Topological central charge from Berry curvature: Gravitational anomalies in trial wave functions for topological phases

Barry Bradlyn and N. Read

Phys. Rev. B **91**, 165306 — Published 20 April 2015

DOI: 10.1103/PhysRevB.91.165306

Topological central charge from Berry curvature: gravitational anomalies in trial wavefunctions for topological phases

Barry Bradlyn* and N. Read
Department of Physics, Yale University, P.O. Box 208120, New Haven, Connecticut 06520-8120, USA
(Dated: April 2, 2015)

We show that the topological central charge of a topological phase can be directly accessed from the ground-state wavefunctions for a system on a surface as a Berry curvature produced by adiabatic variation of the metric on the surface, at least up to addition of another topological invariant that arises in some cases. For trial wavefunctions that are given by conformal blocks (chiral correlation functions) in a conformal field theory (CFT), we carry out this calculation analytically, using the hypothesis of generalized screening. The topological central charge is found to be that of the underlying CFT used in the construction, as expected. The calculation makes use of the gravitational anomaly in the chiral CFT. It is also shown that the Hall conductivity can be obtained in an analogous way from the U(1) gauge anomaly.

I. INTRODUCTION

It has long been appreciated that there exists a deep connection between topologically-protected non-dissipative zero-frequency transport coefficients and Berry curvature. The archetypal example of this is the quantized Hall conductivity in the integer and fractional quantum Hall effects, which can be expressed as a Berry curvature via the Kubo formula[1,2]. Additionally, the Hall viscosity—an analogous non-dissipative contribution to the viscosity tensor of a fluid—can be expressed as a Berry curvature associated with adiabatic changes of the aspect ratio or metric tensor of a system on a torus[3–5], and is related[5,6] to the so-called shift in the number of flux in the ground state on a sphere[7]. Moreover, these properties are related to Chern-Simons terms in the effective (induced) action of the system (the first Wen-Zee term[7] in the case of Hall viscosity). The thermal Hall conductivity is related to the topological central charge of the edge theory[8], that is the difference of the central charges of right and left moving modes on the edge; the topological central charge is a topological property that takes the same value throughout a topological phase of matter. It has long been hoped that the thermal Hall conductivity can be connected with the gravitational Chern-Simons term

$$S_{\mathrm{GCS}} \propto \int d^3x\, \hat{\epsilon}^{\mu\nu\lambda} \left(\Gamma^{\rho}_{\mu\sigma} \partial_\nu \Gamma^{\sigma}_{\lambda\rho} + \frac{2}{3} \Gamma^{\rho}_{\mu\sigma} \Gamma^{\sigma}_{\nu\theta} \Gamma^{\theta}_{\lambda\rho} \right), \quad (1.1)$$

which is expected to appear in the effective action with the topological central charge as its coefficient[8]. Recently, however, it has been shown that the thermal Hall conductivity comes entirely from the edge[9,10]; accessing the topological central charge from the bulk requires more than just the Kubo formula for thermal conductivity.

In addition to Ref. 10, several other recent papers have discussed the form of the effective action for fractional quantum Hall states[11–18], and some of these address the various roles played by the central charge in particular. In contrast to these works, in this paper we show how to obtain the topological central charge as a Berry curvature associated with position-dependent changes in the spatial metric in the bulk of the system. This gives us a method to compute the coefficient of the gravitational Chern-Simons term directly from the ground-state wavefunction for a topological phase; in principle, this method can be applied as a numerical diagnostic tool. (In some cases, the central charge appears in combination with other topological invariants, not in isolation, as we discuss later.) To illustrate and validate the approach, we apply it in detail to trial wavefunctions that can be viewed as conformal blocks[19], in cases in which they represent a topological phase. Using the gravitational anomaly in the chiral conformal field theory (CFT), we calculate the Berry curvature. Thus we find that (as expected) the topological central charge of these states is equal to the central charge of the CFT used to construct the ground-state wavefunction in the bulk. We note that this approach yields the topological central charge as a real number, and not only up to addition of a multiple of 8 or 24, as can be obtained from the modular S-matrix (see e.g. Ref. 20) or from momentum polarization[21], respectively.

First, in Section II we review the connection between Berry phases and the low-energy effective action when all dynamical degrees of freedom have been integrated out (we call this the induced action; it describes the response to classical background fields). We pay particular attention to the induced action relevant for describing fractional quantum Hall states at low energies. Next, in Section III, we introduce some tools to describe the geometry of space in the presence of perturbations of the metric that we have to consider. In Section IV, we review the construction of trial wavefunctions for quantum Hall states from conformal blocks[19], and extend the construction to include perturbations of both the background vector potential and the metric of space. The role of neutralizing background charges in the CFT, and their relation to the external magnetic field, is brought to the forefront. Also in that Section, we introduce the notion of a holomorphic factorization anomaly. Roughly speaking, this is the failure of a CFT correlator to factorize

into holomorphic and antiholomorphic functions of some background fields; both a U(1) gauge field and a metric perturbation cause such anomalies. Using these tools, in Section V we calculate the Berry curvature that is found when the metric of space is changed adiabatically; we show that in this way the topological central charge c of the topological phase of matter can be obtained from Berry curvature and is related to the two-dimensional gravitational anomaly. In parallel with this, we also rederive the Hall conductivity and Hall viscosity from the same point of view, thus connecting the Hall conductivity with the U(1) gauge anomaly. In Sec. VI we make some comments on the form of the results available from the method, on some deficiencies of the Wen-Zee results[7], and on multicomponent states, before concluding.

II. BERRY CURVATURE FROM THE LOW-ENERGY EFFECTIVE THEORY

To begin, we first review the connection between Berry phases and low-energy effective actions. While fairly straightforward, this point of view has not received much attention in the recent literature. Consider a system with action $S[\phi, \mathcal{Q}]$, where ϕ stands for all the quantum fields describing the microscopic or internal degrees of freedom, and we suppose that the system is coupled to some external (classical) fields $\mathcal{Q}$ that we can control. The propagator $U[\phi_i, \phi_f, \mathcal{Q}]$ between an initial state $|\phi_i\rangle$ at time t_i and a final state $|\phi_f\rangle$ at time t_f can be written as the path integral

$$U[\phi_i, \phi_f, \mathcal{Q}] = \int \mathcal{D}\phi \, e^{iS[\phi,\mathcal{Q}]}, \tag{2.1}$$

in which ϕ has initial values ϕ_i, and final value ϕ_f. Let us assume that the action S supports a topological (i.e. gapped) phase as a ground state, and that the gap does not close as $\mathcal{Q}$ is varied. (Here we will ignore edges, for example by putting our system on a closed surface.) In this case, we can take $|\phi_i\rangle$ to be the ground state, and let $\mathcal{Q}$ vary adiabatically in time. In particular, let us consider a variation of $\mathcal{Q}$ around some *closed* path in parameter space, i.e.

$$\mathcal{Q}(t_i) = \mathcal{Q}(t_f). \tag{2.2}$$

Then, by the adiabatic theorem, we will find that

$$U[\phi_i, \phi_i, \mathcal{Q}] = e^{i\Omega}, \tag{2.3}$$

where Ω is the sum of the Berry phase γ_B and the dynamical phase

$$\gamma_D = \int_{t_i}^{t_f} E(t)dt, \tag{2.4}$$

where $E(t)$ is the ground state energy at time t. Using the definition

$$iS_{\text{eff}}[\mathcal{Q}] = \ln \oint \mathcal{D}\phi e^{iS[\phi,\mathcal{Q}]} \tag{2.5}$$

of the effective or induced action, we see immediately that

$$\Omega = S_{\text{eff}}[\mathcal{Q}], \tag{2.6}$$

that is, the change of phase of the ground state wavefunction accompanying an adiabatic change of parameters around a closed path is given by the effective action evaluated on the path. (Here, for simplicity, we assumed that there is a non-degenerate ground state; in the more general case in which the ground state is degenerate, the Berry phase factor is replaced by a unitary matrix. However, in the calculations considered in this paper, such a matrix reduces to a phase factor times the identity matrix.)

There are two sorts of terms that may appear in the effective action for a gapped phase[10]. The first are locally-invariant terms, which can be written as integrals of local expressions that are scalars under all local symmetries of the microscopic theory. Under adiabatic changes of the external fields, the time integral of these terms depend on the rate at which the path in parameter space is traversed. Hence these terms contribute to the dynamical phase γ_D (in fact, these locally-invariant terms are exactly the contributions to the ground state energy density of the system, and hence take the form of Eq. (2.4) explicitly[10]).

The low energy action may also contain Chern-Simons terms. These terms are integrals of local expressions that, rather than being invariant, change by total derivatives under the action of local symmetries. These terms have two very important properties. First, as long as there is a gap in the energy spectrum above the ground state everywhere in spacetime, their coefficients cannot be made position dependent (doing so would destroy the symmetry of the induced action). Consequently, these coefficients are topological properties, which within a topological phase do not change under changes of microscopic parameters. Second, and more directly relevant for us, when it is non-zero, the *value* of a Chern-Simons action as the external fields are varied along a closed path does not depend on the rate at which the fields are varied, because the integrand contains a part first-order in time derivatives. Thus, they contribute directly to the *Berry* phase γ_B.

Using Eq. (2.6), we can extract the Berry *curvature* associated with a given variation of external fields once we know the induced action, by a simple application of Stokes's theorem to a small loop in $\mathcal{Q}$ space. In this work, we will be interested in applying this idea to the low-energy effective action for quantum Hall states. For the precise formulation, we use the non-relativistic set-up of Ref. 10 (for general differential geometry, see e.g. Ref. 22). Briefly, this involves a choice of a frame of $d+1$ vectors with components e^μ_α varying differentiably in spacetime; μ is an ambient spacetime index, $\mu = 0$, $\ldots$, d, and α is an internal index (for the members of the frame) with the same range. The dual set of one-forms e^α_μ obeys $e^\alpha_\mu e^\mu_\beta = \delta^\alpha_\beta$ and $e^\alpha_\mu e^\nu_\alpha = \delta^\nu_\mu$; either the vectors

or the one-forms are referred to as the vielbeins. There is a Christoffel connection Γ in spacetime which is used to form covariant derivatives, and we impose rotation invariance by letting it hold locally in the internal space indices $\alpha = a = 1, \ldots, d$; this involves the introduction of a spin connection $\omega_\mu{}^\alpha{}_\beta$ which is zero for α or $\beta = 0$. We impose covariant constancy of the vielbein,

$$\nabla_\mu e^\alpha_\nu \equiv \partial_\mu e^\alpha_\nu + \omega_\mu{}^\alpha{}_\beta e^\beta_\nu - \Gamma^\lambda{}_{\mu\nu} e^\alpha_\lambda = 0. \tag{2.7}$$

This allows us to express the Christoffel symbols as

$$\Gamma^\lambda{}_{\mu\nu} = e^\lambda_\alpha \partial_\mu e^\alpha_\nu + \omega_\mu{}^a{}_b e^\lambda_a e^b_\nu. \tag{2.8}$$

Note that we do not assume the Christoffel symbols are symmetric on their lower indices, which means the spacetime geometry could possess torsion. For space dimension $d = 2$ of interest here, the spin connection reduces to an internal scalar

$$\omega_\mu = \frac{1}{2}\epsilon_a{}^b \omega_\mu{}^a{}_b, \tag{2.9}$$

as in the approach of Ref. 7. Finally, in addition to the spacetime geometry, we have a background electromagnetic field with potential [U(1) connection] A_μ, corresponding to the fact that particle number is conserved. Except when otherwise noted, this is the only conservation law that we will assume in our system, apart from those for energy, momentum, and orbital angular momentum.

In terms of these external background fields, we can write down the most general induced action at low order in derivatives. The Chern-Simons terms of interest are[7,8,10–12,15,16] (we drop the locally-invariant terms),

$$\begin{aligned} S_{\rm eff} =& \frac{\nu}{4\pi}\int d^3x\, \widehat{\epsilon}^{\mu\nu\lambda}\left(A_\mu \partial_\nu A_\lambda + 2\bar{s}\omega_\mu \partial_\nu A_\lambda + \overline{s^2}\omega_\mu \partial_\nu \omega_\lambda\right) \\ &+ \frac{c}{96\pi}\int d^3x\, \widehat{\epsilon}^{\mu\nu\lambda}\left(\Gamma^\rho_{\mu\sigma}\partial_\nu \Gamma^\sigma_{\lambda\rho} + \frac{2}{3}\Gamma^\rho_{\mu\sigma}\Gamma^\sigma_{\nu\theta}\Gamma^\theta_{\lambda\rho}\right), \end{aligned} \tag{2.10}$$

where A is the electromagnetic potential or connection, ω is a non-relativistic $2+1$ dimensional spin connection, and Γ is the Christoffel connection. Here ν is the filling factor (no confusion with the indices ν should occur), $\bar{s}$ is the mean orbital spin per particle, $\overline{s^2}$ is the mean-squared orbital spin per particle, and c is the topological central charge of the edge theory. The first term here is the U(1) Chern-Simons term. The next two terms are, respectively, the first and second Wen-Zee terms. The last is the gravitational Chern-Simons term; the fact that the coefficient is the topological central charge, which was defined in the previous section as a property of the edge, will be discussed further in Section VI. Note that $\widehat{\epsilon}^{\mu\nu\lambda}$ is the totally-antisymmetric epsilon *symbol*, not tensor, defined by $\widehat{\epsilon}^{012} = 1$.

