of z_m according the arguments in the context of Eqs. (2.8, 2.11). This is another indication that Hertz’s fixed point cannot be stable.

Numerical values of exponents: The values of the commonly measure exponents $\bar{\alpha}_T$, γ_T , and β_T are rather similar at the clean and dirty Hertz fixed points, respectively, in $d = 3$. The same is true for the scaling of the Curie temperature with the control parameter, which is $T_C \propto r^{3/4}$ in the clean case, and $T_C \propto r^{4/5}$ in the disordered one. This needs to be kept in mind when inter-

preting experiments.

Also of interest is the value of the conductivity exponent $s_T = -7/4$ at the disordered Hertz fixed point in $d = 3$. This result is a good candidate for explaining the fact that the resistivity is commonly observed to vary as T^x with $x > 1$ near a ferromagnetic quantum critical point, as the crossover to the asymptotic critical behavior is expected to occur only at very low temperatures.

3. Points related to the physical quantum ferromagnetic fixed point

Significance of $z_c = d$: In Sec. III A 2 we showed that $z_c = d$ follows from the first-order nature of the quantum phase transition. Combined with the origin of the dynamical exponent in a LGW theory, where z_c arises from a combination of the Landau-damping term with the spin susceptibility in the paramagnetic phase, this implies that in a clean Fermi liquid the leading non-analytic wave-number dependence must scale as k^{d-1} . The connection between these seemingly unrelated results is scaling, see Sec. III A 2 and Ref. 47.

In this context we also note that at any first-order quantum phase transition in a fermionic system one expects the specific-heat coefficient to be discontinuous. This implies $\bar{\alpha} = \bar{\alpha}_T = 0$, which in turn requires $z_c = d$, see Eq. (2.33g).

Scale dimension of the conjugate field: We saw in Sec. II C that a dimensionless order parameter implies a scale dimension $[h] = d + z_m$ for the conjugate field. This illustrates the fact that in general it is not possible to simply relate the scaling of h to the scaling of the energy or temperature. For instance, one might argue that the scale dimension of h should be equal to z_c , since h determines the Zeeman energy. While this is sometimes true (for instance, at the disordered critical fixed point), it is not true in general. In this context we also note that the Rushbrooke inequalities, Eqs. (2.22), do not depend on $[h]$, as $[h]$ cancels between the contributions $2\beta_T$ and γ_T (or 2β and γ), respectively, on the left-hand side.

Logarithmic corrections to scaling in a range of dimensions: The theory of Refs. 40,41 yields logarithmic corrections to scaling (in the sense of log-normal terms multiplying power laws) in an entire range of dimensions, $2 < d < 4$. This is unusual; more commonly logarithmic corrections to scaling occur only in a specific dimension. This can be understood within Wegner’s⁴⁸ classification of logarithmic corrections, as has been discussed in Ref. 41. The dynamical critical exponent $z_c = d$ leads to operators that are marginal in a range of dimensions, and this in turn causes the logarithmic terms. We also note that the mathematical problem of determining the critical behavior for the theory of Refs. 40,41 was solved exactly in Ref. 49, although the physical interpretation was unclear at that time. See also Ref. 50.

Scaling of the Grüneisen parameter The Grüneisen parameter $\Gamma = (-\partial s/\partial p)/T(\partial s/\partial T)$, with s the entropy

density, is defined as the ratio of the thermal expansion coefficient $(\partial V/\partial T)_{p,N}/V = -(\partial s/\partial p)_{T,N}$ and the specific heat $c = T\partial s/\partial T$. It was shown in Ref. ? that at a pressure-tuned QPT with a single dynamical exponent z the Grüneisen parameter diverges as $\Gamma \propto T^{-1/\nu z}$; this is readily confirmed by using Eq. (2.5). With two dynamical exponents $z_1 > z_2$ we find from Eq. (2.8) $\Gamma \propto T^{-1/\nu z_1}$. In particular, at the physical fixed point in disordered metallic ferromagnets we have $\Gamma \propto T^{-1/3}$ in $d = 3$. Another useful observation is that, according to its definition, Γ scales as $\Gamma \sim 1/p \sim 1/r$ at a pressure-tuned QPT. Since $r \sim T^{1/\nu z}$ at a critical point, this is equivalent to the result given above. At the first-order transition in clean systems, $\Gamma \sim 1/r$ reflects the same δ -function contribution that was discussed for the compressibility in Sec. IV B 1. This is also apparent from the fact that $s \sim \gamma T$, and hence $\Gamma \sim \partial\gamma/\partial p$, and the specific-heat coefficient γ has a discontinuity across the first-order transition.

