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# Competing chiral orders in the topological Haldane-Hubbard model of spin-1/2 fermions and bosons

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Motivated by experiments on ultracold atoms which have realized the Haldane model for a Chern insulator, we consider its strongly correlated Mott limit with spin-1/2 fermions. We find that slave rotor mean field theory yields gapped or gapless chiral spin liquid Mott insulators. To study competing magnetic orders, we consider the strong coupling effective spin Hamiltonian which includes chiral three-spin exchange. We obtain its classical phase diagram, uncovering various chiral magnetic orders including tetrahedral, cone, and noncoplanar spiral states which can compete with putative chiral quantum spin liquids. We study the effect of thermal fluctuations on these states, identifying crossovers in the spin chirality, and phase transitions associated with lattice symmetry breaking. We also discuss analogous effective spin Hamiltonians for correlated spin-1/2 bosons. Finally, we point out possible experimental implications of our results for cold atom experiments.

## I. INTRODUCTION

Momentum space topology is key to our understanding of phases such as topological insulators<sup>1,2</sup> or quantum anomalous Hall insulators.<sup>3,4</sup> The interplay of momentum space topology and local real space interactions is expected to lead to a rich variety of correlated insulating phases.<sup>5–12</sup> This has motivated an extensive investigation of the time-reversal invariant Kane-Mele-Hubbard model or more realistic variants,<sup>13–19</sup> which provide the simplest examples of interacting quantum spin Hall insulators. Recently, cold atoms in a shaken optical lattice<sup>20</sup> or laser-induced tunneling or transitions<sup>21,22</sup> have been proposed for realizing topological states of matter, and employed to realize the Haldane honeycomb model of a Chern insulator (CI),<sup>23</sup> and Chern bands in the square lattice Hofstadter model with flux  $\pi/2$  per plaquette.<sup>24</sup> Preliminary experiments suggest that interactions in the Haldane model may not lead to excessive heating.<sup>23</sup> Motivated by this, we explore strong correlation effects in the Haldane model for spin-1/2 fermions and bosons.

The Haldane model is defined on the two-dimensional (2D) honeycomb lattice in Fig. 1(a), with a real nearest neighbor hopping amplitude  $t_1$ , and a complex next-neighbor hopping  $t_2 e^{i\phi}$  which breaks time-reversal symmetry.<sup>3</sup> For  $t_2 = 0$ , the energy dispersion is identical to graphene, supporting two inequivalent massless Dirac fermion modes. For small  $t_2 \neq 0$ , and  $\phi \neq 0, \pi$ , the Dirac fermions acquire a mass, leading to a quantized Hall effect  $\sigma_{xy} = \pm e^2/h$  at half-filling. For spin-1/2 fermions, with each species at half-filling,  $\sigma_{xy}^\uparrow = \sigma_{xy}^\downarrow = \pm e^2/h$ . What is the fate of this quantum Hall insulator when interactions lead to Mott localization? Does the Mott insulator support unusual magnetism or spin liquid ground states?

For spinless fermions, nearest-neighbor repulsion induces a topologically trivial charge density wave insulator,<sup>25,26</sup> while spinless bosons with a local Hubbard repulsion form a plaquette Mott insulator with loop currents.<sup>27</sup> A slave-spin approach to the Haldane-

Hubbard model suggests the appearance of topological Néel order at moderate coupling with a staggered sublattice potential.<sup>32</sup> An alternative slave rotor mean field theory of spin-1/2 fermions, in which Hubbard repulsion localizes only the charge degree of freedom, led to a mean field description of a gapped chiral spin liquid (CSL).<sup>28,29</sup> Gutzwiller projected wave function studies show that such a CSL would be in the same phase as the  $\nu = 1/2$  bosonic Laughlin state.<sup>30,31</sup> More generally, such CSLs have also been shown to be induced by chiral 3-spin interactions,<sup>33,34</sup> or appear naturally for  $SU(N)$  fermions<sup>35</sup> at large  $N$ .

However, previous work has not considered the various types of magnetic orders which can compete with quantum spin liquid phases. Indeed, even the type of short-range spin correlations in the correlated Mott insulator, which could potentially be probed in cold atom experiments, have not been explored. Finally, since the topological order associated with such chiral spin liquids will not persist at nonzero temperature, it is important to gain an understanding of the thermal fluctuation effects on the Mott insulating state if we are to make connection with experiments. However, previous studies of this model have all been restricted to the  $T = 0$  limit.

