Classification and description of bosonic symmetry protected topological phases with semiclassical nonlinear sigma models

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Classification and Description of Bosonic Symmetry Protected Topological Phases with semiclassical Nonlinear Sigma models

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In this paper we systematically classify and describe bosonic symmetry protected topological (SPT) phases in all physical spatial dimensions using semiclassical nonlinear Sigma model (NLSM) field theories. All the SPT phases on a $d-$dimensional lattice discussed in this paper can be described by the *same* NLSM, which is an O(d+2) NLSM in (d+1)$-$dimensional space-time, with a topological $\Theta-$term. The field in the NLSM is a semiclassical Landau order parameter with a unit length constraint. The classification of SPT phases discussed in this paper based on their NLSMs is *Completely Identical* to the more mathematical classification based on group cohomology given in Ref. 1,2. Besides the classification, the formalism used in this paper also allows us to explicitly discuss the physics at the boundary of the SPT phases, and it reveals the relation between SPT phases with different symmetries. For example, it gives many of these SPT states a natural "decorated defect" construction.

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I. INTRODUCTION

Symmetry protected topological (SPT) phase is a new type of quantum disordered phase. It is intrinsically different from a trivial direct product state, when and only when the system has certain symmetry G. In terms of its phenomena, a SPT phase on a $d-$dimensional lattice should satisfy at least the following three criteria:

(i). On a $d-$dimensional lattice without boundary, this phase is fully gapped, and nondegenerate;

(ii). On a $d-$dimensional lattice with a $(d-1)-$dimensional boundary, if the Hamiltonian of the entire system (including both bulk and boundary Hamiltonian) preserves certain symmetry G, this phase is either gapless, or gapped but degenerate.

(iii). The boundary state of this $d-$dimensional system cannot be realized as a ($d-1$)-dimensional lattice system with the same symmetry G.

Both the $2d$ quantum spin Hall insulator[3–5] and $3d$ Topological insulator[6–8] are perfect examples of SPT phases protected by time-reversal symmetry and charge U(1) symmetry. In this paper we will focus on bosonic SPT phases. Unlike fermion systems, bosonic SPT phases are always strongly interacting phases of boson systems.

Notice that the second criterion (ii) implies the following two possibilities: On a lattice with a boundary, the system is either gapless, or gapped but degenerate. For example, without interaction, the boundaries of $2d$ QSH insulator and $3d$ TBI are both gapless; but with interaction, the edge states of $2d$ QSH insulator, and $3d$ TBI can both be gapped out through spontaneous time-reversal symmetry breaking at the boundary, and this spontaneous time-reversal symmetry breaking can occur through a boundary transition, without destroying the bulk state[9–11]. When $d \geq 3$, the degeneracy of the boundary can correspond to either spontaneous breaking of G, or correspond to certain topological degeneracy at the boundary. Which case occurs in the system will depend on the detailed Hamiltonian at the boundary of the system. For example, with strong interaction, the boundary of a 3d TBI can be driven into a nontrivial topological phase[12–15].

The concept of SPT phase was pioneered by Wen and his colleagues. A mathematical paradigm was developed in Ref. 1,2 that systematically classified SPT phases based on the group cohomology of their symmetry G. But this approach was unable to reveal all the physical properties of the SPT phases. In the last few years, SPT phase has rapidly developed into a very active and exciting field[1,2,16–31], and besides the general mathematical classification, other approaches of understanding SPT phases were also taken. In 2d, it was demonstrated that the SPT phases can be thoroughly classified by the Chern-Simons field theory[20], although it is unclear how to generalize this approach to 3d. Nonlinear Sigma model field theories were also used to describe some SPT phases in 3d and 2d[21–23], but a complete classification based on this field theory is still demanded.

The goal of this paper is to systematically classify and describe bosonic SPT phases with various continuous and discrete symmetries in *all dimensions*, using semiclassical nonlinear Sigma model (NLSM) field theories. At least in one dimensional systems, semiclassical NLSMs have been proved successful in describing SPT phases. The O(3) NLSM plus a topological $\Theta-$term describes a spin-1 Heisenberg chain when $\Theta = 2\pi$:

$$\mathcal{S}_{1d} = \int dx d\tau \frac{1}{g}(\partial_\mu \vec{n})^2 + \frac{i2\pi}{8\pi}\epsilon_{abc}\epsilon_{\mu\nu} n^a \partial_\mu n^b \partial_\nu n^c, \quad (1)$$

and it is well-known that the spin-1 antiferromagnetic Heisenberg model is a SPT phase with 2-fold degeneracy at each boundary[32–37].

