Two-leg SU(2n) spin ladder: A low-energy effective field theory approach

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Two-leg SU($2n$) spin ladder: A low-energy effective field theory approach

P. Lecheminant[1] and A. M. Tsvelik[2]
[1]*Laboratoire de Physique Théorique et Modélisation, CNRS UMR 8089, Université de Cergy-Pontoise, Site de Saint-Martin, 2 avenue Adolphe Chauvin, 95302 Cergy-Pontoise Cedex, France.*
[2]*Department of Condensed Matter Physics and Materials Science, Brookhaven National Laboratory, Upton, NY 11973-5000, USA*
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We present a field theory analysis of a model of two SU($2n$)-invariant magnetic chains coupled by a generic interaction preserving time reversal and inversion symmetry. Contrary to the SU(2)-invariant case the zero-temperature phase diagram of such two-leg spin ladder does not contain topological phases. Only generalized Valence Bond Solid phases are stabilized when $n > 1$ with different wave vectors and ground-state degeneracies. In particular, we find a phase which is made of a cluster of $2n$ spins put in an SU($2n$) singlet state. For $n = 3$, this cluster phase is relevant to ^{173}Yb ultracold atoms, with an emergent SU(6) symmetry, loaded in double-well optical lattice.

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I. INTRODUCTION

Interacting quantum systems with symmetries higher than the ubiquitous SU(2) one may display new interesting physics.[1] For instance, chiral spin liquids with non-Abelian statistics might emerge in quantum magnets with an extended SU(N) symmetry.[2] Although high symmetries are more difficult to maintain, there are several interesting possibilities of their realization. For instance, in their recent preprint Kugel *et.al.*[3] suggest condensed matter realizations of the antiferromagnetic chains with high symmetry, such as SU(4) in the context of orbital degeneracy.[4–8] The second possibility is related to alkaline-earth or ytterbium ultracold atoms which have a peculiar energy spectrum. The ground state and a metastable first excited state have zero electronic angular momentum so that the nuclear spin I is almost decoupled from the electronic spin. This decoupling paves the way to the experimental realization of magnets with an emergent SU(N) symmetry where $N = 2I + 1$ is the number of nuclear states which can be as large as 10 with strontium and ytterbium atoms.[9–11] Recent experiments with such atoms loading into optical lattices have directly observed the existence of the SU(N) symmetry and have determined the specific form of the interactions between atoms in the ground state and the excited state.[12,13]

The simplest lattice SU(N) model stems from alkaline-earth atoms in their ground state loaded into the lowest band of an 1D optical lattice with a filling of one atom per site which best avoids three-body looses. For large repulsive interaction, the resulting SU(N) symmetric magnet is described by the Hamiltonian:

$$\mathcal{H} = J \sum_n \hat{P}_{n,n+1}, \tag{1}$$

where $\hat{P}_{n,n+1} = \sum_{A=1}^{N^2-1} S_n^A S_{n+1}^A$+const, with S^A being generators of the su(N) Lie algebra, is a permutation operator acting in the tensor product of the N-dimensional Hilbert spaces n and $n+1$. Model (1) is integrable by means of the Bethe-Ansatz approach.[14] For $J > 0$ its excitation spectrum is gapless and at small energies the dispersion is linear. Hence the theory is critical with $N - 1$ gapless relativistic modes; its universality class is the SU(N)$_1$ Wess-Zumino-Novikov-Witten (WZNW) model with central charge $c = N - 1$.[15,16]

When one considers chains coupled by generic interactions the integrability is lost though at the same time the physics becomes more interesting, as one can learn from the example of the problem of two coupled $S = 1/2$ Heisenberg chains.[17–21] The latter model has a phase diagram which includes topological phases containing seeds of superconductivity.[22–27] It also may used to describe experimentally observed confinement of fractional quantum number excitations existing for the single chain problem.[26,28]

In this paper we consider a one-dimensional version of the famous Kugel-Khomskii model,[29] namely a model of two weakly coupled SU(N) chains (labeled $1, 2$):

$$\mathcal{H} = \sum_n \left[\hat{P}^{(1,1)}_{n,n+1} + \hat{P}^{(2,2)}_{n,n+1} + \lambda \hat{V}^{(1,2)}_{n,n} \right], \tag{2}$$

where $\hat{P}^{(a,a)}$ are permutation operators acting on states on chain number $a = 1, 2$. In the simplest case the interaction $\hat{V}$ is just a permutation operator acting between sites of different chains. However, to select for our analysis any specific form of the interaction would be too restrictive. Instead we will use an alternative approach, namely, we will consider weakly coupled SU(N) chains with general interaction, which may go beyond the two-spin exchange. As the result, the problem becomes then a perturbed conformal field theory (CFT) with Hamiltonian:

$$\mathcal{H} = W[SU(N)_1]_1 + W[SU(N)_1]_2 + \sum_{a,b} \lambda_{ab} \int dx\ \mathcal{O}_{1,a}(x)\mathcal{O}_{2,b}(x), \tag{3}$$

where $W[SU(N)_1]_{1,2}$ corresponds to the WZNW Hamiltonian for SU(N)$_1$ CFT on the corresponding chain and $\mathcal{O}_{a,1}$ and $\mathcal{O}_{2,b}$ are operators acting on states of chains 1,2. In the weak coupling limit $|\lambda_{ab}| << J$ one may take into account only relevant perturbations which greatly restricts the number of possible operators. On top of the SU(N) symmetry, our choice will be further restricted by considering only (i) spatially homogeneous ladders and (ii) ladders with inversion symmetry. The latter consideration excludes operators with non-zero conformal spin. In this way a multidimensional phase diagram of the lattice model is projected on a manageable (in fact, in most cases just two-dimensional) phase diagram of the perturbed CFT.

We will investigate the infrared (IR) physics of model (3) when $N = 2n$. Our main conclusions are that the physics of the SU($2n$) ladder with $n > 1$ in some crucial aspects is different from the SU(2) case. In particular, there are no topological non-degenerate phases. All phases contain local order parameters corresponding to Valence Bond Solids (VBS). There are crystals of two types: with wave vectors $(\pi/n, 0)$ or $(\pi/n, \pi)$ ($2k_F$ VBS) and $(2\pi/n, 0)$ ($4k_F$ VBS). The latter possibility did not exist for the spin $S = 1/2$ ladder. It can be viewed as a cluster of $2n$ spins put in an SU($2n$) singlet state. In the simplest case, i.e., $N = 4$, this cluster corresponds to the plaquette phase found in the numerical analysis of Ref. 30 of the SU(4) two-leg spin ladder with an antiferromagnetic interchain coupling. For $N = 6$, we find the emergence of a cluster phase of six spins, leading to trimerization which should occur in the phase diagram of the two-leg SU(6) spin ladder. The latter case is directly relevant to the insulating phase of double tube of ytterbium ^{173}Yb ultracold atoms.

The rest of the paper is organized as follows. In Sec. II, the continuum limit of weakly-coupled SU($2n$) two-leg spin ladder is presented to identify the leading perturbation in Eq. (3). The result is the continuum model (9). In Sec. III this model is analyzed using the conformal embedding approach. As a result model (9) is expressed as a theory of $\mathbb{Z}_N$ parafermions coupled to SU(N)$_2$ WZNW via some relevant operator (27). This new formulation allows us to determine the nature of the phases. The resulting analysis is provided in Sec. IV and our concluding remarks are given in Sec. V. Finally, our paper is supplied with several appendices where some additional technical details are described.

II. THE CONTINUUM LIMIT

In this section, we determine the leading perturbation of model (3) by means of a continuous description of two-leg SU($2n$) spin ladder with generic interactions.

Let us start with the decoupling limit where the lattice model (2) reduces to two decoupled Sutherland models (1). Its low-energy properties can be obtained by starting from the U(N) Hubbard model at $1/N$ filling with large repulsive U interaction.[15,16,31,32] At energies below the charge gap $\sim U$, the SU(N) spin operators in the continuum limit are described by:[15,16,31]

$$S_l^A \simeq J_{lL}^A(x) + J_{lR}^A(x) + \mathrm{e}^{\mathrm{i}2k_F x} N_l^A(x) + \mathrm{e}^{-\mathrm{i}2k_F x} N_l^{A\dagger}(x) + \mathrm{e}^{\mathrm{i}4k_F x} n_l^A(x) + .., \tag{4}$$

where $l = 1, 2$ denotes the two decoupled chains and $k_F = \pi/Na_0$ (a_0 being the lattice spacing) since the underlying Hubbard model is $1/N$ filled (1 electron per site). In Eq. (4), $J_{lL,R}^A$ are the left and right currents which generate the SU(N)$_1$ CFT. They are defined in terms of the underlying left- and right moving Dirac fermions $L_{l\alpha}, R_{l\alpha}$ as

$$J_{lR}^A = R_{l\alpha}^\dagger T_{\alpha\beta}^A R_{l\beta},\ \ J_{lL}^A = L_{l\alpha}^\dagger T_{\alpha\beta}^A L_{l\beta}, \tag{5}$$

where a summation over repeated greek indices (SU(N) indices) is implied. In Eq. (5), T^A is a generator of the su(N) Lie algebra in the fundamental representation, the $2k_F$ and $4k_F$ parts of the spin density are related to primary fields of the SU(N)$_1$ WZNW model. They transform respectively in the N-dimensional fundamental and the $N(N-1)/2$-dimensional antisymmetric representation of the SU(N) group. In particular, we have

$$N_l^A = \lambda\ \mathrm{Tr}(g_l T^A), \tag{6}$$

λ being a constant, related to the charge degrees of freedom, which can be chosen real for a matter of convenience. In Eq. (6), g_l is the SU(N)$_1$ primary field, or WZNW field, for the l th chain with scaling dimension $(N-1)/N$. This operator transforms in the N-dimensional fundamental representation of SU(N) and can be expressed in terms of the fermionic operators through the non-Abelian bosonization approach:[15,33,34]

$$g_{l\beta\alpha} \sim \mathrm{e}^{-i\sqrt{4\pi/N}\Phi_{lc}} L^{\dagger}_{l\alpha} R_{l\beta}, \tag{7}$$

Φ_{lc} being a bosonic field which captures the properties of the charge degrees of freedom of each chain $l = 1, 2$.

