Quasi-one-dimensional pair density wave superconducting state

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We provide a quasi-one-dimensional model which can support a pair-density-wave (PDW) state, in which the superconducting (SC) order parameter modulates periodically in space, with gapless Bogoliubov quasiparticle excitations. The model consists of an array of strongly-interacting one-dimensional systems, where the one-dimensional systems are coupled to each other by local interactions and tunneling of the electrons and Cooper pairs between them. Within the interchain mean-field theory, we find several SC states from the model, including a conventional uniform SC state, PDW SC state, and a coexisting phase of the uniform SC and PDW states. In this quasi-1D regime we can treat the strong correlation physics essentially exactly using bosonization methods and the crossover to the 2D system by means of interchain mean field theory. The resulting critical temperatures of the SC phases generically exhibit a power-law scaling with the coupling constants of the array, instead of the essential singularity found in weak-coupling BCS-type theories. Electronic excitations with an open Fermi surface, which emerge from the electronic Luttinger liquid systems below their crossover temperature to the Fermi liquid, are then coupled to the SC order parameters via the proximity effect. From the Fermi surface thus coupled to the SC order parameters, we calculate the quasiparticle spectrum in detail. We show that the quasiparticle spectrum can be fully gapped or nodal in the uniform SC phase and also in the coexisting phase of the uniform SC and PDW parameters. In the pure PDW state, the excitation spectrum has a reconstructed Fermi surface in the form of Fermi pockets of Bogoliubov quasiparticles.

I. INTRODUCTION

The pair-density-wave (PDW) state is a superconducting (SC) state of matter in which the Cooper pairs have a finite momentum. Due to the finite momentum carried by the pair, the SC order parameter is modulated periodically in space. The PDW state has recently received attention because it can explain the layer decoupling observed in the cuprate $La_{2-x}Ba_xCuO_4$(LBCO), the original high T_c superconductor, at the $x = 1/8$ anomaly.[1] At this doping the T_c of the three-dimensional material is suppressed to temperatures as low as 4K, where the Meissner effect first appears and the system is in a three-dimensional d-wave SC phase. In contrast, away from $x = 1/8$, the SC T_c in LBCO is about 35K. In spite of the low T_c of the uniform d-wave SC state at $x = 1/8$, in this doping regime the CuO planes are already superconducting for a range of temperatures well above T_c.[2,3] With this in mind, Berg and collaborators[1,4] showed that this phenomenon can be explained if the CuO planes are in an inhomogeneous ('striped') SC state, the PDW state, in which charge, spin and SC orders are intertwined with each other. In this state the SC order parameter oscillates along one direction in the CuO planes, and the average of the SC order parameter is zero in the CuO planes.

The PDW state is similar to the traditional Larkin-Ovchinnikov (LO) state[5] where the Cooper pairs have a non-zero center of mass momentum, which in the LO proposal is due to the presence of an external Zeeman field, which breaks the time-reversal symmetry explicitly (for a recent review of the LO state see Ref. [6]). However, the occurrence of the PDW SC state does not nec-essarily require to have a system in which time-reversal symmetry is explicitly broken, nor it does require time-reversal symmetry be spontaneously broken either. Since it was proposed as a candidate competing state to the uniform d-wave SC order,[1,4] the PDW state has been studied extensively. A Landau-Ginzburg (LG) theory of the PDW state provides a simple explanation of much of the observed phenomenology of $La_{2-x}Ba_xCuO_4$,[4,7,8] and of $La_{2-x}Sr_xCuO_4$ in magnetic fields. An outgrowth of these phenomenological theories is a statistical mechanical description of the thermal melting of the PDW phase by proliferation of topological defects which yielded a rich phase diagram which includes, in addition to the PDW phase, a novel charge $4e$ SC state and a CDW phase.[9–11] More recently, Agterberg and Garaud[12] showed that it is possible to have a phase in which a uniform SC and PDW order parameters coexist in the presence of a magnetic field.

The microscopic underpinnings of the PDW state are presently not as well understood as the phenomenologies. Nevertheless, it has been shown that this state can appear in different regimes of several models. In the weak coupling limit, such a state appears naturally in two dimensions (2D) inside an electronic spin-triplet nematic phase.[13] Also at the mean field level, Lee[14] found that it is possible to have a PDW state in his model of 'Amperian' pairing[15] and that the PDW state can explain the pseudo-gap features found in the angle-resolved photoemission experiment.[16] In a series of papers, Loder and collaborators[17,18] found that a PDW superconducting state is preferred in a tight-binding model with strong attractive interactions (although the critical value of the coupling constant above which the PDW is stable is presumably outside the range of validity of the

weak coupling theory). Similarly, PDW states with broken time-reversal invariance and parity have been found recently[19,20] in a 'hot spot' model, which also requires a critical (and typically not small) value of a coupling constant. On the other hand, in one-dimensional systems (1D) the PDW state has been shown to describe the SC state of the Kondo-Heisenberg chain,[9,21,22] and a phase of an extended Hubbard-Heisenberg model on a two-leg ladder.[23] We showed recently that the PDW state appearing in these two 1D models is actually a topological SC which supports Majorana zero modes localized at its boundaries.[24]

There has also been considerable recent effort to determine if the PDW state occurs in simple models of strongly correlated systems. Variational Monte Carlo simulations of the $t - J$ and $t - t' - J$ model on the square lattice at zero magnetic field near doping $x = 1/8$ found that the uniform d-wave SC state is slightly favored over the PDW state.[25–28] Corboz and coworkers,[29] using infinite projected entangled pair-states[30] (iPEPS), found strong evidence in the 2D $t - J$ model that the ground state energies of the uniform d-wave state and the PDW state are numerically indistinguishable (within the error bars) over a broad range of dopings and parameters. This last result indicates that these strongly correlated systems do have a strong tendency to exhibit intertwined orders and that the PDW state occurs more broadly than was anticipated.[11]

In this work, instead of following a conventional weak coupling approach to the PDW states, we will take an alternative path which has the physics of strong correlations as its starting point. Rather than starting from a true 2D system, we will consider a quasi-one dimensional model consisting of weakly coupled (each strongly-interacting) 1D systems. In the decoupled limit we can solve each 1D system non-perturbatively using bosonization methods.[31–33] We will follow a dimensional crossover approach that has been used with considerable success by several authors.[34–41] We will consider a generalization of the model used by Granath and coworkers[36] in which there are two types of 1D subsystems: a set of doped two-leg ladders in the Luther-Emery (LE) liquid regime (which has a single gapless charge sector and a gapped spin sector) and a set of 1D electronic Luttinger liquids (eLL) with both a gapless charge sector and a gapless spin sector. Although the interactions between LE liquids and between LE and eLL liquids will be treated by the interchain mean field theory (MFT) (see, e.g. Carlson *et al.*[34]), the intra LE and intra eLL interactions are treated essentially exactly using bosonization. We will make the reasonable assumption that the interaction between the electronic Luttinger liquids leads to a crossover to a full 2D (anisotropic) Fermi liquid (see, e.g. Ref.[36]). In this fashion, this approach allows to access the strong coupling regime of a strongly correlated system using controlled approximations. In this approximation the resulting superconducting T_c is a power law in the interchain coupling and not exponentially small as

in the usual weak-coupling limit (such as BCS approach).

The main departure of the system that we consider here from previous studies of models on this type is that we will allow for the Josephson couplings between the LE liquids to have either positive or negative signs. A negative sign induces a π phase shift between two neighboring LE liquids. It was shown by Berg *et al.*[7] that two superconductors that are proximity coupled to each other through a 1D weakly doped Hubbard model have a broad regime of parameters (in particular, doping) in which the effective Josephson coupling is negative. Here, in order to incorporate this physics, we will introduce a set of Ising degrees of freedom mediating the interactions between the LE liquids which emulate different doping profiles of the electronic Luttinger liquids. This feature will allow us to consider the interplay between uniform (s-wave or d-wave) superconductivity with PDW superconducting states and coexistence phases, resulting in complex phase diagrams.

We note that inhomogeneous SC states such as the PDW are generally accompanied by a subsidiary charge-ordered state, a charge-density-wave (CDW). The period of the CDW is twice the period of the PDW or equal to the period of the PDW depending on whether this is a pure PDW state or if it is a state in which it coexists with an uniform SC state. The CDW order parameters which describe these states are composite of the PDW order parameters with or without the uniform SC order parameter. The general occurrence of charge-ordered states as subsidiary orders of an inhomogeneous SC state has been emphasized by several authors.[4,7,8,11,14,19] The same should hold in the case of the SC states that we study here.

The experimental consequences of the PDW states have been discussed extensively in the recent literature[1,4,8,11,14,42–44] (including papers by one of the authors) and we will not elaborate further on this questions here. Instead we will focus on microscopic mechanisms behind these inherently strongly interacting states.

The paper is organized as follows. In section II we define our model and summarize our notation for bosonization in 1D. In section III we develop the interchain MFT and discuss the results for the self consistent equations. In section IV we study and discuss the quasiparticle spectrum of the phases, emergent from this quasi-1D system, for the various PDW and uniform SC states found in section III. In section V we summarize other possible phases that could arise in this model using a qualitative scaling dimensional analysis. We finish with our conclusions in section VI.

II. THE MODEL

The quasi-1D model, schematically presented in the Introduction, consists of two different types of 1D systems. One of them is a conventional 1D electronic Luttinger

liquid (eLL) in which both the spin and charge degrees of freedom are gapless. The other type, however, is a 1D system with a spin gap, i.e., it is a 1D Luther-Emery liquid (LE). The presence of the spin gap in the 1D system will bias the full array of 1D systems to a strong tendency to a SC state.

A. 1D Systems and Bosonization

Before we define in detail the quasi-1D model, we start with a short summary on those 1D liquids and their description using bosonization. This material is standard and can be found in several textbooks, e.g. Ref.[31]. Here we will only give some salient results and set up our definitions (and conventions) that we use in later sections.