In this paper we will discuss SPT phases with symmetry Z_2^T, Z_2, $Z_2 \times Z_2$, $Z_2 \times Z_2^T$, $U(1)$, $U(1) \times Z_2$, $U(1) \rtimes Z_2$, $U(1) \times Z_2^T$, $U(1) \rtimes Z_2^T$, Z_m, $Z_m \times Z_2$, $Z_m \rtimes Z_2$,

$Z_m \times Z_2^T$, $Z_m \rtimes Z_2^T$, $SO(3)$, $SO(3) \times Z_2^T$, $Z_2 \times Z_2 \times Z_2$. Here we use the standard notation: Z_2^T stands for time-reversal symmetry, $G \times Z_2^T$ and $G \rtimes Z_2^T$ stand for direct and semidirect product between unitary group G and time-reversal symmetry. A semidirect product between two groups means that these two group actions do not commute with each other. More details will be explained when we discuss the classification of these states. We will demonstrate that a $d-$dimensional SPT phase with any symmetry mentioned above can always be described by an $\mathrm{O}(d+2)$ NLSM in $(d+1)-$dimensional space-time, namely all the 1d SPT phases discussed in this paper can be described by Eq. 1, all the 2d and 3d SPT phases can be described by the following two field theories:

$$\begin{aligned} \mathcal{S}_{2d} &= \int d^2x d\tau \ \frac{1}{g}(\partial_\mu \vec{n})^2 \\ &+ \frac{i2\pi k}{\Omega_3}\epsilon_{abcd} n^a \partial_\tau n^b \partial_x n^c \partial_y n^d, \end{aligned} \tag{2}$$

$$\begin{aligned} \mathcal{S}_{3d} &= \int d^3x d\tau \ \frac{1}{g}(\partial_\mu \vec{n})^2 \\ &+ \frac{i2\pi}{\Omega_4}\epsilon_{abcde} n^a \partial_\tau n^b \partial_x n^c \partial_y n^d \partial_z n^e, \end{aligned} \tag{3}$$

The $\mathrm{O}(d+2)$ vector is a Landau order parameter with a unit length constraint: $(\vec{n})^2 = 1$. Ω_d is the surface area of a $d-$dimensional unit sphere. The $2d$ action Eq. 2 has a level$-k$ in front of its $\Theta-$term, whose reason will be explained later. Different SPT phases in the same dimension are distinguished by the transformation of the $\mathrm{O}(d+2)$ vector under the symmetry. The classification of SPT phases on a $d-$dimensional lattice is given by all the *independent* symmetry transformations of $\vec{n}$ that keep the entire Lagrangian (including the $\Theta-$term) invariant. This classification rule will be further clarified in the next section.

An $\mathrm{O}(d+2)$ NLSM can support maximally $\mathrm{O}(d+2)$ symmetry and other discrete symmetries such as time-reversal. We choose the 17 symmetries listed above, because they can all be embedded into the maximal symmetry of the field theory, and they are the most physically relevant symmetries. Of course, if we want to study an SPT phase with a large Lie group such as SU(N), the above field theories need to be generalized to NLSM defined with a symmetric space of that Lie group. But for all these physically relevant symmetries, our NLSM is already sufficient.

In principle, a NLSM describes a system with a long correlation length. Thus a NLSM plus a $\Theta-$term most precisely describes a SPT phase tuned *close to* a critical point (but still in the SPT phase). When a SPT phase is tuned close to a critical point, the NLSM not only describes its topological properties (*e.g.* edge states *etc.*), but also describes its dynamics, for example excitation spectrum above the energy gap (much smaller than the ultraviolet cut-off). When the system is tuned deep inside the SPT phase, namely the correlation length is comparable with the lattice constant, this NLSM can no longer describe its dynamics accurately, but since the topological properties of this SPT phase is unchanged while tuning, these topological properties (like edge states) can still be described by the NLSM. The NLSM is an effective method of describing the universal topological properties, as long as we ignore the extra nonuniversal information about dynamics, such as the exact dispersion of excitations, which depends on the details of the lattice Hamiltonian and hence is not universal.

