Construction and field theory of bosonic-symmetry-protected topological states beyond group cohomology

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We construct a series of bosonic symmetry protected topological (BSPT) states beyond group cohomology classification using "decorated defects" approach. This construction is based on topological defects of ordinary Landau order parameters, decorated with the bosonic short range entangled (BSRE) states in $(4k + 3)d$ and $(4k + 5)d$ space-time (with k being nonnegative integers), which do not need any symmetry. This approach not only gives these BSPT states an intuitive physical picture, it also allows us to derive the effective field theory for all these BSPT states beyond group cohomology.

1. INTRODUCTION

Bosonic symmetry protected topological (BSPT) states are bosonic analogues of fermionic quantum spin Hall insulator and topological insulator, which have trivial bulk spectrum but nontrivial boundary spectrum, as long as the system preserves certain symmetry. There are roughly two types of BSPT states, their mathematical difference is whether they can be classified and described by group cohomology [1, 2] and semiclassical nonlinear sigma model field theory [3]. For example, the well-known E_8 bosonic short range entangled (BSRE) state [18] [4, 5] in $2d$ space [19], and its higher dimensional generalizations [6] cannot be classified by group cohomology.

Any nontrivial SPT state's boundary state cannot exist by itself, as long as the system preserves the necessary symmetry. This means that the boundary of a SPT state must be "anomalous". The relation between boundary anomaly and bulk SPT states has been studied systematically in Ref. 7. If a nontrivial SRE state does not need any symmetry to protect its boundary, then its boundary must have gravitational anomaly. The $2d$ $p+ip$ topological superconductor, and the $2d$ E_8 state both have chiral edge states, which lead to gravitational anomaly. Analogues of $2d$ E_8 state can be found in all even spatial dimensions. In every $(4k + 2)d$ space (or equivalently $(4k+3)d$ space-time), there is a BSRE state with $\mathbb{Z}$ classification described by action [6]

$$\mathcal{S}_{(4k+3)d} = \int \frac{iK^{IJ}}{4\pi} C^I \wedge dC^J, \qquad (1)$$

where C^I is a $2k+1$ form antisymmetric gauge field, and K^{IJ} is the Cartan matrix of the E_8 group. These states have bosonic $2k-$dimensional membrane excitations in the bulk, and perturbative gravitational anomalies at the boundary [6, 8]; In every $(4k+4)d$ space (or equivalently $(4k + 5)d$ space-time), there is a BSRE state with $\mathbb{Z}_2$ classification described by action

$$\mathcal{S}_{(4k+5)d} = \int \frac{iK^{IJ}}{4\pi} B^I \wedge dB^J, \qquad (2)$$

where B^I is a $2k+2$ form antisymmetric gauge field, and $K^{IJ} = i\sigma^y$. This theory with $k = 0$ ($4d$ space) has been studied carefully in Ref. 9, and it was demonstrated that its $3d$ boundary is an "all fermion" $3d$ QED [10] which cannot be independently realized in $3d$ space, and it has a global gravitational anomaly [11].

As was pointed out by Ref. 12, 13, the state Eq. 2 can also have a time-reversal symmetry. For instance, this action is invariant under $Z_2^T : i \to -i, (B^1, B^2) \to (B^2, B^1)$. But this state is also stable if the time-reversal symmetry is broken. In this paper we will only count this state as a BSRE state without any symmetry.

All these BSRE states in even spatial dimensions have their descendant BSPT states in higher dimensions. All these descendant BSPT states are also beyond the group cohomology classification. Recently, a systematic mathematical formalism for BSRE and BSPT states has been proposed in Ref. 14, which was based on cohomology of $G \times SO(\infty)$, where G is the symmetry group, and $SO(\infty)$ is supposed to describe the gravitational anomaly. The purpose of the current work is to give a physical construction and field theory description of BSPT states beyond the ordinary group cohomology classification. Our results are summarized in Table I.

2. GENERALITIES

We will view the BSRE states without any symmetry in even spatial dimensions (Eq. 1 and Eq. 2) as base states. Our general strategy for constructing other beyond-Group-Cohomology BSPT states, is to first break part or all of the symmetry by condensing an ordinary Landau order parameter, then proliferating/condensing the topological defects of the Landau order parameter. The nontrivial BSPT state and the trivial state are distinguished by the nature of the topological defects: nontrivial BSPT states corresponds to the case where the defects are decorated with the BSRE states in Eq. 1 or Eq. 2. The first example of such beyond-Group-Cohomology BSPT state, which is protected by Time Reversal Symmetry $\mathcal{T}$, was discovered in Ref. 15. This

state can be constructed by proliferating $\mathcal{T}$-breaking domain walls decorated with the $2d$ E_8 state. The topological term in the field theory Lagrangian density that encodes the decoration reads:

$$
\begin{aligned}
\mathcal{L}_{3+1d}^{Z_2^T} &= \frac{i2\pi}{2\pi}ndn \wedge \frac{K_{E_8}^{IJ}}{8\pi^2}C^I \wedge dC^J \\
&= id\theta \wedge \frac{K_{E_8}^{IJ}}{8\pi^2}C^I \wedge dC^J \\
&= -i\theta\frac{K_{E_8}^{IJ}}{8\pi^2}dC^I \wedge dC^J
\end{aligned}
\tag{3}
$$

where the $O(2)$ vector $\vec{n}$ is parametrized as $\vec{n} = (\cos\theta, \sin\theta)$. The $\mathcal{T}$-symmetry transformation is

$$
\begin{aligned}
Z_2^T: \quad & (n_1, n_2) \to (n_1, -n_2), \\
& \theta \to -\theta
\end{aligned}
\tag{4}
$$

One can verify that the Eq. 3 is time-reversal invariant. Also, if we keep time-reversal invariance, then $\langle n_2 \rangle = 0$, namely $\langle\theta\rangle = 0$ or π, which precisely corresponds to the trivial and nontrivial BSPT state discussed in Ref. 15. Meanwhile, across a $\mathcal{T}$-breaking domain wall, θ continuously changes from $-\pi + 0^+$ to $\pi + 0^-$. After integrating over the normal direction, the effective field theory left on the domain wall precisely describes a $2d$ E_8 state.

The idea of "decorated domain wall" construction of SPT states was further explored in Ref. 16. Domain wall of Z_2 or time-reversal symmetry is the simplest kind of topological defect. In our current work we will construct beyond-group-cohomology BSPT states using more general topological defects of other symmetry groups. Here we want to clarify that in our current work the concept "topological defect" refers to a topologically stable configurations of Landau order parameter $\vec{n}$ in $d-$dimensional space $\mathbb{R}^d$ with a singularity $\mathcal{I}$, and the singularity can be viewed as the boundary of $\mathbb{R}^d - \mathcal{I}$. The Landau order parameter $\vec{n}$ has a soliton configuration on $\mathbb{R}^d - \mathcal{I}$, which has no singularity any more. For example, in $2d$ space a vortex core is a singularity at the origin $(0,0)$, and it can be viewed as the boundary of $\mathbb{R}^2 - (0,0)$, which is topologically equivalent to a ring S^1. A vortex configuration corresponds to a $1d$ soliton on S^1, based on the simple fact $\pi_1[S^1] = \mathbb{Z}$. In $3d$ space a hedgehog monopole core is again a singularity at $(0,0,0)$, and a hedgehog monopole corresponds to a soliton on space $\mathbb{R}^3 - (0,0,0)$, based on the fact $\pi_2[S^2] = \mathbb{Z}$.

In general, the field theories we will discuss in this work is a combination of the Θ-term of $\vec{n}$ discussed in Ref. 3 and Chern-Simons form of C^I or B^I in Eq. 1,2. The explicitly form of the topological term in $D-$dimensional space-time is:

$$
\mathcal{L}_{Dd,A} = \frac{i\Theta}{\Omega_{D-(4k+4)}}n \underbrace{dn \wedge ... \wedge dn}_{D-(4k+4)} \wedge \frac{K^{IJ}}{8\pi^2}dC^I \wedge dC^J,
\tag{5}
$$

$$
\mathcal{L}_{Dd,B} = \frac{i\Theta}{\Omega_{D-(4k+6)}}n \underbrace{dn \wedge ... \wedge dn}_{D-(4k+6)} \wedge \frac{(i\sigma^y)^{IJ}}{8\pi^2}dB^I \wedge dB^J,
\tag{6}
$$

where $\vec{n}$ is a Landau order parameter with a unit length. $\Omega_D = V_D \times D!$, V_D is the volume of the unit D-dimensional sphere. Here we assume all components of order parameter $\vec{n}$ transform nontrivially under the symmetry group.

The equations above are also effectively equivalent to the two equations in the follows:

$$
\mathcal{L}_{Dd,A} = \frac{i\Theta}{\Omega_{D-(4k+3)}}n \underbrace{dn \wedge ... \wedge dn}_{D-(4k+3)} \wedge \frac{K^{IJ}}{8\pi^2}C^I \wedge dC^J.
\tag{7}
$$

$$
\mathcal{L}_{Dd,B} = \frac{i\Theta}{\Omega_{D-(4k+5)}}n \underbrace{dn \wedge ... \wedge dn}_{D-(4k+5)} \wedge \frac{(i\sigma^y)^{IJ}}{8\pi^2}B^I \wedge dB^J,
\tag{8}
$$

where the component n_1 does not transform under any symmetry group, but the rest of the components all transform nontrivially. The equivalence between the two descriptions above can be made explicit by parametrizing $\vec{n}$ as: $\vec{n} = (\cos\theta, \sin\theta n_2, \sin\theta n_3, \cdots)$, then following the derivation in Eq. 3, because the desired BSPT state is fully symmetric, $\langle\theta\rangle$ must be either 0 or π, which corresponds to the trivial state and nontrivial BSPT state respectively. And with $\langle\theta\rangle = \pi$, Eq. 7,8 return to Eq. 5 and 6.