We start with a 1D eLL which has a gapless charge sector and a gapless spin sector. The low-energy Hamiltonian is written in terms of the set of the bosonic fields, $\{\phi_a, \theta_a\}$ where $a = c, s$ labels the charge and spin sectors respectively. These fields satisfy canonical equal-time commutation relations

$$[\phi_a(x'), \partial_x \theta_b(x)] = i\delta_{a,b}\delta(x' - x) \qquad (2.1)$$

The effective Hamiltonian for the eLL is

$$H_{eLL}[\phi_a, \theta_a] = \sum_{a=c,s} \frac{v_\alpha}{2} \left[K_a(\partial_x \theta_a)^2 + \frac{1}{K_a}(\partial_x \phi_a)^2 \right], \qquad (2.2)$$

in which K_a (again with $a = c, s$) are the Luttinger parameters for the charge and spin sectors, and v_a are the characteristic speeds for the charge and spin excitations of the liquid. The parameters K_c, K_s, v_c and v_s are determined by the microscopic details of the model. However, for a system with spin-rotational invariance, the resulting SU(2) symmetry restricts the value of the the spin Luttinger parameter to be $K_s = 1$. In this continuum and low-energy limit, we can decompose the electronic field operator in terms of two slowly varying components, with wave vectors near the two Fermi points $\pm k_F$

$$\frac{1}{\sqrt{a}} \psi_\sigma(x) \rightarrow R_\sigma(x) e^{ik_F x} + L_\sigma(x) e^{-ik_F x}, \qquad (2.3)$$

where a is the ultraviolet cut-off (typically the lattice spacing), and where $\sigma = \pm$ denotes the spin of the electron. Here the fermionic fields $R_\sigma(x)$ and $L_\sigma(x)$ are the right- and left-moving components of the electron field ψ_σ, which are slowly varying in space relative to the Fermi momentum k_F.

The right- and left-moving fields can be written in terms of the bosonic charge fields ϕ_c and θ_c, and the spin fields ϕ_s and θ_s, as follows

$$R_\sigma = \frac{F_\sigma}{\sqrt{2\pi a}} e^{i\sqrt{\pi/2}(\theta_c + \sigma\theta_s + \phi_c + \sigma\phi_s)},$$

$$L_\sigma = \frac{F_\sigma}{\sqrt{2\pi a}} e^{i\sqrt{\pi/2}(\theta_c + \sigma\theta_s - \phi_c - \sigma\phi_s)}. \qquad (2.4)$$

The anticommuting Klein factors, F_σ, ensure the fermionic statistics between the right and left moving fermions R_σ and L_σ.

Next we consider a spin-gapped Luttinger liquid, or LE liquid. At energy scales below the spin gap Δ_s, the spin sector can be ignored. Hence, the low-energy Hamiltonian contains only the charge fields ϕ_c and θ_c, and it is given by

$$H_{LE}[\phi_c, \theta_c] = \frac{v_c}{2} \left[K_c(\partial_x \theta_c)^2 + \frac{1}{K_c}(\partial_x \phi_c)^2 \right]. \qquad (2.5)$$

Since the spin sector has been effectively projected-out, we will keep only the charge sector of the LE liquid and drop its c label.

In the LE liquid, all interactions represented by operators that are not spin singlets are irrelevant (and, in fact, with effective scaling dimension infinite). This fact strongly restricts the types of interactions between LE systems and eLL systems. In this case the only fermion bilinears that need to be considered in the LE liquid are the order parameter of the charge-density-wave with momentum $2k_F$ (CDW)

$$O_{CDW}(x) \sim e^{-2ik_F x} R_\sigma^\dagger(x) L_\sigma(x) + h.c., \qquad (2.6)$$

and the (Cooper) pair field spin singlet superconducting order parameter

$$\Delta(x) \sim R_\alpha(x)(i\sigma^y)^{\alpha\beta} L_\beta(x) + (R \leftrightarrow L). \qquad (2.7)$$

Hence the coupling to the LE liquids in the quasi-1D model should involve only the two operators listed above. We note that the suppression of the spin operator and the power-law correlation for the SC operator make the LE liquids the natural building blocks for the quasi-1D SC state. In contrast, the eLL has other observables that need to be considered, including a spin triplet pair field, the $2k_F$ spin-density-wave (SDW) 'Néel' order parameter, the right and left moving spin current operators, and, in tunneling processes, the electron operators.

B. Quasi-1D Model

Given a set of independent 1D LE and eLL systems that were described above, we now define and discuss the full quasi-1D model. The model consists of an array of 1D systems shown in Fig. 1. Each *unit cell* of the array consists of one LE system, labeled by A, and one eLL system, labeled by B. Hence we introduce the bosonic fields $\{\phi_{n,A}, \theta_{n,A}\}$ representing the charge fields in the LE chain of the n-th unit cell and also $\{\phi_{n,B,a}, \theta_{n,B,a}\}$ (with $a = c, s$) representing the charge and spin fields in the eLL chain of the n-th unit cell. Furthermore we assume that the filling of the type B system (an eLL chain) is close to half filling, i.e., $k_F^{(B)} \approx \pi/2$ and $K_c^{(B)} \approx 1/2$. Also the spin rotational symmetry in the B systems is
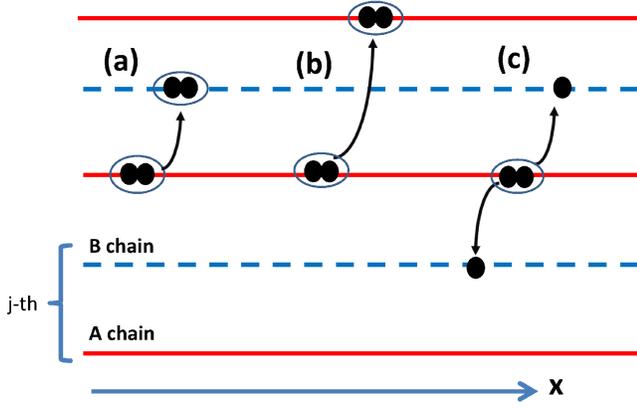
FIG. 1. (color online) The array of LE systems and eLL systems. The A-type LE systems are represented by the solid (red) line. The B-type eLL systems are represented by the dashed (blue) line. Each unit cell consists of one A-type and one B-type systems. Here the black filled dots represent electrons. (a) The conventional Josephson coupling $\mathcal{J}_{AB}$ in Eq.(2.9) (b) The conventional Josephson coupling $\mathcal{J}_{AA}$ in Eq.(2.9) (c) Splitting a Cooper pair in A system into the neighboring B systems $\mathcal{J}'_{AB}$ in Eq.(2.9).

assumed to be unbroken and thus $K_s^{(B)} = 1$. We further assume that the Fermi momenta of the systems A and B are incommensurate to each other.

In the limit in which the LE systems and the eLL systems are decoupled from each other, the effective Hamiltonian of the array is described by the sum of Eq.(2.2) and Eq.(2.5) for each system, and has the form

$$H_0 = \sum_{n \in \mathbb{Z}} \Big(H_{LE}[\phi_{n,A}, \theta_{n,A}] + H_{LL}[\phi_{n,B,a}, \theta_{n,B,a}] \Big).$$
(2.8)

This decoupled limit is an unstable fixed point and the system will eventually flow to the quasi-1D or 2D fixed points under the introduction of the coupling between the 1D systems. We will show that the PDW state, as a quasi-1D fixed point, will emerge from certain couplings.

Following the work of Granath et al.,[36] we write down the possibly relevant local interactions terms between the 1D systems. They are given by

$$H' = \sum_n \int dx \Big\{ -t_{BB} \sum_\sigma [\psi_{B,j,\sigma}^\dagger \psi_{B,j+1,\sigma} + \text{h.c.}]$$
$$+ J_{BB} \boldsymbol{S}_{B,j} \cdot \boldsymbol{S}_{B,j+1}$$
$$- \mathcal{J}_{AA,j}[\Delta_{A,j+1}^\dagger \Delta_{A,j} + \text{h.c.}]$$
$$- \mathcal{J}_{AB}[\Delta_{B,j}^\dagger \Delta_{A,j} + \Delta_{B,j}^\dagger \Delta_{A,j+1} + \text{h.c.}]$$
$$+ \mathcal{J}'_{AB}[\Delta_{A,j}^\dagger(\psi_{B,j,\uparrow}\psi_{B,j-1,\downarrow} + \psi_{B,j-1,\uparrow}\psi_{B,j,\downarrow}) + \text{h.c.}] \Big\}$$
(2.9)

To simplify the analysis, in this paper we will not consider the possible existence of spin-ordered phases (i.e.

spin stripes or SDWs) although these are clearly seen in $\text{La}_{2-x}\text{Ba}_x\text{CuO}_4$ which is the material where the PDW state is most clearly seen. We are mainly concerned about the SC states in which the spins do not play much role, and thus we ignore for now the antiferromagnetic interactions in the discussion. The antiferromagnetic coupling between the eLL chains can also be included in a relatively straightforward extension of the present work. Following Ref. [36] we have ignored the possible CDW couplings between chains. In general the scaling dimensions of the CDW operators become less relevant in the presence of forward scattering interactions between the chains,[35] so they can be neglected. If the interchain CDW couplings were to become relevant we would have bidirectional charge order. In this paper we are only exploring states with unidirectional charge and superconducting order.

In Eq.(2.9), the operator $\Delta_{A,j}(x) \sim \psi_{A,j,\alpha}(x)(i\sigma^y)^{\alpha\beta}\psi_{A,j,\beta}(x)$ represents the density of the spin-singlet Cooper pair of the system A, and $\Delta_{B,j}(x)$ is that of the system B. The effective coupling constants $\mathcal{J}_{AB}$ and $\mathcal{J}_{AA,j}$ are the conventional Josephson coupling between the two neighboring A systems, representing the hopping process of the Cooper pairs (see the (a) and (b) in Fig.1). On the other hand, the local term $\mathcal{J}'_{AB}$ represents the breaking of a Cooper pair in an A system which puts the two single electrons into the nearest neighbor B systems (see the (c) in Fig.1 for a diagram of the process).