Besides the classification, our NLSMs in all dimensions can tell us explicit physical information about this SPT phase. For example, the boundary states of 1d SPT phases can be obtained by explicitly solving the field theory reduced to the 0d boundary. The boundary of a 3d SPT phase could be a 2d topological phase, and the NLSMs can tell us the quantum number of the anyons of the boundary topological phases. The boundary topological phases of 3d SPT phases with $U(1)$ and time-reversal symmetry were discussed in Ref. 21. We will analyze the boundary topological phases for some other 3d SPT phases in the current paper.

Our formalism not only can study each individual SPT phase, it also reveals the relation between different SPT phases. For example, using our formalism we are able to show that there is a very intriguing relation between SPT phases with $U(1) \times (\rtimes) G$ symmetry and SPT phases with $Z_m \times (\rtimes) G$ symmetry, where G is another discrete group such as Z_2, Z_2^T. Our formalism demonstrates that after breaking U(1) to Z_m, whether the SPT phase survives or not depends on the parity of integer m. We also demonstrate that when m is an even number, we can construct some extra SPT phases with $Z_m \times (\rtimes) G$ symmetry that *cannot* be deduced from SPT phases with $U(1) \times (\rtimes) G$ symmetry by breaking U(1) down to Z_m. Our field theory also gives many of these SPT states a natural "decorated defect" construction, which will be discussed in more detail in the next section.

NLSMs with a $\Theta-$term can also give us the illustrative universal bulk ground state wave function of the SPT phases. This was discussed in Ref. 24. These wave functions contain important information for both the boundary and the bulk defects introduced by coupling the NLSM to an external gauge field[24,38]. It was also demonstrated that the NLSMs are useful in classifying and describing symmetry enriched topological (SET) phases[39], but a complete classification of SET phases based on NLSMs will be studied in the future.

In the current paper we will only discuss SPT states within cohomology. It is now understood that the group cohomology classification is incomplete, and in each dimension there are a few examples beyond cohomology classification[40–42]. These beyond-cohomology states all involve gravitational anomalies[43] or mixed gauge-gravitational anomalies[40]. Generalization of our field theory to the cases beyond group cohomology can be found in another paper 44.

II. STRATEGY AND CLARIFICATION

A. Edge states of NLSMs with $\Theta-$term

In $d-$dimensional theories Eq. 1,2 and 3 (d denotes the spatial dimension), when $\Theta = 2\pi$, their boundaries are described by $(d-1)+1-$dimensional O$(d+2)$ NLSMs with a Wess-Zumino-Witten (WZW) term at level-1. When $d = 1$, the boundary of Eq. 1 with $\Theta = 2\pi$ is a 0+1d O(3) NLSM with a Wess-Zumino-Witten term at level $k = 1$[37]:

$$\mathcal{S}_b = \int d\tau \frac{1}{g}(\partial_\tau \vec{n})^2 + \int d\tau du \frac{i2\pi}{8\pi}\epsilon_{abc}\epsilon_{\mu\nu}n^a\partial_\mu n^b\partial_\nu n^c. \quad (4)$$

The WZW term involves an extension of $\vec{n}(\tau)$ to $\vec{n}(\tau, u)$:

$$\vec{n}(\tau, 0) = (0, 0, 1), \quad \vec{n}(\tau, 1) = \vec{n}(\tau). \quad (5)$$

The boundary action $\mathcal{S}_b$ describes a point particle moving on a sphere S^2, with a 2π magnetic flux through the sphere. The ground state of this single particle quantum mechanics problem is two fold degenerate. The two fold degenerate ground states have the following wave functions on the unit sphere:

$$U = (\cos(\theta/2)e^{i\phi/2}, \ \ \sin(\theta/2)e^{-i\phi/2})^t,$$
$$\vec{n} = (\sin(\theta)\cos(\phi), \sin(\theta)\sin(\phi), \cos(\theta)) \,. \quad (6)$$

The boundary doublet U transforms projectively under symmetry of the SPT phase, and its transformation can be derived explicitly from the transformation of $\vec{n}$. For example if $\vec{n}$ transforms as $\vec{n} \to -\vec{n}$ under time-reversal, then this implies that under time-reversal $\phi \to \phi$, $\theta \to \pi + \theta$, and $U \to i\sigma^y U$.