This motivates us to revisit the effect of strong correlations on the Haldane-Hubbard model. Our key results are the following. (i) We extend the slave rotor mean field theory of the fermionic Haldane-Hubbard model to larger  $t_2$ . We show that this parton construction leads to CSLs with gapped or gapless bulk spinon excitations which descend from CI or Chern metal (CM) phases. The gapless spin liquids display multiple spinon Fermi pockets. Both types of spin liquids are expected to support gapless spinon edge states. Realizing the Haldane-Hubbard model with a broad range of  $t_2$  would thus allow one to explore the physics of gapped as well as gapless chiral spin liquid phases. (ii) The gapped or gapless spin liquids uncovered in the slave rotor parton description may be unstable to magnetic ordering due to fluctuation effects

beyond mean field theory. To study magnetic instabilities in the strongly correlated limit, we consider the strong coupling limit of the Haldane-Hubbard model, and derive and study the effective spin Hamiltonian which has chiral 3-spin interactions. This leads us to uncover a rich variety of competing chiral magnetic states such as tetrahedral, cone, and noncoplanar spirals which could compete with spin liquid phases. Quantum melting such phases will typically lead to broken symmetry states. However, unlike other states, we find that the tetrahedral state has completely uniform spin correlations and a large uniform chirality. We tentatively identify this state as the classical parent state for a chiral quantum spin liquid. (iii) Turning to physics at nonzero temperature, we discuss thermal fluctuation effects and lattice symmetry breaking thermal phase transitions associated with many of these states using a combination of analytic Landau theory arguments supplemented by classical Monte Carlo simulations. Such thermal transitions may be observable in experiments. (iv) Finally, we highlight some analogous results for spin-1/2 bosons which can also be studied in cold atom experiments, but which have not been theoretically explored.

## II. FERMI HALDANE-HUBBARD MODEL

For fermions with spin-1/2, the Haldane Hubbard model is described by the Hamiltonian

$$H_{\text{HH}} = -t_1 \sum_{\langle ij \rangle \sigma} (c_{i\sigma}^\dagger c_{j\sigma} + h.c.) - t_2 \sum_{\langle\langle ij \rangle\rangle \sigma} (e^{i\nu_{ij}\phi} c_{i\sigma}^\dagger c_{j\sigma} + h.c.) + U \sum_i n_{i\uparrow} n_{i\downarrow} \quad (1)$$

where  $\langle \cdot \rangle$  and  $\langle\langle \cdot \rangle\rangle$  denote, respectively, first and second neighbors, and  $\nu_{ij} = \pm 1$ , depending on whether we hop along or opposite to the arrows shown in Fig. 1(a), and  $U$  is the local Hubbard repulsion.

For  $U = 0$ , diagonalizing the Hamiltonian leads to Chern bands with Chern numbers  $C = \pm 1$ . Depending on  $t_2/t_1$  and  $\phi$ , these bands may overlap in energy, leading to a CM with a non-quantized Hall effect, or be well separated in energy leading to a CI (equivalently, a quantum Hall insulator) with a Hall conductivity  $\sigma_{xy} = \pm e^2/h$  per spin. This phase diagram is shown in Fig. 1(b), along with  $\sigma_{xy}$  which is quantized in the CI but non-quantized in the CM. (Note that the  $t_2=0$  axis is a Dirac semimetal; on this singular line, the flux plays no role.)

## III. MEAN FIELD CHIRAL SPIN LIQUIDS

Upon increasing the Hubbard repulsion, this topological band metal or band insulator will transition into a correlated Mott insulator once interactions exceed the bandwidth. The critical repulsion  $U_c = 8\eta|K|$ , where

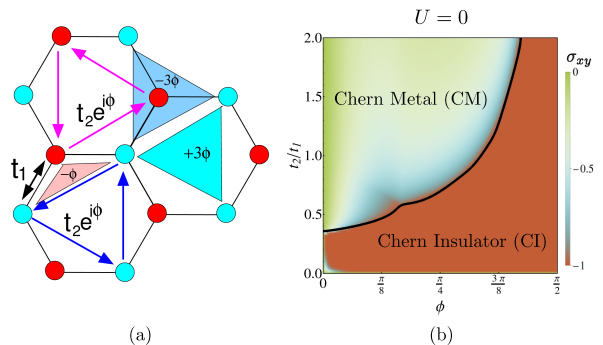


FIG. 1: (a) Honeycomb lattice showing the two sublattices with hoppings  $t_1, t_2$  and phase  $\phi$  defining the noninteracting Haldane model, with large and small triangles enclosing fluxes  $\pm 3\phi$  and  $-\phi$  respectively. (b) Phase diagram at half-filling for each spin species showing Chern band metal and Chern band insulator, and their Hall conductance  $\sigma_{xy}$  in units of  $e^2/h$ .

$K \equiv K(t_1, t_2, \phi)$  is the kinetic energy per site per spin of the noninteracting bands at half-filling.