In view of the relation (2.8), the gravitational Chern-Simons term,

$$S_{\rm GCS} = \frac{c}{96\pi}\int d^3x\, \widehat{\epsilon}^{\mu\nu\lambda}\left(\Gamma^\rho_{\mu\sigma}\partial_\nu \Gamma^\sigma_{\lambda\rho} + \frac{2}{3}\Gamma^\rho_{\mu\sigma}\Gamma^\sigma_{\nu\theta}\Gamma^\theta_{\lambda\rho}\right), \tag{2.11}$$

resembles the second Wen-Zee term. Indeed, substituting that relation into this expression, we find after some algebra (essentially as in Refs. 23 and 24)

$$\begin{aligned} S_{\rm GCS} =& -\frac{c}{48\pi}\int d^3x\, \widehat{\epsilon}^{\mu\nu\lambda}\omega_\mu \partial_\nu \omega_\lambda \\ &+ \frac{c}{288\pi}\int d^3x\, \widehat{\epsilon}^{\mu\nu\lambda} e^\sigma_\beta (\partial_\mu e^\alpha_\sigma)(\partial_\nu e^\beta_\rho)(\partial_\lambda e^\rho_\alpha). \end{aligned} \tag{2.12}$$

Finally, using

$$\partial_\nu\left(e^\alpha_\mu e^\mu_\beta\right) = 0, \tag{2.13}$$

the last term in Eq. (2.12) can be expressed as

$$\begin{aligned} &\widehat{\epsilon}^{\mu\nu\lambda} e^\sigma_\beta (\partial_\mu e^\alpha_\sigma)(\partial_\nu e^\beta_\rho)(\partial_\lambda e^\rho_\alpha) \\ &\quad = \widehat{\epsilon}^{\mu\nu\lambda}\left(e^\alpha_\rho \partial_\mu e^\rho_\beta\right)\left(e^\beta_\sigma \partial_\nu e^\sigma_\gamma\right)\left(e^\gamma_\theta \partial_\lambda e^\theta_\alpha\right). \end{aligned} \tag{2.14}$$

Written in this form, the second term in Eq. (2.12) is a topological invariant (times the topological central charge c), essentially the winding number of the map defined by the vielbein from the spacetime manifold to the general linear group $GL(3,\mathbb{R})$, and so does not vary under small variations of the vielbeins. (For manifolds with boundary, there is a variation on the boundary). We emphasize that while this result is equivalent to that presented in Ref. 18, we derived it here without any constraints on the torsion of the connection. Nonetheless, from this point on we set the reduced torsion[10] to zero, as it plays no role.

Hence the gravitational Chern-Simons term can be rewritten up to boundary contributions as

$$S_{\rm GCS} = -\frac{c}{48\pi}\int d^3x\, \widehat{\epsilon}^{\mu\nu\lambda}\omega_\mu \partial_\nu \omega_\lambda, \tag{2.15}$$

and thus, *as far as the bulk of the system is concerned*, it can be combined with the second Wen-Zee term. The resulting term contains the coefficient

$$c_{\rm app} = c - 12\nu\overline{s^2} \tag{2.16}$$

which we term the "apparent central charge." It has appeared before in special cases[12,14,15,25]; in particular for the $\nu = 1/Q$ Laughlin state, it gives $c_{\rm app} = 1 - 3Q$ when the expected values $c = 1$ and $\overline{s^2} = Q^2/4$ are used.

We see that in the induced action there are terms in which the explicit derivative is with respect to time, and the indices on the connections A_μ or ω_μ are spatial; these terms determine the Berry phase for a loop and hence the Berry curvature for those types of variation of the external fields for quantum Hall systems. Conversely, if we can compute from the ground-state wavefunction the Berry curvature associated with some perturbations of the background fields, we can calculate the coefficients in Eq. (2.10) directly; this will be carried out for some trial states in this paper. We note that the Berry curvature can also be termed the "symplectic form" for the

background fields in question, and is similar to what occurs in relation to phase space in classical mechanics.

To illustrate this procedure, let us consider the U(1) Chern-Simons action for A_μ. We take $A_0 = 0$ fixed for all time, while A_1 and A_2 evolve in time around some closed path in function space. We can then write the Chern-Simons action as

$$S_{CS} = \frac{\nu}{4\pi} \int d^2x \int dt \, [A_2(\mathbf{x},t)\partial_0 A_1(\mathbf{x},t) - A_1(\mathbf{x},t)\partial_0 A_2(\mathbf{x},t)] \, . \quad (2.17)$$

Now, examining the above, we note that $dt\partial_0 A_\mu(\mathbf{x},t)$ is independent of the parametrization t of the path in function space. In fact, this combination is simply $dA_\mu(\mathbf{x})$, the differential of $A_\mu(\mathbf{x})$ along the path at any $\mathbf{x}$, with t used as the path parameter. We may then identify the integral above with

$$S_{CS} = \int d^2x \int dA_\mu(\mathbf{x}) \mathcal{A}^\mu(\mathbf{x}), \quad (2.18)$$

where μ is summed from 1 to 2, and

$$\mathcal{A}^1 = \frac{\nu}{4\pi} A_2, \qquad \mathcal{A}^2 = -\frac{\nu}{4\pi} A_1. \quad (2.19)$$

We can view $\mathcal{A}^\mu = \delta S_{CS}/\delta\partial_0 A_\mu$ as the functional derivative of the action with respect to the time derivative $\partial_0 A_\mu$; it plays the role of a functional Berry connection. By a further functional derivative, we can form the functional Berry curvature

$$\mathcal{F}_{A_1,A_2} = \frac{\delta \mathcal{A}^2(\mathbf{x})}{\delta A_1(\mathbf{y})} - \frac{\delta \mathcal{A}^1(\mathbf{x})}{\delta A_2(\mathbf{y})} \quad (2.20)$$

($\mathcal{F}_{A_2,A_1} = -\mathcal{F}_{A_1,A_2}$), which here gives

$$\mathcal{F}_{A_1,A_2} = -\frac{\nu}{2\pi}\delta(\mathbf{x}-\mathbf{y}). \quad (2.21)$$

Then applying Stokes's theorem in parameter space we obtain

$$S_{CS} = \int d^2x \int d^2y \int\int dA_1(\mathbf{x}) dA_2(\mathbf{y}) \mathcal{F}_{A_1,A_2}(\mathbf{x},\mathbf{y}). \quad (2.22)$$

(This involves a double integral over parameters, say t, t', that cover the interior of the closed path in function space.)

One can carry out a very similar calculation in the simpler case of A_μ constant in space on a 2-torus (equivalent to a twisted boundary condition), and the result is directly related to writing the Hall conductivity as a Berry curvature or Chern number in the space of boundary conditions[2]. On the other hand, if in the same geometry we consider $A_\mu = \partial_\mu \Lambda$ ($\mu = 1, 2$) and $A_0 = 0$ globally, with Λ a globally defined function, so that the space components are those of a pure gauge, the U(1) Chern-Simons term for a closed loop in $\Lambda(\mathbf{x})$-space vanishes. This may raise some concerns as to whether we can in fact obtain the central charge as a Berry curvature in this way from the gravitational Chern-Simons term using variations of the spatial metric. Nonetheless it turns out that we can, as we will see.

III. DIFFERENTIAL-GEOMETRIC PRELIMINARIES

The results of the preceding section tell us how in principle to compute the coefficients of the Chern-Simons terms in the effective action, including the topological central charge: we should find the ground state in the presence of a perturbation of the background fields, and by varying that along a closed path, the Berry curvature. We will carry out such a calculation for conformal-block wavefunctions, but first we explain some further techniques for describing the background geometry, in particular the spatial metric. These tools could be useful in calculations for general wavefunctions as well.

In the non-relativistic formalism of Ref. 10, the metric is $g_{\mu\nu} = e^a_\mu e^b_\nu \eta_{ab}$, where $\eta_{ab} = \delta_{ab}$ is the standard metric on the internal space indices only; thus $g_{\mu\nu}$ is degenerate. From this point on, we further assume that the timelike vielbein is trivial: $e^0_\mu = \delta^0_\mu$, and similarly for e^μ_0. With this convention we can view the indices μ, ν, ..., as running over 1, 2 only; we adopt this convention when dealing with the two-dimensional spatial geometry from here on. (In Ref. 10, i, j, ..., were used, but in this paper we reserve these for particle labels.) Then the metric can be viewed as $g_{\mu\nu}$, the spatial metric in each time slice.

We assume that the space at each time is an orientable surface without boundary; at some points it will be convenient to assume the surface is a torus, but usually it can be any surface. We assume that we have fixed a choice of an atlas of coordinate charts on the surface[22], and use x^1, x^2 for the coordinates in one chart (region). For the purposes of CFT, complex coordinates are useful, but the complex structure involved is not unique, even given the fixed coordinate system. First, an *almost-complex structure* on an orientable even-dimensional manifold is a (real) tensor field $\mathcal{J}^\mu_\nu$ satisfying $\mathcal{J}^\mu_\lambda \mathcal{J}^\lambda_\nu = -\delta^\mu_\nu$ (see Ref. 26). In two dimensions, and if $\mathcal{J}$ varies sufficiently smoothly with position (which we will tacitly assume to hold), it is always possible locally to find a system of *holomorphic* complex coordinates F, $\bar{F}$ such that in F, $\bar{F}$ components $\mathcal{J}$ takes the form

$$\mathcal{J} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}. \quad (3.1)$$

F is unique modulo conformal transformations that replace F by a holomorphic function of F. We can find an atlas of such local coordinate charts[22] related by local conformal mappings in the overlap regions. A *complex*

structure is defined as the equivalence class of such atlases modulo globally-defined conformal mappings; thus in two dimensions, any globally-defined $\mathcal{J}$ determines a complex structure. On our standard coordinate chart, a particular almost-complex structure, which we call the standard one, is determined by the standard complex coordinates $z = x^1 + ix^2$, $\bar{z} = x^1 - ix^2$. In these coordinates we define complex components for vector fields and one-forms by

$$B^z = B^1 + iB^2,\ B^{\bar{z}} = B^1 - iB^2, \tag{3.2}$$

$$A_z = \frac{1}{2}(A_1 - iA_2),\ A_{\bar{z}} = \frac{1}{2}(A_1 + iA_2); \tag{3.3}$$

for such components in the standard coordinates, we frequently omit the indices z, $\bar{z}$, and write for example $A = A_z$, $\bar{A} = A_{\bar{z}}$. Similarly, we write ∂ for $\partial_z = \frac{1}{2}(\partial_1 - i\partial_2)$ and likewise for $\bar{\partial}$.

The general almost-complex structure $\mathcal{J}$ can alternatively be described using the standard one and a position-dependent *Beltrami differential* $\mu = \mu^z_{\bar{z}}$, which does not transform as a tensor (no confusion of the Beltrami differential with a spacetime index should occur). Then a holomorphic coordinate F is defined by the property that its differential dF is proportional to $dz + \mu d\bar{z}$ at each point, with $|\mu| < 1$. This means that F is a solution to the Beltrami equation (see e.g. Ref. 27)

$$\bar{\partial}F = \mu\partial F; \tag{3.4}$$

this equation always has solutions for F. We note that

$$\frac{\partial}{\partial \bar{F}} = \frac{1}{(1-|\mu|^2)\bar{\partial}\bar{F}}\left(\bar{\partial} - \mu\partial\right), \tag{3.5}$$

such that $\partial F/\partial\bar{F} = 0$, $\partial\bar{F}/\partial\bar{F} = 1$; the first condition is equivalent to the Beltrami equation.

If we change from our coordinates z, $\bar{z}$ to another set ζ, $\bar{\zeta}$ (which are functions of z, $\bar{z}$), but leave the almost-complex structure unchanged, then dF is unchanged, but μ is replaced by $\mu^\zeta_{\bar{\zeta}}$, with

$$\mu^\zeta_{\bar{\zeta}} = \frac{\frac{\partial z}{\partial\zeta} + \mu\frac{\partial\bar{z}}{\partial\zeta}}{\frac{\partial z}{\partial\zeta} + \mu\frac{\partial\bar{z}}{\partial\zeta}} \tag{3.6}$$

$$= \frac{\mu\partial\zeta - \bar{\partial}\zeta}{\bar{\partial}\bar{\zeta} - \mu\partial\bar{\zeta}}. \tag{3.7}$$

This shows that μ does not transform as a tensor, and that a holomorphic coordinate system can be viewed as a choice $\zeta = F$ such that $\mu^F_{\bar{F}} = 0$. If we begin with the zero Beltrami, $\mu = 0$, and make an infinitesimal coordinate transformation $\zeta = z + f$, where f is small, then in the ζ coordinates $\mu^\zeta_{\bar{\zeta}} = -\bar{\partial}f$ to first order.

Sometimes it is instead convenient to say that dF is proportional to $dx^1 + \tau dx^2$ at each point, $\mathrm{Im}\,\tau > 0$ (with τ depending on position). It is easy to show that

$$\mu = \frac{i - \tau}{i + \tau} \tag{3.8}$$

(this expression already appeared in Ref. 6). As an example of distinct complex structures, consider the torus described in the standard coordinates as the square region $0 \le x^1 \le L$, $0 \le x^2 \le L$ with periodic boundary conditions. If the almost-complex structure is described in these coordinates by a position-independent τ, and we define $F = x^1 + \tau x^2$, then in the complex F plane the region becomes a parallelogram, and the ratio of the two sides adjacent to the origin (i.e. of the two distinct complex displacements along the sides in the complex F plane) is τ. This is a standard way of describing the torus, used in the theory of elliptic functions (and in Refs. 5 and 6). Torii with different constant τ have inequivalent complex structures (they cannot be related by global conformal maps), and for the torus (a surface of genus one) the single complex parameter τ is the only parameter needed to determine the complex structure[26–28]. For compact surfaces of genus $\mathcal{G} > 1$, complex structures are parametrized by $3\mathcal{G} - 3$ complex parameters (so-called moduli), while for the sphere ($\mathcal{G} = 0$), there is a unique complex structure.