From the continuum representation (4), we derive a transformation of the SU(N)$_1$ WZNW fields with respect to the one-step translation symmetry T_{a_0}:

$$T_{a_0}: \quad g_{1,2} \rightarrow \mathrm{e}^{2i\pi/N} g_{1,2}. \tag{8}$$

The next step of the approach is to find the leading perturbation in Eq. (3). One can identify it by means a of a symmetry analysis. To this end, let us recall what symmetry restrictions we adopt. First, we have the SU(N) symmetry: $g_l \rightarrow U g_l U^{\dagger}$, with U belonging to SU(N). Second, we consider the spatially uniform model. Then since matrix operator g_l is not invariant with respect to translations (8) the perturbation can include only products of $g_1 g_2^{+}$ (or $g_1^{+} g_2$). From the inversion symmetry it follows that there are no operators with nonzero conformal spin such as, for example, $\mathrm{Tr} g_1^{+} \partial_x g_2$. These considerations yield the following Hamiltonian:

$$\mathcal{H} = W[SU(N)_1; g_1] + W[SU(N)_1; g_2] + \lambda_1 \int dx \left[\mathrm{Tr}(g_1 g_2^{+}) + H.c. \right] + \lambda_2 \int dx \left[\mathrm{Tr} g_1 \mathrm{Tr} g_2^{+} + H.c. \right] + ..., \tag{9}$$

where the dots stand for less relevant operators like marginal current-current interaction. For the case when the interchain interaction includes only two spins, i.e., $\hat{V}^{(1,2)}_{n,n} = J_{\perp} \sum_A S^A_{1,n} S^A_{2,n}$, one can check directly the form of Eq. (9). By substituting Eqs. (4,6) into the interaction term of Eq. (2) and taking into account that the individual chains are described by the SU(N)$_1$ WZNW model, we find model (9) with $\lambda_2 = -\lambda_1/N$ and $\lambda_1 = J_{\perp}\lambda^2/2$. It is important to note that the ratio of the naive continuum limit $\lambda_2 = -\lambda_1/N$ is not universal and will be modified by higher-order contributions. For $N = 2n$ matrix $-g$ still belongs to the SU(N) group. Then the transformation $g_1 \rightarrow -g_1$ leaves the WZNW part of (9) unchanged and changes the sign of the coupling constants of the perturbation. This fact will enable us to identify the corresponding parts of the phase diagram of model (9).

III. CONFORMAL EMBEDDING APPROACH

In the weak-coupling regime, two-leg SU(N) spin ladders are thus described by a model of two coupled SU(N)$_1$ WZNW models perturbed by two strongly relevant perturbations with scaling dimension $2(N-1)/N$. The phase diagram results thus from the competition between these two terms. Though they have the same scaling dimension, the two perturbations are of very different nature. The one with coupling constant λ_1 is invariant under an SU(N)$_{\mathrm{L}}$ $\times$ SU(N)$_{\mathrm{R}}$ symmetry: $g_l \rightarrow U_{\mathrm{L}} g_l U_{\mathrm{R}}$, $U_{\mathrm{L,R}}$ being two independent SU(N) matrices. In stark contrast, the second with coupling constant λ_2 is only SU(N) invariant. It turns out that at $\lambda_2 = 0$, as it will be seen, theory (9) is integrable and therefore it makes sense to consider this model at $|\lambda_1| >> |\lambda_2|$ and treat the λ_2-term as a perturbation.

A. Fateev model: $\lambda_2 = 0$ case

In this respect, we now use the following conformal embedding which singles out the SU(N) symmetry:

$$SU(N)_1 \times SU(N)_1 \sim SU(N)_2 \times \mathbb{Z}_N, \tag{10}$$

where the SU(N)$_2$ CFT has central charge $c = 2(N^2-1)/(N+2)$ and $\mathbb{Z}_N$ is the parafermionic CFT with central charge $c = 2(N-1)/(N+2)$.[35,36] The latter captures the universal properties of the critical point of the $\mathbb{Z}_N$ generalization of two-dimensional (2D) Ising models. The low-temperature and high-temperature phases are described respectively by order and disorder parameters σ_k and μ_k ($k = 1, .., N-1$) which are dual to each other under the Kramers-Wannier (KW) transformation. The latter maps the $\mathbb{Z}_N$ symmetry, spontaneously broken in the low-temperature phase ($\langle\sigma_k\rangle \neq 0$ and $\langle\mu_k\rangle = 0$), onto a $\tilde{\mathbb{Z}}_N$ symmetry which is broken in the high-temperature phase where $\langle\mu_k\rangle \neq 0$ and $\langle\sigma_k\rangle = 0$. The order and disorder operators carry respectively a $(k, 0)$ and $(0, k)$ charges under the $\mathbb{Z}_N \times \tilde{\mathbb{Z}}_N$ symmetry:

$$\begin{aligned} \sigma_k &\rightarrow \mathrm{e}^{2\pi imk/N} \sigma_k \ \text{ under } \ \mathbb{Z}_N, \ \sigma_k \rightarrow \sigma_k \ \text{ under } \ \tilde{\mathbb{Z}}_N \\ \mu_k &\rightarrow \mathrm{e}^{2\pi imk/N} \mu_k \ \text{ under } \ \tilde{\mathbb{Z}}_N, \ \mu_k \rightarrow \mu_k \ \text{ under } \ \mathbb{Z}_N, \end{aligned} \tag{11}$$

with $m = 0, \ldots, N-1$. At the critical point, the theory is self-dual with a $\mathbb{Z}_N \times \tilde{\mathbb{Z}}_N$ symmetry and σ_k, μ_k become primary fields with scaling dimension $d_k = k(N-k)/N(N+2)$.[35] The $\mathbb{Z}_N$ CFT is generated by chiral right and left parafermionic currents $\Psi_{kR,L}$ ($\Psi^\dagger_{kR,L} = \Psi_{N-kR,L}$, $k = 1, \ldots, N-1$) with scaling dimension $\Delta_k = k(N-k)/N$ which are the generalization of the Majorana fermions of the $\mathbb{Z}_2$ Ising model. Under the $\mathbb{Z}_N \times \tilde{\mathbb{Z}}_N$ symmetry, Ψ_{kL} (respectively Ψ_{kR}) carries a (k,k) (respectively $(k,-k)$) charge which means:

$$
\begin{aligned}
\Psi_{kL,R} &\rightarrow \mathrm{e}^{2i\pi mk/N} \Psi_{kL,R} \ \text{ under } \ \mathbb{Z}_N \\
\Psi_{kL,R} &\rightarrow \mathrm{e}^{\pm 2i\pi mk/N} \Psi_{kL,R} \ \text{ under } \ \tilde{\mathbb{Z}}_N .
\end{aligned} \tag{12}
$$

The next step of the approach is to observe that model (9) at $\lambda_2 = 0$, being $\mathrm{SU}(N)_\mathrm{L} \times \mathrm{SU}(N)_\mathrm{R}$ invariant, is independent of the $\mathrm{SU}(N)_2$ sector of the embedding (10) but depends only on the $\mathbb{Z}_N$ parafermionic currents. One way to see that is to relate the λ_1 term of Eq. (9), i.e. V_1, in terms of the underlying Dirac fermions $R_{l\alpha}, L_{l\alpha}$ of the continuum limit:

$$
\begin{aligned}
V_1 &= -\frac{\lambda_1}{\lambda^2} \int dx \left[L^\dagger_{1\alpha} L_{2\alpha} R^\dagger_{2\beta} R_{1\beta} + H.c. \right] \\
&= -\frac{\lambda_1}{\lambda^2} \int dx \left[j^+_L j^-_R + H.c. \right],
\end{aligned} \tag{13}
$$

where we have introduced a chiral $\mathrm{SU}(2)_N$ $\boldsymbol{j}_{L,R}$ current. As shown in Ref. 35, there is a free-field representation of an $\mathrm{SU}(2)_N$ current in terms of a bosonic field and the first $\mathbb{Z}_N$ parafermion current:

$$
\begin{aligned}
j^+_L &= \frac{\sqrt{N}}{2\pi} \mathrm{e}^{\mathrm{i}\sqrt{8\pi/N}\Phi_{-cL}} \Psi_{1L} \\
j^z_L &= \sqrt{\frac{N}{2\pi}} \partial_x \Phi_{-cL},
\end{aligned} \tag{14}
$$

$\Phi_{-c} = (\Phi_{1c} - \Phi_{2c})/\sqrt{2}$ being the relative charge bosonic field. For the right sector, we have a similar expression:

$$
\begin{aligned}
j^+_R &= -\frac{\sqrt{N}}{2\pi} \mathrm{e}^{-\mathrm{i}\sqrt{8\pi/N}\Phi_{-cR}} \Psi^\dagger_{1R} \\
j^z_R &= \sqrt{\frac{N}{2\pi}} \partial_x \Phi_{-cR},
\end{aligned} \tag{15}
$$

where the KW duality symmetry $\Psi_{1R} \rightarrow -\Psi^\dagger_{1R}$ has been used for future convenience.