All the terms above are "topological" in the sense that they are invariant under local coordinate transformation, because they do not involve the metric. We only wrote down the most important topological terms explicitly, but the readers should be reminded that there are other terms that guarantee the system is in a fully gapped and nondegenerate phase. For example, we need a term $1/g(\partial_\mu \vec{n})^2$ in the field theory to control the dynamics of $\vec{n}$, and we must keep g large enough to disorder $\vec{n}$; we also need a BF theory term[15] $\sim (d\mathcal{B})^2 + \frac{1}{2\pi}\mathcal{B} \wedge dC$ to gap out all the excitations of the C^I field.

Naively, we can also write down the following field theory, with all $\vec{n}$ components transforming non-trivially under symmetry:

$$
\mathcal{L} = \frac{i\Theta}{\Omega}ndn \wedge ... \wedge dn \wedge \frac{K^{IJ}}{8\pi^2}C^I \wedge dC^J.
$$

For example, we can write down such field theory in $3+1d$ space-time, with $\vec{n}$ being an $O(2)$ vector, and C^I a one form vector gauge field. Then the physical meaning of this field theory is that, the vortex core of $\vec{n}$ will host the boundary state of the $2d$ E_8 state, which must be gapless. Then this means that we can never achieve a fully gapped nondegenerate state by proliferating the vortex loops. Thus this field theory will always be gapless, unless we explicitly break the $U(1)$ symmetry of $\vec{n}$. Therefore this field theory describes the boundary of a $4d$ space, rather than a $3d$ bulk state.

For field theories in Eq. 5 and 6, in general we consider fixed points $\Theta = 2\pi p$ with $p \in \mathbb{Z}$. However, this does not mean that we have a $\mathbb{Z}$ classified state. If we can show that two field theories, $\Theta = 0$ and $\Theta = 2\pi q$ with certain $q \in \mathbb{Z}$, can be smoothly connected without closing the bulk gap, then they must be in the same phase. In that case, the classification will be reduced to $\mathbb{Z}_q$.

Because our field theory is constructed with order parameter $\vec{n}$ and Chern-Simons form of C^I or B^I, the classification will depend on both sectors.

For pure $C \wedge dC$ theory, the classification is $\mathbb{Z}$, because its boundary state has perturbative gravitational anomaly [6, 8], which has $\mathbb{Z}$ classification. Then the classification of the mixed field theory of $\vec{n}$ and C^I only depends on the $\vec{n}$ sector.

For instance, we can take Eq. 3 as an example. Take two copies of the field theories and couple them to each other:

$$\mathcal{L} = \frac{i2\pi}{2\pi} n_{(1)} dn_{(1)} \wedge \frac{K_{E_8}^{IJ}}{8\pi^2} C_{(1)}^I \wedge dC_{(1)}^J + (1 \to 2)$$
$$+ \beta n_{2,(1)} \cdot n_{2,(2)} + \lambda dC_{(1)}^I \wedge \star dC_{(2)}^I, \quad (9)$$

where $\star$ is the Hodge star operator. Now we fix λ at a negative value, and tune β from negative to positive. With negative β, effectively $\vec{n}_{(1)}$ and $\vec{n}_{(2)}$ will align with each other, thus $n_{2,(1)} = n_{2,(2)}$, $C_{(1)} = C_{(2)}$, then the two theories will "constructively interfere" with each other, and the final theory effectively has $\Theta = 4\pi$; with positive β, effectively $n_{2,(1)} = -n_{2,(2)}$, $C_{(1)} = C_{(2)}$, thus the two theories will "destructively interfere" with each other, and the final theory effectively has $\Theta = 0$. Because both theories are fully gapped and nondegenerate in the bulk, tuning the coupling between them does not close the bulk gap (as long as the coupling is not too strong to overcome the bulk gap), thus the two effective coupled theories with $\Theta = 0$ and $\Theta = 4\pi$ are smoothly connected without going through a bulk phase transition. Therefore the classification for the state Eq. 3 is $\mathbb{Z}_2$.