Any spatial line element on the surface, $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$ in the standard coordinates, can be written as[28]

$$ds^2 = e^\Phi\,|dz + \mu d\bar{z}|^2 \tag{3.9}$$

for some choice of μ. Here $e^{\Phi(z,\bar{z})}$ is the so-called conformal factor (Φ is real, and called the Liouville field). Just as μ is not a tensor, Φ does not transform as a scalar: in coordinates $\zeta(z,\bar{z})$, it becomes

$$\Phi' = \Phi + \ln\left|\frac{\partial z}{\partial\zeta} + \mu\frac{\partial\bar{z}}{\partial\zeta}\right|^2 \tag{3.10}$$

$$= \Phi - \ln\left|\partial\zeta + \mu^\zeta_{\bar{\zeta}}\partial\bar{\zeta}\right|^2. \tag{3.11}$$

The line element Eq. (3.9) corresponds to a metric tensor g which in complex coordinates has components

$$g_{zz} = e^\Phi\bar{\mu}, \tag{3.12}$$

$$g_{z\bar{z}} = \frac{e^\Phi}{2}\left(1 + |\mu|^2\right), \tag{3.13}$$

and $g_{\bar{z}\bar{z}} = \overline{g_{zz}}$, $g_{\bar{z}z} = g_{z\bar{z}}$. Let us notice that

$$\sqrt{\det\, g} = e^\Phi\left(1 - |\mu|^2\right). \tag{3.14}$$

Later, in Sec. V, it will be convenient to assume that $\det g = 1$, but in general we will not assume this, and so μ and Φ are independent variables. We note that in holomorphic coordinates, the line element becomes $ds^2 = e^\Phi|dF|^2$, where Φ is not the same one as in the standard coordinates. A nice example is the case of a sphere. In the quantum Hall literature, wavefunctions on the sphere are commonly written in stereographic coordinates, which are holomorphic coordinates $F = z$ covering all of the sphere except the south pole at $|z| \to \infty$. In

those coordinates, $\mu = 0$ and the rotation- [i.e. SO(3)-] invariant metric is determined by

$$e^{\Phi(z,\bar{z})} = \frac{1}{[1+|z|^2/(4\mathcal{R}^2)]^2}, \tag{3.15}$$

where $\mathcal{R}$ is the radius of the sphere (see e.g. Ref. 29).

Returning to the general case, we can associate with the metric (3.12), (3.13) a "canonical" choice of vielbeins. We can introduce internal complex components for internal vector and one-form fields as

$$\begin{aligned} e^+ &= e^1 + ie^2, \qquad e^- = e^1 - ie^2, \\ f_+ &= \tfrac{1}{2}(f_1 - if_2), \;\; f_- = \tfrac{1}{2}(f_1 + if_2). \end{aligned} \tag{3.16}$$

With these conventions we can rewrite the line element ds^2 in terms of the vielbeins $e^{\pm}_{\nu}$, defined by

$$\begin{aligned} e^+_\nu dx^\nu &= e^{\Phi/2}(dz + \mu d\bar{z}), \\ e^-_\nu dx^\nu &= e^{\Phi/2}(d\bar{z} + \bar{\mu} dz), \end{aligned} \tag{3.17}$$

as $ds^2 = |e^+_\nu dx^\nu|^2$. From the formalism of Ref. 10, the spin connection ω_μ (see Sec. II) in z, $\bar{z}$ space components is

$$\begin{aligned} \omega_z &= \frac{i}{1-|\mu|^2}\left(\frac{1+|\mu|^2}{2}\partial\Phi - \bar{\mu}\bar{\partial}\Phi + \bar{\mu}\partial\mu - \bar{\partial}\bar{\mu}\right), \\ \omega_{\bar{z}} &= \overline{\omega_z}. \end{aligned} \tag{3.18}$$

It will also be useful later to know that

$$\omega_{\bar{z}} - \mu\omega_z = i\left[\partial\mu - \frac{1}{2}\left(\bar{\partial}\Phi - \mu\partial\Phi\right)\right], \tag{3.19}$$

and this is proportional to $\omega_{\bar{F}}$ at each point (as for $\partial/\partial\bar{F}$ above). We should notice that if we construct the "canonical" combination $e^+_\nu dx^\nu$ in another coordinate system, then it changes by a phase, the local rotation of the coordinates. This is unconventional, as a combination $e^a_\nu dx^\nu$ should be invariant under coordinate transformations, but transform by rotation on the internal space index a under an internal rotation. The construction of the "canonical" vielbeins involves a choice of gauge for internal rotations that is related to the ambient space coordinates, such that there now *is* a change of phase induced by a coordinate transformation when this gauge choice is used. This induces an inhomogeneous term in the transformation of ω_μ also.

A further geometric structure arises because we consider charged particles in a magnetic field; the field is described by a vector potential A_μ on the surface (after choosing a gauge). A wavefunction for such a particle (it is sufficient here to consider a single-particle wavefunction; the generalization to many particles is performed in the usual way) transforms as a scalar under coordinate transformations, and by a phase factor under a gauge transformation. We can define lowest Landau level (LLL) wavefunctions ψ in our geometry analogously to their usual form, as functions that in terms of holomorphic coordinates F are annihilated by the covariant derivative with respect to $\bar{F}$. This condition reduces to

$$(\bar{\partial} - \mu\partial - i\bar{A} + i\mu A)\psi = 0. \tag{3.20}$$

For $\mu = 0$, this reduces further to the standard form often used in the quantum Hall effect (we are assuming that the magnetic field strength $B(\mathbf{x}) = \partial_1 A_2 - \partial_2 A_1$ is positive everywhere). In the coordinate patch considered, the equation has solutions of the form of a "particular" solution, a function of F and $\bar{F}$ (or z and $\bar{z}$), times any function holomorphic in F. (These must be patched together using coordinate and gauge transformations to obtain global solutions; on a compact surface, the space of solutions is finite dimensional.) Usually the LLL functions are defined as the lowest-energy states for the single-particle Hamiltonian $H_0 = -g^{\mu\nu}(\nabla_\mu - iA_\mu)(\nabla_\nu - iA_\nu)/(2m_p)$, where m_p is the mass of the particle. With the non-trivial geometry, these states may not all be degenerate, and may not be the same as we have defined. But by addition of a suitable potential term to H_0 [namely, $-B(\mathbf{x})/(2m_p\sqrt{\det g})$], the LLL states defined here can be obtained as zero-energy eigenstates (when studying this, it is convenient to use holomorphic coordinates). As we study topological properties of a gapped phase of matter, and because if a topological phase occurs in the LLL for such a Hamiltonian it can presumably be connected continuously with the same phase when it occurs for the more conventional Hamiltonian, we are free to use such a definition and consider states in which all particles are in the LLL as defined here.

IV. CONFORMAL BLOCK WAVEFUNCTIONS AND HOLOMORPHIC FACTORIZATION

Our goal is to construct the ground states of a system in a topological phase on a surface with a perturbation in the background metric, and also, both for the sake of completeness and for comparison, with a perturbation in the background vector potential. We employ the conformal block construction of Moore and Read to achieve this end. The details of the construction can be found in Refs. 5 and 19; we first summarize the main points of those works here and introduce the related issues of holomorphic factorization and anomalies, which arise when we introduce perturbations in the background fields. These subjects are explained in detail in the following subsections, first in the case of the U(1) gauge potential, then in the case of a metric perturbation.

A. Basic construction and issues arising

We begin with the case of a system forming a disk in the plane with the standard complex coordinates, without perturbations of the background fields. We represent the (non-normalized) wavefunction $\psi_e(z_1, z_2, ...z_N)$

for a quantum Hall state at filling factor ν as a chiral correlator[30] (conformal block)

$$\lim_{\alpha\to 0}\left\langle \prod_{i=1}^{N} a(z_i) \prod_{j=1}^{N/\alpha} \mathcal{O}_{-\alpha/\sqrt{\nu}}(w_j) \right\rangle_0 . \tag{4.1}$$

Here the expectation $\langle\cdots\rangle_0$ is taken in the vacuum of the CFT without background field perturbations, $i = 1$ to N labels the N particles, and $a(z)$ is the operator in the CFT that represents a physical particle (the electron or other underlying particle). The CFT is the product of two other CFTs. The first is the theory of a scalar field φ with action

$$S = \frac{1}{8\pi}\int d^2x \left(\nabla\varphi\right)^2 . \tag{4.2}$$

This theory has a U(1) current, which is $J = -i\partial\varphi(z)$ and a corresponding expression for $\overline{J}$, and this current corresponds to the conserved "charge" or particle number, so this CFT is called the charge sector. The second theory is the "statistics sector", which can be used to obtain non-Abelian quantum Hall states (for example, the Moore-Read state was obtained with a statistics sector given by a free Majorana fermion theory[19]). The particle operator is then defined as the product of a chiral vertex operator

$$\mathcal{O}_{1/\sqrt{\nu}}(z) = e^{i\varphi(z)/\sqrt{\nu}} \tag{4.3}$$

(in which $\varphi(z)$ is the chiral part of the scalar field[30]) which creates charge $1/\sqrt{\nu}$ in the charge sector (corresponding to charge 1 in physical units), and a primary field σ in the statistics sector: $a(z) = \sigma(z)\mathcal{O}_{1/\sqrt{\nu}}(z)$. (The field σ has Abelian monodromy and is a simple current, but these points will not be important for our discussion; the value of ν is determined so that the correlator is single valued as one z_i encircles another. In the Moore-Read example, σ is the Majorana field.) Because the CFT enforces neutrality in the charge sector, we have also inserted into the correlator N/α vertex operators $\mathcal{O}_{-\alpha/\sqrt{\nu}}(w_j)$ of charge $-\alpha/\sqrt{\nu}$ at positions w_j; we refer to these as "background charges". With the w_j arranged on, say, a uniform grid with spacing $\propto\sqrt{\alpha}$, and taking a limit $\alpha\to 0$, these negative charges create a continuous uniform background charge distribution, and the function (using also a singular gauge transformation) represents a wavefunction for particles in the lowest Landau level in a uniform magnetic field[5,19]. These "conformal block" trial wavefunctions include as examples the Laughlin and Moore-Read states[19], and the Read-Rezayi series[31]. We note that, in line with Sec. II, in this work we consider only one-component quantum Hall states; the particles carry no spin or other quantum numbers. There are straightforward generalizations to multicomponent conformal block states, to which our results can be extended; we return to this briefly at the end of Sec. VI.

Next we will begin in earnest to calculate the wavefunctions for a quantum Hall state on a surface in the presence of the perturbed background metric specified by Eqs. (3.12-3.13), and also a perturbed vector potential δA. Our final goal, as mentioned earlier, is to compute the Berry curvature associated with perturbations of the Beltrami differential μ. This requires that we calculate the inner products of functional derivatives with respect to the Beltrami of the states. This can be simplified if the states are normalized and depend holomorphically, or at least in an explicitly known way, on the parameters that are varied adiabatically (see e.g. Ref. 5). Therefore, we begin by examining the norms of the conformal block states in the presence of the background fields, and for this we need the conformal blocks in the presence of the backgrounds. The latter are the subject of the remainder of this section.

When we require the norms of the conformal block wavefunctions with respect to the usual inner product that involves the integral over coordinates z_i of the mod-square wavefunction, it is natural to invoke a corresponding correlator of the non-chiral (i.e. left-right symmetric) CFT whose chiral parts were used above; this is in fact the starting point for the construction of chiral CFTs. Then in the most basic cases, this non-chiral correlator can be factored into pieces respectively holomorphic (anti-holomorphic) in the particle coordinates z. It would be natural to expect similar holomorphy properties in the presence of perturbations of the background. However, when the almost-complex structure is perturbed by introducing nonzero μ, holomorphy in holomorphic coordinates like F would be more natural than in the standard coordinate z, and even then the wavefunctions are not expected to be holomorphic, but only up to the non-holomorphic factor due to the background magnetic field (such as the familiar $e^{-|z|^2/4}$ in the case of the plane) and (as we will see) the curvature of space in the case of conformal primary fields with non-zero spin. In relation to the background fields μ ($\bar{\mu}$) and $\delta\bar{A}$ (δA) (the relation of the latter with the physical potentials A is determined later in the paper, and the reason for the bar on δA is conventional), one would expect these conformal blocks to be holomorphic in their dependence on them also. However, it has been known for a long time that factorization into a factor holomorphic in those fields times its conjugate is not quite possible in general[32–34], and that this "factorization anomaly" is due to the anomalies in the gauge and coordinate-transformation invariance (or in current and stress correlations) in the chiral CFTs. To achieve factorization, and to ensure that the non-chiral theory has no anomalies, there are additional factors multiplied into the blocks, that are exponentials of integrals of local expressions (counterterms) that are neither holomorphic nor antiholomorphic in the background fields[35–37]. These can be viewed as being somewhat analogous to what occurs in the dependence on the particle coordinates due to the background magnetic field.

Below, we will first consider factorization in the pres-

ence of a U(1) gauge field rather explicitly in Subsection IV B. With the techniques demonstrated there, we will then move on to analyze the gravitational case in Subsection IV C. Finally, in Subsection IV D we will remark on the interplay between spatial curvature and the neutralizing background charges needed for quantum Hall wavefunctions.