We thus find that the λ_1 term in model (9) is directly related to an integrable perturbation of $\mathbb{Z}_N$ parafermions introduced by Fateev with euclidean action:[37,38]

$$
\mathcal{S} = \mathcal{S}_{\mathbb{Z}_N} - \tilde{\lambda} \int \mathrm{d}^2 x \left(\Psi_{1L} \Psi_{1R} + H.c. \right), \tag{16}
$$

$\mathcal{S}_{\mathbb{Z}_N}$ being the action of the $\mathbb{Z}_N$ CFT and $\tilde{\lambda} = -\lambda_1 N / 4\pi^2$. This perturbation is invariant under the $\tilde{\mathbb{Z}}_N$ symmetry but explicitly breaks the $\mathbb{Z}_N$ symmetry as seen from Eq. (12).

B. SU(N)$_2$ perturbed CFT

Our next step is to express the λ_2-term in Eq. (9) in the $\mathrm{SU}(N)_2 \times \mathbb{Z}_N$ basis. The expression of the two $\mathrm{SU}(N)_1$ WZNW fields $g_{1,2}$ in the $\mathrm{SU}(N)_2 \times \mathbb{Z}_N$ basis ($N > 2$) was obtained in Ref. 39. We will justify their results from simple arguments based on symmetries.

To perform such analysis we need the representation of the $\mathbb{Z}_N \times \tilde{\mathbb{Z}}_N$ symmetry in terms of the underlying Dirac fermions. This can be done thanks to the definitions (14,15). Since Ψ_{1L} and Ψ_{1R} have $(1,1)$ and $(1,-1)$ charges under the $\mathbb{Z}_N \times \tilde{\mathbb{Z}}_N$ symmetry, we find that the $\mathbb{Z}_N$ symmetry is implemented as follows on the fermions:

$$
L_{1\alpha} \rightarrow \mathrm{e}^{-i\pi m/N} L_{1\alpha}, \ \ L_{2\alpha} \rightarrow \mathrm{e}^{i\pi m/N} L_{2\alpha}, \ \ R_{1\alpha} \rightarrow \mathrm{e}^{i\pi m/N} R_{1\alpha}, \ \ R_{2\alpha} \rightarrow \mathrm{e}^{-i\pi m/N} R_{2\alpha}, \tag{17}
$$

while under $\tilde{\mathbb{Z}}_N$ we have:

$$
L_{1\alpha} \rightarrow \mathrm{e}^{-i\pi m/N} L_{1\alpha}, \ \ L_{2\alpha} \rightarrow \mathrm{e}^{i\pi m/N} L_{2\alpha}, \ \ R_{1\alpha} \rightarrow \mathrm{e}^{-i\pi m/N} R_{1\alpha}, \ \ R_{2\alpha} \rightarrow \mathrm{e}^{i\pi m/N} R_{2\alpha}. \tag{18}
$$

From these results and the definition (7), we deduce the transformation of the two original SU(N)$_1$ WZNW fields. Under the $\mathbb{Z}_N$ symmetry, we have

$$g_1 \rightarrow \mathrm{e}^{2i\pi m/N} g_1, \;\; g_2 \rightarrow \mathrm{e}^{-2i\pi m/N} g_2, \tag{19}$$

whereas $g_{1,2}$ are invariant under the $\tilde{\mathbb{Z}}_N$ symmetry.

Now we are in a position to reproduce the results of Ref. 39 which was based on CFT consistencies. First of all, since $g_{1,2}$ transform in the fundamental representation of SU(N), they should be related to the SU(N)$_2$ WZNW primary field G which transforms in the same representation and has scaling dimension $(N^2-1)/N(N+2)$ (see Appendix A). Since the scaling dimension of $g_{1,2}$ is $1-1/N$, we need an additional operator in the $\mathbb{Z}_N$ CFT with scaling dimension $(N-1)/N(N+2)$, i.e., σ_1, $\sigma_1^{\dagger}$ or the disorder fields μ_1, $\mu_1^{\dagger}$. One way to eliminate the ambiguity is to use the transformation of the different fields under the $\mathbb{Z}_N \times \tilde{\mathbb{Z}}_N$ symmetry. This suggests the following identification:

$$\begin{aligned} (g_1)_{\alpha\beta} &\sim G_{\alpha\beta}\sigma_1 \\ (g_2)_{\alpha\beta} &\sim G_{\alpha\beta}\sigma_1^{\dagger}. \end{aligned} \tag{20}$$

Indeed, the disorder operators μ_1, $\mu_1^{\dagger}$ cannot appear in the decomposition since $g_{1,2}$ are singlets under the $\tilde{\mathbb{Z}}_N$ symmetry. The occurence of σ_1 and $\sigma_1^{\dagger}$ in Eq. (20) are consistent with the transformation law of $g_{1,2}$ under the $\mathbb{Z}_N$ symmetry (19). We note that the results (20) do not hold for $N=2$ which is a special case because the fundamental representation of SU(2) is self-conjugate. In that case, the expression can be obtained using the fact that SU(2)$_2$ CFT is related to three decoupled 2D Ising models (see for instance Ref. 20). One important consequence of the identity (20) is that the one-step translation symmetry (8) becomes now:

$$T_{a_0}: \;\; G \rightarrow \mathrm{e}^{2i\pi/N} G. \tag{21}$$

Finally, the identification (20) can be generalized for all SU(N)$_1$ primary fields φ_l which transform in the antisymmetric representation of SU(N) described by a Young tableau with a single column and l boxes $(1 \leq l \leq N-1)$.[39] For the first SU(N)$_1$ theory, these primary fields, i.e. φ_{1l}, are obtained by l fusion of g_1 by itself. Using the result (20) and the fusion rules of the $\mathbb{Z}_N$ parafermionic CFT, one can derive the correspondence between $\varphi_{1,2l}$ and SU(N)$_2$ and $\mathbb{Z}_N$ primaries. If we denote Φ_l the SU(N)$_2$ primary field with scaling dimension $l(N+1)(N-l)/N(N+2)$ (see Appendix A) which transforms in the lth antisymmetric representation of SU(N), we find:

$$\begin{aligned} \varphi_{1l} &\sim \Phi_l \sigma_l \\ \varphi_{2l} &\sim \Phi_l \sigma_l^{\dagger}, \end{aligned} \tag{22}$$

which is, of course, fine at the level of the scaling dimension since $l(N-l)/N = l(N+1)(N-l)/N(N+2) + l(N-l)/N(N+2)$.

We are now ready to find to express the λ_2-perturbation of Eq. (9) in the new basis. Since $\sigma_1\sigma_1 \sim \sigma_2$, we obtain:

$$\mathrm{Tr}\; g_1 \mathrm{Tr}\; g_2^{+} \sim : \mathrm{Tr}\; G\; \mathrm{Tr}\; G^{+} : \sigma_2 \sim \mathrm{Tr}(\Phi_{\mathrm{adj}})\sigma_2, \tag{23}$$

where Φ_{adj} is the SU(N)$_2$ primary field with scaling dimension $2N/(N+2)$, which transforms in the adjoint representation of SU(N). In this derivation, we have used the definition of the adjoint primary field:[33]

$$\mathrm{Tr}(\Phi_{\mathrm{adj}}) = \mathrm{Tr}(G^{+}T^{A}GT^{A}) = \frac{1}{2}\left(\mathrm{Tr}\; G\; \mathrm{Tr}\; G^{+} - \frac{1}{N}\mathrm{Tr}(GG^{+})\right), \tag{24}$$

and the identity

$$\sum_A T^A_{\alpha\beta}T^A_{\gamma\rho} = \frac{1}{2}\left(\delta_{\alpha\rho}\delta_{\beta\gamma} - \frac{1}{N}\;\delta_{\alpha\beta}\delta_{\gamma\rho}\right). \tag{25}$$

As the result of Eq. (23) we obtain the following expression for λ_2-perturbation, V_2, of model (9):

$$V_2 \simeq \lambda_2 \int dx\; \mathrm{Tr}(\Phi_{\mathrm{adj}})\left(\sigma_2 + \sigma_2^{\dagger}\right). \tag{26}$$

In summary, the low-energy effective field theory (9) of weakly coupled SU(N) Heisenberg chains can be reformulated in the basis (10) with Hamiltonian:

$$\mathcal{H} = W[SU(N)_2; G] + \mathcal{H}_{\mathbb{Z}_N} - \tilde{\lambda}\int dx\;\; (\Psi_{1L}\Psi_{1R} + H.c.) + \lambda_2 \int dx\; \mathrm{Tr}(\Phi_{\mathrm{adj}})\left(\sigma_2 + \sigma_2^{\dagger}\right). \tag{27}$$

The crucial difference between $N = 2$ and $N > 2$ cases is that for $N = 2$ there is no $\sigma_2, \sigma_2^\dagger$ operators and therefore two sectors of the theory decouple from each other. This does not happen for $N > 2$ and this determines the difference in physics. Recall now that for $N = 2n$ the spectrum of the original model (9) is invariant under a change of sign of the both coupling constants. Such sign change should be compensated by field transformations in Eq. (27), but we managed to find them only for $N = 4p$.