The behavior at the clean Hertz fixed point has been discussed in Ref. ? . Up to logarithmic corrections, $\Gamma \sim r^{-1} \sim T^{-2/3}$ in agreement with the above arguments and $\nu = 1/2$, $z_c = 3$ from Table I. At the disordered Hertz fixed point the corresponding result is $\Gamma \sim r^{-1} \sim T^{-1/2}$.

Acknowledgments

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Appendix A: Definitions of critical exponents

Let T be the temperature, h the field conjugate to the order parameter, r the control parameter, i.e., the dimensionless distance from criticality at $T = h = 0$, and f a suitable free-energy density. Consider the correlation length ξ , the order parameter $m = \partial f/\partial h$, the order-parameter susceptibility $\chi_m = \partial^2 f/\partial h^2$, and the specific-heat coefficient $\gamma = c/T = -\partial^2 f/\partial T^2$ as functions of r , T , and h , and the susceptibility also as a function of the wave number k . Further consider the susceptibility $\chi_r = -\partial^2 f/\partial r^2$, which we refer to as the control-parameter susceptibility. (More precisely, χ_r is the susceptibility of the thermodynamic quantity whose conjugate field is the control parameter.) We define critical exponents at a quantum phase transition as follows.

Correlation length:

$$\xi(r \rightarrow 0, T = 0) \propto |r|^{-\nu} \quad , \quad \xi(r = 0, T \rightarrow 0) \propto T^{-\nu_T} \quad . \quad (\text{A1})$$

Order parameter:

$$\begin{aligned} m(r \rightarrow 0, T = 0, h = 0) &\propto (-r)^\beta \quad , \\ m(r = 0, T = 0, h \rightarrow 0) &\propto h^{1/\delta} \quad , \\ m(r = 0, T \rightarrow 0, h = 0) &\propto T^{\beta_T} \quad . \end{aligned} \quad (\text{A2})$$

The last definition is purely formal: Since the magnetization is nonzero only in the ordered phase, the T^{β_T} in the last line has a zero prefactor. See the discussion in Sec. IV B 1 of how to interpret the exponent β_T .

Order-parameter susceptibility:

$$\begin{aligned} \chi_m(r \rightarrow 0, T = 0; k = 0) &\propto |r|^{-\gamma} \quad , \\ \chi_m(r = 0, T \rightarrow 0; k = 0) &\propto T^{-\gamma_T} \quad , \\ \chi_m(r = 0, T = 0, k \rightarrow 0) &\propto 1/k^{2-\eta} \quad . \end{aligned} \quad (\text{A3})$$

Specific-heat coefficient:

$$\gamma(r \rightarrow 0, T = 0) \propto |r|^{-\bar{\alpha}} \quad , \quad \gamma(r = 0, T \rightarrow 0) \propto T^{-\bar{\alpha}_T} \quad . \quad (\text{A4})$$

Control-parameter susceptibility

$$\chi_r(r \rightarrow 0, T = 0) \propto |r|^{-\alpha} \quad , \quad \chi_r(r = 0, T \rightarrow 0) \propto T^{-\alpha_T} \quad . \quad (\text{A5})$$