B. Perturbed gauge field and gauge anomaly

The simplest example of a holomorphic factorization anomaly is that associated with U(1) gauge symmetry in a chiral theory; we will discuss this case in detail. Consider a chiral conformal field theory with a global U(1) symmetry. There is a current J (normalized so that

$$J = 2\pi i \frac{\delta S}{\delta\, \delta \bar{A}} \tag{4.4}$$

if S is the action that depends on the corresponding vector potential δA_μ) associated to this symmetry by Noether's theorem. Generally, the current obeys the operator product expansion[30] (or OPE; OPEs apply inside of expectation values, giving correlation functions)

$$J(z)J(0) \sim \frac{k}{z^2} + \ldots, \tag{4.5}$$

where $\ldots$ represent terms suppressed by positive integer powers of z, and the constant k is called the level of the current algebra. Similar relations hold for the anti-chiral copy of the theory, with the substitutions $J \to \bar{J}$ and $z \to \bar{z}$. We will also make use of the OPE

$$J(z)\phi(z') \sim \frac{q}{z-z'}\phi(z') + \ldots, \tag{4.6}$$

as $z \to z'$ which holds within correlation functions. This means that ϕ_i is a primary field[30] for the current algebra as well as for the Virasoro algebra; q is its U(1) charge.

Let us now couple the chiral current to a gauge field $\delta\bar{A}$. In the chiral CFT, this has the effect on (unnormalized) chiral correlators of inserting $e^{\frac{i}{2\pi}\int d^2x\, \delta\bar{A}(w)J(w)}$ into any correlator. Then for the current expectation in the presence of the background gauge field we have, using Eq. (4.5),

$$\begin{aligned}\bar{\partial}\langle J(z)\rangle &= \bar{\partial}\left\langle J(z) e^{\frac{i}{2\pi}\int d^2x\, \delta\bar{A}(w)J(w)}\right\rangle_0, \\ &\sim \frac{i}{2\pi}\int d^2x \delta\bar{A}(w) Z_{\delta\bar{A}} \bar{\partial}\langle J(z)J(w)\rangle_0, \\ &= -\frac{ik}{2} Z_{\delta\bar{A}} \partial \delta\bar{A},\end{aligned} \tag{4.7}$$

where

$$Z_{\delta\bar{A}} = \left\langle e^{\frac{i}{2\pi}\int d^2x\, \delta\bar{A}(w)J(w)}\right\rangle_0, \tag{4.8}$$

and we have used the relation

$$\bar{\partial}\frac{1}{z} = \pi\delta(\mathbf{x}). \tag{4.9}$$

This result is the gauge anomaly: the current J is not holomorphic, or conserved (divergenceless), as one might have expected naively, but instead $\bar{\partial}J = -i\frac{k}{2}\partial\delta\bar{A}$. In addition, any charged primary fields in the chiral correlator act as δ-function sources of $\bar{\partial}J$.

Let us now consider what happens in the full non-chiral theory. In the left-moving part (if the chiral or holomorphic part is right-moving), the OPEs have the same form, with the addition of bars over Js, zs, qs and k. δA is included by inserting $e^{-\frac{i}{2\pi}\int d^2x\, \delta A(w)\bar{J}(w)}$, and then $\partial\bar{J} = i\frac{\bar{k}}{2}\bar{\partial}\delta A$. Even when we set $\bar{k} = k$ (otherwise there is no possibility for the non-chiral theory to be anomaly-free), the two pieces do not cancel in $\bar{\partial}J - \partial\bar{J}$ (nor is the result gauge invariant). However, if the logarithm of the partition function (unnormalized vacuum amplitude), or induced action, of the non-chiral theory contains the additional local counterterm[35,37]

$$\delta S = \frac{k}{4\pi}\int d^2x\, \delta A \delta\bar{A}, \tag{4.10}$$

which mixes the holomorphic and antiholomorphic components of the gauge field, then the net divergence of the current is zero. We note that we can view this counterterm as producing an additional term proportional to δA_μ in the expression for the current operator, and a delta-function term in the current-current two-point correlator. Such terms are quite familiar in condensed matter physics and include, for example, the diamagnetic term in the Kubo formula for electrical conductivity. Just like in the non-chiral CFT, those terms are sometimes needed to satisfy the Ward identities which are consequences of gauge invariance or of the divergencelessness of the current.

The counterterm has implications for the factorization properties of CFT correlators. The correlator of a set of primary fields on a Riemann surface factorizes as

$$\begin{aligned}\left\langle \prod_j \phi_j(z_j, \bar{z}_j)\right\rangle = \exp\left(-\frac{k}{4\pi}\int d^2x\, \delta A \delta\bar{A}\right) \\ \times \sum_{e,e'} N_{ee'} \Psi_e(\{z_j\}|\delta\bar{A}) \bar{\Psi}_{e'}(\{\bar{z}_j\}|\delta A),\end{aligned} \tag{4.11}$$

where the conformal block functions Ψ_e are independent of δA (that is, the functional derivative with respect to δA vanishes) and are defined as

$$\Psi_e(\{z_j\}|\delta\bar{A}) = \left\langle \exp\left(\frac{i}{2\pi}\int d^2x\, \delta\bar{A}(z)J(z)\right)\prod_j \phi_j(z_j)\right\rangle_{0,e}, \tag{4.12}$$

and we denote by $\langle\ldots\rangle_{0,e}$ the conformal block in the unperturbed theory with $\delta A = 0$, with e and e' as indices for a basis of conformal blocks in that system. Also, we assume throughout that the left and right moving CFTs are the same, that the non-chiral theory is "diagonal", and so are the non-chiral fields $\phi_j(z_j, \bar{z}_j)$ that we use, that is $\bar{q}_j = q_j$. Then $\bar{\Psi}_{e'}(\{\bar{z}_j\}|\delta A) = \overline{\Psi_e(\{z_j\}|\delta\bar{A})}$, and $N_{e,e'} = \delta_{e,e'}$.

We now show how to obtain wavefunctions in the presence of a gauge field perturbation from factorization. As we wish to postpone discussion of the background charges, and thus of true quantum Hall wavefunctions, we will here continue to use a general correlator as we have so far. From Eq. (4.12), crucially we include the square root of the non-holomorphic factor into the conformal block to obtain what we can call a wavefunction ψ_e, in the presence of a background gauge field δA:

$$\psi_e(\{z_j\}|\delta\bar{A},\delta A) = e^{-\frac{k}{8\pi}\int d^2x \delta\bar{A}\delta A}\Psi_e(\{z_j\}|\delta\bar{A}). \quad (4.13)$$

The results about the cancellation of the anomaly in the non-chiral correlator can now be rephrased in terms of the behavior of these wavefunctions under a gauge transformation. As the expression is supposed to give the wavefunction for any choice of δA, $\delta\bar{A}$, it should be correct to simply substitute $\delta\bar{A}+\bar{\partial}\Lambda$ for $\delta\bar{A}$, and similarly for its conjugate. (Λ is a real function of z, $\bar{z}$.) Using integration by parts, a similar calculation as that above for $\bar{\partial}J$ shows that the wavefunction transforms by a phase that is one-half the charge of each primary field times Λ at the position of the field, plus an additional phase due to the anomaly:

$$\begin{aligned}&\psi_e(\{z_j\}|\delta\bar{A}+\bar{\partial}\Lambda,\delta A+\partial\Lambda) = \\ &\exp\left(-\frac{i}{2}\sum_j q_j\Lambda(z_j,\bar{z}_j) - \frac{ik}{16\pi}\int d^2x\,\Lambda\delta B\right) \\ &\times\psi_e(\{z_j\}|\delta\bar{A},\delta A), \end{aligned} \quad (4.14)$$

where $\delta B = \partial_x\delta A_y - \partial_y\delta A_x$ appears because of the anomaly. Note that, if the non-holomorphic term were absent, the anomaly contribution would not be a phase factor; only a transformation by a phase factor can cancel when the wavefunction is multiplied by its conjugate.

Finally, we check that the wavefunctions are holomorphic, in the sense that their covariant derivative with respect to any of the $\bar{z}_j$'s vanishes. This will be useful in ensuring that the conformal block indeed represents a lowest Landau level electron wavefunction, and in correctly relating δA in the CFT to perturbations of the physical vector potential A_μ. Taking the derivative and applying Eq. (4.6), we find for each j

$$\left(\partial_{\bar{z}_j} + \frac{i}{2}q_j\delta\bar{A}(z_j,\bar{z}_j)\right)\psi_e = 0, \quad (4.15)$$

provided z_j is not equal to any $z_{j'}$, $j'\neq j$.

C. Perturbed metric and gravitational anomaly

When we consider perturbations of the background metric rather than in a U(1) gauge field, the relevant anomaly is the gravitational or coordinate-transformation anomaly. The analog of the current is now the stress tensor T, which can be defined as[28]

$$T(z) = \pi\frac{\delta S}{\delta\mu}. \quad (4.16)$$

and similarly for the antiholomorphic $\bar{T}$. The logic we used above to compute the gauge anomaly holds essentially unchanged[35], except that now the relevant OPEs are

$$\begin{aligned}T(z)T(0) &\sim \frac{c}{2z^4} + \frac{2}{z^2}T(0) + \frac{1}{z}\partial T(0) + \dots, \\ T(z)\phi(0) &\sim \frac{h}{z^2}\phi(0) + \frac{1}{z}\partial\phi_i(0) + \dots, \end{aligned} \quad (4.17)$$

where $c = c_{\rm CFT}$ is the (right-moving) central charge of the chiral CFT, and ϕ_i is a primary field with conformal weight h_i; for brevity, we will refer to $c_{\rm CFT}$ simply as c throughout this Section and the next, Sec. V. Similar to the U(1) case above, the c-number term (with coefficient c) in the TT OPE implies the existence of an anomaly in purely chiral theories. As in the previous case, when we combine the chiral and antichiral theories, we look for a counterterm to remove the gravitational anomaly. There are two main complications here as compared to the U(1) case. First, the "gauge" group is not Abelian as it was for the U(1) gauge case, and so the local counterterms are not simply bilinear in μ, $\bar{\mu}$. Also, when the chiral and antichiral theories are combined with the appropriate counterterm, there is still a relic of the anomaly remaining in the trace anomaly associated with metric rescalings[28].

H. Verlinde compiled a particularly concrete and general form for the nonholomorphic prefactors in a correlator of primary fields in a non-chiral theory[38]:

$$\begin{aligned}\left\langle\prod_j\phi_j(z_j,\bar{z}_j)\right\rangle_g &= \exp\left(\frac{c}{12\pi}K[\mu,\bar{\mu},\Phi] - \sum_j\Phi(z_j)h_j\right) \\ &\times\sum_{e,e'}N_{ee'}\Psi_e(\{z_j\}|\mu)\bar{\Psi}'_e(\{\bar{z}_j\}|\bar{\mu}),\end{aligned} \quad (4.18)$$

where h_j are the conformal weights of the primary fields ϕ_j (as before, we have $h_j=\bar{h}_j$). Similarly, we assume the left-moving central charge $\bar{c}$ is equal to c, otherwise the non-chiral theory would be anomalous. Ψ_e is a conformal block given by

$$\Psi_e(\{z_j\}|\mu) = \left\langle\exp\left(-\frac{1}{\pi}\int d^2x\,\mu(z,\bar{z})T(z)\right)\prod_j\phi_j\right\rangle_{0,e}, \quad (4.19)$$

We note that Ψ_e is by definition independent of $\bar{\mu}$. The functional $K[\mu,\bar{\mu},\Phi]$ is the Belavin-Knizhnik counterterm[33]

$$\begin{aligned}K = &\int d^2x\,(1-|\mu|^2)^{-1}\left(\partial\mu\bar{\partial}\bar{\mu} - \frac{1}{2}\mu(\bar{\partial}\bar{\mu})^2 - \frac{1}{2}\bar{\mu}(\partial\mu)^2\right) \\ &+\frac{1}{4}\int d^2x(1-|\mu|^2)\left(\frac{1}{2}e^{\Phi}g^{\nu\lambda}\partial_\nu\Phi\partial_\lambda\Phi + \Phi R_*\right),\end{aligned} \quad (4.20)$$

where R_* is the Ricci scalar for the metric $g_* = e^{-\Phi}g$. K is the local counterterm needed to make the partition function invariant under coordinate transformations, analogous to the U(1) counterterm Eq. (4.10).