Slave rotor mean field theory<sup>36,37</sup>, provides a concrete realization of the Mott phase and the Mott transition; it is a parton construction in which the spin and charge degrees of freedom are carried by two partons, a neutral spin-1/2 fermion and a charged spinless rotor. The localization of the rotor angular momentum corresponds to the transition into the Mott insulator phase. A single-site slave rotor mean field theory<sup>36,37</sup> yields  $\eta = 1$ . However, a comparison with quantum Monte Carlo studies of the honeycomb lattice Hubbard model<sup>38,39</sup> at  $t_2 = 0$ , and variational numerical studies of the triangular lattice Hubbard model<sup>40</sup> with  $t_1 = \phi = 0$  show that the single site slave rotor theory overestimates  $\eta$ ; the Mott transition occurs at a smaller renormalized  $\eta \approx 0.6-0.7$ .

Fig. 2 shows the phase boundaries which delineate the topological CM or CI from the correlated Mott insulator (assuming  $U_c = 8\eta|K|$  with  $\eta = 0.65$ ) at an interaction strength  $U/t_1 = 10$ . Within slave rotor mean field theory, the spinon band structure in the Mott insulator resembles the CI or CM band structure from which the Mott insulator descends. Thus, depending on  $t_2, \phi$ , one could have gapped or gapless spinon excitations in the bulk, accompanied by gapless chiral edge modes; we indicate these mean field states as ‘gapped CSL’ and ‘gapless CSL’. The spinon band structure mean field theory is simply inherited from the parent noninteracting fermion dispersion. Since the parent Chern metal phase exhibits multiple Fermi pockets, as shown in Fig. 2, the gapless CSL will inherit these as spinon Fermi pockets. Gapless chiral spin liquids have also been recently discussed in kagome antiferromagnets.<sup>43</sup>

Going beyond mean field theory, we expect the ‘gapped CSL’ to become a genuine spin liquid with semion excitations and topological degeneracy.<sup>30,31</sup> While these CSLs are expected to be stable for weak gauge fluctuations, strong gauge fluctuations might drive magnetic ordering

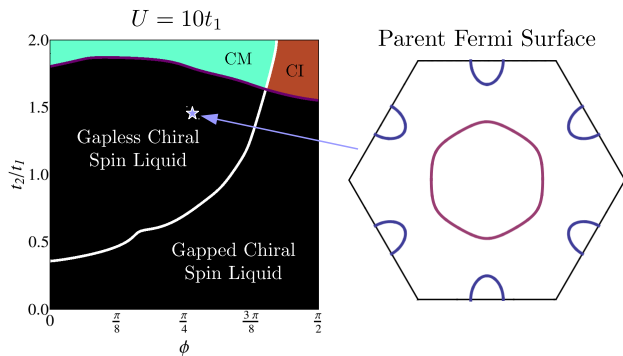


FIG. 2: Slave rotor mean field theory of the Haldane-Hubbard model showing the topological Chern metal (CM, green), Chern insulator (CI, brown), and topological spin liquid Mott insulators (black) at  $U/t_1 = 10$ . The topological Mott insulator could have gapless or gapped spinon excitations depending on whether it descends from a CM or CI phase, leading to gapless or gapped chiral spin liquids. An example of the parent Fermi surface of the non-interacting model, which is inherited by the spinons at mean-field level, is shown at a particular point. The slave rotor result has been rescaled by  $\eta = 0.65$  (see text).

instabilities, or, spinon pairing instabilities in the case of the gapless CSL.<sup>41,42</sup> Since slave rotor theory does not capture competing magnetic ordering tendencies of the Mott insulator, we address such potential competing phases using a strong coupling expansion.<sup>44</sup>

#### IV. COMPETING MAGNETIC ORDERS

At large  $U$ , a standard derivation<sup>44,45</sup> leads to an effective spin Hamiltonian

$$H_{\text{spin}} = \frac{4t_1^2}{U} \sum_{\langle ij \rangle} \mathbf{S}_i \cdot \mathbf{S}_j + \frac{4t_2^2}{U} \sum_{\langle\langle ij \rangle\rangle} \mathbf{S}_i \cdot \mathbf{S}_j - \frac{24t_1^2 t_2}{U^2} \sum_{\text{small}-\Delta} \hat{\chi}_{\Delta} \sin \Phi_{\Delta} - \frac{24t_2^3}{U^2} \sum_{\text{big}-\Delta} \hat{\chi}_{\Delta} \sin \Phi_{\Delta} \quad (2)$$

where  $\hat{\chi}_{\Delta} \equiv \mathbf{S}_i \cdot \mathbf{S}_j \times \mathbf{S}_k$  is the scalar spin chirality operator defined around triangular plaquettes, with the sites  $\{ijk\}$  being labelled going anticlockwise around the triangles.  $\Phi_{\Delta}$  being the flux enclosed by the corresponding triangle. As shown in Fig. 1(a), the set of triangles includes small triangles that enclose flux  $\Phi_{\Delta} = -\phi$ , big triangles within a hexagon which enclose a flux  $\Phi_{\Delta} = 3\phi$ , and big triangles defined around a site which enclose a flux  $\Phi_{\Delta} = -3\phi$ .