In the Hamiltonian H', Eq.(2.9), the most relevant term is the electron tunneling term, with coupling strength t_{BB}, between two nearest-neighbor B systems. Under this perturbation, the decoupled B systems flow to the 2D Fermi liquid fixed point, which, in turn, becomes coupled to the superconducting state emergent from A systems.[36] Due to this dimensional crossover of the B systems it is difficult to apply the conventional interchain MFT to analyze Eq.(2.9). In order to make progress, we ignore at first the B systems, as the first order approximation to the problem, and perform the interchain MFT only with the A systems, which embodies the strong-coupling nature of the superconductivity emergent in the quasi-1D models. We should stress that in the A systems, there are no electron-like quasiparticles due the existence of the spin gap which leads to a fully gapped 2D SC phases when the coupling between the A systems are turned on. Technically speaking, we solve first for an array of B (eLL) systems coupled by t_{BB} and for an array of A systems coupled only by $\mathcal{J}_{AA,j}$ in Eq.(2.9), and take $\mathcal{J}'_{AB}$ and $\mathcal{J}_{AB}$ as perturbations. At this level of the approximation, the emergent SC state is determined by the Josephson coupling $\mathcal{J}_{AA,j}$ and the subsequent SC state of the full system follows by proximity effect between the A and the B subsystems. This was the strategy used by Granth et al..[36] The main difference between this work and that of Granath and coworkers is the inclusion of an additional, Ising-like, degree of freedom in the coupling between the A systems, as we already

discussed in the Introduction.

C. Coupled LE Systems

It is clear that $\mathcal{J}_{AA}$ will determine the nature of the emergent 2D SC state from the quasi-1D model. More precisely, the spatial pattern of $\mathcal{J}_{AA,j}$ determines that of the Cooper pair. For example, if the Josephson coupling $\mathcal{J}_{AA,j}$ in Eq.(2.9) is uniform and positive, it is clear that the uniform spin-singlet SC will emerge. However, in the strongly-correlated quasi-1D system, the Josephson coupling $\mathcal{J}_{AA,j}$ may not be always uniform and positive. In Ref. [4], the Josephson coupling between two systems with an intermediate chain (which is close to the insulator phase) has been calculated by a numerical density matrix renormalization group (DMRG) method and it was found that it can be negative, i.e., forming a π-Josephson junction between the two A systems. From this, it is not difficult to imagine that there might be more complicated patterns, depending on the microscopic details, than the uniform π-Josephson junction.

To reflect this physics and to consider a broader possible patterns of the Josephson coupling $\mathcal{J}_{AA,j}$, we introduce a *phenomenological* Ising degree of freedom σ_j which can change the magnitude and possibly the sign of effective Josephson coupling. This Ising degree of freedom can be regarded as a local change in the doping of the intervening B system between two neighboring A systems. In this sense the Ising degree of freedom should be regarded as reflecting the tendency to frustrated phase separation of a doped strongly correlated system.[45,46] To this effect, we consider the following interaction between the Ising degrees of freedom and the LE liquid

$$\mathcal{H}_{\text{int}} = - \mathcal{J}_{AA} \sum_i \cos[\sqrt{2\pi}(\theta_{A,i} - \theta_{A,i+1})]$$
$$- \mathcal{J}'_{AA} \sum_i \sigma_i \sigma_{i+1} \cos[\sqrt{2\pi}(\theta_{A,i} - \theta_{A,i+1})]$$
$$+ \mathcal{H}_{\text{Ising}}[\sigma_i], \qquad (2.10)$$

in which we write $\mathcal{J}_{AA,i}$ in Eq.(2.9) as $\mathcal{J}_{AA,i} = \mathcal{J}_{AA} - \mathcal{J}'_{AA}\sigma_i\sigma_{i+1}$. The factor $\sim \cos[\sqrt{2\pi}(\theta_i - \theta_{i+1})]$ is the Josephson coupling between the LE systems because of $\Delta(x) \sim e^{i\sqrt{2\pi}\theta_{A,i}}$.

In Eq.(2.10), the Ising interaction Hamiltonian $\mathcal{H}_{\text{Ising}}[\sigma_i]$ is assumed to have several phases depending on the parameters in $\mathcal{H}_{\text{Ising}}$ and temperature, e.g. paramagnetic phase $\langle \sigma_i \rangle = 0$, and various symmetry-broken phases. In this paper, we further assume that the Ising variable σ_i orders at a much higher temperature (or energy scale) than the spin gap Δ_s in the LE liquid. Hence, we ignore any correction to the Ising variable due to the fluctuations of the SC states emergent from LE liquids. To simplify the analysis we have assumed that the Ising variables are constant along the direction of the 1D systems and are classical (i.e. we did not include a transverse field term). The first assumption is not a prob-

lem since we will do mean field theory assuming that the resulting modulation (if any) is unidirectional. More microscopically we will need to assume that the Ising model has frustrated nearest and next nearest neighbor interactions along one direction only. This is the so-called anisotropic next-nearest-neighbor Ising (ANNNI) model which is well known to have a host of modulated phases.[47] Similar physics, with a rich structure of periodic and quasi-periodic states, is obtained from the Coulomb-frustrated phase separation mechanism.[46,48]

In what follows we will not specify the form of $\mathcal{H}_{\text{Ising}}$ and assume that its ground state is encoded in a specific pattern of order for the Ising variables. In this picture an inhomogeneous charge-ordered state occurs first (and hence has a higher critical temperature) and this pattern causes the effective Josephson couplings to have an "antiferromagnetic" sign (i.e. π junctions).[4,7] Nevertheless, as noted in Ref. [4], once the PDW state sets in there is always a (subdominant) CDW order state with twice the ordering wave-vector as that of the PDW.

The symmetry breaking patterns that we to study are: i) Uniform configuration $\langle \sigma_i \rangle = \pm$, $\forall i$, ii) Staggered configuration $\langle \sigma_i \rangle = (-1)^i$, and iii) Period 4 configurations (which will become clear soon below II C 4). Thus, when the Ising variables order and spontaneously break the translational symmetry, the effective Josephson coupling between the different A systems will be modulated too. For concreteness, throughout this work, we will take $\mathcal{J}_{AA}$ and $\mathcal{J}'_{AA}$ to be positive. This condition is not necessary and the following arguments can be easily extended to the other signs of $\mathcal{J}_{AA}$ and $\mathcal{J}'_{AA}$.

We will start by analyzing the ground state (or the mean field (MF) state) of the LE systems coupled to Ising variables. We will do this for different configurations of the Ising variables and see what are the possible phases that arise in the system of coupled LE liquids.

1. Ising Paramagnetic Configuration

Before proceeding to the symmetry-broken phases of the Ising variable, we first briefly comment on the case with the paramagnetic phase of the Ising variable σ_i. In the Ising paramagnetic phase, we first note that $\mathcal{J}'_{AA}\sigma_i\sigma_{i+1}\cos[\sqrt{2\pi}(\theta_i - \theta_{i+1})]$ is effectively zero at the level of mean field theory and can be ignored. Thus Eq.(2.10) will become at the low energy

$$\mathcal{H}_{\text{int}} \rightarrow -\mathcal{J}_{AA} \sum_i \cos[\sqrt{2\pi}(\theta_{A,i} - \theta_{A,i+1})] + \cdots \quad (2.11)$$

in which $\cdots$ are the terms generated by integrating the fluctuations of the Ising variables in the paramagnetic phase, e.g., $\sim \cos[2\sqrt{2\pi}(\theta_{A,i} - \theta_{A,i+1})]$, which is strictly less relevant than $-\mathcal{J}_{AA}\cos[\sqrt{2\pi}(\theta_{A,i} - \theta_{A,i+1})]$ appearing in $\mathcal{H}_{\text{int}}$. It is well-known that Eq.(2.11) induces an uniform 2D superconducting state.[34,40]

2. Uniform Ising Configuration

We now analyze the simplest case with $\langle \sigma_i \rangle \neq 0$, where all the σ_i have the same value, $\sigma_i = \sigma = \pm$. In this case H_{int} is just given by

$$\mathcal{H}_{\text{int}} = -(\mathcal{J}_{AA} + \mathcal{J}'_{AA}) \sum_i \cos[\sqrt{2\pi}(\theta_{A,i} - \theta_{A,i+1})]$$

$$\equiv -\mathcal{J}_T \sum_i \cos[\sqrt{2\pi}(\theta_{A,i} - \theta_{A,i+1})] \qquad (2.12)$$

The system of coupled LE systems can be treated in interchain MFT, where all the systems are in phase, since $\mathcal{J}_T > 0$. In this case we just have a uniform SC state in the direction perpendicular to the systems $\Delta_j = \Delta$, where Δ includes the spin gap and the MFT value for $\langle \cos \sqrt{2\pi}\theta_{A,i} \rangle$. We will show in the following section how to compute the value $\langle \cos \sqrt{2\pi}\theta_{A,i} \rangle$. Thus, this is the same phase as in the Ising paramagnetic case but with a larger value of the effective Josephson coupling.

3. Staggered (Period 2) Ising Configuration

Let us now consider $\sigma_i = (-1)^i$. In this case $\mathcal{H}_{\text{int}}$ is given by

$$\mathcal{H}_{\text{int}} = -(\mathcal{J}_{AA} - \mathcal{J}'_{AA}) \sum_i \cos[\sqrt{2\pi}(\theta_{A,i} - \theta_{A,i+1})]$$

$$\equiv -\delta\mathcal{J} \sum_i \cos[\sqrt{2\pi}(\theta_{A,i} - \theta_{A,i+1})] \qquad (2.13)$$

Again, the system of coupled LE systems can be treated in interchain MFT. However, we need to be careful about the sign of $\delta\mathcal{J}$. If $\delta\mathcal{J} > 0$ the SC order parameter in all the systems are in phase. It is important to emphasize that although all the systems are in phase as in the uniform Ising configuration, the expectation value $\langle \cos \sqrt{2\pi}\theta_{A,i} \rangle$ is different in both cases, since as we will see below, it explicitly depends on the coupling between the systems, in this case $\mathcal{J}_T$ or $\delta\mathcal{J}$. On the other hand, if $\delta\mathcal{J} < 0$ the phase of SC order parameter has a π phase shift between nearest neighbors. In the former case we just have a uniform superconducting state in the direction perpendicular to the systems, while in the second case we have a PDW state $\Delta_{A,j} \sim (-1)^j$. There is a direct transition from the uniform SC state to the PDW SC state at $\mathcal{J}_{AA}/\mathcal{J}'_{AA} = 1$. In this simple period 2 Ising configuration there is no room for coexistence between the uniform SC and the PDW state.