As above, we can define "wavefunctions" in the presence of a nontrivial metric as

$$\psi_e(\{z_j\}|\mu,\bar{\mu},\Phi) = \exp\left(\frac{c}{24\pi}K[\mu,\bar{\mu},\Phi] - \sum_j \Phi(z_j)h_j/2\right) \times \Psi_e(\{z_j\}|\mu). \tag{4.21}$$

To cement the identification with electron wavefunctions, we check that the functions Eq. (4.21) are annihilated by an antiholomorphic covariant derivative. We find, for each j,

$$\left[\bar{\partial} - \mu\partial + h_j\left(-\partial\mu + \frac{1}{2}\bar{\partial}\Phi - \frac{\mu}{2}\partial\Phi\right)\right]\Big|_{z_j,\bar{z}_j} \psi_e = 0 \tag{4.22}$$

(the derivatives act on, and the functions are evaluated at, z_j, $\bar{z}_j$). The terms multiplied by ih_j agree with Eq. (3.19), so this equation is equivalent to

$$(\partial_{\bar{F}} + ih_j\omega_{\bar{F}})\,|_{F_j,\bar{F}_j}\psi_e = 0. \tag{4.23}$$

As in the gauge case, the nonholomorphic prefactors ensure that the wavefunction transforms under change of coordinates only by a phase factor. In more detail, we first point out that, on calculating the expectation of $\bar{\partial}T(z)$ in the conformal block, we obtain the anomaly equation[36,37]

$$\bar{\partial}T - \mu\partial T - 2\partial\mu T = \frac{c}{12}\partial^3\mu \tag{4.24}$$

(plus terms that appear when z is at the location of one of the primary fields ϕ_j). Then including the counterterms, we find that under the infinitesimal change of coordinates $z \to \zeta = z + f$ we have, through first order in f, μ, and Φ,

$$\psi_e(\{\zeta_j\}|\mu_\zeta^\zeta,\bar{\mu}_\zeta^{\bar{\zeta}},\Phi') = \exp\left(\frac{ic}{12\pi}\int d^2x\,\mathrm{Im}(\partial f\partial^2\mu) - i\sum_j h_j \mathrm{Im}(\partial f - \bar{\mu}\bar{\partial}f)\big|_{z_j,\bar{z}_j}\right)\psi_e(\{z_j\}|\mu,\bar{\mu},\Phi), \tag{4.25}$$

where μ_ζ^ζ and Φ' are μ and Φ in the ζ coordinates (see Sec. III for their form). Here $\mathrm{Im}\,(\partial f - \bar{\mu}\bar{\partial}f)$ is the local rotation angle (where by a rotation we mean a transformation that leaves the local metric invariant). These terms underline another way to view the non-holomorphic factors $e^{-h_j\Phi/2}$ included in the wavefunctions, as follows. Usually, one considers conformal blocks and uses holomorphic coordinates, in which the primary fields transform as tensors, with the conformal weight as the (fractional) number of lower minus the number of upper indices; thus a field of weight one transforms as the components of a one-form under a conformal coordinate transformation[30]. To obtain the coordinate-invariant non-chiral correlators, each lower z index on the block must be contracted with a lower $\bar{z}$ index on the conjugate block using the inverse metric tensor $g^{\mu\nu}$, which contains a factor $e^{-\Phi}$; we choose to absorb the square root of each factor into the wavefunction. As the inverse vielbein e_a^ν is effectively a square root of the inverse metric tensor, including the factors into the wavefunctions in our case (in general, with $\mu \neq 0$) can be viewed as transforming the blocks into tensors with only internal indices, because the (inverse) vielbein converts the indices. The precise form corresponds to the way that the factor $e^{\Phi/2}$ turns $dz + \mu d\bar{z}$ (which has an upper index) into the "canonical" vielbein one-form in Sec. III, in terms of which ds^2 is simply a modulus square, just like the non-chiral correlators. Because of the use of the "canonical" vielbeins (and corresponding spin connection), our wavefunctions transform by the local rotation (phase factor) under coordinate transformation, which would not occur if they were genuine scalars. Genuine scalar behavior (up to the anomaly contribution, anyway) can be restored by combining the coordinate transformation with an internal rotation (i.e. by removing the one that is a consequence of the "canonical" vielbeins—see Sec. III).

In these expressions, we set $\delta A = \delta\bar{A} = 0$; if we wish to turn on both perturbations, we can simply include both sets of effects in a fairly straightforward way. The absence of mixed gauge-gravitational anomalies in two dimensions ensures that the factorization formulas still hold[35]. We note that then the conformal blocks depend on $\delta A_{\bar{F}}$ (a combination of δA and $\delta\bar{A}$) rather than on δA only, just as they are functions of particle coordinates z_j and $\bar{z}_j$, not z_j only; however, we will suppress both points in our notation $\Psi_e(\{z_j\}|\delta\bar{A},\mu)$ for these blocks, to lighten notation. We note that in the following Berry curvature calculations, we will in fact consider the two perturbations separately, so that the explicit form of the combined expressions will not be required.

D. Background charges, curvature, and magnetic field

We now show what happens when background charges are introduced into the wavefunctions of the previous subsections, so that we obtain true quantum Hall wavefunctions. As in the Moore-Read construction with unperturbed background fields, the positions of these primary fields will be written as w_k, reserving z_j for particle coordinates. Using the wavefunctions from the previous

subsections, we take a limit to obtain wavefunctions

$$\psi_e(\{z_j\}|\delta\bar{A},\delta A,\mu,\bar{\mu},\Phi,\rho) = \exp\left(-\frac{1}{8\pi}\int d^2x\sqrt{\det g}\delta\bar{A}\delta A + \frac{c}{24\pi}K[\mu,\bar{\mu},\Phi] - s\sum_{j'}\Phi(z_{j'})/2\right)\lim_{\alpha\to 0}\Psi_e(\{z_j\},\{w_k\}|\delta\bar{A},\mu). \quad (4.26)$$

Here we identified $h_j = s$ (the spin) for the primary fields a that represent the particles, and put $k = 1$ to agree with the Moore-Read construction. Also, $-\rho/\sqrt{\nu}$ is the density of background charge (in the z coordinates and in the CFT units); the limit is taken so that this approaches a continuous density. We note that the part of the exponential factor containing Φ evaluated at the positions w_k drops out, because the conformal weights h_k for the background charges are $O(\alpha^2)$. Taking the limit actually involves use of a singular gauge transformation[19], as we explain next; this transformation is left implicit in this expression.

Before the limit is taken, the wavefunction is not single valued as a z_i makes a circuit (under analytic continuation) around some w_js[5,19]; without loss of generality, we can think of a circuit that is not self-intersecting, contains no other z_js, and is traversed in the positive direction. The resulting phase factor is, in the limit,

$$\exp(i\theta_c) = \exp\left(-2\pi i\nu^{-1}\int d^2x\ \rho\right), \quad (4.27)$$

where the integral is over the interior of the curve, by using Stokes's Theorem. The phase factor is removed from the wavefunctions by a singular gauge transformation, whose form is clear before the limit (after the limit, it becomes very singular). The resulting wavefunction, which is the one just defined, is single valued under analytic continuation, but due to the singular gauge transformation there is an additional U(1) connection (vector potential) A^0_μ experienced by the particles, whose curl is the density (here we choose to use units where the particle has charge 1): $\partial_1 A^0_2 - \partial_2 A^0_1 = 2\pi\rho/\nu$.

Combining the last connection with those due to δA_μ and ω_μ, as found earlier in eqs. (4.15), (4.23), and comparing with the physical vector potential A_μ for particles with charge 1 (by convention) and zero orbital spin as in Eq. (3.20), we arrive at the identification of the space components of the vector potential, for our wavefunctions,

$$A_\mu = A^0_\mu - \frac{1}{2\sqrt{\nu}}\delta A_\mu - s\omega_\mu. \quad (4.28)$$

The wavefunctions $\psi_e(\{z_j\}|\delta\bar{A},\delta A,\mu,\bar{\mu},\Phi,\rho)$ are LLL functions for this A_μ, as defined earlier in Sec. III (a similar result was found in Ref. 39).

Taking the curl of the last equation, we obtain for the densities (or two-forms)

$$B = \frac{2\pi}{\nu}\rho - \frac{1}{2\sqrt{\nu}}\delta B - sR, \quad (4.29)$$

where the density $R = \partial_1\omega_2 - \partial_2\omega_1$ is essentially the Riemann tensor in two dimensions[10]. Rearranging, and setting $\delta B = 0$, we have

$$\rho = \frac{\nu}{2\pi}(B + sR), \quad (4.30)$$

This relation was recently derived using a different (but related) approach in Refs. 15, 16, and 25. Anticipating the generalized screening arguments to be made in the following section, in a wavefunction for a gapped quantum Hall state, $\rho(z,\bar{z})$ can be identified with the local particle density (in units where the particles have charge 1), up to possible corrections higher order in gradients of the background fields. Integrating over the surface and using the Gauss-Bonnet theorem then gives

$$N_\phi = \nu^{-1}N - s\chi \quad (4.31)$$

where $2\pi N_\phi$ is the integral of B (N_ϕ is the number of flux quanta), and $\chi = 2 - 2\mathcal{G}$ is the Euler characteristic ($\chi = 2$ for the sphere). This result contains the insight of Wen and Zee[7], that the shift $\mathcal{S}$ in the number of flux quanta N_ϕ ($\mathcal{S} = 2s$ for the sphere) resembles the effect of an intrinsic spin s carried by the particles, as spin couples to spatial curvature R.

V. BERRY CURVATURE OF TRIAL WAVEFUNCTIONS

In this section, we will exploit the holomorphic factorization properties of conformal block trial wavefunctions to compute Berry curvatures for quantum Hall states. In Subsection V A we consider as a warm-up exercise the response of a quantum Hall system to a uniform perturbation in the vector potential (equivalent to using twisted boundary conditions) on the torus with the standard metric, and from this derive the Hall conductivity. Using similar reasoning, in Subsection V B we will show how the topological central charge can be extracted as a Berry curvature.

A. Hall conductivity

Consider a quantum Hall system at filling factor ν on the torus. We take the metric of the torus to be the standard one, $ds^2 = |dz|^2$, and $z = x^1 + ix^2$ with x^1, x^2 running from 0 to L. As is well known, addition of a constant vector potential δA_μ on the torus is equivalent (after a gauge transformation) to twisting the boundary condition on the particles. We use the trial wavefunctions $\psi_e(\{z_j\}|\delta\bar{A},\delta A,\rho)$ obtained above, in which we

have dropped μ, $\bar{\mu}$, Φ as they are all zero. We take the background charge density, or ρ, to be uniform in these coordinates: $\rho = N/L^2$. For Laughlin states, the wavefunctions ψ_e for δA_μ constant can be expressed in terms of certain elliptic theta functions—see for example Ref. 31.

We now proceed to compute the Berry curvature associated with adiabatic transport in δA_μ. Here we invoke the hypothesis of generalized screening, that is, we suppose that in these wavefunctions all correlations of local operators decay exponentially with distance. (Note that here we mean quantum-mechanical expectations in the states with wavefunctions; in this section, angle brackets $\langle \cdots \rangle$ always mean such expectations or inner products, defined by multiplying a wavefunction and a complex conjugate wavefunction and integrating over particle coordinates with weight $\sqrt{\det g}$ for each.) In the Laughlin states, generalized screening is simply screening in the plasma; generalized screening and its consequences were discussed in Refs. 5 and 40. (Further, in Ref. 41, mappings of some states onto actual plasmas were obtained.) When the hypothesis holds, it implies[5] that the trial ground state wavefunctions, as we have constructed them, are normalized *independently of* δA_μ in the present case, at least up to higher derivative terms that we can drop. (Locally, the wavefunctions are gauge equivalent to those with $\delta A_\mu = 0$, and so the normalization is unchanged, as it is insensitive to global effects according to the hypothesis.) Then the Berry connection is

$$\mathcal{A}_{\delta A} = i \left\langle \psi_e \middle| \frac{\partial \psi_e}{\partial \delta A} \right\rangle = -\frac{iL^2}{8\pi} \delta \bar{A},$$
$$\mathcal{A}_{\delta \bar{A}} = i \left\langle \psi_e \middle| \frac{\partial \psi_e}{\partial \delta \bar{A}} \right\rangle = -i \left\langle \frac{\partial \psi_e}{\partial \delta \bar{A}} \middle| \psi_e \right\rangle = \frac{iL^2}{8\pi} \delta A, \quad (5.1)$$

where we have exploited the fact that the only dependence of the wavefunctions on δA (i.e. non-holomorphic in $\delta \bar{A}$) comes from the prefactor $\exp\left(-\frac{1}{8\pi} L^2 \delta \bar{A} \delta A\right)$. (The Berry connection is really a matrix in e, e', but off-diagonal elements are zero, and the diagonal elements are independent of e, as shown.) Taking the exterior derivative (essentially, curl) with respect to δA, we find for the components of the Berry curvature

$$\mathcal{F}_{\delta A, \delta \bar{A}} = -\mathcal{F}_{\delta \bar{A}, \delta A} = iL^2 \frac{1}{4\pi}. \quad (5.2)$$

Finally, using the correspondence Eq. (4.28), we have in terms of the physical vector potential and Cartesian components

$$\mathcal{F}_{A_1, A_2} = -L^2 \frac{\nu}{2\pi}. \quad (5.3)$$

This agrees exactly with the Berry curvature in Eq. (2.21) for this perturbation, which was obtained from the U(1) Chern-Simons term in the induced action Eq. (2.10). In fact, because the vector potential A enters only through the combination $A_\mu + s\omega_\mu$, which here for space components we can identify with $A^0_\mu - \frac{1}{2\sqrt{\nu}} \delta A_\mu$, this calculation (in which only the space components of $A_\mu + s\omega_\mu$ are perturbed adiabatically) implies that the induced action also contains the first and second Wen-Zee terms with coefficients as in that equation if we identify $\overline{s} = s$, $\overline{s^2} = s^2$, similar to Ref. 7. However, because the change in metric can also enter in other ways, we cannot yet make the last of these identifications rigorously.

Physically, we note that since a spatially uniform but temporally varying vector potential on the torus corresponds to an electric field, this Berry curvature gives us the Hall conductivity upon dividing by the area of the system, giving[2] (with the correct sign)

$$\sigma^H = \frac{\nu}{2\pi}. \quad (5.4)$$

As far as we are aware, this direct derivation of the Hall conductivity from conformal block wavefunctions using the factorization behavior of the blocks, while immediate, has not previously appeared in the literature.

We want to comment that, instead of constant δA, we could have considered functional derivatives with respect to spatially-varying δA, as in Sec. II. The derivation would be similar, and resemble more closely that in the following Subsection. Some care would be necessary in connection with normalization and screening if this involved $\delta B \neq 0$ with ρ fixed, but we will not enter into this here.