IV. PHASES OF THE GENERALIZED TWO-LEG SU($2n$) SPIN LADDER

In this section, we will investigate the IR physics of model (27) to deduce the nature of the possible phases of generalized two-leg SU($2n$) spin ladder.

A. Field theory strong coupling approach

To shed light on the possible phases, it might be interesting to first perform a strong-coupling approach directly to the continuum model (9) when both coupling constants $\lambda_{1,2}$ are of the order of the ultraviolet cut-off and $|\lambda_1| >> |\lambda_2|$. In this respect, let us consider the euclidean action corresponding to Eq. (9):[33,40]

$$\mathcal{S} = \mathcal{S}[SU(N)_1; g_1] + \mathcal{S}[SU(N)_1; g_2] + \lambda_1 \int d^2x \left[\mathrm{Tr}(g_1 g_2^+) + H.c.\right] + \lambda_2 \int d^2x \left[\mathrm{Tr} g_1 \mathrm{Tr} g_2^+ + H.c.\right], \tag{28}$$

where the action of the SU(N)$_k$ WZNW model is given by:

$$\begin{aligned} \mathcal{S}[SU(N)_k; g] &= \frac{k}{8\pi} \int d^2x \; \mathrm{Tr}\; (\partial^\mu g^+ \partial_\mu g) + \Gamma(g) \\ \Gamma(g) &= \frac{k}{12\pi} \int_B d^3y \; \epsilon^{\alpha\beta\gamma} \mathrm{Tr}\; (g^+ \partial_\alpha g g^+ \partial_\beta g g^+ \partial_\gamma g), \end{aligned} \tag{29}$$

g being an SU(N) matrix field and $\Gamma(g)$ the WZNW topological term.

The results of the strong-coupling approach depend on the sign of λ_1 and we assume $N = 2n$.

1. $\lambda_1 < 0$

Then the minimization of the λ_1 term in action (28) gives $g_1 = g_2 = G$ and the WZNW topological term (29) is doubled. The resulting effective action is therefore

$$\mathcal{S}_{\text{eff}} = \mathcal{S}[SU(N)_2; G] + 2\lambda_2 \int d^2x \; |\mathrm{Tr}\; G|^2. \tag{30}$$

Now, if $\lambda_2 < 0$ then $|\mathrm{Tr}\; G|$ should be maximal. For an SU(N) matrix it leads to the conclusion that G belongs to the center of SU(N), i.e. $\mathbb{Z}_N$, with $G = \mathrm{e}^{2i\pi k/N} I$, $k = 0, 1, \ldots, N-1$ and I is the $N \times N$ identity matrix. The one-step translation T_{a_0} is spontaneously broken and the system has a finite order parameter with wave vector $2k_F = \pi/n$. If $\lambda_2 > 0$, as this would be the case for the two-spin interchain exchange ladder where $\lambda_2 = -\lambda_1/N$, we get the condition $\mathrm{Tr}\; G = 0$. Here, as we will see, the order parameter has $4k_F$ wave vector.

2. $\lambda_1 > 0$

A similar approach leads to $g_1 = -g_2 = G$ which is still an SU(N) matrix if N is even. The resulting model becomes then

$$\mathcal{S}_{\text{eff}} = \mathcal{S}[SU(N)_2; G] - 2\lambda_2 \int d^2x \; |\mathrm{Tr}\; G|^2. \tag{31}$$

For $\lambda_2 > 0$ we have again $G = \mathrm{e}^{2i\pi k/N} I$ which corresponds to $2k_F$ order which is now staggered between the chains since $g_1 = -g_2$. For $\lambda_2 < 0$ we get the condition $\mathrm{Tr}\; G = 0$ corresponding to an $4k_F$ ordering.

B. Integrable point $\lambda_2 = 0$.

A weak-coupling approach can be performed by exploiting the fact that when $\lambda_2 = 0$ model (9) becomes integrable and is related to the Fateev model (16). According to Ref. 37, the integrable deformation of the $\mathbb{Z}_N$ parafermions (16) is a massive field theory with a mass gap Δ for any sign of its coupling constant $\tilde{\lambda}$ when N is even. The action explicitly breaks the $\mathbb{Z}_N$ symmetry and the exact spectrum consists of massive kink excitations that result from degenerate ground states labelled by an odd integer $s = 1, 3, \ldots, N+1$.[37,38] One can then average over high-energy degrees of freedom represented by the theory (16) and obtain an effective field theory for the SU$(N)_2$ sector in the low-energy limit $E \ll \Delta$. This theory describes magnetic excitations carrying quantum numbers of the SU(N) group. Using the result (27) of the previous section, we obtain the Hamiltonian density of this effective field theory:

$$\mathcal{H}_{\rm eff} = \frac{2\pi v}{N+2}\Big(:I^A_R I^A_R: + :I^A_L I^A_L: \Big) + \eta_s \mathrm{Tr}\ \Phi_{adj}, \tag{32}$$

where $I^A_{R,L}$ are chiral SU$(N)_2$ currents and the mass M of the $\mathbb{Z}_N$ particle serves as the upper cut-off. The coupling constant of the adjoint primary field Φ_{adj} is

$$\eta_s = \lambda_2 \langle \sigma_2 + \sigma_2^\dagger \rangle_s M^{d_\Phi}, \tag{33}$$

$d_\Phi = 2N/(N+2)$ being the scaling dimension of the adjoint primary field (see Appendix A). When $\tilde{\lambda} > 0$, the vacuum average of the spin fields σ_j in the ground state s is known from Ref. 41:

$$\langle \sigma_j \rangle_s = \langle 0_s | \sigma_j | 0_s \rangle = \frac{\sin\left[\frac{\pi(j+1)s}{N+2}\right]}{\sin\left(\frac{\pi s}{N+2}\right)} (M/4)^{2h_j} e^{Q_j} \tag{34}$$

$$Q_j = \int_0^\infty \frac{dt}{t} \Big\{ \frac{\sinh(tj)\sinh[(N-j)t]}{\sinh(Nt)\sinh[(N+2)t]} - 2h_j e^{-2t} \Big\},$$

where $h_j = j(N-j)/N(N+2)$. In particular, from Eq. (34) we see that $\langle \sigma_1 \rangle_s = \langle \sigma_{N-1} \rangle_s \equiv \langle \sigma_1^\dagger \rangle_s$. According to Eq. (20), this means that for $\lambda_1 < 0$, i.e., $\tilde{\lambda} \sim -\lambda_1 > 0$, we have $g_1 = g_2$, as we envisaged from the strong-coupling approach of the previous subsection.

Let us now return to the problem of finite λ_2. Our task is to analyse the infrared physics of the SU$(N)_2$ model (32) perturbed by its adjoint primary field. Since it is a strongly relevant perturbation, we may expect that the magnetic SU(N) sector is always gapped.

Fine details of the spectrum, however, depend on the sign and magnitude of coupling constant η_s (33). According to the result (34) these depend on the ground state of model (16) $|0\rangle_s$. We suggest that s is selected in such a way that the ground-state energy of (32) is the lowest. It is reasonable to think that this corresponds to largest spectral gaps.

To determine a qualitative dependence of the spectrum on η_s it is useful to consider a direct semiclassical approach to the interacting Hamiltonian density (32) using the identity (24):

$$\mathcal{H}_{\rm int} = \eta_s \mathrm{Tr}\ G\ \mathrm{Tr}\ G^+, \tag{35}$$

where G is now an SU(N) matrix. When $\eta_s < 0$, the minimization selects the center group of SU(N):

$$G = \exp(\mathrm{i}2\pi k/N) I, \;\; \eta_s < 0, \tag{36}$$

where $k = 0, \ldots, N-1$ and the solution breaks spontaneously the one-step translation symmetry T_{a_0} (21). The ground state is thus N-fold degenerate.

When $\eta_s > 0$, the minimization gives then the condition that G is an SU(N) matrix with the constraint: Tr $G = 0$. The general solution for $N = 2n$ takes then a Grassmanian form:[16]

$$G = \exp(\mathrm{i}2\pi k/N)\ U^+ \mathrm{diag}(1, ...1, -1, ... -1) U, \;\; \eta_s > 0, \tag{37}$$

U being a general unitary U(N) matrix. As shown in Ref. 16, within this semiclassical approach, model (32) with even N becomes equivalent to the Grassmanian sigma model on $U(N)/[U(N/2) \times U(N/2)]$ manifold with a trivial topological term $\theta = 2\pi$. This model describes a fully gapped phase.

Although for both signs of η_s we obtain gapped spectra, the gaps for the Grassmanian sigma model are expected to be smaller. This becomes clear at large $N >> 1$ since $1/N$ serves as a coupling constant for the sigma model and hence the gaps are exponentially small in N: $m \sim \exp(-\mathrm{const}N)$. At the same time for $\eta_s < 0$ the gaps are algebraic in $1/N$.