By contrast, let us consider a $U(1)$ BSPT in $4d$ space with the following field theory:

$$\mathcal{L}_{4+1d}^{U(1)} = \frac{i2\pi}{2\pi} n dn \wedge \frac{K_{E_8}^{IJ}}{8\pi^2} dC^I \wedge dC^J. \quad (10)$$

The $U(1)$ symmetry acts as $U(1): (n_1 + in_2) \to e^{i\phi}(n_1 + in_2)$. Imagine we have two copies of the theory, the only $U(1)$ symmetry allowed coupling between these two theories would be $\beta \vec{n}_{(1)} \cdot \vec{n}_{(2)}$. Then for either sign of β, $i.e.$ for either $\vec{n}_{(1)} \sim \vec{n}_{(2)}$ or $\vec{n}_{(1)} \sim -\vec{n}_{(2)}$, the final effective theory always has $\Theta = 4\pi$ (simply because $(-1)^2 = +1$). Thus there is no symmetry allowed coupling that can continuously connect $\Theta = 4\pi$ to $\Theta = 0$. Therefore the classification for this $U(1)$ BSPT state is $\mathbb{Z}$.

For pure $B \wedge dB$ theory, the classification is $\mathbb{Z}_2$ [9], therefore the classification of the mixed state can only be $\mathbb{Z}_2$ or trivial depending on the classification on the $\vec{n}$ sector.

Symmetry	$3+1d$	$4+1d$	$5+1d$	$6+1d$
$U(1)$	0	$\mathbb{Z}$	0	$\mathbb{Z} \times \mathbb{Z}_2$
Z_2	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2^2$
Z_2^T	$\mathbb{Z}_2$	0	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^2$
$U(1) \rtimes Z_2^T$	$\mathbb{Z}_2$	$\mathbb{Z}$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^3$ ($\mathbb{Z}_2^4$)
$U(1) \times Z_2^T$	$\mathbb{Z}_2$	0	$\mathbb{Z}_2^3$	$\mathbb{Z}_2^2$ ($\mathbb{Z}_2^3$)
$U(1) \rtimes Z_2$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2^2$	$\mathbb{Z} \times \mathbb{Z}_2^3$
$U(1) \times Z_2$	0	$\mathbb{Z} \times \mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z} \times \mathbb{Z}_2^4$

TABLE I: BSPT beyond Group Cohomology constructed from decorated topological defects. Please note that the states within group cohomology classification is not listed here. The case for $U(1) \times Z_2$ symmetry was not discussed in Ref. 14. Our results largely agree with Ref. 14. The results in Ref. 14 that do not fully agree with ours are highlighted in red.

3. EXAMPLES OF BSPT BEYOND GROUP COHOMOLOGY

In this section we study examples of beyond-group-cohomology BSPT states with various symmetries up to $6 + 1$ space-time dimensions. All these states are constructed with Landau order parameters and the $2d$ E_8 state or the $4d$ BSRE state in Eq. 2. Our results are summarized in TABLE I. Our results are mostly consistent with results in Ref. 14, exceptions are highlighted in red in the table.

A. $U(1)$ Symmetry

• In $4d$ space, there is a series of BSPT states with $U(1)$ symmetry that is beyond the group cohomology, their field theory is given by:

$$\mathcal{L}_{4+1d}^{U(1)} = \frac{i2\pi k}{2\pi} n dn \wedge \frac{K_{E_8}^{IJ}}{8\pi^2} dC^I \wedge dC^J, \quad (11)$$

where C^I's are rank-1 gauge field, and k can take arbitrary integer value. Physically this state can be viewed as decorating the 2π vortex of $U(1)$ order parameter $\vec{n} = (n_1, n_2)$ (which is a $2d$ membrane in this dimension) with the E_8 state, and then proliferating the vortices. As we have shown in the previous section, this phase has $\mathbb{Z}$ classification.

The $3 + 1d$ boundary of this state can be a superfluid phase with spontaneously U(1) symmetry breaking, whose vortex line hosts the edge states of the $2d$ E_8 state, $i.e.$ a chiral conformal field theory with central charge $c = 8$. If we couple $\vec{n}$ to a $U(1)$ gauge field, then after we integrate out the gapped matter field $\vec{n}$, the boundary of the system will have a mixed $U(1)$-gravitational anomaly, namely the stress tensor of the system is no longer conserved inside the $U(1)$ flux at the boundary. A

similar mixed gauge-gravity anomaly was also studied in Ref. 17.

• In $6 + 1d$ space-time, there are two root states for $U(1)$ BSPT states beyond group cohomology, the first state is described by the following field theory:

$$\mathcal{L}_{6+1d,A}^{U(1)} = \frac{i2\pi k}{12\pi^2} ndn \wedge dn \wedge dn \wedge \frac{K_{E_8}^{IJ}}{8\pi^2} dC^I \wedge dC^J \quad (12)$$

with

$$U(1): \quad (n_1 + in_2) \rightarrow e^{i\phi}(n_1 + in_2),$$
$$(n_3 + in_4) \rightarrow e^{i\phi}(n_3 + in_4). \quad (13)$$

This state has $\mathbb{Z}$ classification. The state is constructed by decorating the E_8 states on the intersection of two $U(1)$ vortices, and then proliferate the vortices (the two-vortex intersection is now a $2d$ brane in $6d$ space).