B. Central charge and Hall viscosity

Next, we consider the effect of metric perturbations in the wavefunctions $\psi_e(\{z_j\}|\delta \bar{A}, \delta A, \mu, \bar{\mu}, \Phi, \rho)$. We work again on the torus, as in the previous Subsection; we set $\delta A_\mu = 0$; and we will hold the positions of the background charges fixed in the z coordinates as the metric is varied, so $\rho = N/L^2$ is constant and fixed. In fact, this means we can hold $A^0_\mu = A_\mu + s\omega_\mu$ fixed as we vary the metric. This prevents us from picking up any Berry phase, like that in the previous Subsection, from the Hall conductivity. We consider adiabatic variation of the metric away from the standard flat one $g_{\mu\nu} = \delta_{\mu\nu}$, viewing the Beltrami differentials μ, $\bar{\mu}$ as the small independent variations; they are always assumed to vary slowly on the scale of the particle spacing. We further choose to fix $\det g = 1$, which from Eq. (3.14) implies that

$$e^\Phi = \frac{1}{1 - |\mu|^2}, \quad (5.5)$$

so Φ is determined by μ also. We emphasize that all of this means that our variations do *not* fix the magnetic field strength B; instead $B + sR$ is fixed, R can be nonzero, and hence there can be a change in B. These choices are simply the most convenient for our purpose.

Now we can again make use of the hypothesis of generalized screening. As the wavefunction transforms as a scalar, up to phase factors, under a change of coordinates, we can go to holomorphic coordinates in any local

patch, and in these the conformal block has the same dependence on the coordinates F as it does in z for $\mu = 0$. (However, in these coordinates $\rho \propto e^{\Phi} = \sqrt{\det g}$ and these are not independent of position in general; note that the wavefunctions include the effect of the interactions between the background charges, and these depend on μ and ρ.) Then the norm-square of the wavefunction is like a partition function of a two-dimensional system, which according to the hypothesis of generalized screening is in a massive phase. As for the case of an ordinary non-chiral CFT subjected to a perturbation that makes it massive (such as free fermions with a mass term, which also arise below) and in a perturbed background, the logarithm of the norm-square or partition function can be expanded as a series of integrals of local expressions, in which each local expression is a coordinate-transformation invariant constructed from the background fields and their derivatives, that is in the present case, from the metric and the curvature density (two-form) R. Thus it has the form, in an arbitrary system of coordinates ζ^μ,

$$\int d^2\zeta \left[a_0 \sqrt{\det g} + a_1 R + \ldots \right]. \tag{5.6}$$

Here the constants a_0, a_1, ... are scalars, independent of the μ that we used, but dependent on the CFT used in constructing the state. (The fact that a_0 is independent of μ can be seen for μ independent of position by using holomorphic coordinates and arguments from Section III in Ref. 6, together with the $e^{-s\Phi/2}$ factors included in the wavefunctions here. We emphasize the importance of the interactions among the background charges included in that calculation; they are presumably also the reason why our result appears different from those in Refs. 14–16, and 25, where those factors are not included in the wavefunctions.) The terms omitted are higher than second order in derivatives of $g_{\mu\nu}$; for example, one such term contains R^2 (it can be more easily expressed as the square of the Ricci scalar, times $\sqrt{\det g}$). Furthermore, $\int R$ is a topological invariant (the Gauss-Bonnet Theorem again), and so does not change as μ is varied. Hence we conclude that the norm-squares of our wavefunctions are independent of μ under the conditions we specified, through second order in derivatives. Similarly, the wavefunctions for distinct e are orthogonal. We can proceed to the calculation of the Berry curvature similarly as in the case of the Hall conductivity above, and of the Hall viscosity[5]. In fact, our approach reduces precisely to the approach of Ref. 5 if we take μ (and hence Φ) constant in space. In the wavefunctions, the factors $e^{-s\sum_j \Phi(z_j)/2}$, which are non-holomorphic in μ or τ, contain the same non-holomorphic factors $(\mathrm{Im}\,\tau)^{Ns/2}$ that were crucial in the calculation there.

Given that the wavefunctions are normalized up to a factor that is independent of μ, we will see in a moment that in order to compute the Berry curvature in the $\mu = 0$ limit, we need only retain nonholomorphic terms in the wavefunctions, and these only to order $|\mu|^2$. Expanding the counterterms, these are

$$\begin{aligned}\psi_e(\{z_i\}|\mu,\bar\mu,\Phi,\rho) \simeq \\ \left(1 + \frac{c}{24\pi}\int d^2z\, \partial\mu\bar\partial\bar\mu\right)\prod_i \left(1 - \frac{s}{2}|\mu(z_i)|^2\right) \\ \times \lim_{\alpha\to 0} \Psi_e(\{z_i\},\{w_k\}|\mu),\end{aligned} \tag{5.7}$$

and we recall that the conformal block Ψ_e is holomorphic in μ. (Here again we left the singular gauge transformation implicit; readers who have concerns about the limit and gauge transformation can simply postpone those operations to the end of the calculation.)

We define a Berry connection associated to general variations in μ in the usual way, though here we do it using *functional* derivatives with μ, $\bar\mu$, as

$$\mathcal{A}_\mu^{ee'}(z,\bar z) = i\left\langle \psi_e \left| \frac{\delta}{\delta\mu(z,\bar z)} \right| \psi_{e'} \right\rangle, \tag{5.8}$$

$$\mathcal{A}_{\bar\mu}^{ee'}(z,\bar z) = i\left\langle \psi_e \left| \frac{\delta}{\delta\bar\mu(z,\bar z)} \right| \psi_{e'} \right\rangle. \tag{5.9}$$

(Here it is important that the subscript μ or $\bar\mu$ means the function μ, not an ordinary index.) Using the orthonormality of the wavefunctions, we can see that the derivative in Eq. (5.9) is only sensitive to the nonholomorphic dependence of the chiral wavefunction on $\bar\mu$. Similarly, to evaluate Eq. (5.8) we can use the constancy of the norm of the wavefunction to move the derivative with respect to μ onto the conjugate wavefunction. With these observations, we find to first order that

$$\begin{aligned}\mathcal{A}_\mu(z,\bar z) &= \frac{i}{2}\rho s\bar\mu(z,\bar z) + \frac{ic}{24\pi}\partial\bar\partial\bar\mu(z,\bar z), \\ \mathcal{A}_{\bar\mu}(z,\bar z) &= -\frac{i}{2}\rho s\mu(z,\bar z) - \frac{ic}{24\pi}\partial\bar\partial\mu(z,\bar z).\end{aligned} \tag{5.10}$$

In the first term of each expression, we used the fact that the expectation of the particle density operator is ρ, independent of position. (We dropped the indices e, e', as once again $\mathcal{A}$ is diagonal in these indices, and independent of e.) Finally, upon taking a variational exterior derivative with respect to $\mu(z',\bar z')$ and its conjugate, we arrive at the functional Berry curvature at $\mu = \bar\mu = 0$:

$$\mathcal{F}_{\bar\mu\mu} = -\mathcal{F}_{\mu\bar\mu} = i\rho s\delta(z-z') + \frac{ic}{12\pi}\partial\bar\partial\delta(z-z'). \tag{5.11}$$

This is our first main result. The first term in the curvature is nonzero even for μ constant in space. It is exactly the Berry curvature associated with adiabatic deformations of the modular parameter τ of the torus as computed in Ref. 5, which gives the Hall viscosity[3]. The second term contains the central charge c of the (chiral) CFT.

To compare with the induced action in Eq. (2.10), we first note that we have fixed the space components of A_μ^0, so by design the contribution of the U(1) Chern-Simons and Wen-Zee terms of the form found in the Hall conductivity calculation above is absent. However, uniquely, the

first Wen-Zee term also contributes to Berry curvature through terms in the action of quadratic order in μ that arise from the *time* component ω_0 of the spin connection. To the order needed, we have for ω in terms of μ

$$\omega_z = -i\bar{\partial}\bar{\mu} + O(\mu^2), \tag{5.12}$$

$$\omega_{\bar{z}} = i\partial\mu + O(\mu^2), \tag{5.13}$$

$$\omega_0 = \mathrm{Im}\,(\mu\partial_0\bar{\mu}) + O(\mu^3). \tag{5.14}$$

Inserting these expressions in the induced action, we find that the Berry curvature above corresponds with that obtained from the Chern-Simons terms under the conditions as stated, with the Hall viscosity that agrees with the coefficient in the first Wen-Zee term[42,43], while if we assume that $\overline{s^2} = s^2$ then the topological central charge in the gravitational Chern-Simons term is exactly the chiral central charge in the CFT underlying the trial wavefunctions, as expected. This is our second main result; it is discussed further in the following Section VI.

We want to point out that the calculation in this subsection also applies to conformal-block trial wavefunctions for particle systems without a background magnetic field; such wavefunctions can even be used for systems of anyons. In this case, there are no complications related to a neutralizing background charge density. One example is a $p-ip$ paired state of fermions, represented by a conformal block in a free chiral Majorana fermion theory; it corresponds to the Moore-Read state without the charge sector. (In this example the norm-square of the wavefunction is precisely the partition function of a massive Majorana field in a background metric, which we used above to illustrate the case with background charges.) Then in all such examples the induced action lacks the U(1) gauge Chern-Simons and both Wen-Zee terms, because the filling factor is infinity and cannot occur. For the $p-ip$ example, by a calculation as here, the gravitational Chern-Simons term occurs with coefficient $c = 1/2$, while the Hall viscosity (the value of which contains the mean particle density in place of ρ, and $s = 1/2$) must now be accounted for through an "Euler current" term in the induced action[44].

VI. DISCUSSION

In this Section, we comment on our results and their applications, and discuss some of the remaining issues. We concentrate on the one-component quantum Hall examples as before, but discuss multicomponent generalizations at the end.

A. General discussion for one-component systems

First, we emphasize again that we have made the general point that the Chern-Simons terms (the terms that are not locally invariant) in the induced action can be related to Berry curvature calculations for the ground state in the presence of perturbed background fields. For nonrelativistic systems, we described the geometric aspects for metric perturbations, and how the LLL condition can be formulated. We showed that for trial wavefunctions obtained from conformal blocks, the topological central charge c is equal to the chiral central charge c_{CFT} of the CFT, up to one assumption we discuss below; this validates our procedure. But the initial point and the set-up is much more general. It can be implemented numerically, if the ground state wavefunction in the presence of (small) perturbations in the backgrounds can be found: in particular, of the metric (or vielbein), to obtain the central charge. The most straightforward way to do so is to take $\mu = \mu_0 \exp(ik_\nu x^\nu)$, with μ_0 constant, and the wavevector $\mathbf{k}$ compatible with the geometry of the torus, in which case we see that there will be a Berry curvature which for the conformal-block trial states takes the form

$$\mathcal{F}_{\bar{\mu}_0,\mu_0} = i\left(\rho s - \frac{c|\mathbf{k}|^2}{48\pi}\right) L^2 \tag{6.1}$$

in μ_0 space (when holding $A_\mu + s\omega_\mu$ fixed), where $c = c_{\mathrm{CFT}}$. We mention that this expression allows us to see immediately (by comparing with Refs. 3 and 5) that the topological central charge contributes to the $O(|\mathbf{k}|^2)$ part of the Hall viscosity[11,15], under the conditions we used to calculate it. Similarly, one could work on the sphere, using spherical harmonics. Further, the method is not restricted to the topological phases considered here, that can be modeled by wavefunctions obtained from conformal blocks. The approach also allows considerable flexibility. For example, it is not necessary to vary the magnetic field B so that $B + sR$ remains fixed: one could work at fixed B (or A_μ) instead.

There is however one important point about what we have said. First we remark that in the general case we have written the coefficient of the second Wen-Zee term as $\overline{s^2}$, which is natural if it is viewed as a Hall conductivity for orbital spin, and not all particles have the same spin. Likewise, the coefficient of the first Wen-Zee term is written in terms of $\overline{s}$, which is also related to the shift on the sphere[7], $\mathcal{S} = 2\overline{s}$. Now as discussed in Sec. II, from a bulk point of view the second Wen-Zee and the gravitational Chern-Simons terms cannot be distinguished. Then one can expect to obtain only the apparent central charge $c - 12\nu\overline{s^2}$ from the bulk induced action or Berry curvature, after the coefficients of the U(1) Chern-Simons and the first Wen-Zee term have been determined from the Hall conductivity and Hall viscosity (which we know how to obtain from Berry curvature calculations that do not involve spatial variation of μ). If we use the same technique as in the preceding calculation, of varying the metric holding $A_\mu + \overline{s}\omega_\mu$ fixed, then in general the Berry curvature contains the coefficient

$$c - 12\nu\left(\overline{s^2} - \overline{s}^2\right) \tag{6.2}$$

in place of c. (The combination $\overline{s^2} - \overline{s}^2 = \mathrm{var}\, s$ can be called the variance of the orbital spin.) Likewise,

when working with A_μ fixed, the coefficient will be $c_{\rm app} = c - 12\nu\overline{s^2}$. In general, if we assume that the coefficients of the distinct terms are as written in Sec. II, then in a numerical study we will need additional information to determine $\mathrm{var}\, s$ separately from c, where these are defined using the coefficients of terms that can only be distinguished using information from the edge of the system. Hence an additional bulk Berry curvature calculation cannot resolve this issue, even in principle.

In the context of our calculation for trial states, strictly speaking we find only that $c - 12\nu \mathrm{var}\, s$ is equal to $c_{\rm CFT}$, the central charge of the chiral CFT used in the construction of the states. If we make the natural assumption that for these states $\overline{s^2} = s^2$ (as well as using $\overline{s} = s$—see Ref. 5), then $c = c_{\rm CFT}$, as stated above.