Treating the spin as a classical vector, we have obtained the magnetically ordered ground states of  $H_{\text{spin}}$  using numerical simulated annealing and variational spin configurations. This approximation may be formally set up by scaling the 2-spin exchange couplings by  $1/(4S^2)$  and the 3-spin exchange couplings by  $1/(8S^3)$  and then

taking  $S \rightarrow \infty$  which leads to a classical spin model with quantum fluctuations suppressed by  $\mathcal{O}(1/S)$ . Below we will use this classical approximation for  $S = 1/2$  as an uncontrolled approximation, which is nevertheless known to work well in many cases in predicting the correct magnetic order in 2D quantum spin models. For plotting the ground state spin configurations and structure factors, we view the honeycomb lattice as a square lattice with  $1/4$  deleted bonds (‘brickwall’ lattice), similar to the geometry realized in optical lattice experiments.<sup>23</sup> The resulting phase diagram, shown in Fig. 3 for  $U/t = 10$ , contains six phases: Néel, Tetrahedral, Cantellated Tetrahedral, Spiral, Cone-I and Cone-II, with a line indicating the regime below which this strong coupling description is appropriate.

Our main finding, atleast for  $t_2/t_1 \lesssim 1.5$ , as discussed below, is that the gapped spin liquid phase found in the slave rotor mean field theory in Section III may be potentially unstable to *commensurate* magnetic orders such as Neel or Tetrahedral order. On the other hand, the regime we identify as a gapless spin liquid with Fermi pockets in the slave rotor mean field theory appears to be dominantly unstable to various *incommensurate* spiral phases such as Spiral or Cone phases. Since the phase diagrams obtained in Fig. 2 and Fig. 3 involve different approximations, neither one is fully reliable; further numerical work is necessary to definitely identify whether the spin liquid phases are indeed stable or give way to magnetic or other symmetry breaking orders. However, the true phase diagram is expected to display phases shown in the phase diagrams in Fig. 2 and Fig. 3.

We describe these various competing magnetic ground states below. In the Appendix, we include “Common origin” plots<sup>46</sup> for these states for further insights.

(i) Néel: The Néel state is collinear with spins pointing opposite to each other on the two sublattices of the brick-wall lattice, with  $\langle \hat{\chi}_{\Delta} \rangle = 0$  on all triangular plaquettes. Its structure factor  $\mathcal{S}(\mathbf{q}) = \frac{1}{N} \sum_{i,j} \langle \mathbf{S}_i \cdot \mathbf{S}_j \rangle e^{i\mathbf{q} \cdot (\mathbf{r}_i - \mathbf{r}_j)}$  exhibits a Bragg peak at  $\mathbf{q} = (\pi, \pi)$ .

(ii) Tetrahedral: In this state, the spins point from the origin towards the four corners of a tetrahedron, and are tiled on the lattice as shown in Fig. 3. This noncoplanar state has a uniform chirality  $\langle \hat{\chi}_{\Delta} \rangle = -1/6\sqrt{3}$  on each small- $\Delta$ . The tetrahedral state is a triple- $\mathbf{q}$  state, with the structure factor  $\mathcal{S}(\mathbf{q})$  exhibiting Bragg peaks at  $\mathbf{q} = \pm(\pi/2, \pm\pi/2)$  and  $\mathbf{q} = (0, \pi)$ .

(iii) Cantellated Tetrahedral: The Cantellated Tetrahedral (CT) state descends from a parent tetrahedral state. It is reminiscent of the cuboctahedral state in frustrated kagome antiferromagnets.<sup>47–49</sup> In the parent tetrahedral configuration, the spins marked  $A$  (or  $B, C, D$ ) in Fig. 3 form a honeycomb (or brickwall) lattice, with a larger unit cell. We can deform these ferromagnetically aligned spins by dividing the larger honeycomb lattice into two sublattices, and then allowing for a  $\sqrt{3} \times \sqrt{3}$  canting on each sublattice; this splits each vertex of the tetrahedron into six vertices (forming a small hexagon) leading to the CT state with a 24-site unit cell. Here, the CT state ex-