ν , β , γ , δ , and η are defined in analogy to the corresponding exponents at a classical phase transition.¹ In the main text we refer to these exponents, and also to $\bar{\alpha}$ and α , as the r -exponents. The definition of $\bar{\alpha}$ deviates from the one of the classical exponent customarily denoted by α , which is defined in terms of the specific heat rather than the specific-heat coefficient. This is necessary in order to factor out the factor of T in the relation between the specific heat and the specific-heat coefficient,¹⁷ which makes no difference at a thermal phase transition, but goes to zero at a QCP. For instance, the thermodynamic identity that underlies the Rushbrooke inequality, Eq. (B5), has no explicit T -dependence only if it is formulated in terms of specific-heat coefficients rather than the specific heats. At a classical phase transition, $\bar{\alpha}$ coincides with α . Our definition of α at a QPT is analogous to the definition of the classical exponent α from a scaling point of view. However, the physical interpretation of α and the susceptibility χ_r depends on the nature of the control parameter. For instance, if the control parameter is hydrostatic pressure, and f is the Gibbs free energy density, then χ_r is proportional to the compressibility of the system. At a thermal transition, where $r \propto T - T_c$, χ_r is identical with the specific-heat coefficient γ , and α has its usual meaning. α_T , $\bar{\alpha}_T$, ν_T , β_T , and γ_T reflect the fact that a QPT can be approached either in the $T = 0$ plane, or from $T > 0$. These exponents are referred to as the T -exponents in the main text.

We finally define exponents s and s_T that describe the behavior of the electrical conductivity σ at the QPT. With $\Delta\sigma$ the scaling part of σ , these are defined as follows:

Electrical conductivity:

$$\Delta\sigma(r \rightarrow 0, T = 0) \propto |r|^s \quad , \quad \Delta\sigma(r = 0, T \rightarrow 0) \propto T^{s_T} \quad . \quad (\text{A6})$$

The exponent s obviously makes sense only in systems with quenched disorder.

Appendix B: Classical scaling relations

For the convenience of the reader we recall some well-known classical exponent relations (see, e.g., Refs. 1,4):

$$\gamma = \beta(\delta - 1) \quad (\text{Widom}) \quad (\text{B1a})$$

$$\alpha + 2\beta + \gamma = 2 \quad (\text{Essam-Fisher}) \quad (\text{B1b})$$

$$\gamma = (2 - \eta)\nu \quad (\text{Fisher}) \quad (\text{B1c})$$

$$\nu d = 2 - \alpha \quad (\text{“hyperscaling”}) \quad (\text{B1d})$$

where α is the ordinary specific-heat exponent. The first two equalities follow from a weak scaling assumption, in the sense of Sec. II A, for the singular part f of the free-energy density. This has the effects of dangerous irrelevant variables, if any, built in and can be written, for instance, as

$$f(r, h) = b^{-(2-\alpha)/\nu} \Phi_f(r b^{1/\nu}, h b^{\beta\delta/\nu}), \quad (\text{B2})$$

with Φ_f a scaling function. The Fisher scaling relation, Eq. (B1c), requires an additional (weak) scaling assumption,⁴ namely, that scaling works for correlation functions as well as for thermodynamic quantities. If we use a homogeneity law for the wave-number dependent order-parameter susceptibility,

$$\chi(r, k) = b^{\gamma/\nu} \Phi_\chi(r b^{1/\nu}, k b), \quad (\text{B3})$$

and put $r = 0$ we obtain Eq. (B1c). The hyperscaling relation requires yet another assumption,⁴ which is a strong scaling assumption in the sense of Sec. II A and which is equivalent to saying that the scale dimension of the free energy density is equal to d :

$$f(r, h) = b^{-d} \Phi_f(r b^{1/\nu}, h b^{\beta\delta/\nu}). \quad (\text{B4})$$

If hyperscaling holds, the six exponents $\alpha, \beta, \gamma, \delta, \eta$, and ν are constrained by the four relations in Eqs. (B1), and only two of the exponents are independent.

We also list the Rushbrooke inequality

$$\alpha + 2\beta + \gamma \geq 2, \quad (\text{B5})$$

which holds rigorously, as it only depends on thermodynamic stability conditions. There are other inequalities that depend on various assumptions.¹