When $d = 2$, the boundary is a 1+1-dimensional O(4) NLSM with a WZW term at level $k = 1$, and it is well-known that this theory is a gapless conformal field theory if the system has a full O(4) symmetry[45,46]. The 1d boundary could be gapped but still degenerate if the symmetry of $\vec{n}$ is discrete (the degeneracy corresponds to spontaneous discrete symmetry breaking); when $d = 3$, the boundary is a 2+1d O(5) NLSM with a WZW at level $k = 1$, which can be reduced to a 2+1d O(4) NLSM with $\Theta = \pi$ after the fifth component of $\vec{n}$ is integrated out[21]. This $2+1d$ boundary theory should either be gapless or degenerate, and one particularly interesting possibility is that it can become a topological order, which will be discussed in more detail in section IIF. Starting with this topological order, we can prove that this $2+1d$ boundary system can never be gapped without degeneracy.

All components of $\vec{n}$ in Eq. 1,2 and 3 must have a nontrivial transformation under the symmetry group G, namely it is not allowed to turn on a linear "Zeeman" term that polarizes any component of $\vec{n}$. Otherwise the edge states can be trivially gapped, and the bulk $\Theta-$term plays no role.

B. Phase diagram of NLSMs with a $\Theta-$term

In our classification, the NLSM including its $\Theta-$term is invariant under the symmetry of the SPT phase, for arbitrary value of Θ. For special values of Θ, such as $\Theta = k\pi$ with integer k, some extra discrete symmetry may emerge, but these symmetries are *unimportant* to the SPT phase. However, these extra symmetries guarantee that $\Theta = k\pi$ is a fixed point under renormalization group (RG) flow. In 1+1d NLSMs, the RG flow of Θ was calculated explicitly in Ref. 47,48 and it was shown that $\Theta = 2\pi k$ are stable fixed points, while $\Theta = (2k+1)\pi$ are instable fixed points, which correspond to phase transitions; in higher dimensions, similar explicit calculations are possible, but for our purposes, we just need to argue that $\Theta = 2\pi k$ are stable fixed points under RG flow. The bulk spectrum of the NLSM with $\Theta = 2\pi k$ is identical to the case with $\Theta = 0$: in the quantum disordered phase the bulk of the system is fully gapped without degeneracy. Now if Θ is tuned away from $2\pi k$: $\Theta = 2\pi k \pm \epsilon$, this perturbation cannot close the bulk gap, and since the essential symmetry of the SPT phase is unchanged, the SPT phase including its edge states should be stable against this perturbation. Thus a SPT phase corresponds to a finite phase $\Theta \in (2\pi k - \delta_1, 2\pi k + \delta_2)$ in the phase diagram.

There is a major difference between $\Theta-$term in NLSM and the $\Theta-$term in the response action of the external gauge field. In our description, a SPT phase corresponds to the entire phase whose stable fixed point is at $\Theta = 2\pi$ (or $2\pi k$ with integer k). Tuning slightly away from these stable fixed points will not break any essential symmetry that protects the SPT state, and hence it does not change the main physics. The theory will always flow back to these stable fixed points under RG (this RG flow was computed explicitly in $1+1d$ in Ref. 47,48, and a similar RG flow was proposed for higher dimensional cases[49]). The $\Theta-$term of the external gauge field after integrating out the matter fields is protected by the symmetry of the SPT phase to be certain discrete value. For example $\Theta = \pi$ for the ordinary 3d topological insulator[50,51] is protected by time-reversal symmetry. Tuning Θ away from π will necessarily break the time-reversal symmetry.

C. $\mathbb{Z}_k$ or $\mathbb{Z}$ classification?

In the classification table in Ref. 1,2, one can see that in even dimensions, there are many SPT states with $\mathbb{Z}$ classifications, but in odd dimensions, $\mathbb{Z}$ classification never appears. This fact was a consequence of mathematical calculations in Ref. 1,2, but in this section we will give a very simple explanation based on our field theories.

The manifold of O$(d+2)$ NLSM is S^{d+1}, which has a $\Theta-$term in $(d+1)-$dimensional space-time due to homotopy group $\pi_{d+1}[S^{d+1}] = \mathbb{Z}$. However, this does *not* mean that the $\Theta-$term will always give us $\mathbb{Z}$ classification, because more often than not we can show that $\Theta = 0$ and $\Theta = 2\pi k$ with certain nonzero integer k can be connected to each other without any bulk transition.