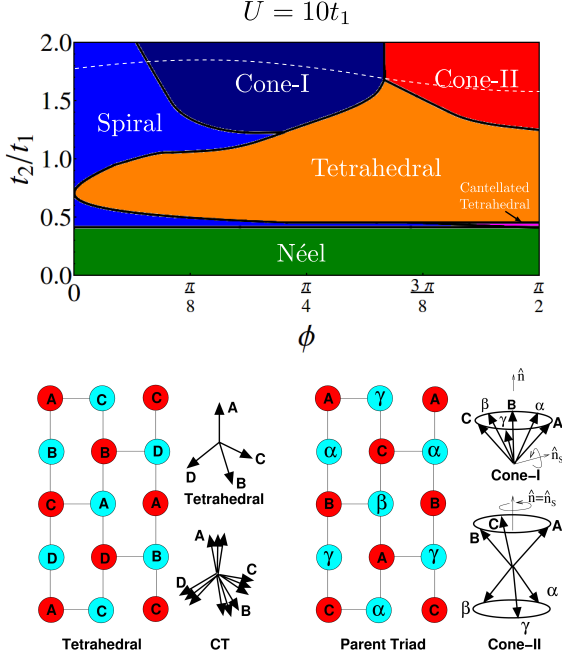


FIG. 3: Top: Phase diagram of  $H_{\text{spin}}$  showing various classical magnetically ordered ground states at  $U/t = 10$ : Néel, Spiral, Tetrahedral, Cantellated Tetrahedral (CT), Cone-I, and Cone-II. Below the dashed (white) line, the slave rotor result shows that we are in the Mott insulator, so the strong coupling expansion is expected to be valid. Stars indicate points where we numerically study the effect of thermal fluctuations. Bottom: Spin configurations in the Tetrahedral and Cone states (viewing the honeycomb lattice as a ‘brickwall’ for visual convenience). The indicated Triad states act as parent states for Cone-I and Cone-II, which are obtained by spiralling the Triads about the indicated  $\hat{n}_s$  axis. The CT descends from the Tetrahedral by a  $\sqrt{3} \times \sqrt{3}$  canting pattern.

ists over an extremely narrow window between the Néel and Tetrahedral states. Third-neighbor  $\mathbf{S}_i \cdot \mathbf{S}_j$  terms, which favor the cuboctahedral state on the kagome lattice, could enhance the regime of CT order.

(iv) *Cone-I*: This state descends from a Triad-I state in which spins on the large- $\Delta$  tend to form a triad on each sublattice, leading to a nonzero chirality. The organization of the spins is shown in Fig. 3. In Cone-I, which occurs for  $\phi < \pi/3$ , we start with triads on the two sublattices that are lined up with a common axis  $\hat{n}$ , with spins making an angle  $\theta$  with  $\hat{n}$ . Forming an orthonormal basis  $\{\hat{n}, \hat{n}_{\perp 1}, \hat{n}_{\perp 2}\}$ , we parameterize the three spins of the triad on sublattice-A,  $\mathbf{S}_j^A$  ( $j = 0, 1, 2$ ), as  $\mathbf{S}_j^A = \cos \theta \hat{n} + \sin \theta \cos(2\pi j/3) \hat{n}_{\perp 1} + \sin \theta \sin(2\pi j/3) \hat{n}_{\perp 2}$ , while the triad on sublattice-B is rotated by  $\chi$  about  $\hat{n}$ , with  $\mathbf{S}_j^B = \cos \theta \hat{n} + \sin \theta \cos(\varphi + 2\pi j/3) \hat{n}_{\perp 1} + \sin \theta \sin(\varphi + 2\pi j/3) \hat{n}_{\perp 2}$ . This state has a nonzero net magnetization. The ground state energy is independent of  $\varphi$ . The triad angle  $\theta$  varies with  $t_2, \phi$ ; at large  $t_2/t_1$ , the spins on each large- $\Delta$  tend to form an orthonormal triad, with  $\theta = \tan^{-1}(1/\sqrt{2})$ . The Triad-I is a triple- $\mathbf{q}$  state;  $\mathcal{S}(\mathbf{q})$

exhibits Bragg peaks at  $\mathbf{q} = (0, \pm 2\pi/3)$ ,  $\mathbf{q} = (\pi, \pm \pi/3)$ , and  $\mathbf{q} = (0, 0)$ . Let us now consider an axis  $\hat{n}_s$  which is perpendicular to the plane formed by  $\hat{n}$  and any one of the spins. The Cone-I state is obtained by rotating all the spins about this axis by an angle  $\theta(\mathbf{r}) = \mathbf{q} \cdot \mathbf{r}$  where  $\mathbf{q}$  is an incommensurate spiral wavevector; moving along  $\mathbf{q}$ , each spin of the triad again traces out a cone, and the Bragg peaks acquire a weak incommensuration.