From these results, obtained within the semiclassical analysis, we can now determine a phase diagram of the generalized two-leg SU$(2n)$ spin ladder.

C. $\lambda_1 < 0$

We first assume that $\lambda_1 < 0$ so that $\tilde{\lambda} \sim -\lambda_1 > 0$ and we can use the result (34) for the vacuum expectation values of the σ_j fields. As we have mentioned above, the coupling constant η_s (33) of the low-energy theory (32) depends on the vacuum of model (16). The degeneracy is removed by a selection of η_s which yields the lowest energy ground state energy for model (32).

To establish a relationship between s and η_s we recall that the vacuum expectation values (34) enjoy the property: $\langle \sigma_j \rangle_s = (-1)^{1+s} \langle \sigma_j^\dagger \rangle_s = \langle \sigma_j^\dagger \rangle_s$ since s is odd. Then substituting Eq. (34) into Eq. (33) we get for $N = 2n$:

$$\eta_{s=2k+1} \sim \lambda_2 \Big(1 + 2\cos\Big[\frac{\pi(2k+1)}{n+1}\Big]\Big), \quad k = 0, 1, ...n. \tag{38}$$

1. $\lambda_2 < 0$

When $\lambda_2 < 0$, the result (38) suggests that the true ground state of model (32) corresponds to $s = 1$. Indeed, (i) this state has the largest value of the coupling η_s, (ii) $\eta_{s=1} < 0$ which according to the semiclassical arguments presented above, corresponds to a larger gap than for a positive coupling. The semiclassical analysis conducted above predicts a spin gap phase with N-fold ground-state degeneracy which stems from the complete breaking the one-step translation symmetry T_{a_0}. Using the identification (20), we deduce the result:

$$\text{Tr } g_1 \pm \text{Tr } g_2 \sim \text{Tr } G\ (\sigma_1 \pm \sigma_1^\dagger), \tag{39}$$

and since $\langle \sigma_1 \rangle_{s=1} = \langle \sigma_1^\dagger \rangle_{s=1} \neq 0$ from Eq. (34), we have: $\langle \text{Tr } g_1 \rangle = \langle \text{Tr } g_2 \rangle \neq 0$. A lattice order parameter can then be deduced by means of the result of Appendix B (see Eq. (B7) with $m = 1$):

$$\langle \mathcal{O}^{\mathrm{u}\dagger}_{2k_F}(n) \rangle = \mathrm{e}^{-i\frac{2\pi n}{N}} \langle \sum_{l=1}^{2} S^A_{l,n} S^A_{l,n+1} \rangle \sim \langle \text{Tr } g_1 + \text{Tr } g_2 \rangle \neq 0. \tag{40}$$

Thus we have a uniform $2k_F$ VBS phase which is in phase between the legs of the two-leg ladder and the wave vector is thus $(\pi/n, 0)$. This $N = 2n$-fold degenerate phase breaks spontaneously the T_{a_0} symmetry as expected from the semiclassical approach.

2. $\lambda_2 > 0$

In this case according to Eq. (38) for even n the most negative coupling constant corresponds to a single vacuum $k = n/2$. For odd n there are two degenerate vacua $k = (n \pm 1)/2$.

Although the spectrum of model (32) remains the same as for $\lambda_2 < 0$, the order parameters change. Indeed, from Eq. (34) we have

$$\langle \sigma_1 \rangle_{2k+1} \sim \cos\Big(\frac{\pi(2k+1)}{2n+2} \Big), \tag{41}$$

and for $k = n/2$ this vacuum expectation value vanishes. From the correspondence (39), we deduce that the $2k_F$ order parameter vanishes as well: $\langle \text{Tr } g_1 + \text{Tr } g_2 \rangle = 0$, in agreement with the strong-coupling approach. However, since $G^2 = \exp(\mathrm{i}4\pi k/N) I$ from Eq. (36) and $\langle \sigma_2 \rangle_{s=n+1} \neq 0$, for $N = 4p$ we have instead the formation of a $4k_F$ VBS phase:

$$\langle \mathcal{O}^{\dagger}_{4k_F}(n) \rangle = \mathrm{e}^{-i\frac{4\pi n}{N}} \langle \sum_{l=1}^{2} S^A_{l,n} S^A_{l,n+1} \rangle \sim \langle \text{Tr } \varphi_{11} + \text{Tr } \varphi_{21} \rangle \sim \langle \text{Tr } G^2 \rangle \neq 0, \tag{42}$$

where we have exploited the result (B7) with $m = 2$ of Appendix B and Eq. (22). The $4k_F$ VBS phase has wave vector $(2\pi/n, 0)$ and a n-fold ground-state degeneracy.

For n odd we have

$$\langle \sigma_1 \rangle_{n,n+2} \sim \pm \sin\Big(\frac{\pi}{2n+2} \Big), \tag{43}$$

and $\langle\sigma_1\rangle$ does not vanish for odd n. In this respect, one might expect the existence of a $2k_F$ VBS phase described by the order parameter (40). In contrast, the strong-coupling approach of Sec. IV A leads to the formation of a $4k_F$ VBS phase as in the n even case. One cannot exclude the occurence of a quantum phase transition between weak and strong coupling regimes when n is odd. However, the degeneracy between the $s = n$ and $s = n + 2$ states might not be protected and with a result that the true ground state is a symmetric combination of the two states with opposite signs of σ_1 so that the resulting average vanishes: $\langle\sigma_1\rangle = 0$. Such scenario would support the $4k_F$ VBS phase as in the n even case. Numerical calculations on the two-leg SU(6) spin ladder are clearly called for to shed light on the issue of a quantum phase transition.

D. $\lambda_1 > 0$

We now turn to $\lambda_1 > 0$ case ($\tilde{\lambda} < 0$). The $\mathbb{Z}_N$ model (16) is still a massive field theory. Unfortunately, the vacuum expectation values of the order parameters (34) for $\tilde{\lambda} < 0$ are not known. However, when $N = 4p$ (even n), one can perform the transformation $\Psi_{1L,R} \to \mathrm{i}\Psi_{1L,R}$ to change the sign of $\tilde{\lambda}$ in model (16). From the fusion rules of the $\mathbb{Z}_N$ parafermionic theory[35] we have $\sigma_1\mu_1 \sim \Psi_{1L}$ and $\sigma_1\mu_1^\dagger \sim \Psi_{1R}$, we deduce that

$$\sigma_1 \to \mathrm{i}\sigma_1, \quad \sigma_2 \to -\sigma_2, \tag{44}$$

and thus $\langle\sigma_1\rangle_s = -\langle\sigma_1^\dagger\rangle_s$. As shown in Appendix C, the $\mathbb{Z}_4$ parafermions CFT can be described by a bosonic field theory with central charge $c = 1$. In particular, one can establish the mapping $\sigma_2 \to -\sigma_2$ in the $N = 4$ case directly by means of this bosonized description.

The transformation (44) maps the spectrum of model (9) with couplings $\lambda_1 > 0, \lambda_2$ onto the spectrum with couplings $-\lambda_1, -\lambda_2$. The former one in Eq. (44) means that the order parameter in the region $\lambda_1 > 0, \lambda_2 > 0$ is $\langle\mathrm{Tr}(g_1 - g_2)\rangle$ using the correspondence (39). It leads to the formation of a staggered $2k_F$ VBS phase with a N-fold degeneracy:

$$\langle\mathcal{O}_{2k_F}^{\mathrm{stag}\dagger}(n)\rangle = \mathrm{e}^{-\frac{2i\pi n}{N}}\langle\sum_{l=1}^{2}(-1)^{l+1}S_{l,n}^A S_{l,n+1}^A\rangle \sim \langle\mathrm{Tr}g_1 - \mathrm{Tr}g_2\rangle \neq 0. \tag{45}$$

The emergence of a staggered phase is consistent with the strong-coupling approach in the $\lambda_1 > 0, \lambda_2 > 0$ region.

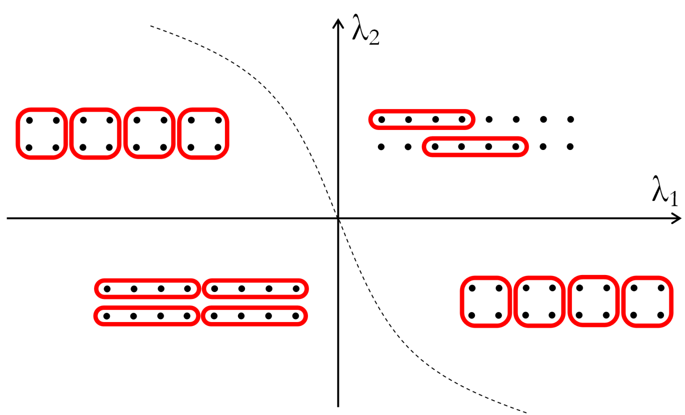

FIG. 1: Phase diagram of model (9) for $N = 4$ by means of a field-theory analysis when $|\lambda_1| >> |\lambda_2|$ and a mean-field approach for $|\lambda_2| >> |\lambda_1|$. A similar phase diagram is obtained in the even n case where the plaquette phase is a n-site rectangular singlet phase.