For example, let us couple two Haldane phases to each other:

$$\mathcal{L} = \frac{1}{g}(\partial_\mu \vec{n}^{(1)})^2 + \frac{i2\pi}{8\pi}\epsilon_{abc}\epsilon_{\mu\nu} n_a^{(1)}\partial_\mu n_b^{(1)}\partial_\nu n_c^{(1)} + 1 \to 2 + A(\vec{n}^{(1)} \cdot \vec{n}^{(2)}). \quad (7)$$

When $A < 0$, effectively $\vec{n}^{(1)} = \vec{n}^{(2)} = \vec{n}$, then the system is effectively described by one O(3) NLSM with $\Theta = 4\pi$; while when $A > 0$, effectively $\vec{n}^{(1)} = -\vec{n}^{(2)} = \vec{n}$, the effective NLSM for the system has $\Theta = 0$. When parameter A is tuned from negative to positive, the bulk gap does not close. The reason is that, since $\Theta = 2\pi$ in both Haldane phases, the $\Theta-$term does not affect the bulk spectrum at all. To analyze the bulk spectrum (and bulk phase transition) while tuning A, we can just ignore the $\Theta-$term. Without the $\Theta-$term, both theories are just trivial gapped phases, and an inter-chain coupling can not qualitatively change the bulk spectrum unless it is strong enough to overcome the bulk gap in each chain. We have explicitly checked this phase diagram using a Monte Carlo simulation of two coupled O(3) NLSMs, and the result is exactly the same to what we would expect from the argument above. Thus the theory with $\Theta = 4\pi$ and $\Theta = 0$ are equivalent.

By contrast, if we couple two chains with $\Theta = \pi$ each, then the cases $A > 0$ and < 0 correspond to effective $\Theta = 0$ and 2π respectively, and these two limits are separated by a bulk phase transition point $A = 0$, when the system becomes two decoupled chains with $\Theta = \pi$ each. And it is well-known that a $1 + 1d$ O(3) NLSM with $\Theta = \pi$ is the effective field theory that describes a spin-1/2 chain[32,33], and according to the Lieb-Shultz-Matthis theorem, this theory must be either gapless or degenerate[52]. This conclusion is consistent with the RG calculation in Ref. 47,48, and a general nonperturbative argument in Ref. 49.

In fact when $\Theta = 4\pi$ the boundary state of Eq. 1 is a spin-1 triplet, and by tuning A, at the boundary there is a level crossing between triplet and singlet, while there is no bulk transition. This analysis implies that with SO(3) symmetry, $1d$ spin systems have two different classes: there is a trivial class with $\Theta = 4\pi k$, and a nontrivial Haldane class with $\Theta = (4k + 2)\pi$.

If we cannot connect $\Theta = 4\pi$ to $\Theta = 0$ without closing the bulk gap, then the classification would be bigger than $\mathbb{Z}_2$. For example, let us consider the 2d SPT phase with U(1) symmetry which was first studied in Ref. 17. This phase is described by Eq. 2. $B \sim n_1 + in_2$ and $B' \sim n_3 + in_4$ ($n_1 \cdots n_4$ are the four components of O(4) vector $\vec{n}$ in Eq. 2) are two complex boson (rotor) fields that transform identically under the global U(1) symmetry. Now suppose we couple two copies of this systems together through symmetry allowed interactions:

$$\mathcal{S} = \mathcal{S}_1 + \mathcal{S}_2 + A_1 B_1 B_2^\dagger + A_2 B_1 B_2'^\dagger + A_3 B_1' B_2^\dagger + A_4 B_1' B_2'^\dagger + H.c. \quad (8)$$

No matter how we tune the parameters A_i, the resulting effective NLSM *always* has $\Theta = 4\pi$ instead of $\Theta = 0$ (this is simply because $(-1)^2 = (-1)^4 = +1$). This implies that we cannot smoothly connect $\Theta = 4\pi$ to 0 without any bulk transition. Thus the classification of 2d SPT phases with U(1) symmetry is $\mathbb{Z}$ instead of $\mathbb{Z}_2$. This is why in $2d$ (and all even dimensions), many SPT states have $\mathbb{Z}$ classification, while in odd dimensions there is no $\mathbb{Z}$ classification at all, namely all the nontrivial SPT phases in odd dimensions correspond to $\Theta = 2\pi$. Thus in Eq. 2 we added a level$-k$ in the $\Theta-$term.