(v) *Cone-II*: The Cone-II state occurs for  $\phi > \pi/3$ , where the  $\sin 3\phi$  term in Eq. 2 changes sign. Here, the parent state is a Triad-II state in which the spins on sublattice-A are reordered, with  $\mathbf{S}_j^A = \cos \theta \hat{n} + \sin \theta \cos(4\pi j/3) \hat{n}_{\perp 1} + \sin \theta \sin(4\pi j/3) \hat{n}_{\perp 2}$ , while the cone on sublattice-B is simply flipped, so that  $\mathbf{S}_j^B = -\cos \theta \hat{n} + \sin \theta \cos(\varphi + 2\pi j/3) \hat{n}_{\perp 1} + \sin \theta \sin(\varphi + 2\pi j/3) \hat{n}_{\perp 2}$ , with the ground state energy being independent of  $\varphi$ . The Triad-II state has a nonzero staggered magnetization, with Bragg peaks in  $\mathcal{S}(\mathbf{q})$  at  $\mathbf{q} = (0, \pm 2\pi/3)$ ,  $\mathbf{q} = (\pi, \pm \pi/3)$ , and  $\mathbf{q} = \pm(\pi, \pi)$ . The Cone-II state is obtained by rotating all the spins of Triad-II about the  $\hat{n}_s = \hat{n}$  axis by  $\theta(\mathbf{r}) = \mathbf{q} \cdot \mathbf{r}$  where  $\mathbf{q}$  is an incommensurate spiral wavevector, leading to weakly incommensurate Bragg peaks.

(vi) *Noncoplanar Spiral*: At  $\phi = 0$ , the Hamiltonian  $H_{\text{spin}}$  is the frustrated  $J_1$ - $J_2$  Heisenberg antiferromagnet on the honeycomb lattice. The classical ground states of this model consists of coplanar incommensurate spirals for  $J_2 > J_1/6$ .<sup>50</sup> For  $\phi \neq 0$ , chiral terms in  $H_{\text{spin}}$  cause spins on the A and B sublattices to cant in opposite directions away from the spiral plane, leading to a noncoplanar spiral with a uniform  $\chi_{\Delta} \neq 0$  on all big- $\Delta$ s.

## V. THERMAL FLUCTUATIONS

In 2D, the Mermin-Wagner theorem<sup>51</sup> precludes spin  $SU(2)$  symmetry breaking at  $T > 0$ . However, discrete orders associated with lattice symmetry breaking can survive thermal fluctuations.

The Néel and Tetrahedral orders break  $SU(2)$  symmetry, but their  $SU(2)$  invariant spin correlations, such as  $\langle \mathbf{S}_i \cdot \mathbf{S}_j \rangle$  and  $\langle \mathbf{S}_i \cdot \mathbf{S}_j \times \mathbf{S}_k \rangle$ , respect all lattice symmetries. These states at  $T > 0$  can thus be smoothly connected to the high- $T$  paramagnetic state. As shown in Fig. 4(a), for the Tetrahedral state at  $U/t_1 = 10$ , there is a crossover temperature  $\sim 0.05t_1$  below which the spin chirality  $\chi_{\Delta}$  on small triangles becomes large, saturating to  $1/6\sqrt{3}$  as  $T \rightarrow 0$ . The  $T > 0$  regime here can thus be viewed as a *classical CSL*. By contrast,  $\chi_{\Delta}$  vanishes at  $T = 0$  in the Néel state, but it exhibits a peak at nonzero temperature.

Previous work has shown that at  $\phi = 0$ , the spiral ground states of the frustrated  $J_1$ - $J_2$  Heisenberg antiferromagnet have nematic order associated with broken  $C_3$  honeycomb lattice symmetry which survives up to a thermal  $Z_3$ -clock transition.<sup>50</sup> The noncoplanar spiral does not involve any additional symmetry breaking, so it undergoes a similar  $Z_3$ -clock transition at  $\phi \neq 0$ .

In the Cone states, the choice of the common axis  $\hat{n}$  represents a spontaneously broken  $SU(2)$  symmetry;



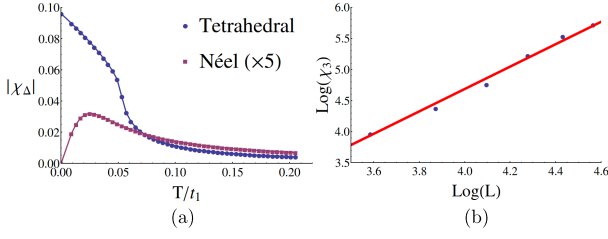


FIG. 4: (a) Temperature dependence of the spin chirality  $\chi_\Delta$  on small triangles for  $U/t = 10$  and  $\phi = \pi/3$ , showing its low temperature saturation in the Tetrahedral phase ( $t_2/t_1 = 1$ ) and its vanishing in the  $T = 0$  Néel phase ( $t_2/t_1 = 0.4$ ). We have plotted  $5 \times \chi_\Delta$  in the Néel state for clarity. (b) Log plot of finite size scaling of critical susceptibility for the  $Z_3$ -clock ordering transition in the Cone-II state at  $U/t_1 = 10$ ,  $\phi = 9\pi/20$ ,  $t_2/t_1 = 1.5$  showing  $\chi_3 \sim L^{\gamma/\nu}$ , with  $\gamma/\nu = 1.80(8)$ .