4. Longer Period Ising Configurations

We can generalize the phases obtained with period 2 Ising configurations to cases with longer periods of the Ising variables. For instance for a period 4 of the Ising

variables,

$$\cdots, \uparrow, \uparrow, \downarrow, \downarrow, \cdots, \qquad (2.14)$$

the effective Josephson couplings will have a period 2 modulation. In this case we will find either an uniform SC state or a period 4 PDW SC state, but no coexistence phase.

However, we will see that for Ising configurations with period n, with $n > 2$, we can have a richer phase diagram, including a coexistence phase if $n \geq 3$. For example, for a period 3 structure of the Ising variables, the allowed SC state is a coexistence phase, whereas for period 8 with the following spatial pattern of the Ising degrees of freedom

$$\cdots \downarrow, \uparrow, \uparrow, \uparrow, \uparrow, \downarrow, \downarrow, \downarrow, \downarrow, \uparrow, \uparrow, \cdots, \qquad (2.15)$$

we will find either a coexistence phase with period 4 or a PDW SC with period 8. It is straightforward to generalize this to more intricate configurations of the Ising variables.

III. INTERCHAIN MFT ON THE LE SYSTEMS

Keeping the quasi-1D model of the previous section in mind, we now solve the coupled LE system problem using the interchain MFT. In this section, we generalize the works of Lukyanov and Zamolodchikov,[49] and Carr and Tsvelik[38] to the patterns of the Josephson coupling between the LE systems emergent from various symmetry-breaking phases of the Ising variables.

A. Uniform SC and Period 2 PDW SC Phases

We first review the uniform configuration of the Ising variable (and also the paramagnetic phase of the Ising variable), in which the SC operator will develop the same expectation value for all the LE systems[38,49]. For the staggered (period 2) Ising configuration, there are two phases, depending on the sign of $\delta\mathcal{J}$, a uniform SC state and a PDW state. We will solve the self-consistency equations for the both phases, the uniform SC state and a PDW state. Although the equations have the same form, they correspond to different phases. The case of a period 4 Ising configuration of the form $\uparrow, \uparrow, \downarrow, \downarrow$ can be treated in the same manner. The only difference is that the two phases will be an uniform SC state or a period 4 PDW SC state. Here we will focus in the simpler period 2 case.

1. Uniform SC Phase

In the uniform configuration of the Ising variable, the effective Josephson interaction between neighboring A

subsystems (the LE liquids) is

$$\mathcal{H}_{\text{int}} = -\mathcal{J}_T \sum_i \cos[\sqrt{2\pi}(\theta_{A,i} - \theta_{A,i+1})] \qquad (3.1)$$

To perform the interchain MFT, we consider only the terms involving the i-th type-A system among $\mathcal{H}_{\text{int}}$. Using standard interchain MFT[34,38,40] we can approximate eq. (3.1) by:

$$\mathcal{H}_{\text{int}} = -2\mu \int d^2x \cos(\sqrt{2\pi}\theta_{A,i}), \qquad (3.2)$$

with $2\mu = \mathcal{J}_T[\langle\cos(\sqrt{2\pi}\theta_{i+1})\rangle + \langle\cos(\sqrt{2\pi}\theta_{i-1})\rangle]$. The self-consistency of the MFT then requires that

$$\langle\cos(\sqrt{2\pi}\theta_{A,i})\rangle = \frac{\mu}{\mathcal{J}_T}. \qquad (3.3)$$

Following Refs. [49,38] the self-consistency equation can be solved from the following two expressions:

$$\langle\cos(\sqrt{2\pi}\theta_{A,i})\rangle = \frac{(1+\xi)\pi\Gamma(1-d/2)}{16\sin\pi\xi\ \Gamma(d/2)} \times$$
$$\left(\frac{\Gamma(\frac{1}{2}+\frac{\xi}{2})\Gamma(1-\frac{\xi}{2})}{4\sqrt{\pi}}\right)^{(d-2)} \left(2\sin\frac{\pi\xi}{2}\right)^d M^d \qquad (3.4)$$

where M, the soliton mass in the $1+1$-dimensional sine-Gordon model, is related to μ by

$$\mu = \frac{\Gamma(d/2)}{\pi\Gamma(1-d/2)} \left(\frac{2\Gamma(\xi/2)}{\sqrt{\pi}\Gamma(\frac{1}{2}+\frac{\xi}{2})}\right)^{d-2} M^{2-d} \qquad (3.5)$$

In these equations $d = 1/(2K_c)$ is the scaling dimension of the vertex operator $e^{i\sqrt{2\pi}\theta_{A,i}}$ and $\xi = \frac{1}{2-d}$. Using equations Eq.(3.4) and Eq.(3.5), we can compute explicitly the value of $\langle\cos(\sqrt{2\pi}\theta_{A,i})\rangle$ for a given value of $\mathcal{J}_T$ and K_c. This completely determines, at least at the mean field level, the solution of the coupled LE systems[34,40].

2. Period 2 PDW SC Phase

In the staggered configuration of the Ising variable, the interaction term between the A systems is

$$\mathcal{H}_{\text{int}} = -\delta\mathcal{J} \sum_i \cos[\sqrt{2\pi}(\theta_{A,i} - \theta_{A,i+1})], \qquad (3.6)$$

which is identical to that of the uniform configuration case, Eq.(3.1), if $\delta\mathcal{J} > 0$. Hence if $\delta\mathcal{J} > 0$, we can simply replace $\mathcal{J}_T$ by $\delta\mathcal{J}$ to find the MF solution. This will give a uniform SC state.

If $\delta\mathcal{J} < 0$, then we can perform a transformation on the even sites, $\sqrt{2\pi}\theta_{A,2i} \to \sqrt{2\pi}\theta_{A,2i} + \pi$, effectively changing the sign of δJ and coming back to the first case. Though the form of the equation is identical to that of the uniform SC state, it is important to remember that

the MF solution doubles the unit cell, due to the transformation $\sqrt{2\pi}\theta_{A,2i} \to \sqrt{2\pi}\theta_{A,2i} + \pi$ acting only on the even sites. Thus, SC order parameter oscillates in space

$$\Delta_j(x) \sim (-1)^j \langle\cos(\sqrt{2\pi}\theta_A)\rangle, \qquad (3.7)$$

corresponding to a period-2 PDW SC state.

Before moving onto the coexistence phase in the next section, let us mention what is the dependence of T_c with $\delta\mathcal{J}$ (or $\mathcal{J}_T$, depending on the Ising configuration). We can think of 2μ in eq. (3.2) effectively as an external field due to the mean field value of $m_j = \langle\cos(\sqrt{2\pi}\theta_j)\rangle$ in the nearest neighbor systems. We can write then

$$H_j = H_j^{(0)} - h_j \int dx \cos(\sqrt{2\pi}\theta_j) \qquad (3.8)$$

in which $h_j = \mathcal{J}(m_{j+1} + m_{j-1})$ and $H_j^{(0)}$ is the conventional kinetic term for the Luther-Emery liquid. As we saw above, for the uniform or staggered configuration the value of m_j is the same in all the systems, or effectively the same for $\delta\mathcal{J} < 0$ since we can perform a transformation on the even sites $\sqrt{2\pi}\theta_{A,2i} \to \sqrt{2\pi}\theta_{A,2i} + \pi$.

In summary, we can write just $m = m_j = \langle\cos(\sqrt{2\pi}\theta_j)\rangle$ and $h = h_j = 2\mathcal{J}m$ (where $\mathcal{J} = \delta\mathcal{J}$ or $\mathcal{J}_T$ depending on the case). For $h \to 0$ we have that self-consistency implies

$$m = \chi_{SC}h = 2\mathcal{J}\chi_{SC}m, \qquad (3.9)$$

which has the trivial solution $m = 0$ or a non-trivial solution $m \neq 0$ if $2\mathcal{J}\chi_{SC} = 1$ (which determines the critical temperature). Using that for a Lutter-Emery liquid

$$\chi_{SC}(T) \sim \frac{\Delta_s}{T^{2-1/K_c}}, \qquad (3.10)$$

we have that:

$$T_c \sim \Delta_s \mathcal{J}^\alpha \qquad (3.11)$$

where the exponent is $\alpha = \dfrac{1}{2 - 1/K_c}$. Although the resulting T_c is small when $\mathcal{J}$ is small, what is important is that is only power-law small, instead of exponentially small as in the BCS case.

B. Uniform SC and Period 4 PDW SC state coexistence phase

Now we consider the period 8 states of the Ising variables $\sigma_i = (-1)^{\lfloor i/4 \rfloor}$. Then the Josephson coupling also modulates in space with period 4, and thus we need to solve four coupled self-consistent equations in MFT. The effective MF Hamiltonian for each A system is given by[38]

$$H_{\text{int}}^{(i)} = -2\mu_i \int d^2x \cos(\sqrt{2\pi}\theta_{A,i}) \qquad (3.12)$$

with

$$2\mu_i = [\mathcal{J}_i\langle\cos(\sqrt{2\pi}\theta_{A,i+1})\rangle + \mathcal{J}_{i-1}\langle\cos(\sqrt{2\pi}\theta_{A,i-1})\rangle] \qquad (3.13)$$

where $\mathcal{J}_i = \mathcal{J}_{AA} - \mathcal{J}'_{AA}\sigma_i\sigma_{i+1}$ in which σ_i is in the period 4 structure.

Using $\mathcal{J}_T = \mathcal{J}_{AA} + \mathcal{J}'_{AA}$ and $\delta\mathcal{J} = \mathcal{J}_{AA} - \mathcal{J}'_{AA}$ and defining $m_i = \langle\cos(\sqrt{2\pi}\theta_{A,i})\rangle$, it is clear that we need to solve only for the four systems $i = 0, 1, 2, 3$ in this MFT by assuming that the MF solution does not break the translational symmetry $i \sim i + 4$ of the pattern of the Josephson coupling.