The field theory of the second root phase is

$$\mathcal{L}_{6+1d,B}^{U(1)} = \frac{i2\pi}{2\pi} ndn \wedge \frac{(i\sigma^y)^{IJ}}{8\pi^2} dB^I \wedge dB^J \quad (14)$$

where

$$U(1): (n_1 + in_2) \rightarrow e^{i\phi}(n_1 + in_2) \quad (15)$$

and B's are 2-form fields. The state has $\mathbb{Z}_2$ classification according to our rules. And physically this field theory corresponds to decorating the $U(1)$ vortex with the $4d$ BSRE state in Eq. 2.

B. Z_2 Symmetry

• In $4 + 1d$ space-time, there is one nontrivial beyond-cohomology BSPT state with Z_2 symmetry, and this state is the descendant of the $U(1)$ beyond Group Cohomology state in the same dimension in the sense that it can be obtained by breaking the $U(1)$ symmetry to its subgroup Z_2 from Eq. 11:

$$\mathcal{L}_{4+1d}^{Z_2} = \frac{i2\pi}{2\pi} ndn \wedge \frac{K_{E_8}^{IJ}}{8\pi^2} dC^I \wedge dC^J \quad (16)$$

with

$$Z_2: (n_1, n_2) \rightarrow -(n_1, n_2) \quad (17)$$

while the classification of the state is now reduced to $\mathbb{Z}_2$ because the n-sector is now $\mathbb{Z}_2$ classified.

• In $5 + 1d$ space-time, there is a $\mathbb{Z}_2$ classified new state which is not a descendant of any $U(1)$ state discussed in the previous subsection. Physically this state is constructed by decorating the Z_2 domain wall with $4d$ BSRE state:

$$\begin{aligned}\mathcal{L}_{5+1d}^{Z_2} &= \frac{i2\pi}{2\pi} ndn \wedge \frac{(i\sigma^y)^{IJ}}{8\pi^2} B^I \wedge dB^J \\ &= id\theta \wedge \frac{(i\sigma^y)^{IJ}}{8\pi^2} B^I \wedge dB^J \\ &= -i\theta \frac{(i\sigma^y)^{IJ}}{8\pi^2} dB^I \wedge dB^J \end{aligned} \quad (18)$$

Here we parametrize $\vec{n}$ as $\vec{n} = (\cos\theta, \sin\theta)$. The symmetry transformation is:

$$\begin{aligned} Z_2: \quad &(n_1, n_2) \rightarrow (n_1, -n_2) \\ &(B^1, B^2) \rightarrow (B^2, B^1) \\ &\theta \rightarrow -\theta. \end{aligned} \quad (19)$$

Notice that B^I must transform nontrivially under Z_2 symmetry, in order to guarantee that the field theory is Z_2 invariant. We can also choose a different transformation for B^I: $Z_2: B \rightarrow \sigma^z B$, but this transformation is equivalent to the previous after a basis change. In a Z_2 invariant state, $\langle n_2 \rangle = 0$, i.e. $\langle \theta \rangle = 0$ or π, which corresponds to the trivial and BSPT state respectively.

• In $6 + 1d$ space-time, there are two root states, both of which are descendants of $U(1)$ BSPT states, and both have $\mathbb{Z}_2$ classification:

$$\mathcal{L}_{6+1d,A}^{Z_2} = \frac{i2\pi}{12\pi^2} ndn \wedge dn \wedge dn \wedge \frac{K_{E_8}^{IJ}}{8\pi^2} dC^I \wedge dC^J \quad (20)$$

with

$$Z_2: (n_1, n_2, n_3, n_4) \rightarrow -(n_1, n_2, n_3, n_4). \quad (21)$$

$$\mathcal{L}_{6+1d,B}^{Z_2} = \frac{i2\pi}{2\pi} ndn \wedge \frac{(i\sigma^y)^{IJ}}{8\pi^2} dB^I \wedge dB^J \quad (22)$$

with

$$Z_2: (n_1, n_2) \rightarrow -(n_1, n_2). \quad (23)$$

C. Z_2^T Symmetry

• In $3 + 1d$ space-time, it is well-known that there is a BSPT state beyond Group Cohomology [15]. The state can be understood by decorating Z_2^T domain walls with the $2d$ E_8 state:

$$\begin{aligned}\mathcal{L}_{3+1d}^{Z_2^T} &= \frac{i2\pi}{2\pi} ndn \wedge \frac{K_{E_8}^{IJ}}{8\pi^2} C^I \wedge dC^J \\ &= -i\theta \frac{K_{E_8}^{IJ}}{8\pi^2} dC^I \wedge dC^J \end{aligned} \quad (24)$$

with

$$\begin{aligned} Z_2^T: \quad &(n_1, n_2) \rightarrow (n_1, -n_2) \\ &\theta \rightarrow -\theta. \end{aligned} \quad (25)$$

θ is defined as before, $\langle \theta \rangle = 0$ and π correspond to the trivial and BSPT state respectively. This state has $\mathbb{Z}_2$ classification.