In contrast, the region $\lambda_1 > 0, \lambda_2 < 0$ is occupied by $4k_F$ VBS with a $N/2$-fold degeneracy as in Eq. (42). For $N = 4$, it corresponds to a plaquette phase which is known to appear in the standard two-leg SU(4) spin ladder.[30,42] A summary of the results obtained for even n can be found in Fig. 1.

When $N = 4p + 2$, the vacuum expectation values (34) of the Fateev model (16) with $\tilde{\lambda} < 0$ are not known so we cannot conclude as before unfortunately. However, from Eq. (9), we saw that the change of sign $\lambda_{1,2} \to -\lambda_{1,2}$ can be absorbed by the redefinition $g_1 \to -g_1$. The spectrum of model (9) for $\lambda_1 > 0, \lambda_2$ is thus related to that of with couplings $\lambda_1 < 0, -\lambda_2$. We thus expect the formation of a staggered $2k_F$ (respectively $4k_F$) VBS phase when $\lambda_2 > 0$ (respectively $\lambda_2 < 0$) as in the even n case.

E. Mean-field analysis in the $|\lambda_2| >> |\lambda_1|$ regime

So far, we have exploited the existence of an integrable line $\lambda_2 = 0$ in model (9) to find the possible phases of generalized two-leg SU($2n$) spin ladders. As emphasized before, such approach is valid in the $|\lambda_1| >> |\lambda_2|$ regime. When $\lambda_1 = 0$, the strongly relevant perturbation in Eq. (27) is not an integrable and is thus difficult to analyse. In the regime $|\lambda_2| >> |\lambda_1|$, we expect the formation of spectral gaps in the problem. One can then investigate the resulting IR physics in this regime by means of a simple mean-field decoupling approach: $\mathcal{H}_{\rm mf} = \mathcal{H}_1 + \mathcal{H}_2$ with:

$$\begin{aligned} \mathcal{H}_1 &= \mathcal{H}_{\mathbb{Z}_N} + \lambda_2 \int dx \, \langle \mathrm{Tr}(\Phi_{\rm adj}) \rangle \left(\sigma_2 + \sigma_2^\dagger \right) \\ \mathcal{H}_2 &= \frac{2\pi v}{N+2} \Big(: I_R^A I_R^A : + : I_L^A I_L^A : \Big) + \lambda_2 \int dx \, \langle \sigma_2 + \sigma_2^\dagger \rangle \, \mathrm{Tr}(\Phi_{\rm adj}). \end{aligned} \tag{46}$$

The perturbation in the $\mathbb{Z}_N$ sector with the σ_2 spin field with scaling dimension $2(N-2)/N(N+2)$ is more relevant than the one in the Fateev model (16). In the regime $|\lambda_2| >> |\lambda_1|$, one can thus ignore the latter perturbation. The field theory $\mathcal{H}_1$ is not integrable except in the $N = 4$ case where it is equivalent to the sine-Gordon model with $\beta^2 = 2\pi/3$ (see Appendix C). The σ_2 spin perturbation gives a mass gap to the $\mathbb{Z}_N$ degrees of freedom and, when $\lambda_2 < 0$, we have $\langle \sigma_2 + \sigma_2^\dagger \rangle > 0$ since $\langle \mathrm{Tr}(\Phi_{\rm adj}) \rangle > 0$. The perturbation in the SU(N) sector, described by $\mathcal{H}_2$, becomes equivalent to model (32) with $\eta_s < 0$. The one-step translation symmetry is then spontaneously broken and a 2k$_F$ VBS phase emerges in the regime $|\lambda_2| >> |\lambda_1|$. When $\lambda_1 > 0$ for instance, there is thus a competition between 4k$_F$ and 2k$_F$ VBS phases which is marked by the transition line in Fig. 1. The full determination of the transition lines is beyond the scope of our approach.

A strong-coupling analysis of the lattice standard antiferromagnetic two-leg SU($2n$) ladder is presented in Appendix D. In Fig. 1, it corresponds to the line $\lambda_2 = -\lambda_1/N$ with $\lambda_1 \to \infty$. For $N = 4$ and $N = 6$, we find respectively plaquette and trimerized phases, i.e., $4k_F$ VBS phases. The lattice strong-coupling analysis confirms thus the prediction of our field-theory approach presented in Sec. IV when $|\lambda_1| >> |\lambda_2|$. We thus conclude that the $4k_F$ VBS phase occurs in the phase diagram of the two-leg SU(4) and SU(6) spin ladders for a sufficiently strong $J_\perp > 0$. Whether this phase is stabilized upon switching on a small interchain coupling or remplaced by a competing $2k_F$ VBS phase is an interesting question which cannot be answered unfortunately within our approach. In this respect, large-scale numerical simulations are necessary to shed light on this intriguing possibility. When $n > 4$, an SU($2n$)$_1$ quantum criticality is expected from the lattice strong-coupling approach of Appendix D. A quantum phase transition should occur between weak and strong coupling regimes of the standard SU($2n$) spin ladder with $J_\perp > 0$.

V. CONCLUSIONS

We have studied the zero-temperature phase diagram of the SU($2n$) Sutherland antiferromagnetic ladder with weak but generic interchain interactions. We have found that for a translationally invariant ladder with inversion symmetry this phase diagram is controlled by only two parameters $\lambda_{1,2}$ in the continuum limit. These are coupling constants of the two most relevant operators allowed by the symmetries. Our field-theory analysis exploits the existence of an integrable massive field theory of $\mathbb{Z}_N$ parafermions along the $\lambda_2 = 0$ line. In the vicinity of that line, i.e. $|\lambda_1| >> |\lambda_2|$, our approach leads to the conclusion that the phase diagram contains only VBS phases characterized by local order parameters. In this respect, there are no topological phases as for $n = 1$,i.e., a two-leg SU(2) spin ladder. There are two types of VBS phase with wave vectors $(\pi/n, 0)$ or $(\pi/n, \pi)$ ($2k_F$ VBS) and $(2\pi/n, 0)$ ($4k_F$ VBS). The $4k_F$ VBS can be viewed as a cluster of $2n$ spins put in an SU($2n$) singlet with a n-fold ground-state degeneracy. For $N = 4$, it corresponds to a plaquette phase which is known to appear in the standard two-leg SU(4) spin ladder.[30,42] Our results for $N = 6$ suggest that a cluster phase of six spins, leading to trimerization, should occur in the phase diagram of the two-leg SU(6) spin ladder. The latter case is directly relevant to the insulating phase of double tube of ytterbium ^{173}Yb ultracold atoms which can be engineered by considering double-well optical lattices.[43] In this respect, it will be interesting to investigate numerically the latter system to confirm the existence of such a cluster phase for $N = 6$ by means of numerical methods for SU(N) magnet as in Ref. 44. Finally, the field-theory analysis of two-leg spin ladder with SU($2n + 1$) spins is different and will be presented elsewhere.

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Appendix A: Quadratic Casimir of SU(N)

In this Appendix, we give some conformal data of some SU$(N)_k$ primary fields which appear in the conformal embedding approach of Sec. III. The scaling dimension of a SU$(N)_k$ primary field which transforms in some representation R of the SU(N) group is given by:[33]

$$\Delta_R = \frac{2C_R}{N+k}, \tag{A1}$$

where C_R is the quadratic Casimir in the representation R. Its expression can be obtained from the general formula where R is written as a Young tableau:

$$C_R = T^a T^a = \frac{1}{2}[l(N - l/N) + \sum_{i=1}^{n_{row}} b_i^2 - \sum_{i=1}^{n_{col}} a_i^2] \tag{A2}$$

for Young tableau of l boxes consisting of n_{row} rows of length b_i each and n_{col} columns of length a_i each. For instance, we get $C_R = (N^2-1)/2N$ for the fundamental representation, $C_R = N$ for the adjoint representation, $C_R(k) = k(N+1)(N-k)/2N$ for the kth basic antisymmetric representation made of a Young tableau with a single column and k boxes, and $C_R = N - 2/N + 1$ for the symmetric representation with dimension $N(N+1)/2$.

Appendix B: Lattice representation of SU$(N)_1$ primary fields

We investigate here the lattice representation of SU$(N)_1$ primary fields which transform in the basic antisymmetric representations of SU(N). The results are important for the identification of the phases of the generalized two-leg SU($2n$) spin ladder.