Upon implementing the MFT analysis from the previous section we have the following set of coupled equations:

$$m_0 = f(d)\left(\frac{m_3\delta\mathcal{J} + m_1\mathcal{J}_T}{2}\right)^{d/(2-d)},$$

$$m_1 = f(d)\left(\frac{m_0\mathcal{J}_T + m_2\mathcal{J}_T}{2}\right)^{d/(2-d)},$$

$$m_2 = f(d)\left(\frac{m_1\mathcal{J}_T + m_3\mathcal{J}_T}{2}\right)^{d/(2-d)},$$

$$m_3 = f(d)\left(\frac{m_2\mathcal{J}_T + m_0\delta\mathcal{J}}{2}\right)^{d/(2-d)}, \qquad (3.14)$$

where $f(d)$ is a constant that only depends on the scaling dimension $d = \frac{1}{2K_c}$. The explicit expression for $f(d)$ is:

$$f(d) = \frac{(1+\xi)\pi\Gamma(1-d/2)}{16\sin\pi\xi\ \Gamma(d/2)}\left(\frac{\Gamma(\frac{1}{2} + \frac{\xi}{2})\Gamma(1 - \frac{\xi}{2})}{4\sqrt{\pi}}\right)^{(d-2)}$$

$$\times\left(2\sin\frac{\pi\xi}{2}\right)^d\left(\frac{\pi\Gamma(1-d/2)}{\Gamma(d/2)}\right)^{d/(2-d)}$$

$$\times\left(\frac{2\Gamma(\xi/2)}{\sqrt{\pi}\Gamma(\frac{1}{2} + \frac{\xi}{2})}\right)^d \qquad (3.15)$$

Notice that the system of Eqs. (3.14) is non-linear. Nevertheless it is easy to see that m_0 and m_3 (m_1 and m_2) will take the same value ($m_0 = m_3$ and $m_1 = m_2$). We can therefore reduce Eq. (3.14) to a system of only two coupled equations:

$$m_0 = f(d)\left(\frac{m_0\delta\mathcal{J} + m_1\mathcal{J}_T}{2}\right)^{d/(2-d)} \qquad (3.16)$$

$$m_1 = f(d)\left(\frac{m_0\mathcal{J}_T + m_1\mathcal{J}_T}{2}\right)^{d/(2-d)} \qquad (3.17)$$

Taking the ratio of Eq. (3.16) and Eq. (3.17) we get:

$$x = \left(\frac{\lambda x + 1}{x + 1}\right)^{d/(2-d)} \qquad (3.18)$$

where $\lambda = \delta\mathcal{J}/\mathcal{J}_T$.

We can solve numerically the previous transcendental Eq.(3.14), or directly solve the system Eqs. (3.16)-(3.17). Before solving the system of equations (3.16)-(3.17) numerically for some values of the parameters, let us comment on Eq. (3.18).

In the limiting case where $\mathcal{J}_T = \delta\mathcal{J}$ (i.e. $\mathcal{J}'_{AA} = 0$) Eq. (3.18) has the trivial solution $x = 1$. In this case all the SC order parameters are in phase in the case $\delta\mathcal{J} > 0$. On the other hand, for $\delta\mathcal{J} < 0$, there is a shift of π every four lattice spacings. So, in this case, the periodicity of the PDW order parameter will be eight (and not four), although the self-consistency equations actually will take the same form.

For now we will assume $\delta\mathcal{J} > 0$ (see section IV C for the $\delta\mathcal{J} < 0$ case). Then in the pattern that we consider here, we find $x < 1$ and so there is a coexistence between the uniform SC and the period 4 PDW order parameters. Let us now solve the system of equations (3.16)-(3.17) numerically for some values of the parameters. The results are summarize in table I.

We now compute T_c for this case. Following the same steps as in the previous section we have that:

$$H_0 = H_0^{(0)} - h_0\int dx\,\cos(\sqrt{2\pi}\theta_0)$$

$$H_1 = H_1^{(0)} - h_1\int dx\,\cos(\sqrt{2\pi}\theta_1) \qquad (3.19)$$

and

$$h_0 = \delta\mathcal{J}m_0 + \mathcal{J}_Tm_1$$

$$h_1 = \mathcal{J}_Tm_0 + \mathcal{J}_Tm_2 \qquad (3.20)$$

where we have used that $m_0 = m_3$ and $m_1 = m_2$. Since all the A-systems are equivalent, they have the same SC susceptibility χ. Then, the self-consistency equations are

$$m_0 = \chi_{SC}h_0, \qquad m_1 = \chi_{SC}h_1 \qquad (3.21)$$

We can write this as a system of linear equations,

$$\begin{pmatrix} 1 - \chi_{SC}\delta\mathcal{J} & -\chi_{SC}\mathcal{J}_T \\ -\chi_{SC}\mathcal{J}_T & 1 - \chi_{SC}\mathcal{J}_T \end{pmatrix}\begin{pmatrix} m_0 \\ m_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad (3.22)$$

which has a non-trivial solution if and only if the determinant of the 2×2 matrix is zero. This gives us a quadratic equation in for χ_{SC}. Choosing the positive solution we find that the critical temperature for the coexisting state is

$$T_c = \Delta_s\left(\frac{2\mathcal{J}_T(\mathcal{J}_T - \delta\mathcal{J})}{-\mathcal{J}_T - \delta\mathcal{J} + \sqrt{5\mathcal{J}_T^2 - 2\mathcal{J}_T\delta\mathcal{J} + \delta\mathcal{J}^2}}\right)^\alpha \qquad (3.23)$$

where we recall that the exponent is given by $\alpha = \dfrac{1}{2 - 1/K_c}$. Notice that, in the limit $\delta\mathcal{J} \to \mathcal{J}_T$, we recover Eq. (3.11). Thus, as in the uniform or pure period 2 PDW state, T_c has a power law behavior in $\mathcal{J}_T$ and $\delta\mathcal{J}$, and it is not exponentially small as it would be in a weak coupling BCS type theory.

$\mathcal{J}_T$	$\delta\mathcal{J}$	d	m_0	m_1	$\overline{m}$	m_{PDW}
1	1	1/4	0.890893	0.890893	0.890893	0
1	0.8	1/4	0.876601	0.889789	0.883195	0.0065943
1	0.5	1/4	0.853007	0.887947	0.870477	0.0174703
1	0	1/4	0.806035	0.884205	0.845120	0.0390853

TABLE I. Numerical solution for the system of equations
(3.17) for different values of the parameters $\mathcal{J}_T$, $\delta\mathcal{J}$ and
$d = 1/2K_c$. We also define $\overline{m} = (m_1 + m_0)/2$ and $m_{\mathrm{PDW}} =
(m_1 - m_0)/2$, which correspond to the uniform and PDW part
of the SC order parameter.

IV. FERMIONIC QUASIPARTICLES OF THE SUPERCONDUCTING STATES

So far, we have solved the coupled LE systems in the
limit $|\mathcal{J}_{AA,j}| \gg |\mathcal{J}_{AB}|$ and $|\mathcal{J}_{AA,j}| \gg |\mathcal{J}'_{AB}|$ in Eq.(2.9)
so that the couplings of the LE systems to eLL systems
can be taken as the perturbation. In this limit, we have
ignored the type-B eLL systems and shown that the various
SC states can emerge. Now we include the eLL systems
back and investigate the nature of the full emergent
SC state by looking at the SC proximity effect. First of
all, we note that the eLL systems themselves will flow to
the 2D Fermi liquid fixed point (at low enough temperatures)
under the effect of the hopping amplitude t_{BB}.
This is the most relevant coupling in Eq.(2.9). The result,
for t_{BB} small enough, is an anisotropic Fermi liquid
with an open Fermi surface, shown as the dashed curves
in Fig.2.

Having solved the largest energy scales in Eq.(2.9), set
by t_{BB} and $\mathcal{J}_{AA}$, we now include the effect of the pair
tunneling processes mixing the systems A with the systems
B, presented in Eq.(2.9), and parametrized by the
coupling constants $\mathcal{J}_{AB}$ and $\mathcal{J}'_{AB}$, respectively. We will
study the effects of the SC states on the A systems on
the B systems by treating the pair-tunneling terms to the
lowest non-trivial order in perturbation theory in these
coupling constants. Hence, we are assuming that the interaction
with the type-B eLL systems does not back react
to considerably change the MFT value of the SC gap
in the LE systems. As in Ref.[36], under the proximity
effect mechanism the B systems become superconducting
and provide the quasiparticles for the combined A-B
system.

Since we are interested in the effect of the SC order parameters
on the electronic spectrum, we replace the pair
density $\Delta_{A,j}(x)$ of the type-A LE systems in Eq.(2.9) by
its MF value $\langle\Delta_{A,j}\rangle$ determined by the interchain MFT
discussed in the previous Section III. In this approximation,
we find that Eq.(2.9) reduces to

$$H' \to \sum_j \int dx \Big\{ -t_{BB} \sum_\sigma [\psi_{B,j,\sigma}^\dagger \psi_{B,j+1,\sigma} + \text{h.c.}]$$
$$- \mathcal{J}_{AB}[\Delta_{B,j}^\dagger \langle\Delta_{A,j}\rangle + \Delta_{B,j}^\dagger \langle\Delta_{A,j+1}\rangle + \text{h.c.}]$$
$$+ \mathcal{J}'_{AB}[\langle\Delta_{A,j}^*\rangle(\psi_{B,j,\uparrow}\psi_{B,j-1,\downarrow} + \psi_{B,j-1,\uparrow}\psi_{B,j,\downarrow}) + \text{h.c.}]\Big\},$$
$$(4.1)$$

which is simply a theory of a Fermi surface coupled to the
SC via a proximity coupling. Since Eq.(4.1) is quadratic
in the electron fields, we can readily diagonalize the effective
Hamiltonian, and obtain the quasiparticle spectrum
for the different SC states found in Section III.