• In $5 + 1d$ space-time, there are two root states, both have $\mathbb{Z}_2$ classification. The field theory for the first state reads:

$$\mathcal{L}_{5+1d,A}^{Z_2^T} = \frac{i2\pi}{8\pi} ndn \wedge dn \wedge \frac{K_{E_8}^{IJ}}{8\pi^2} dC^I \wedge dC^J \quad (26)$$

with

$$Z_2^T : (n_1, n_2, n_3) \rightarrow -(n_1, n_2, n_3). \quad (27)$$

The physical meaning of this state is most transparent if we start with a system with an enlarged $SO(3) \times Z_2^T$ symmetry, and $\vec{n}$ forms a vector under the $SO(3)$ symmetry. Then Eq. 26 can be viewed as decoration of the hedgehog monopole of $\vec{n}$ with the $2d$ E_8 state. Weakly breaking the $SO(3)$ symmetry while preserving the Z_2^T symmetry does not change the nature of this state. Alternatively, we can view the hedgehog monopole as the intersection of three Z_2^T domain walls.

The field theory for the second root state is

$$\mathcal{L}_{5+1d,B}^{Z_2^T} = \frac{i2\pi}{2\pi} ndn \wedge \frac{(i\sigma^y)^{IJ}}{8\pi^2} B^I \wedge dB^J$$
$$= -i\theta \frac{(i\sigma^y)^{IJ}}{8\pi^2} dB^I \wedge dB^J \quad (28)$$

with

$$Z_2^T : \quad (n_1, n_2) \rightarrow (n_1, -n_2)$$
$$\theta \rightarrow -\theta. \quad (29)$$

This state can be viewed as decoration of Z_2^T domain wall with the $4d$ BSRE state.

• In $6 + 1d$ space-time, there is one new state with $\mathbb{Z}_2$ classification:

$$\mathcal{L}_{6+1d}^{Z_2^T} = \frac{i2\pi}{2\pi} ndn \wedge \frac{(i\sigma^y)^{IJ}}{8\pi^2} dB^I \wedge dB^J \quad (30)$$

with

$$Z_2^T : \quad (n_1, n_2) \rightarrow -(n_1, n_2)$$
$$(B^1, B^2) \rightarrow (B^2, B^1). \quad (31)$$

The state is constructed by decorating the vortex of $\vec{n}$ (or the intersection of two Z_2^T domain walls) with the $4d$ BSRE state.

D. $U(1) \rtimes Z_2^T$ Symmetry

• In $3 + 1d$ space-time, there is one nontrivial beyond-cohomology BSPT state with $U(1) \rtimes Z_2^T$ symmetry, but it is identical to the Z_2^T state in the same dimension, $U(1)$ symmetry simply acts trivially.

• In $4 + 1d$ space-time, there is one root state with $\mathbb{Z}$ classification:

$$\mathcal{L}_{4+1d}^{U(1) \rtimes Z_2^T} = \frac{i2\pi k}{2\pi} ndn \wedge \frac{K_{E_8}^{IJ}}{8\pi^2} dC^I \wedge dC^J. \quad (32)$$

with

$$U(1) : \quad (n_1 + in_2) \rightarrow e^{i\phi}(n_1 + in_2)$$
$$Z_2^T : \quad (n_1, n_2) \rightarrow (n_1, -n_2). \quad (33)$$

• In $5 + 1d$ space-time, there are two root states, both are identical to the Z_2^T state in the same dimension with trivial $U(1)$ symmetry transformation, and both are $\mathbb{Z}_2$ classified.

• In $6 + 1d$ space-time, in Ref. 14 there are *four* root states, all $\mathbb{Z}_2$ classified. However, we can only find *three* $\mathbb{Z}_2$ classified root states by our construction. The first one is identical to the Z_2^T state in $6 + 1d$. The other two root states are given by:

$$\mathcal{L}_{6+1d,A}^{U(1) \rtimes Z_2^T} = \frac{i2\pi}{12\pi^2} ndn \wedge dn \wedge dn \wedge \frac{K_{E_8}^{IJ}}{8\pi^2} dC^I \wedge dC^J \quad (34)$$

with

$$U(1) : \quad (n_1 + in_2) \rightarrow e^{i\phi}(n_1 + in_2),$$
$$Z_2^T : \quad (n_1, n_2, n_3, n_4) \rightarrow (n_1, -n_2, -n_3, -n_4) \quad (35)$$

and

$$\mathcal{L}_{6+1d,B}^{U(1) \rtimes Z_2^T} = \frac{i2\pi}{2\pi} ndn \wedge \frac{(i\sigma^y)^{IJ}}{8\pi^2} dB^I \wedge dB^J \quad (36)$$

with

$$U(1) : \quad (n_1 + in_2) \rightarrow e^{i\phi}(n_1 + in_2),$$
$$Z_2^T : \quad (n_1, n_2) \rightarrow (n_1, -n_2) \quad (37)$$

We suspect the state we missed here is the mixed SPT state described by $E^d(G)$ in Ref. 14.