Let us consider an SU(N) spin chain in the fundamental representation of SU(N). As recalled in the introduction, the model is integrable and displays a quantum critical behavior in the SU$(N)_1$ universality class with central charge $c = N - 1$.[14,16] At low-energy, the lattice SU(N) operators in the continuum limit are described by:[16,31]

$$S_n^A \simeq J_L^A + J_R^A + \sum_{m=1}^{N-1} \mathrm{e}^{\mathrm{i}2mk_F x} N_m^A, \tag{B1}$$

where $k_F = \pi/Na_0$, $x = na_0$, and $J_{L,R}^A$ are the left and right SU$(N)_1$ currents. The $2mk_F$ parts of this decomposition are related to the $m = 1, \ldots, N-1$ SU$(N)_1$ primary field Φ_m with scaling dimension $m(N-m)/N$ which transforms in the antisymmetric representation of SU(N) made of a Young tableau with a single column and m lines:

$$N_m^A = \lambda_m \, \mathrm{Tr}(\Phi_m T_m^A), \tag{B2}$$

where T_m^A are SU(N) generators in the mth basic antisymetric representation of the SU(N) group and λ_m non-universal real constants. It is interesting to get some lattice interpretation of Tr Φ_m in terms of the original lattice SU(N) spin operators. In this respect, let us introduce lattice $2mk_F$ bond SU(N) operators:

$$\mathcal{O}^\dagger_{2mk_F}(n) = \mathrm{e}^{-\mathrm{i}\frac{2\pi mn}{N}} S_n^A S_{n+1}^A, \tag{B3}$$

with $m = 1, \ldots, N-1$. The continuum description of these operators can be obtained from the identification (B1):

$$\mathcal{O}^+_{2mk_F}(n) \simeq N_m^A(x)\left[J_L^A(x+a_0) + J_R^A(x+a_0)\right] + \mathrm{e}^{\mathrm{i}\frac{2\pi}{N}}\left[J_L^A(x) + J_R^A(x)\right] N_m^A(x+a_0). \tag{B4}$$

At this point, it is useful to recall the defining operator product expansions (OPE) for the SU$(N)_1$ primary fields:[40]

$$\begin{aligned} J_L^A(z)\left(\Phi_m\right)_{r,s}(w,\bar{w}) &\sim -\frac{1}{2\pi(z-w)}\left(T_m^A\right)_{r,p}\left(\Phi_m\right)_{p,s}(w,\bar{w}) \\ J_R^A(\bar{z})\left(\Phi_m\right)_{r,s}(w,\bar{w}) &\sim \frac{1}{2\pi(\bar{z}-\bar{w})}\left(T_m^A\right)_{p,s}\left(\Phi_m\right)_{r,p}(w,\bar{w}). \end{aligned} \tag{B5}$$

Using this result together with the definition (B2), we get the following OPE:

$$e^{i\frac{2\pi}{N}}\left[J_L^A(z)+J_R^A(\bar z)\right]N_m^A(w,\bar w)+N_m^A(z,\bar z)\left[J_L^A(w)+J_R^A(\bar w)\right]\ \sim$$
$$-\frac{\lambda_m C_m}{2\pi}\left(e^{i\frac{2\pi}{N}}-1\right)\left[\frac{1}{z-w}-\frac{1}{\bar z-\bar w}\right]\mathrm{Tr}\ \Phi_m(w,\bar w), \tag{B6}$$

where C_m is the quadratic Casimir of the mth antisymmetric representation of the SU(N) group. We then deduce the lattice representation of Tr Φ_m in terms of the $2mk_F$ bond SU(N) operators:

$$\mathcal{O}^{\dagger}_{2mk_F}(n)=e^{-i\frac{2\pi mn}{N}}S_n^A S_{n+1}^A\sim \mathrm{Tr}\ \Phi_m(x). \tag{B7}$$

Appendix C: A special case of the SU(4) group

For $N=4$ one can exploit the fact that SU(4) group is isomorphic to O(6). The Kac-Moody algebras SU(4)$_k$ and O(6)$_k$ are equivalent and for $k=1$ and $k=2$ it is possible to employ Abelian bosonization and Majorana fermions techniques.[20]

1. $\mathbb{Z}_4$ Fateev model and sine-Gordon model

In particular, the $\mathbb{Z}_4$ parafermion CFT has central charge $c=1$ and admits a bosonized description in terms of a compactified boson field χ at radius $R=\sqrt{3/2\pi}$ ($\chi\sim\chi+\sqrt{6\pi}$) which lives on a $\mathbb{Z}_2$ orbifold: $\chi\sim-\chi$.[45] Some primary fields of the $\mathbb{Z}_4$ CFT have a simple bosonic representation in terms of a vertex operator as the σ_2 field and the first thermal operator ϵ_1:

$$\sigma_2\sim\cos(\sqrt{2\pi/3}\,\chi),\quad \epsilon_1\sim\cos(\sqrt{8\pi/3}\,\chi), \tag{C1}$$

which have respectively scaling dimension 1/6 and 2/3 as it should. The spin field operators σ_1,σ_1^+ with scaling dimension 1/8 are related to the twist fields of the $c=1$ $\mathbb{Z}_2$ orbifold CFT.[40]

Using this bosonization approach, the Fateev model (16) can be shown to be equivalent to the $\beta^2=6\pi$ sine-Gordon theory with Hamiltonian density:[37,38,46]

$$\mathcal{H}_{\mathbb{Z}_4}=\frac{1}{2}\left((\partial_x\chi)^2+(\partial_x\tilde\chi)^2\right)-\tilde\lambda\cos(\sqrt{6\pi}\,\chi), \tag{C2}$$

$\tilde\chi$ being the dual field of χ.

According to the exact solution of model (16), there are three degenerate vacua where σ_2 acquires expectation values (see Eq. (34)):

$$\langle 0|\sigma_2|0\rangle_s\sim 1+2\cos(\pi s/3),\quad s=1,3,5, \tag{C3}$$

corresponding to $2,-1,2$. The sine-Gordon model (C2) at $\beta^2=6\pi$ leads to the pinning of the bosonic field at the minima of the potential when $\tilde\lambda>0$: $\langle\chi\rangle=p\sqrt{2\pi/3}$, p being integer. Taking into account the radius of the boson which leads to the identification $\chi\sim\chi+\sqrt{6\pi}$, we see that the sine-Gordon model (C2) has three degenerate ground states with: $\langle\chi\rangle=0,\sqrt{2\pi/3},2\sqrt{2\pi/3}$. Using the bosonic representation (C1), we find that the vacuum expectation values of the σ_2 operator give three values with two of them being equal as the exact result (C3). Using this bosonic approach, we can also determine how result (C3) is modified when $\tilde\lambda<0$. The change of sign $\tilde\lambda\to-\tilde\lambda$ can be obtained by the mapping: $\chi\to\chi+\sqrt{\pi/6}+q\sqrt{2\pi/3}$, q being integer. The value of q is fixed by the requirement that the first thermal operator should be invariant under the transformation since the $\mathbb{Z}_4$ symmetry should not be broken when $\tilde\lambda<0$. The bosonization result (C1) gives $q=1$ and therefore:

$$\chi\to\chi+\sqrt{3\pi/2}. \tag{C4}$$

We thus deduce that under the change of sign $\tilde\lambda\to-\tilde\lambda$, the σ_2 operator transform as: $\sigma_2\to-\sigma_2$. We recover the result (44) from the bosonization approach for $N=4$.

2. Abelian bosonization approach in the $N = 4$ case

Using this bosonization description of $\mathbb{Z}_4$ parafermions, one can investigate a free-field representation of model (27) when $N = 4$. The central charge of the $SU(4)_2$ is $c = 5$ and this model might be described in terms of five bosonic fields.

First of all, the currents of the $SU(4)_1 \sim SO(6)_1$ CFT, which has central charge $c = 3$, can be expressed in terms of six Majorana fermions χ_a:

$$j^{ab} = \mathrm{i}\chi_a\chi_b, \quad 1 \leq a < b \leq 6. \tag{C5}$$

Then the $SU(4)_2$ currents J^{ab}, being the sum of two $SU(4)_1$ ones, can be bosonized since two Majorana fermions can be expressed in terms of a single bosonic field:

$$J^{ab} = \mathrm{i}\chi_a\chi_b + \mathrm{i}\eta_a\eta_b = \frac{\mathrm{i}\kappa_a\kappa_b}{\pi a_0}[\cos(\sqrt{4\pi}\varphi_a)\cos(\sqrt{4\pi}\varphi_b) + \sin(\sqrt{4\pi}\varphi_a)\sin(\sqrt{4\pi}\varphi_b)] =$$
$$\frac{\mathrm{i}\kappa_a\kappa_b}{\pi a_0}\cos[\sqrt{4\pi}(\varphi_a - \varphi_b)], \tag{C6}$$

where κ_a are Klein factors. Since the center of mass of these bosonic fields drops out from this expression, we get the central charge 5. Thus we have a proof that $SU(4)_2$ can be bosonized. We suggest the following bosonic form of model (32):

$$\mathcal{L}_{eff} = \frac{1}{2}\sum_{a=1}^{6}(\partial_\mu\Phi_a)^2 - \gamma\sum_{a>b}\cos\left[\sqrt{\frac{8\pi}{3}}(\Phi_a - \Phi_b)\right]. \tag{C7}$$

The center of mass field

$$\Phi_0 = \frac{1}{\sqrt{6}}\sum_a \Phi_a \tag{C8}$$

is decoupled from the interaction and is redundant. Here, however, it is important to obsverve that the fields cannot be the same as in Eq. (C6) because otherwise $\mathrm{Tr}\Phi_{adj}$ will not be local with respect to J. Using Eq. (C1) and the identification (C2), we deduce a bosonic description of model (27) when $N = 4$ with Lagrangian density:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\chi)^2 + \frac{1}{2}\sum_{a=1}^{6}(\partial_\mu\Phi_a)^2 - \lambda_1\cos(\sqrt{6\pi}\chi) - \lambda_2\cos(\sqrt{2\pi/3}\chi)\sum_{a>b}\cos\left[\sqrt{\frac{8\pi}{3}}(\Phi_a - \Phi_b)\right]. \tag{C9}$$