A more general argument is also available. The second Wen-Zee term and the gravitational Chern-Simons term, while equivalent in the bulk, do differ at the edge of the system. The expectation is that the topological central charge c multiplies the gravitational Chern-Simons terms (as we have written in Sec. II). It then describes the inflow of energy and momentum onto the edge under perturbations of the vielbeins, an effect that cancels the gravitational anomaly (i.e. in energy and momentum conservation) in the edge theory[45] (see also Ref. 9). On the other hand, the Wen-Zee terms contain the spin connection ω_μ, and the corresponding gauge invariance under internal rotations is used to account for rotation invariance in ambient space (in a local time slice). The latter symmetry is obviously lost at an edge, and in the edge theory there is in general no Lorentz invariance that can take its place, even at low energies. Hence for the second-Wen-Zee term, in particular, it is not at all clear that there should be any well-defined corresponding anomaly in the boundary effective theory. [We should point out that if a system has full relativistic Lorentz invariance, both in the bulk and the edge, then the gravitational and internal Lorentz-symmetry anomalies are essentially interchangeable[23] (but even in this case, the Lorentz invariance on the boundary is in one dimension fewer than in the bulk). We cannot expect the same to hold in the non-relativistic situation considered here.] Then we can make the field-theoretical argument that the coefficient of the gravitational Chern-Simons term *has* to be the topological central charge c of the edge theory (just as written in Sec. II). For the conformal-block trial states, the topological central charge of the edge theory is supposed to be the same as the central charge of the CFT used in the construction (this is part of a conjecture that goes back to Moore and Read[19], and corresponds to a result of Witten[46]), and so our conclusion from the calculation is that

$$\mathrm{var}\, s = 0. \tag{6.3}$$

In general, this argument shifts the discussion from one about the topological central charge to one about the value of $\overline{s^2}$: $\overline{s^2}$ can be obtained from $c_{\rm app}$ if the value of c is taken as known. We remark that our result $\mathrm{var}\, s = 0$ for the trial states considered here disagrees in a number of cases with the recent results of Ref. 18 (in which $c = c_{\rm CFT}$ was also obtained); after the appearance of the first version of the present paper, these authors discovered an error in their calculation, and their revised result[47] agrees with ours.

It may be thought that for Abelian quantum Hall states, the value of $\overline{s^2}$, as well as that of $\overline{s}$, or the shift, can be predicted analytically from the work of Wen and Zee (WZ)[7]. In the case of the shift, its value is easily obtained, at least in many cases, from other approaches such as trial wavefunctions, but the situation is less clear for $\overline{s^2}$. For conformal-block states, $\overline{s^2} = \overline{s}^2 = s^2$, where s is the conformal weight of the field a that represents the particles, seems to be natural as we have said, and is supported by our results under an assumption. In particular, for the Laughlin state at filling $\nu = 1/Q$, we agree with WZ that $\overline{s^2} = Q^2/4$. In certain other cases, obtaining it is straightforward. In the case of non-interacting fermions filling (integer) ν Landau levels, the orbital spin is $\mathcal{N} + 1/2$ for the $\mathcal{N}$th Landau level ($\mathcal{N} = 0, 1, \ldots$; note that the angular momentum is minus this spin in our conventions), which agrees with the angular momentum due to the cyclotron motion. In this case $\overline{s^2}$ is calculable, and the result should be correct (it seems to have motivated WZ[7]). Some further examples are discussed at the end of this section.

For general Abelian states, WZ used an approach in which there are in effect several types of particles, with different charges under the U(1) gauge field. (These types of particles may correspond to the use of hierarchical constructions, or other techniques.) They argued that by analogy with the charge, these particles can have different spins, through which they couple to spatial curvature. They begin with an Abelian Chern-Simons effective (not induced) action (which unlike the induced action we use in this paper) describes the dynamics of the low-energy states of the systems, which are not yet integrated out. It contains as its dynamical variables one or more gauge fields a_μ, and the coupling to the background gauge field A_μ (spin connection ω_μ) is through a charge vector t (spin vector s) that has components in the index space of the fields a_μ (which we suppress in our notation). On integrating out the dynamical gauge fields a_μ, they obtained an induced action of the same form as our $S_{\rm eff}$, in which ν, $\overline{s}$, and $\overline{s^2}$ are given by expressions containing the K-matrix of the Abelian state and the charge and spin vectors t and s. (However, they omitted the gravitational Chern-Simons term entirely, which should arise from the derivation; this has been corrected recently[18].) Their main result was explaining the shift $\mathcal{S}$ in terms of $\overline{s}$, as we have already mentioned.

However, WZ also obtained expressions for the spin of the quasiparticle excitations of the Abelian states (as well as expressions for the charge and statistics that are not in question). They pointed out that, with their expressions, the spin-statistics theorem does not hold (this is reaffirmed in their Erratum[48]). In two dimensions, for

quasiparticles with Abelian statistics, the spin-statistics theorem says that

$$\theta = 2\pi s \ (\mathrm{mod}\, 2\pi), \tag{6.4}$$

where θ, which is only defined modulo 2π, is the Berry phase picked up on adiabatically exchanging two identical such quasiparticles along a counter-clockwise semicircle (and removing the Aharonov-Bohm phase[49]), and s is the spin of one of the quasiparticles, under the same sign convention as in this paper. For a simple topological derivation that holds both relativistically and nonrelativistically, see Ref. 50. In Abelian Chern-Simons gauge theories, as well as in related CFTs, the statistics of the quasiparticles, as well as their spins, are quadratic expressions in their quantum numbers (the fluxes in the dynamical gauge fields). However, while for the statistics WZ agree that the expression is quadratic, WZ's spins contain a term linear in the quantum numbers, that is also linear in the spin vector; this is the source of the disagreement. It is also a requirement that if the statistics of a quasiparticle of type α is θ_α, then the statistics of its antiparticle type $\bar{\alpha}$ (which has the opposite sign quantum numbers) is $\theta_{\bar{\alpha}} = \theta_\alpha$ (all of these are modulo 2π). If the expression for the spin contains a term linear in the quantum numbers (and in the spin vector), then it is not possible to satisfy this relation either. We note that, in the modular tensor category point of view, which captures the strictly topological properties of a topological phase, spin is only defined modulo integers, just as statistics is only defined modulo 2π for Abelian quasiparticles. (These statements were reviewed in e.g. Ref. 5.) This makes sense, because it is always possible in principle to make some local, bosonic, excitation near a quasiparticle that changes its angular momentum by an integer, without changing anything topologically. For quasiholes in conformal-block states on the sphere, the spins were shown to agree with the conformal weights[29], and not only modulo integers, and to obey the spin-statistics theorem. The Laughlin state is one of these, and the result for that is in direct disagreement with WZ.

The exception to the difficulty in satisfying the requirements in the presence of a linear term in the spin is that, in some cases, the quantities can be equal modulo 2π, even though they are not equal as real numbers. This is what occurs for the case of ν filled Landau levels. There, the spin of a hole in one of the filled Landau levels is 1/2 (modulo an integer), and spin-statistics is satisfied. The question of antiparticles does not arise, because additional particles cannot be created in the filled Landau levels (other than to destroy a hole) to make the antiparticle of the hole. Particles can, however, be created in one of the empty Landau levels above, and are again fermions with half-integer spin. It is not clear if they are described by the ν-component dynamical Chern-Simons gauge theory, however.

In view of these observations, we do not accept in general the spin values assigned to quasiparticles by WZ's theory. This may also cast some doubt on the values of $\overline{s^2}$ that were obtained by WZ; at the least, aspects of the derivation they gave (some of which was also used in the recent Refs. 12 and 18) seem questionable.

B. Multicomponent systems

Finally, we return briefly to the topic of extending the results for conformal-block trial states to multicomponent conformal-block states. This can mean either that the underlying particles, whose wavefunction we are describing, are identical, but carry additional quantum numbers or "internal" indices, which can represent internal (not orbital) spin or layer indices, and so on, or that there are simply different types of particles present, so that particles of distinct types can be distinguished from one another. (In the remainder of this section, we use indices a, b, $\ldots = 1, \ldots, \mathcal{N}$ to represent particle types, and emphasize that these do not in general correspond to quasiparticle types which were mentioned just above.) In addition, there can be background magnetic fields, which may be different for different types of particles, and not all derived from a single U(1) magnetic field as was the case previously. In the conformal-block trial wavefunctions, each particle of type a has an orbital spin s_a, equal to the conformal weight of the primary field that represents that particle type in the CFT; clearly, particle types that are related by a symmetry will have the same s_a. We begin with the Berry curvature for varying the metric, with the background charge densities held fixed, and additional vector potentials (corresponding to δA_μ in Sec. V; both the magnetic fields and the vector potentials are discussed further below) set to zero. Then it is almost immediate that the Berry curvature has the same form as obtained in the one-component cases in Sec. V B, with the Hall viscosity equal to one half the density of orbital spin (i.e. to $\frac{1}{2}\sum_a n_a s_a$, where n_a is the number density for each type a), and $c = c_{\mathrm{CFT}}$ is the central charge of the CFT used in the construction. In this sense, the preceding result generalizes straightforwardly.

For the remainder of the induced action, or of the responses to background fields, the structure is more involved, and here a brief sketch will have to suffice. In general, the presence of more than one particle type will imply that there are additional conserved quantities in the system, such as the number of particles of each type or for each index value (these correspond to U(1) symmetries), and there could be further symmetries that map one index value to another [so that some U(1)s are in fact part of a non-Abelian group, such as SU(2)], as in the case of internal (not orbital) spin. If the largest possible group of continuous "internal" Lie-group symmetries is identified, then we will assume that it has the form of a direct product of one or more U(1) factors with one or more simple Lie groups, each of which might be SU(n), Spin(n), Sp($2n$) (each for some value of n), or one of the exceptional simple Lie groups, and possibly a quotient by some discrete subgroup must also be taken, to pre-

cisely describe the group. (We ignore the latter, as we usually focus on the Lie algebra level of description.) We have mentioned that we can include background magnetic fields; we assume that these take the form of U(1) field strengths only. As we intend the symmetry group we identified to be a symmetry of both the CFT used and of the states, the background magnetic fields must respect the symmetries; if one begins with a non-Abelian group and introduces a background field strength this breaks the symmetry down to some U(1)s by assumption, and it is only the remaining unbroken symmetry to which we will refer as the "symmetry".

The induced action should now be extended to include vector potentials corresponding to each of these continuous symmetries. Apart from the gravitational Chern-Simons term and second Wen-Zee term, whose form is unchanged from the one-component case, the Chern-Simons terms allowed in the induced action are Abelian and non-Abelian Chern-Simons terms that involve only the gauge (vector) potentials, and the first Wen-Zee term, which involves ω_μ in addition to a fixed linear combination of the U(1) vector potentials. The U(1) potentials will be important to us, so (choosing a particular basis) we will denote them by $A_\mu^{(a)}$, one for each particle type a.

Next we consider the conformal-block trial wavefunctions obtained by a Moore-Read construction subject to the assumptions so far mentioned. In general, we may introduce into the construction both the background spatial metric and corresponding spin connection, and for each continuous symmetry (whether Abelian or non-Abelian) *for which there is a conserved current in the CFT*, a corresponding vector potential (generalizing δA_μ in Sec. V). We note that there may not be such currents in the CFT for every conserved quantity (or symmetry) in the system; for example, in the Majorana field theory addressed at the end of Sec. V, there is a conserved particle type, but no U(1) current. But for each current operator in the CFT, there are non-zero gauge anomalies, and these produce corresponding Berry curvatures (obtained with the remaining vector potentials and metric held fixed). For the U(1) vector potentials, these can take the form of a matrix of terms, as we will describe in a moment. It is only for these U(1)s that background magnetic fields can be present in the state, because such a field has to be simulated by background charge densities as in the one-component case addressed so far.

We must also relate the "physical" vector potentials to the corresponding ones used in the conformal blocks. We focus on the U(1) vector potentials; the non-Abelian ones can be handled in a similar way, but with fewer complications. For the U(1) vector potentials, we first make a further assumption about what our trial states describe, for definiteness: namely, we assume that the underlying particles all have zero orbital spin. (This was the assumption above, also.) We will label the U(1) vector potentials $\delta A_\mu^{(u)}$ (corresponding to U(1) current operators in the CFT) by indices u, v, $\ldots = 1, \ldots, \mathcal{M}$ ($\mathcal{M} \leq \mathcal{N}$). When we examine the holomorphy condition obeyed by the wavefunction in each particle coordinate, similar to Sec. IV D it contains the background magnetic field seen by that particle type, the spin connection times s_a for that type, and some combination of the background gauge field perturbations consisting of the U(1) potential $\delta A_\mu^{(u)}$ (and possibly also some non-Abelian vector potentials), and this combination must be identified with the "physical" potential, which consists of $A_\mu^{(a)}$ and some non-Abelian potentials. Rearranging slightly, for the U(1) parts this takes the form

$$A_\mu^{(a)} + s_a \omega_\mu = P_{au} \delta A_\mu^{(u)}, \tag{6.5}$$

where P is an $\mathcal{N} \times \mathcal{M}$ matrix, and we omit the vector potentials A_μ^{0u} representing background magnetic fields, as they do not affect the argument. This relation implies that the shift in the number of flux seen by particles of type a is always $2s_a$, even when there is no extensive background magnetic field (for the $p-ip$ example, this was pointed out in Ref. 8.) We can assume that P is an injective map (it has rank $\mathcal{M}$), because otherwise some of the U(1) currents and vector potentials can be eliminated by a change of basis, reducing $\mathcal{M}$.