long wavelength fluctuations will disorder  $\hat{n}$  at arbitrarily small  $T > 0$ , restoring  $SU(2)$  symmetry. Using Monte Carlo simulations, we find that the nearest neighbor spin correlations  $\langle \mathbf{S}_i \cdot \mathbf{S}_j \rangle$  are modulated in the Cone states, leading to energy modulations on the bonds of the lattice. In the Cone-I state, these energy modulations resemble a  $\sqrt{3} \times \sqrt{3}$  columnar valence-bond solid pattern, while in the Cone-II state the energy modulations on the bonds resemble a staggered valence-bond solid bond pattern. This leads to a  $Z_3$  symmetry breaking in both states. We thus expect the Cone states to undergo a thermal  $Z_3$ -clock transition into the high temperature paramagnetic state. In the Cone-II state at  $U/t_1 = 10$ ,  $\phi = 9\pi/20$ ,  $t_2/t_1 = 1.5$ , the computed peak susceptibility  $\chi_3(L)$  for this transition is plotted in Fig. 4(b); its finite size scaling shows  $T_c \approx 0.17t_1$ , with  $\chi_3(L) \sim L^{\gamma/\nu}$ , with  $\gamma/\nu = 1.80(8)$ , consistent with the exact  $Z_3$ -clock result<sup>52</sup> for the exponent ratio  $\gamma/\nu = 26/15$ .

## VI. MOTT INSULATOR OF SPINOR BOSONS IN THE HALDANE HUBBARD MODEL

Cold atom experiments can also study interacting bosons in the Haldane-Hubbard model. Previous work on spinless bosons has shown the emergence of chiral superfluids and plaquette Mott insulators with loop currents.<sup>27</sup> For pseudospin-1/2 bosons, the Haldane-Hubbard model is

$$H_{\text{HH}}^{\text{Bose}} = -t_1 \sum_{\langle ij \rangle \sigma} (b_{i\sigma}^\dagger b_{j\sigma} + h.c.) - t_2 \sum_{\langle\langle ij \rangle\rangle \sigma} (e^{i\nu_{ij}\phi} b_{i\sigma}^\dagger b_{j\sigma} + h.c.) + \frac{1}{2} \sum_{i\sigma\sigma'} U_{\sigma\sigma'} n_{i\sigma} n_{i\sigma'} \quad (3)$$

For atoms such as  $^{87}\text{Rb}$ , the background scattering lengths are nearly isotropic, so we set  $U_{\sigma\sigma'} = U$ .

Here, we focus on the strong coupling limit  $U/t_1, U/t_2 \gg 1$ . The spin Hamiltonian in the resulting Mott insulator can be derived in a manner similar to

the fermion case, although it is more tedious to calculate the terms to  $\mathcal{O}(t^3/U^2)$ . We find

$$H_{\text{spin}}^{\text{Bose}} = J_1(\phi) \sum_{\langle ij \rangle} \mathbf{S}_i \cdot \mathbf{S}_j + J_2(\phi) \sum_{\langle\langle ij \rangle\rangle} \mathbf{S}_i \cdot \mathbf{S}_j + \frac{24t_1^2 t_2}{U^2} \sum_{\text{small-}\Delta} \hat{\chi}_\Delta \sin \Phi_\Delta + \frac{24t_2^3}{U^2} \sum_{\text{big-}\Delta} \hat{\chi}_\Delta \sin \Phi_\Delta$$

with  $J_1(\phi) = -4\frac{t_1^2}{U} - 24\frac{t_1^2 t_2}{U^2} \cos \phi$  and  $J_2(\phi) = -\frac{4t_2^2}{U} - 6\frac{t_1^2 t_2}{U^2} \cos \phi - 12\frac{t_2^3}{U^2} \cos 3\phi$ . Similar to the fermion case, the set of triangles includes small triangles that enclose flux  $\Phi_\Delta = -\phi$ , big triangles within a hexagon which enclose a flux  $\Phi_\Delta = 3\phi$ , and big triangles defined around a site which enclose a flux  $\Phi_\Delta = -3\phi$ . As expected, the two-spin exchanges are ferromagnetic at  $\phi = 0$ , and the chiral terms also change sign relative to the fermions.

In order to localize bosons, it is well known that the required repulsive interactions are much stronger. Thus,  $t/U$  is much smaller in the Mott insulating phase of spinor bosons, so that the chiral terms are much weaker in magnitude compared to the fermion case. In addition, unlike the fermion case, both  $J_1$  and  $J_2$  are ferromagnetic, which means these two-spin exchange couplings do not frustrate each other. Due to both these reasons, we find that ferromagnetic order almost completely dominates the magnetic phase diagram in the Mott insulator of spinor bosons.