A. Uniform SC phase and pure PDW phase

As we saw in section III A 2, for the staggered (period
2) configurations of the Ising variables is possible to have
either a pure uniform SC state or a pure PDW state. The
case of the uniform SC was studied by Granath, et al.[36]
who showed that, depending on the values of $\mathcal{J}_{AB}$ and
$\mathcal{J}'_{AB}$, it is possible to have either a d-wave SC state with a
fully gapped spectrum of quasiparticles or a conventional
d-wave SC state one with a nodal quasiparticle spectrum.
We refer the reader to their paper for further details.[36]

On the other hand, for the pure PDW state, even
though the MF equation for the SC gap has the same
form as for the uniform SC gap, the quasiparticle spectrum
is quite different. We will study this spectrum in
detail here. Let us start by defining the period 2 PDW order
parameter, i.e. with ordering wave vector $\boldsymbol{Q} = (0,\pi)$,

$$\Delta_j^A = \Delta_{\boldsymbol{Q}} e^{i\pi j} \qquad (4.2)$$

where $\Delta_{\boldsymbol{Q}}$ is given by the spin gap and the interchain
MFT value for $\langle\cos\sqrt{2\pi}\theta\rangle$ which is given in Section III A
for the period 2 configuration of the Ising variables. Notice
that for a period 2 state $\Delta_{\boldsymbol{Q}} = \Delta_{-\boldsymbol{Q}}$, since for a
period 2 state $\boldsymbol{Q}$ and $-\boldsymbol{Q}$ differ by a reciprocal lattice
vector.

To find the quasiparticle spectrum we first write down
the Hamiltonian of Eq. (4.1) in momentum space. Defining
the Nambu basis (here we dropped the B label in the
electronic operators, since it is understood that we are
referring to the eLL systems) as:

$$\Psi_{\mathbf{k}}^\dagger = (\psi_{\mathbf{k}\uparrow}^\dagger, \psi_{\mathbf{k}+(0,\pi)\uparrow}^\dagger, \psi_{-\mathbf{k}\downarrow}, \psi_{-\mathbf{k}-(0,\pi)\downarrow}) \qquad (4.3)$$

we can write the Bogoliubov de-Gennes (BdG) Hamiltonian
as

$$H = \sum_{\mathbf{k}} \psi_{\mathbf{k}}^\dagger \hat{H}_{\mathbf{k}} \psi_{\mathbf{k}} \qquad (4.4)$$

where the one-particle Hamiltonian $\hat{H}_{\mathbf{k}}$ is given by

$$\hat{H}_{\boldsymbol{k}} = \begin{pmatrix} \varepsilon(k_x) - t_{BB}\cos(k_y) & 0 & 0 & 2i\mathcal{J}'_{AB}\Delta_{\boldsymbol{Q}}^*\sin(k_y) \\ 0 & \varepsilon(k_x) + t_{BB}\cos(k_y) & -2i\mathcal{J}'_{AB}\Delta_{\boldsymbol{Q}}^* sin(k_y) & 0 \\ 0 & 2i\mathcal{J}'_{AB}\Delta_{\boldsymbol{Q}}\sin(k_y) & -\varepsilon(k_x) + t_{BB}\cos(k_y) & 0 \\ -2i\mathcal{J}'_{AB}\Delta_{\boldsymbol{Q}}\sin(k_y) & 0 & 0 & -\varepsilon(k_x) - t_{BB}\cos(k_y) \end{pmatrix} \tag{4.5}$$

From this one-particle Hamiltonian we find the quasiparticle spectrum

$$E(\mathbf{k}) = \pm t_{BB}\cos(k_y) \pm \sqrt{\varepsilon^2(k_x) + 4\mathcal{J}'^2_{AB}|\Delta_{\boldsymbol{Q}}|^2\sin^2(k_y)} \tag{4.6}$$

In Fig. 2 we plot the Fermi surface of the Bogoliubov
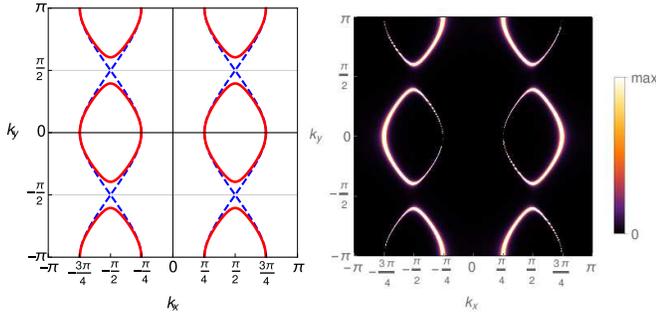
FIG. 2. (Color online) On the left, Fermi surface for the pure period 2 PDW state. The dashed (blue) line corresponds to the original FS in the absence of superconductivity. The solid (red) line corresponds to the new FS after the superconducting proximity state is established. On the right, the spectral function $A(\mathbf{k}, 0)$ corresponding to the pockets on the left. We used $\mathcal{J}'_{AB}\Delta_{\boldsymbol{Q}} = 0.12t$, $t_{BB} = 0.7t$, $\varepsilon(k_x) = -t\cos k_x$ and $\delta = 10^{-4}t$

quasiparticles of this period 2 PDW state for some values of the parameters. In contrast to the pure uniform SC state, whose spectrum can be either nodal or fully gapped, we find that this PDW state ($\Delta_{\boldsymbol{Q}} \neq 0$ in Eq. (4.2)) has pockets of Bogoliubov quasiparticles, as it is also found in the weak coupling theories.[4,14,17,44,50,51] The size of the pockets depends on the strength of the SC gap. In addition, we compute the spectral function given by (see for instance Ref. [52]):

$$A(\mathbf{k}, \omega) = -\frac{1}{\pi}\text{Im}[\hat{G}_{11}(\mathbf{k}, \omega)] \tag{4.7}$$

where

$$\hat{G}(\mathbf{k}, \omega) = \frac{1}{\omega + i\delta - \hat{H}_{\boldsymbol{k}}} \tag{4.8}$$

is the retarded Green function and $\delta = 0^+$. The spectral function $A(\mathbf{k}, \omega = 0)$ for this pure period 2 PDW state is shown in Fig. 2. In Fig. 3 we plot the dispersion relation of the Bogoliubov excitations for several values of k_y.
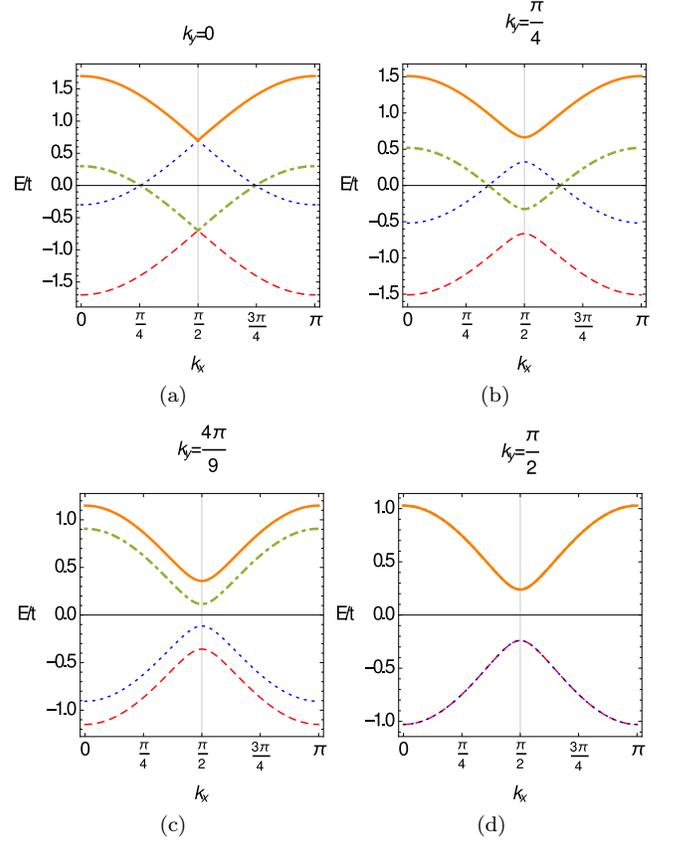

FIG. 3. (Color online) a-d: Dispersion relation of the Bogoliubov quasiparticles for $k_y = 0, \frac{\pi}{4}, \frac{4\pi}{9}, \frac{\pi}{2}$, respectively, for the period 2 PDW state. Here we used $\mathcal{J}'_{AB}\Delta_{\boldsymbol{Q}} = 0.12t$, $t_{BB} = 0.7t$ and $\varepsilon(k_x) = -t\cos k_x$

B. Coexistence Phase of a Period 4 PDW and a uniform SC: the Striped Superconductor

We start by writing the SC order parameter, which includes both the uniform SC and the PDW order parameters as an expansion of the form

$$\Delta_j^A = \Delta_0 + \sqrt{2}\Delta_{\boldsymbol{Q}}\cos\left(\frac{\pi j}{2} + \frac{\pi}{4}\right) \tag{4.9}$$

where the expectation values of the order parameters Δ_0 and $\Delta_{\boldsymbol{Q}}$, where $\boldsymbol{Q} = (0, \frac{\pi}{2})$ is the ordering wave vector, are set jointly by the spin gap of the LE systems and by the interchain MFT value for $\langle\cos\sqrt{2\pi}\theta\rangle$ found in the previous section for the period 4 state of the Ising degrees of freedom.