E. $U(1) \times Z_2^T$ Symmetry

• In $3 + 1d$ space-time, there is a state identical to the pure Z_2^T state with trivial $U(1)$ symmetry transformation.

• In $5 + 1d$ space-time, we find three $\mathbb{Z}_2$ classified root states. Two of them are identical to the Z_2^T states in $5 + 1d$ space-time, with trivial $U(1)$ symmetry transformation. The third state is given by:

$$\mathcal{L}_{5+1d}^{U(1) \times Z_2^T} = \frac{i2\pi}{8\pi} ndn \wedge dn \wedge \frac{K_{E_8}^{IJ}}{8\pi^2} dC^I \wedge dC^J \quad (38)$$

with

$$U(1) : \quad (n_1 + in_2) \rightarrow e^{i\phi}(n_1 + in_2),$$
$$Z_2^T : \quad (n_1, n_2, n_3) \rightarrow (n_1, n_2, -n_3). \quad (39)$$

This state can be viewed as decorating the $2d$ E_8 state on the intersection of a Z_2^T domain wall and a $U(1)$ vortex (it can also be viewed as the hedgehog monopole of $\vec{n}$), then proliferating both the domain walls and vortices.

• In $6 + 1d$ space-time, in Ref. 14 there are *three* $\mathbb{Z}_2$ classified root states. However, using our method we can only construct *two* $\mathbb{Z}_2$ classified root states. The first one

is identical to the Z_2^T state with trivial $U(1)$ symmetry transformation. The other one is:

$$\mathcal{L}_{6+1d}^{U(1)\times Z_2^T} = \frac{i2\pi}{2\pi}ndn \wedge \frac{(i\sigma^y)^{IJ}}{8\pi^2}dB^I \wedge dB^J \qquad (40)$$

with

$$\begin{aligned} U(1): \quad & (n_1 + in_2) \to e^{i\phi}(n_1 + in_2), \\ Z_2^T: \quad & (n_1, n_2) \to -(n_1, n_2), \quad B^{1(2)} \to B^{2(1)}. \end{aligned} \qquad (41)$$

One may ask whether field theory like Eq. 34 could correspond to a new root state. However, there is no consistent way to assign the U(1)$\times Z_2^T$ symmetry transformations on Eq. 34, namely Eq. 34 cannot be invariant under U(1)$\times Z_2^T$ symmetry, although it is invariant under U(1)$\rtimes Z_2^T$ symmetry.

F. $U(1) \rtimes Z_2 = O_2$ Symmetry

• In $4 + 1d$ space-time, there is one root state identical to the BSPT state with Z_2 symmetry in the same dimension, the $U(1)$ symmetry simply acts trivially.

• In $5 + 1d$ space-time, there are two root states, both $\mathbb{Z}_2$ classified. One is the same Z_2 state with trivial $U(1)$ action. The other one is given by:

$$\mathcal{L}_{5+1d}^{U(1)\rtimes Z_2} = \frac{i2\pi}{8\pi}ndn \wedge dn \wedge \frac{K_{E_8}^{IJ}}{8\pi^2}dC^I \wedge dC^J \qquad (42)$$

with

$$\begin{aligned} U(1): \quad & (n_1 + in_2) \to e^{i\phi}(n_1 + in_2), \\ Z_2: \quad & (n_1, n_2, n_3) \to (n_1, -n_2, -n_3). \end{aligned} \qquad (43)$$

This state can be viewed as decorating the $2d$ E_8 state on the intersection of $U(1)$ vortex and Z_2 domain wall. Also, the O_2 symmetry is a subgroup of $SO(3)$ symmetry, thus the vortex-domain wall intersection is simply the hedgehog monopole of the $SO(3)$ vector $\vec{n}$.