From Eq. (C3), we see that the operator $\cos(\sqrt{2\pi/3}\chi)$ has three vacuum expectation values (two of which are equal to each other). The minimal ground-state energy is achieved when the sign of $\gamma = \lambda_2\langle\cos(\sqrt{2\pi/3}\chi)\rangle$ is positive. At $|\lambda_1| >> |\lambda_2|$ we average over χ field first and obtain the effective theory (C7) with $\gamma > 0$. The latter can be studied either by $1/N$-expansion when we artificially extend the summation over isotopic indices to N. Then we decouple the interaction via Hubbard-Stratonovich transformation and consider a saddle point:

$$\mathcal{L}_{\text{eff}} = \frac{|\Delta|^2}{2\gamma} + \sum_a\left[\left(\Delta e^{\mathrm{i}\sqrt{8\pi/3}\Phi_a} + H.c.\right) + \frac{1}{2}(\partial_\mu\Phi_a)^2\right], \tag{C10}$$

($\gamma \sim \lambda_2$). The saddle point represents N-copies of the $\beta^2 = 8\pi/3$ sine-Gordon theory; its spectrum consists of kink, antikink and one breather with mass $\sqrt{3}$ of the kink's mass.

a. *Confinement of the heavy particles*

The structure of model (C9) and especially the λ_2 term critically affects the spectrum of the $\mathbb{Z}_4$ Fateev model (which in the given case is equivalent to the sine-Gordon model with $\beta^2 = 6\pi$ (C2)). Now we will try to discern the nature of the new excitations.

As we have said, the operator $\cos(\sqrt{2\pi/3}\chi)$ has two degenerate vacuum values. When $\lambda_2 < 0$ the values $\chi = \pm\sqrt{\pi/6}$ correspond to vacua with $\Phi_a = 0$. So one can suggest there are two types of kinks and antikinks of χ-field: short kinks where $\sqrt{6\pi}\chi$ shifts by 2π and long kinks where it shifts by 4π. Both types of kinks are $SU(4)$ neutral.

For $\lambda_2 > 0$ it is not clear what the vacuum configuration is. It may be the vacua $\chi = \pm\sqrt{\pi/6}$ and nonzero Φ_a or the single vacuum $\chi = 0$. Which situation is realized is determined by the energetics. In the first case there are short and long kinks as before, in the second one there is only one type of kinks, the one where $\sqrt{6\pi}\chi$ shifts by 6π.

The second case corresponds to three particle confinement which has not been explored in the literature. This makes it worth commenting on which we do below. Averaging by small fluctuations of Φ_a fields in (C9) we obtain the effective theory in the form of two-frequency sime-Gordon model:

$$\mathcal{L}_{\rm para} = \frac{1}{2}(\partial_\mu\chi)^2 - \lambda_1\cos(\sqrt{6\pi}\chi) - g\cos(\sqrt{2\pi/3}\chi). \tag{C11}$$

At $g = 0$ the sine-Gordon model has only kinks and no breathers. When $g > 0$ the situation changes. The second term in Eq. (C11) has three vacua: $\beta\chi = 0, 2\pi, -2\pi$. In the presence of the last term the last two vacua become false. This leads to confinement of $(0, 2\pi)$ and $(0, -2\pi)$ kinks with formations of breathers (mesons). The kinks now are between 0 and 6π vacua (hadrons) and are formed by confinement of three kinks $(0, 2\pi), (2\pi, 4\pi), (4\pi, 6\pi)$. Both mesons and hadrons are SU(4) singlets. Being topological excitations they will not experience decay from interaction with the SU(4) sector.

The confinement of two kinks can be described in a standard way. The spectrum of mesons starts from above $2M$ threshould, where M is the kink's mass in the $\beta^2 = 6\pi$ sine-Gordon model.

The confinement of three kinks can be approximately described as follows. Only $(0, 2\pi)$ and $(4\pi, 6\pi)$ kinks interpolate between the vacua with different value of $\langle\cos(\sqrt{2\pi/3}\chi)\rangle$. Kinks $(2\pi, 4\pi)$ do not change this expectation value. Therefore the linear potential exists only between the former kinks and the latter one can move freely in the space between the two. In the adiabatic approximation this movement contributes the amount

$$\delta E = \frac{\pi^2}{2M|x_{12}|^2}, \tag{C12}$$

to the total energy. As a result the Schroedinger equation for the former kinks is

$$\Big[-\frac{1}{2M}\partial_1^2 - \frac{1}{2M}\partial_2^2 + \eta|x_{12}| + \frac{\pi^2}{2M|x_{12}|^2}\Big]\Psi = (E - 3M)\Psi, \tag{C13}$$

where M is the kink's mass and $\eta \sim gM^{1/6}$ is the energy density of the false vacuum.

b. *The SU(4) sector*

Now let us return to the SU(4) sector. The kinks transform according to the fundamental representation of the SU(6) group, not the adjoint representation of the SU(4). Apparently, the above vector bosons are unstable and decay into the kinks, antikinks.

How to get the SU(4) out of this? It is possible that since one has to project out the center of mass mode (C8), the physical states are made of *confined* antisymmetric kink-antikink pairs which comprise 15-dimensional adjoint representation of the SU(4) group.

The easiest task is to calculate the breather contribution to the $\langle gg^+\rangle$ since the neutral breather can be made as a linear conbination of breathers from different sine-Gordon copies. Such excitation just corresponds to the expansion of the cosines in (C7) around their minima and hence does not contain the zero mode. So the contribution is

$$\frac{Z}{\omega^2 - p^2 - 3m^2}. \tag{C14}$$

Since the kinks and antikinks appear in pairs, the energy threshould for their emission is $2m$ or even higher if there is a real confinement and not just a condition of global neutrality. So it seems that the vector particles are indeed coherent as it has been found before.

Appendix D: Strong-coupling analysis of the lattice two-leg SU($2n$) spin ladder

In this Appendix, we consider the standard two-leg SU($2n$) ladder model (2) with $\hat{V} = \hat{P}^{(1,2)}$ (SU($2n$) permutation operator between the two chains) in the strong-coupling regime when $\lambda \to \infty$. In that limit, the model is equivalent to a single SU($2n$) spin chain where the spin operators belong to the antisymmetric representation defined by a Young tableau with a single column of two boxes. For $N = 4$, the physics of the latter model is well understood and a dimerized phase with a two-fold ground-state degeneracy occurs.[47–51] It corresponds to the plaquette phase of Fig. 1 found in our field theory analysis in the weak-coupling regime.

When $2n > 4$, the nature of the phase might be inferred by means of a non-Abelian bosonization approach starting from the underlying U($2n$) Hubbard model at a filling of 2 atoms per site ($k_F = \pi/na_0$).[16] At such a filling, there is an umklapp operator which couples the charge degrees of freedom with the non-Abelian ones:

$$\mathcal{V}_u \sim e^{2ink_F x} \prod_{i=1}^{n} L^{\dagger}_{a_i} R_{a_i} + H.c., \tag{D1}$$

where L_{a_i} and R_{a_i} are respectively the $2n$ left and right-moving Dirac fermions with $a_i = 1, \ldots, 2n$. Using the non-Abelian bosonization rule similar to Eq. (7), we get a regular contribution:

$$\mathcal{V}_u \sim -g_u \cos\left(\sqrt{2\pi n}\Phi_c\right) : (\mathrm{Tr} g)^N :, \tag{D2}$$

where Φ_c is the charge bosonic field and g is the SU(2n)$_1$ WZNW primary field with scaling dimension $1 - 1/2n$. The operator $: (\mathrm{Tr} g)^N :$ corresponds to the WZNW primary field with scaling dimension $n/2$ which transforms in the self-conjugate antisymmetric representation of the SU($2n$) group with a Young tableau of a single column with n boxes. Assuming that we have a charge gap Δ_c (with a Luttinger parameter $K_c < 4/n$), we have in the low-energy limit $E \ll \Delta_c$ the effective interacting Hamiltonian for the non-Abelian degrees of freedom:

$$\mathcal{H}_{\mathrm{eff}} \simeq -g_u : (\mathrm{Tr} g)^N :, \tag{D3}$$

which is a strongly relevant perturbation when $n \leq 3$, marginal for $n = 4$, and is irrelevant for $n \geq 5$. A straightforward semiclassical analysis gives rise, at $g_u > 0$ and $n \leq 3$, to an n-fold degenerate phase which breaks spontaneously the one-step translation symmetry: $g = e^{ik\pi/n} I$, $k = 0, \ldots, n-1$. In this respect, a dimerized (respectively trimerized) phase is expected for $n = 2$ (respectively, $n = 3$), i.e., $N = 4$ (respectively, $N = 6$). In particular, the trimerized phase for $N = 6$ where the spins form a 6-site rectangular cluster, found in the weak-coupling regime, extends to the strong-coupling case. Interestingly enough, when $n > 4$, the interaction in Eq. (D3) becomes irrelevant and an SU($2n$)$_1$ quantum criticality is then expected. For $n > 4$, we thus a expect a quantum phase transition, in the SU($2n$)$_1$ universality class, for a finite value of the interchain coupling λ. A recent variational Monte-carlo numerical analysis confirms these predictions and has shown that the SU(8) case displays a quantum critical behavior.[52]

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