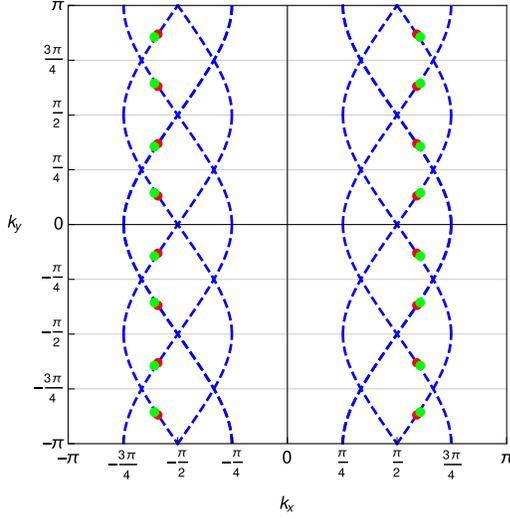We now write down the Hamiltonian in momentum

FIG. 4. (Color online) Quasiparticle spectra with nodal points in the coexistence phase. The dashed (blue) line corresponds to the original FS in the absence of superconductivity. The red points correspond to the position of the nodes in the absence of the PDW state ($\Delta_Q = 0$). The green points correspond to the position of the nodes in the presence of the PDW state with $\Delta_Q = 0.2$. We have chosen the parameters $\mathcal{J}'_{AB} = 0.5t$, $\mathcal{J}_{AB} = 0.2t$, $t_{BB} = 0.7t$, $\Delta_0 = 0.2$ and $\varepsilon(k_x) = -t \cos k_x$

.

space following the notation of Ref. [50]. We define the Nambu spinor as:

$$\Psi^\dagger_{\mathbf{k}} = (\psi^\dagger_{\mathbf{k}\uparrow}, \psi^\dagger_{\mathbf{k}+\mathbf{q}\uparrow}, \ldots, \psi_{-\mathbf{k}\downarrow}, \psi_{-(\mathbf{k}+\mathbf{q})\downarrow}, \ldots) \quad (4.10)$$

where $\mathbf{q}$ is the ordering wavevector. In our case $\mathbf{q} = (0, \pi/2)$ and $\mathbf{k}$ is taking values over the reduced Brillouin zone (RBZ) associated with the ordered state, which in this case is $k_x \in [-\pi, \pi)$ and $k_y \in [-\pi/4, \pi/4)$. In this basis the Hamiltonian is given by:

$$H = \sum_{\mathbf{k} \in RBZ} \psi^\dagger_{\mathbf{k}} \hat{H}_{\mathbf{k}} \psi_{\mathbf{k}}, \quad (4.11)$$

where the BdG Hamiltonian $\hat{H}_{\mathbf{k}}$ in the Nambu basis of Eq.(4.10) is given by:

$$\hat{H}_{\mathbf{k}} = \begin{pmatrix} \mathcal{A}_{\mathbf{k}} & \mathcal{C}_{\mathbf{k}} \\ \mathcal{C}^\dagger_{\mathbf{k}} & -\mathcal{A}_{\mathbf{k}} \end{pmatrix} \quad (4.12)$$

where $\mathcal{A}_{\mathbf{k}} = \mathrm{diag}(\varepsilon(\mathbf{k}), \varepsilon(\mathbf{k}+\mathbf{q}), \ldots)$ is a diagonal matrix, and the square matrix $\mathcal{C}_{\mathbf{k}}$ contains the SC order parameters. Since the ordering vector is $\pi/2$ along the k_y direction, our matrix $\mathcal{C}_{\mathbf{k}}$ is given by a 4×4 matrix with the form:

$$\mathcal{C}_{\mathbf{k}} = \begin{pmatrix} f_0(\mathbf{k}) & f_1(\mathbf{k}) & f_2(\mathbf{k}) & f_3(\mathbf{k}) \\ f_1^*(\mathbf{k}) & f_0(\mathbf{k}+\mathbf{q}) & f_1(\mathbf{k}+\mathbf{q}) & f_2(\mathbf{k}+\mathbf{q}) \\ f_2^*(\mathbf{k}) & f_1^*(\mathbf{k}+\mathbf{q}) & f_0(\mathbf{k}+2\mathbf{q}) & f_1(\mathbf{k}+2\mathbf{q}) \\ f_3^*(\mathbf{k}) & f_2^*(\mathbf{k}+\mathbf{q}) & f_1^*(\mathbf{k}+2\mathbf{q}) & f_0(\mathbf{k}+3\mathbf{q}) \end{pmatrix} \quad (4.13)$$

where f_0 corresponds to uniform pairing and f_1, f_2, f_3 to the finite momentum pairing. The explicit expressions are the following:

$$f_0(\mathbf{k}) = 2\Delta_0(\mathcal{J}_{AB} - \mathcal{J}'_{AB} \cos k_y)$$
$$f_1(\mathbf{k}) = -i\Delta_Q(\mathcal{J}_{AB} - \sqrt{2}\mathcal{J}'_{AB} \cos(k_y + q_y/2))$$
$$f_2(\mathbf{k}) = 0$$
$$f_3(\mathbf{k}) = i\Delta_Q(\mathcal{J}_{AB} - \sqrt{2}\mathcal{J}'_{AB} \cos(k_y - q_y/2)) \quad (4.14)$$

where we recall that $\mathbf{q} = (0, \pi/2)$, so $q_y = \pi/2$.

First of all, due to the periodicity of the PDW SC state, it is necessary to the fold original FS. Let us first analyze the case of the pure uniform SC state. In this case ($\Delta_Q = 0$) the spectrum can be easily calculated from the Hamiltonian given in eq. (4.12):

$$E^2_{1,\pm} = (\varepsilon(k_x) \pm t_{BB} \cos(k_y))^2 + \Delta_0^2(\mathcal{J}_{AB} \pm \mathcal{J}'_{AB} \cos(k_y))^2$$
$$E^2_{2,\pm} = (\varepsilon(k_x) \pm t_{BB} \sin(k_y))^2 + \Delta_0^2(\mathcal{J}_{AB} \pm \mathcal{J}'_{AB} \sin(k_y))^2 \quad (4.15)$$

We can see that this SC state will have a quasiparticle spectrum with nodes if $|\mathcal{J}_{AB}| < |\mathcal{J}'_{AB}|$. Now, even in the coexistence phase, where both $\Delta_Q \neq 0$ and $\Delta_0 \neq 0$, the quasiparticle spectrum may still can have nodes. For the pure uniform SC state, the position of the nodes depends on the values $\mathcal{J}_{AB}/\mathcal{J}'_{AB}$ and t_{BB}. In the coexistence phase the position of the nodes will depends on Δ_Q as well (see Fig. 4). As in the case of pure period 2 PDW state, we show in Fig. 5 the dispersion relation of the quasiparticles for several values of k_y.

C. Period 8 PDW state

Above we focused on the coexistence phase for the period 4 case. This was the case when $\delta\mathcal{J} > 0$. However, if $\delta\mathcal{J} < 0$ (i.e. for $\mathcal{J}_{AA} < \mathcal{J}'_{AA}$) the case is different and we find a PDW state. There is a shift of π every four lattice sizes, so in this case the periodicity of the PDW order parameter is actually eight (not four!). Nevertheless, the self-consistency equations will have the same form:

$$m_0 = f(d) \left(\frac{m_0|\delta\mathcal{J}| + m_1\mathcal{J}_T}{2} \right)^{d/(2-d)}$$
$$m_1 = f(d) \left(\frac{m_0\mathcal{J}_T + m_1\mathcal{J}_T}{2} \right)^{d/(2-d)} \quad (4.16)$$

The pattern of the SC order parameter is now that of a pure period 8 PDW SC state:

$$\Delta = (\Delta_1, \Delta_2, \Delta_2, \Delta_1, -\Delta_1, -\Delta_2, -\Delta_2, -\Delta_1, \Delta_1, \ldots)$$

We can write the previous pattern using the following SC order parameter:

$$\Delta^A_j = \Delta \sin\left(\frac{\pi j}{4} + \frac{\pi}{8} \right) + \tilde{\Delta} \sin\left(\frac{3\pi j}{4} + \frac{3\pi}{8} \right) \quad (4.17)$$
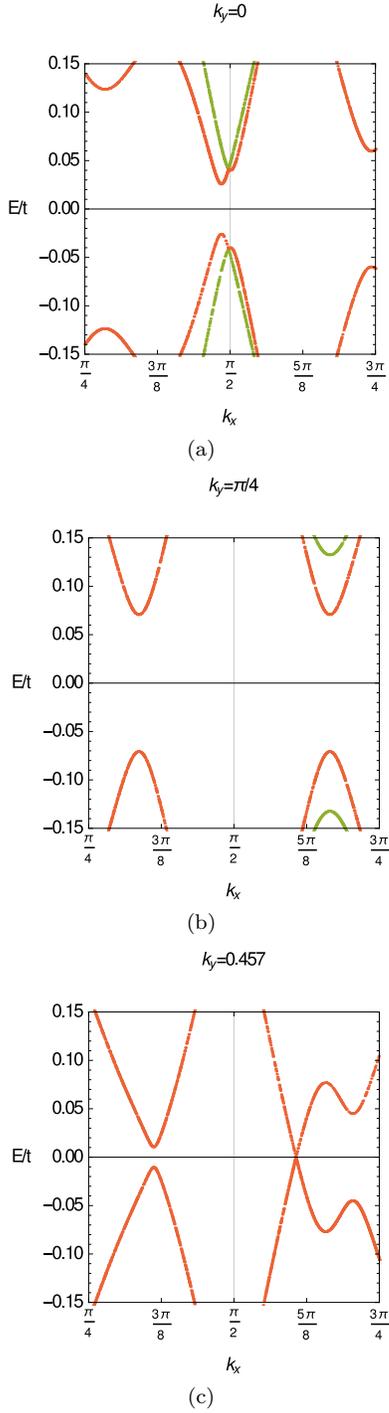
FIG. 5. (Color online) Dispersion relations of the quasi-particles in the coexistence phase shown for several values of k_y. Notice that the dispersion relation is only gapless for $k_y \approx 0.457$, which corresponds to the position of the nodal point for the same set of parameters used in Fig. 4. $\mathcal{J}'_{AB} = 0.5t$, $\mathcal{J}_{AB} = 0.2t$, $t_{BB} = 0.7t$, $\Delta_0 = \Delta_{\mathbf{Q}} = 0.2$ and $\varepsilon(k_x) = -t \cos k_x$
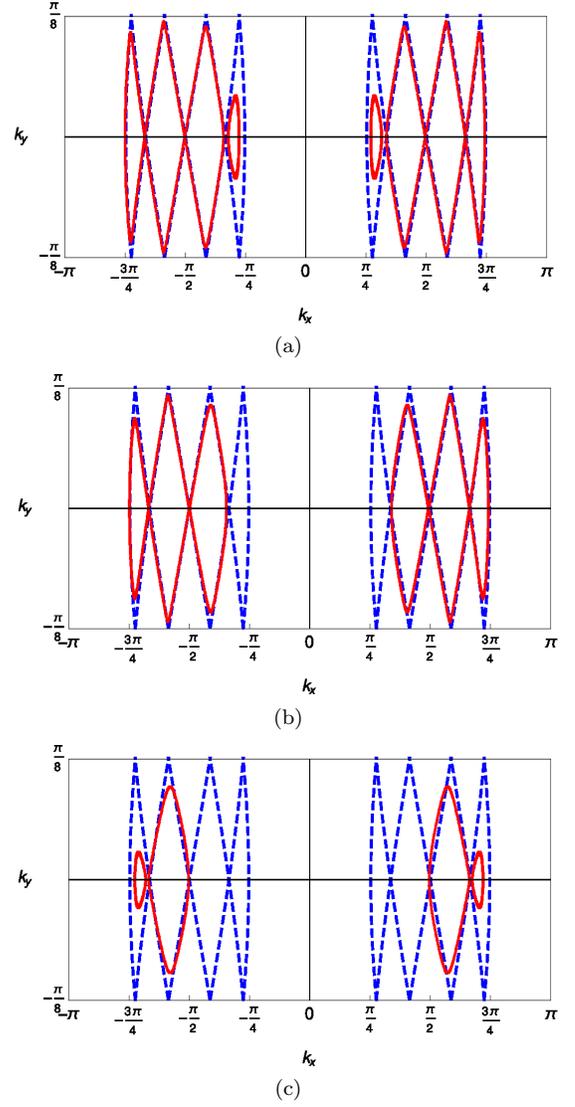