In the ferromagnetic Mott insulator, the spin of the bosons plays no role. Thus, our model should yield results identical to that for spinless bosons. Indeed, assuming ferromagnetic order in our model, the chiral terms vanish. Nevertheless, we find that the bond energies displays a flux dependence arising from the flux-dependence of the two-spin exchange couplings. This leads to a weak nonzero loop currents  $\sim t^3/U^2$  around triangular plaquettes, consistent with weak loop currents predicted in the plaquette Mott insulator of spinless bosons using quite different approaches.<sup>27</sup> This result for spinor bosons is in striking contrast to the fermion case, where the only flux dependence arises from the chiral terms, so loop currents to  $\mathcal{O}(t^3/U^2)$  only arise in fermion Mott insulators which have non-coplanar spins with a nonzero scalar chirality.

## VII. DISCUSSION

We have discussed spin-1/2 fermions or bosons in the strongly correlated Haldane-Hubbard model. Our main finding is the rich variety of competing chiral magnetic orders with varying flux and hopping. Such orders, even if they are short-ranged, could be probed via magnetic Bragg scattering,<sup>53,54</sup> which has been shown to be a useful probe in atomic gases. For fermions, since  $\partial H_{\text{spin}}/\partial \phi \sim \hat{\chi}_\Delta$ , measuring the energy change upon an adiabatic change of flux is an indirect way to measure  $\chi_\Delta$ . This relation also ties the spin chirality to (weak) charge currents in the Mott insulator which could potentially

be detected using quantum quenches.<sup>55–57</sup> More direct routes to detect the spin chirality of atoms are desirable. Doping these magnetically ordered states leads to non-trivial Chern bands and a nonzero charge Hall effect, similar to that in certain frustrated metallic magnets.<sup>58–60</sup> Quantum melting the Tetrahedral state, which is uniform with a large chirality, may lead to a quantum CSL ground state with short range magnetic correlations at the same wavevectors. Further work is needed to clarify the energetic competition between CSLs and chiral magnetic orders. Finally, although the Mott insulating phase of spinor bosons is dominated by the ferromagnetic phase, lowering  $U$  might lead to superfluids with exotic magnetic orders; this will be explored elsewhere.

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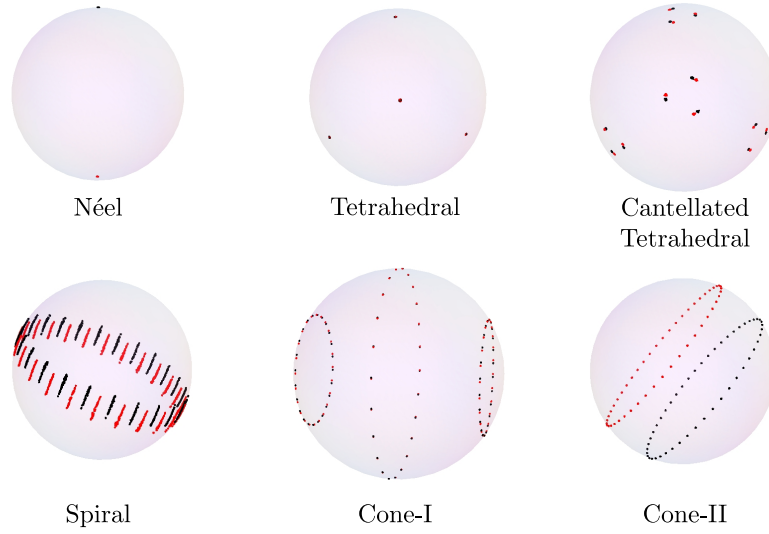


FIG. 5: Common origin plots for the Néel, Tetrahedral, Cantellated Tetrahedral, Spiral, Cone-I and Cone-II states, with red and black dots representing spins on the two sublattices of the honeycomb (or 'brickwall') lattice.

#### Appendix A: Common origin plots for the various magnetic orders

For visualizing spin configurations, we employ common origin plots<sup>46</sup> where all spins on the lattice are plotted with their tails at the center of a unit sphere and heads marked as dots on the surface of the sphere. We mark the spins on the two sublattices of the honeycomb (i.e., brickwall) lattice using different colors, red and black. Doing this for the Tetrahedral state, for instance, we see that all spins point along one of four directions, towards the vertices of a tetrahedron. For the Cantellated Tetrahedral, the vertices split into six points, three on each sublattice. For the Spiral state, we find that the spins on the two sublattices which support incommensurate spirals cant out of the plane of the coplanar spiral in opposite directions. The Cone-I state, which is obtained by spinning a triad state about a specific axis leads to one great circle and two small circles, while Cone-II leads to two parallel small circles.