• In $6 + 1d$ space-time, we find *four* root states, which is *more* than the results in Ref. 14. Two of them are the same as the BSPT states with Z_2 symmetry, both of which are $\mathbb{Z}_2$ classified. The third root state is described by

$$\mathcal{L}_{6+1d,A}^{U(1)\rtimes Z_2} = \frac{i2\pi k}{12\pi^2}ndn \wedge dn \wedge dn \wedge \frac{K_{E_8}^{IJ}}{8\pi^2}dC^I \wedge dC^J \qquad (44)$$

with

$$\begin{aligned} U(1): \quad & (n_1 + in_2) \to e^{i\phi}(n_1 + in_2), \\ & (n_3 + in_4) \to e^{i\phi}(n_3 + in_4), \\ Z_2: \quad & (n_1, n_2, n_3, n_4) \to (n_1, -n_2, n_3, -n_4) \end{aligned} \qquad (45)$$

This state is $\mathbb{Z}$ classified. This state can be viewed as decorating the $2d$ E_8 state on the intersection of two vortices, then proliferate the vortices afterwards.

The last root state in $6 + 1d$ space-time is described by

$$\mathcal{L}_{6+1d,B}^{U(1)\rtimes Z_2} = \frac{i2\pi}{2\pi}ndn \wedge \frac{(i\sigma^y)^{IJ}}{8\pi^2}dB^I \wedge dB^J \qquad (46)$$

with

$$\begin{aligned} U(1): \quad & (n_1 + in_2) \to e^{i\phi}(n_1 + in_2), \\ Z_2: \quad & (n_1, n_2) \to (n_1, -n_2), \\ & (B^1, B^2) \to (B^2, B^1). \end{aligned} \qquad (47)$$

This state has $\mathbb{Z}_2$ classification.

G. $U(1) \times Z_2$ Symmetry

• In $4 + 1d$ space-time, we have two root states, both of which are descendants from pure $U(1)$ state and pure Z_2 state respectively.

• In $5 + 1d$ space-time, there is only one root state, which is the same as the state with Z_2 symmetry only.

• In $6 + 1d$ space-time, there are five root states. The first three states can all be described by the same field theory:

$$\mathcal{L}_{6+1d,A}^{U(1)\times Z_2} = \frac{i2\pi k}{12\pi^2}ndn \wedge dn \wedge dn \wedge \frac{K_{E_8}^{IJ}}{8\pi^2}dC^I \wedge dC^J \qquad (48)$$

These three different states have the same form of Lagrangian, but they are distinguished from each other by their symmetry transformations:

$$\begin{aligned} (1) \quad U(1): \quad & trivial, \\ Z_2: \quad & (n_1, n_2, n_3, n_4) \to -(n_1, n_2, n_3, n_4). \end{aligned} \qquad (49)$$

$$\begin{aligned} (2) \quad U(1): \quad & (n_1 + in_2) \to e^{i\phi}(n_1 + in_2), \\ Z_2: \quad & (n_1, n_2, n_3, n_4) \to -(n_1, n_2, n_3, n_4). \end{aligned} \qquad (50)$$

$$\begin{aligned} (3) \quad U(1): \quad & (n_1 + in_2) \to e^{i\phi}(n_1 + in_2), \\ & (n_3 + in_4) \to e^{i\phi}(n_3 + in_4), \\ Z_2: \quad & (n_1, n_2, n_3, n_4) \to -(n_1, n_2, n_3, n_4). \end{aligned} \qquad (51)$$

The classification of the three states are $\mathbb{Z}_2$, $\mathbb{Z}_2$ and $\mathbb{Z}$ respectively.

The other two states are described by the following field theory:

$$\mathcal{L}_{6+1d,B}^{U(1)\times Z_2} = \frac{i2\pi}{2\pi}ndn \wedge \frac{(i\sigma^y)^{IJ}}{8\pi^2}dB^I \wedge dB^J \qquad (52)$$

again, these two states have different transformations under symmetry groups:

$$\begin{aligned} (4) \quad U(1): \quad & trivial, \\ Z_2: \quad & (n_1, n_2) \to -(n_1, n_2). \end{aligned} \qquad (53)$$

$$\begin{aligned} (5) \quad U(1): \quad & (n_1 + in_2) \to e^{i\phi}(n_1 + in_2), \\ Z_2: \quad & trivial. \end{aligned} \qquad (54)$$

The classification of the two states are both $\mathbb{Z}_2$.

4. SUMMARY

In this work, we construct field theories of beyond-Group-Cohomology BSPT states based on decorated topological defect picture. Our results are largely consistent with Ref. 14, with a few exceptions. We listed examples of BSPT states below six dimensional space, but our construction can be straightforwardly generalized to all higher dimensions, as long as we use the generalized base states in Eq. 1,2 for $k \geq 1$. We also note that Ref. 17 proposed the SPT states beyond group cohomology should have mixed gauge-gravity responses, which is also consistent with the formalism used in our current work.

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[18] In this paper we define short range entangled state as systems with gapped and nondegenerate bulk spectrum, namely it has no topological entanglement entropy. In our definition, SRE states include SPT states as a subset.
[19] Throughout the paper, the term "space" always refers to the real physical space. The phrase space-time will always be written explicitly.