FIG. 6. (Color online) FS for the period 8 PDW state. The dashed (blue) line corresponds to the original FS in the absence of superconductivity. The solid (red) line corresponds to the new FS (pockets) after the introduction of superconductivity ($\Delta_1 \neq 0$ and $\Delta_2 \neq 0$). In (a) $\Delta_1 = \Delta_2 = 0.05$, in (b) $\Delta_1 = 0.08$ and $\Delta_2 = 0.1$ and in (c) $\Delta_1 = 0.25$ and $\Delta_2 = 0.3$ In all figures $\mathcal{J}'_{AB} = 0.6t$, $\mathcal{J}_{AB} = 0.4t$, $t_{BB} = 0.7t$, $\Delta_0 = 0$ and $\varepsilon(k_x) = -t \cos k_x$

where we have defined:

$$\Delta = \Delta_1 \sin\left(\frac{\pi}{8}\right) + \Delta_2 \cos\left(\frac{\pi}{8}\right)$$
$$\tilde{\Delta} = \Delta_1 \cos\left(\frac{\pi}{8}\right) - \Delta_2 \sin\left(\frac{\pi}{8}\right) \tag{4.18}$$

where Δ_1 and Δ_2 are given by the spin gap and the interchain MFT value for $\langle \cos \sqrt{2\pi}\theta \rangle$ in eq. (4.16).

Since we are dealing with a period 8 SC state, the reduced Brillouin Zone is now case is $k_x \in [-\pi, \pi)$ and $k_y \in [-\pi/8, \pi/8)$ and $\mathbf{q} = (0, \pi/4)$. The difference between the period 4 and the period 8 is that the definition

of the $\mathcal{C}_{\mathbf{k}}$ matrix is different, since is now an 8×8 matrix.

$$\mathcal{C}_{\mathbf{k}} = \begin{pmatrix} f_0(\mathbf{k}) & f_1(\mathbf{k}) & \cdots & f_7(\mathbf{k}) \\ f_1^*(\mathbf{k}) & f_0(\mathbf{k}+\mathbf{q}) & \cdots & f_6(\mathbf{k}+\mathbf{q}) \\ \vdots & & \ddots & \\ f_7^*(\mathbf{k}) & & & f_0(\mathbf{k}+7\mathbf{q}) \end{pmatrix} \quad (4.19)$$

where the $f_i(\mathbf{k})$'s are given by the following expressions:

$$f_0(\mathbf{k}) = f_2(\mathbf{k}) = f_4(\mathbf{k}) = f_6(\mathbf{k}) = 0$$

$$f_1(\mathbf{k}) = i\Delta \left(\frac{1}{2} \mathcal{J}_{AB} (e^{-i\pi/8} + e^{-i3\pi/8}) \right.$$
$$\left. - \mathcal{J}'_{AB} e^{-i\pi/4} \cos(k_y + q_y/2) \right)$$

$$f_3(\mathbf{k}) = i\tilde{\Delta} \left(\frac{1}{2} \mathcal{J}_{AB} (e^{-3i\pi/8} + e^{-i9\pi/8}) \right.$$
$$\left. - \mathcal{J}'_{AB} e^{-i3\pi/4} \cos(k_y + 3q_y/2) \right)$$

$$f_5(\mathbf{k}) = -i\tilde{\Delta} \left(\frac{1}{2} \mathcal{J}_{AB} (e^{3i\pi/8} + e^{i9\pi/8}) \right.$$
$$\left. - \mathcal{J}'_{AB} e^{i3\pi/4} \cos(k_y - 3q_y/2) \right)$$

$$f_7(\mathbf{k}) = -i\Delta \left(\frac{1}{2} \mathcal{J}_{AB} (e^{i\pi/8} + e^{i3\pi/8}) \right.$$
$$\left. - \mathcal{J}'_{AB} e^{i\pi/4} \cos(k_y - q_y/2) \right)$$

where we recall that $\mathbf{q} = (0, \pi/4)$, so $q_y = \pi/4$. Having $\mathcal{C}_{\mathbf{k}}$ we can write down our BdG Hamiltonian as in eq. (4.12). In Fig. (6) we show the FS for some values of Δ_1 and Δ_2. As in the pure period 2 PDW state, we see the formation of pockets due to the folding of the FS.

V. OTHER PHASES

For completeness we summarize the other possible phases occurring in the system. Following closely Granath et al.[36] we treat the interactions appearing in eq. (2.9) perturbatively around the so called decoupled fixed point. At this fixed point (FP) the systems are completely decoupled, and each one of the systems corresponds to a 1D system that can be solved using bosonization. Around the decoupled FP a perturbation with coupling constant g is relevant (irrelevant) if its scaling dimension $d_g < 2$ ($d_g > 2$). The scaling dimensions for the operators appearing in Eq. (2.9) are given in the work of Granath et al.[36] The phases found by Granath et al. are:

1. Typically, the couplings between the eLL and LE systems are irrelevant or less relevant than the coupling between AA and BB systems separately. In this case the RG flows to the point where all the AB couplings go to zero. At this FP the system is made of two (independent) interpenetrating systems, A and B.

2. The $\mathcal{J}_{AA}$ ($\mathcal{J}'_{AA}$) term is relevant for $K_c^{(A)} > 1/2$. In this case the A systems develop long-range order and a full spin gap. Since the BB electron tunneling operator has lower scaling dimension than the BB spin exchange interaction, in the absence of a charge gap in the B subsystem, most probably the B subsystem is in a anisotropic Fermi liquid phase. However, this two fluid FP is unstable due to the proximity effect. Depending on the parameters in the Hamiltonian of Eq.(2.9) the quasiparticle spectrum can be gapless (present nodes or pockets in the pure PDW state) or fully gapped. This means that we can have several possible stable SC phases, a SC state with Fermi pockets, a nodal SC state, or a fully gapped SC state. These were the phases studied in the previous sections using interchain MFT and coupling the eLL systems to the LE systems.

3. If $\Delta_c^{(B)} > 0$, the B subsystem can develop a antiferromagnetic phase. At this FP will be a coexistence between superconductivity (in the A subsystem) and antiferromagnetism (in the B subsystem). This FP is stable, due to the spin gap in the SC (A) and the charge gap in the antiferromagnet (B). The quasiparticle spectrum is therefore fully gapped as is also found in BCS-type theories.[18]

VI. CONCLUDING REMARKS

We have investigated a model of an array of two inequivalent systems in the quasi-one dimensional limit. In this limit we have treated the interactions between the different systems in the array exactly using bosonization methods and interchain mean field theory. The phases that we found are either a uniform d-wave superconductor, a striped superconductor (in which the uniform SC and the PDW SC state coexist), and a PDW state. To simplify the analysis we only looked at the case in which the modulation of the SC state is commensurate.

The resulting critical temperatures are, as expected, upper bounds on the actual physical critical temperatures. As emphasized in Refs.[40] and [53], the analytic dependence of these mean field T_c's on the coupling constants obeys the exact power-law scaling behavior predicted by a renormalization group analysis of the dimensional crossover from the 1D regime to the full (but anisotropic) 2D phases, albeit with an overestimate of the prefactor.

On the other hand, the actual critical temperatures are significantly suppressed from the values quoted here due to the the two-dimensional nature of the array. Hence we expect the ground states that we found here to undergo a sequence of thermodynamic phase transitions leading to a complex phase diagram of the type discussed by Berg et al.[42] (and by Agterberg and Tsunetsugu.[8]) It is well known from classical critical phenomena of 2D

commensurate systems that states of the type we discuss here may become incommensurate at finite temperatures due to thermal fluctuations if the period of the ordered state is longer than a critical value (typically equal to four), see, e.g. Ref.[54].

We have shown that a high energy scales (of the order of the spin gap), we can first determine the SC phases of one set of systems (in our notation, the Luther-Emery liquid systems A). At these energy scales we showed that it is possible to have, in addition to a uniform SC phase, a pure PDW state and a coexistence phase of a uniform and a PDW state. Having determined the SC in the LE systems, we proceeded to incorporate the electronic Luttinger liquid systems perturbatively. We found that the quasiparticle spectrum arising from the eLL systems can present Fermi pockets if the SC state is a pure PDW state. In the case of coexistence uniform SC and PDW state or pure uniform SC (i.e. a striped superconductor) the quasiparticle spectrum can have nodes or be fully gapped depending on the value of the coupling in the model. We should stress, as it was done recently in Ref.

[11], that in this quasi-1D approach the superconducting state evolves from a local high energy scale, the spin gap, which hence has magnetic origin. For temperature T higher that the spin gap, the system is a quasi 1D system which does not have quasiparticles in the spectrum up to a scale, determined by an electron tunneling scale, to a crossover to a Fermi liquid type system. Hence, at least qualitatively, systems of this type behave as 'high T_c superconductors.'

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