The subsequent analysis is most conveniently handled in distinct cases. First we describe what we will call the non-degenerate case. In this case, P is invertible, so in particular $\mathcal{N} = \mathcal{M}$, and a similar statement holds for the non-Abelian gauge fields, if any. In this case we obtain states that describe true multicomponent quantum Hall-like states that have a gap in the bulk excitation spectrum for all excitations, so that the induced action is local. As P is invertible, for the U(1) gauge fields we can choose the basis for the space with indices u such that P is the identity, and then use only a, b indices. Then the part of the induced action containing the U(1) vector potentials takes the form

$$\frac{1}{4\pi} \int d^3x \, \widehat{\epsilon}^{\mu\nu\lambda} \sum_{a,b} K_{ab}^{-1} (A_\mu^{(a)} + s_a \omega_\mu) \partial_\nu (A_\lambda^{(b)} + s_b \omega_\lambda). \tag{6.6}$$

where K_{ab}^{-1} is the matrix of gauge anomalies in the U(1) currents (suitably normalized), or Hall conductivities, with the a, b entry corresponding to currents of types a, b in the CFT; it is real, symmetric, and positive definite, because the CFT used is chiral and should be unitary[5]. This form resembles the induced action in WZ, but has additional gauge potentials. (We point out that the WZ effective action[7] actually has additional U(1) symmetries, so that such background gauge potentials could have been introduced there also.) The remaining Chern-Simons terms in the induced action are the gravitational Chern-Simons term, which was already discussed, and the non-Abelian (pure gauge) Chern-Simons terms. Thus we have determined the coefficients of all Chern-Simons–type terms in the induced action, using Berry curvature. We see that the analog of $\nu \overline{s^2}$ is

$$\nu \overline{s^2} \longrightarrow \sum_{a,b} K_{ab}^{-1} s_a s_b, \tag{6.7}$$

just as in WZ[7], however in the present very general point of view, it is not clear if there is always a unique way to identify ν and so separate it from this expression. (There are different types of particles, and different magnetic fields for each, and thus different filling factors; indeed the filling factor has become a matrix.) Then if one calculates the Berry curvature for varying the background metric with all the vector potentials including $A_\mu^{(a)}$ (instead of $A_\mu^{(a)} + s_a\omega_\mu$) for all a held fixed, the coefficient will be $c_{\rm app} = c - 12\sum_{a,b} K^{-1}_{ab} s_a s_b$ (instead of c). In these more general examples, we cannot necessarily obtain a simple relation between some $\bar{s}$ and $\overline{s^2}$. On the other hand, we have seen that an analog of $\operatorname{var} s = 0$ does hold in these states, in the sense that the Berry curvature with $A_\mu^{(a)} + s_a\omega_\mu$ (for all a) and non-Abelian vector potentials held fixed contains just the coefficient c, with no correction like $\operatorname{var} s$.

In the degenerate case (we assume it is the U(1) sector that causes the degeneracy; again, the non-Abelian analog is similar), the trial wavefunctions are expected to possess long-range correlations in some off-diagonal local operators, and so for a Hamiltonian that is short-range and respects the symmetries they describe gapless phases of matter that exhibit broken symmetries; particular examples include quantum Hall ferromagnets, the $p - ip$ paired superfluid as described in Sec. V B, anyon superfluids, and combinations of these. Accordingly, the full induced action may not be local. We can nonetheless analyze the Berry curvatures possessed by the trial states, and we will describe these in terms of Chern-Simons terms as before; these terms are likely to appear in any induced action for the system. Now the relations (6.5) act as constraints on some combinations of $A_\mu^{(a)}$, which cannot be varied freely at fixed ω_μ (by varying $\delta A_\mu^{(u)}$) as they could in the non-degenerate case. P can be inverted only for those $A_\mu^{(a)} + s_a\omega_\mu$ that satisfy these constraints. For such background gauge fields, the U(1) part of the induced action takes the form

$$\frac{1}{4\pi}\int d^3x\, \hat{\epsilon}^{\mu\nu\lambda} \sum_{u,v} K^{-1}_{uv} [P^{-1}(A_\mu + s\omega_\mu)]_u \times \partial_\nu [P^{-1}(A_\lambda + s\omega_\lambda)]_v. \quad (6.8)$$

Here K^{-1}_{uv} is a real symmetric positive-definite matrix of the gauge anomalies as before, and $[P^{-1}(A_\mu + s\omega_\mu)]_u$ is the inverse image of the set of $A_\mu^{(a)} + s_a\omega_\mu$, which exists because of the constraint, and is unique because P is injective. We have already described the Berry curvature for varying the metric with $\delta A_\mu^{(u)}$ fixed for all a (which gives the Hall viscosity and c), and for varying the $\delta A_\mu^{(u)}$ with fixed metric (which gives the matrix K^{-1}_{uv}). (We should also point out that in these cases the Hall viscosity and the shifts are not fully accounted for by these terms in the induced action; an example of this was already seen at the end of Sec. V B.) It remains to discuss the analog of $\nu\overline{s^2}$. Previously, this could be obtained by varying the metric with $A_\mu^{(a)}$ held fixed, at least by using the induced action, and probably also as a Berry curvature. In the present case, the same can be done when the spin vector with components s_a lies in the image space of P (i.e. it is a linear combination of the columns of P). Then it is possible to hold $A_\mu^{(a)}$ fixed as the metric is varied, and still satisfy the constraint. In this special case, the analog of $\nu\overline{s^2}$ is $\sum_{u,v} K^{-1}_{uv}(P^{-1}s)_u(P^{-1}s)_v$. (This case subsumes the non-degenerate cases as well.) But in the general case, in which the spin vector s_a does not lie in the subspace, when ω_μ varies (because of the variation in metric), $A_\mu^{(a)}$ must change somewhat, so as to remain in the constrained subspace at each point in space. While varying with $\delta A_\mu^{(u)}$ held fixed makes sense, keeping $A_\mu^{(a)}$ fixed except so as to satisfy the constraint requires a projection of the $A_\mu^{(a)}$ before the variation into the constrained space which has been translated because of the change in ω_μ. In the absence of an inner product on the space with indices a, there is no unique way to do this, or to define $\nu\overline{s^2}$; the answer depends on exactly what is held fixed, and there is no preferred choice.

For cases in which degeneracy occurs in the non-Abelian sector, a different phenomenon can occur, in which the symmetries generated by the currents in the CFT form a proper subgroup of the non-Abelian symmetries of the particles. An example is the n-component version of the one-component $p - ip$ state discussed at the end of Sec. V B. The CFT consists of n chiral Majorana fields, and has SO(n) symmetry and currents, which give rise to an SO(n) level-1 Chern-Simons term in the induced action, while the particle system may have U(n) symmetry. This makes sense if we understand the trial state as a paired superfluid of n particle types (each type pairing with itself), and the pairing as breaking the symmetry down from U(n) to an SO(n) subgroup. The system possesses non-Abelian Hall-conductivity responses in the unbroken SO(n) subalgebra.

It might be interesting to pursue further extensions to conformal-block trial states in which the CFT has higher-spin current algebras, such as W- and superconformal algebras[30], for which there are sometimes further coefficients that we expect can be obtained from Berry curvature. These algebras should correspond to additional structures on the spacetime, and to additional terms in the induced action.

VII. CONCLUSION

To summarize, we have shown that the topological central charge can be obtained from the Berry curvature arising from the ground state wavefunction when the spatial metric is varied adiabatically. This method can be applied in a numerical calculation. We showed analytically that it does produce the expected result, equal to the central charge of the underlying CFT, when applied to topological phases that possess conformal-block trial

wavefunctions in which the generalized screening hypothesis holds. We emphasized that this is a *bulk* approach, which does not involve an edge, and that the result is a topological invariant that characterizes a topological phase. More precisely, in some cases the central charge is obtained only in combination with another topological property νs^2, as emphasized in Sec. VI. We explained how in general, if the value of the topological central charge is assumed known from the edge theory, then the Berry curvature yields instead the value of νs^2.

ACKNOWLEDGMENTS

We would like to thank J. Dubail and G. Moore for helpful discussions, and we are grateful for the hospitality of the Simons Center for Geometry and Physics in Stony Brook, NY, and of the Erwin Schrödinger Institute in Vienna, Austria, at each of which parts of this work were conceived. We are also grateful to an anonymous referee, who suggested that we give more details for multicomponent states. The work of B.B. and N.R. was supported by NSF grants, Nos. DMR-1005895 and DMR-1408916.

* barry.bradlyn@yale.edu

[1] D. J. Thouless, M. Kohmoto, M. P. Nightingale, and M. den Nijs, Phys. Rev. Lett. **49**, 405 (1982).
[2] Q. Niu, D. J. Thouless, and Y.-S. Wu, Phys. Rev. B **31**, 3372 (1985).
[3] J. E. Avron, R. Seiler, and P. G. Zograf, Phys. Rev. Lett. **75**, 697 (1995).
[4] P. Lévay, J. Math. Phys. **36**, 2792 (1995).
[5] N. Read, Phys. Rev. B **79**, 045308 (2009).
[6] N. Read and E. H. Rezayi, Phys, Rev. B **84**, 085316 (2011).
[7] X.-G. Wen and A. Zee, Phys. Rev. Lett. **69**, 953 (1992).
[8] N. Read and D. Green, Phys. Rev. B **61**, 10267 (2000).
[9] M. Stone, Phys. Rev. B **85**, 184503 (2012).
[10] B. Bradlyn and N. Read, Phys. Rev. B **91**, 125303 (2015).
[11] A. G. Abanov and A. Gromov, Phys. Rev. B **90**, 014435 (2014).
[12] A. Gromov and A. G. Abanov, Phys. Rev. Lett. **113**, 266802 (2014).
[13] A. Gromov and A. G. Abanov, Phys. Rev. Lett. **114**, 016802 (2015).
[14] T. Can, M. Laskin, and P. Wiegmann, Phys. Rev. Lett. **113**, 046803 (2014).
[15] T. Can, M. Laskin, and P. Wiegmann, (2014), arXiv:1411.3105.
[16] T. Can, M. Laskin, and P. Wiegmann, (2014), arXiv:1412.8716.
[17] G. Y. Cho, Y. You, and E. Fradkin, Phys. Rev. B **90**, 115139 (2014).
[18] A. Gromov, G. Y. Cho, Y. You, A. G. Abanov, and E. Fradkin, Phys. Rev. Lett. **114**, 016805 (2015).
[19] G. Moore and N. Read, Nucl. Phys. B **360**, 362 (1991).
[20] Y. Zhang, T. Grover, A. Turner, M. Oshikawa, and A. Vishwanath, Phys. Rev. B **85**, 235151 (2012).
[21] H.-H. Tu, Y. Zhang, and X.-L. Qi, Phys. Rev. B **88**, 195412 (2013).
[22] S. Carroll, *Spacetime and Geometry: an Introduction to General Relativity* (Addison Wesley, San Francisco, CA, 2004).
[23] W. A. Bardeen and B. Zumino, Nucl. Phys. B **244**, 421 (1984).
[24] L. Alvarez-Gaume, S. Della Pietra, and G. Moore, Annals of Physics **163**, 288 (1985).
[25] F. Ferrari and S. Klevtsov, JHEP **12**, 086 (2014).
[26] M. B. Green, J. H. Schwarz, and E. Witten, *Superstring Theory*, Vol. 2 (Cambridge University Press, Cambridge, 1987).
[27] L. Bers, Bull. London Math. Soc. **4**, 257 (1972).
[28] D. Friedan and S. Shenker, Nucl. Phys. B **281**, 509 (1987).
[29] N. Read, (2008), arXiv:0807.3107.
[30] P. DiFrancesco, P. Mathieu, and D. Senechal, *Conformal Field Theory* (Springer, New York, 1997).
[31] N. Read and E. Rezayi, Phys. Rev. B **59**, 8084 (1999).
[32] D. Quillen, Functional Analysis and Its Applications **19**, 31 (1985).
[33] A. A. Belavin and V. G. Knizhnik, Phys. Lett. B **168**, 201 (1986).
[34] L. Alvarez-Gaumé, G. Moore, and C. Vafa, Commun. Math. Phys. **106**, 1 (1986).
[35] L. Alvarez-Gaumé and E. Witten, Nucl. Phys. B **234**, 269 (1984).
[36] A. M. Polyakov, Mod. Phys. Lett. **A2**, 893 (1987).
[37] A. M. Polyakov, in *Fields, Strings, and Critical Phenomena* (Les Houches, North Holland, Amsterdam, 1988) pp. 305–368.
[38] H. Verlinde, Nucl. Phys. B **337**, 652 (1990).
[39] S. Klevtsov, JHEP **2014**, 133 (2014).
[40] C. Nayak and F. Wilczek, Nucl. Phys. B **479**, 529 (1996).
[41] P. Bonderson, V. Gurarie, and C. Nayak, Phys. Rev. B **83**, 075303 (2011).
[42] N. Read and W. Goldberger, (2009), unpublished.
[43] C. Hoyos and D. T. Son, Phys. Rev. Lett. **108**, 066805 (2012).
[44] S. Golkar, M. M. Roberts, and D. T. Son, (2014), arXiv:1407.7540.
[45] C. G. Callan and J. A. Harvey, Nucl. Phys. B **250**, 427 (1985).
[46] E. Witten, Commun. Math. Phys. **121**, 351 (1989).
[47] A. Gromov, G. Y. Cho, Y. You, A. G. Abanov, and E. Fradkin, (2015), arXiv:1410.6812, Erratum.
[48] X.-G. Wen and A. Zee, Phys. Rev. Lett. **69**, 3000 (1992).
[49] D. Arovas, J. R. Schrieffer, and F. Wilczek, Phys. Rev. Lett. **53**, 722 (1984).
[50] F. Wilczek and A. Zee, Phys. Rev. Lett. **51**, 2250 (1983).