Appendix C: The Harris criterion

Harris considered the effects of quenched disorder on classical critical points. He found that the fixed point describing the transition in the clean system is stable with respect to disorder, i.e., the critical behavior is unaffected by the disorder, as long as the specific-heat exponent is negative, $\alpha < 0$.⁵¹ This statement is referred to as the Harris criterion. Chayes et al⁴³ generalized Harris’s argument and made it rigorous to show that, under very

general conditions, the correlation-length exponent at a critical point in a disordered system, classical or quantum, must obey

$$\nu \geq 2/d \quad (\text{Chayes-Chayes-Fisher-Spencer}) . \quad (\text{C1})$$

- ¹ E. Stanley, *Introduction to Phase Transitions and Critical Phenomena* (Oxford University Press, Oxford, 1971).
- ² K. G. Wilson and J. Kogut, Phys. Rep. **12**, 75 (1974).
- ³ S.-K. Ma, *Modern Theory of Critical Phenomena* (Benjamin, Reading, MA, 1976).
- ⁴ M. E. Fisher, in *Advanced Course on Critical Phenomena*, edited by F. W. Hahne (Springer, Berlin, 1983), p. 1.
- ⁵ Throughout this paper we restrict ourselves to phase transitions for which a local order parameter exists. See, however, the last paragraph in Sec. IV B 1.
- ⁶ For a ferromagnetic transition, this means approaching the Curie temperature in zero external field; for the liquid-gas transition, it means approaching the critical point along the critical isochore.
- ⁷ Throughout this paper we will assume that exponents that are defined on either side of the phase transition have the same value on either side, as is almost always the case in real systems.
- ⁸ J. Hertz, Phys. Rev. B **14**, 1165 (1976).
- ⁹ P. C. Hohenberg and B. I. Halperin, Rev. Mod. Phys. **49**, 435 (1977).
- ¹⁰ C. DeDominicis and L. Peliti, Phys. Rev. B **18**, 353 (1978).
- ¹¹ M. E. Fisher and A. N. Berker, Phys. Rev. B **26**, 2507 (1982).
- ¹² Here, and in all other homogeneity laws, it is tacitly understood that it is only the scaling part of the observable in question that obeys the homogeneity law. In general, there also is a non-scaling part that shows no singular behavior at the phase transition.
- ¹³ There is no guarantee that even weak scaling is valid once strong scaling is violated, and to our knowledge no general criteria are known. An example where it is not is provided by spin glasses, Ref. 52.
- ¹⁴ D. Belitz, T. R. Kirkpatrick, and T. Vojta, Rev. Mod. Phys. **77**, 579 (2005).
- ¹⁵ This argument hinges on strong scaling. DIVs can in principle invalidate it, but in that case the system is above its upper critical dimensionality and the problem can usually be solved exactly.
- ¹⁶ G. S. Rushbrooke, J. Chem. Phys. **39**, 842 (1963).
- ¹⁷ Here we assume that far from the transition the specific heat for $T \rightarrow 0$ is given by $c = \gamma T$ as is the case, for instance, in a Fermi liquid. If one deals with a system where $c = \tilde{\gamma} T^a$, with $a \neq 1$, then one should formulate the identity expressed by Eq. (2.21) in terms of $\tilde{\gamma}$ instead of γ .
- ¹⁸ F. J. Wegner, Z. Phys. B **25**, 327 (1976).
- ¹⁹ E. Abrahams, P. W. Anderson, D. C. Licciardello, and T. V. Ramakrishnan, Phys. Rev. Lett. **42**, 673 (1979).
- ²⁰ We also note that the strong-scaling argument that leads to Eq. (2.28) works for the conductivity, but not for the resistivity if a non-scaling part is present. This is related to the fact that only the conductivity can be written in terms of a time-correlation function via a Kubo formula.
- ²¹ This argument, while useful, is of a purely formal nature since a dimensionless OP means that a Landau expansion in powers of the OP is not valid.
- ²² V. Privman and M. E. Fisher, J. Stat. Phys. **33**, 385 (1983).
- ²³ D. B. Abraham, A. Maciolek, and O. Vasilyev, Phys. Rev. Lett. **113**, 077204 (2014).
- ²⁴ M. Brando, D. Belitz, F. M. Grosche, and T. R. Kirkpatrick, arXiv:1502.02898.
- ²⁵ This statement assumes that the action in the form of Eq. (3.1) remains valid in $d = 1$. This is not the case, since the Landau-damping term $|\Omega_n|/|\mathbf{k}|$ reflects the properties of a Fermi liquid, which does not exist in $d = 1$. Nevertheless, it is illustrative to consider Hertz's action as a model that can be analyzed in any spatial dimension, which is the attitude we are taking here.
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