D. NLSM and "decorated defect" construction of SPT states

Ref. 27 has given us a physical construction of some of the SPT states in terms of the "decorated domain wall" picture. For example, one of the $3d$ $Z_2^A \times Z_2^B$ SPT state can be constructed as follows: we first break the Z_2^B symmetry, then restore the Z_2^B symmetry by proliferating the domain wall of Z_2^B, and each Z_2^B domain wall is decorated with a $2d$ SPT state with Z_2^A symmetry. This state is described by Eq. 3 with transformation

$$Z_2^B \ : \ n_{1,2} \to -n_{1,2}, \quad n_a \to n_a (a = 3, 4, 5);$$
$$Z_2^A \ : \ n_1, \to n_1, \quad n_a \to -n_a (a = 2, \cdots 5). \quad (9)$$

Here n_i is the ith component of vector $\vec{n}$. To visualize the "decorated domain" wall picture, we can literally make a domain wall of n_1, and consider the following configuration of vector $\vec{n}$: $\vec{n} = (\cos\theta, \sin\theta N_2, \sin\theta N_3, \sin\theta N_4, \sin\theta N_5)$, where $\vec{N}$ is a O(4) vector with unit length, and θ is a function of coordinate z only:

$$\theta(z = +\infty) = \pi, \quad \theta(z = -\infty) = 0. \quad (10)$$

Plug this parametrization of $\vec{n}$ into Eq. 3, and integrate along z direction, the $\Theta-$term in Eq. 3 precisely reduces to the $\Theta-$term in Eq. 2 with $k = 1$, and the O(4) vector $\vec{n} = \vec{N}$. This is precisely the $2d$ SPT with Z_2 symmetry. This implies that the Z_2^B domain wall is decorated with a $2d$ SPT state with Z_2^A symmetry.

Many SPT states can be constructed with this decorated domain wall picture. Some $3d$ SPT states can also be understood as "decorated vortex", which was first discussed in 21. This state has $U(1) \times Z_2^T$ symmetry, and the vector $\vec{n}$ transforms as

$$U(1) \ : \ (n_1 + in_2) \to (n_1 + in_2)e^{i\theta}, \quad n_{3,4,5} \to n_{3,4,5},$$
$$Z_2^T \ : \ \vec{n} \to -\vec{n}. \quad (11)$$

If we make a vortex of the U(1) order parameter (n_1, n_2), Eq. 3 reduces to Eq. 1 with O(3) order parameter

(n_3, n_4, n_5). Thus this SPT can be viewed as decorating the U(1) vortex with a $1d$ Haldane phase, and then proliferating the vortices.

E. Independent NLSMs

Let us take the example of 1d SPT phases with $Z_2 \times Z_2^T$ symmetry. As we claimed, all 1d SPT phases in this paper are described by the same NLSM Eq. 1. With $Z_2 \times Z_2^T$ symmetry, there seems to be three different ways of assigning transformations to $\vec{n}$ that make the entire Lagrangian invariant:

$$\begin{aligned}
(1) &: \ Z_2 : \vec{n} \to \vec{n}, \ \ Z_2^T : \vec{n} \to -\vec{n}. \\
(2) &: \ Z_2 : n_{1,2} \to -n_{1,2}, \ \ n_3 \to n_3 \\
& \ \ \ Z_2^T : \vec{n} \to -\vec{n} \\
(3) &: \ Z_2 : n_{1,2} \to -n_{1,2}, \ \ n_3 \to n_3 \\
& \ \ \ Z_2^T : n_3 \to -n_3, \ \ n_{1,2} \to n_{1,2}.
\end{aligned} \tag{12}$$

However the NLSMs defined with these three different transformations are not totally independent from each other, which means that if all three theories exist in one system, although each theory is a nontrivial SPT phase individually, we can turn on some symmetry allowed couplings between these NLSMs and cancel the bulk topological terms completely, and drive the entire coupled system to a trivial state. For example, let us take O(3) vectors $\vec{n}^{(i)}$ with transformations (1), (2) and (3) respectively:

$$\vec{n}^{(i)}(\vec{r}) = (n_1^{(i)}, n_2^{(i)}, n_3^{(i)}) = \left(\sin(\theta_{\vec{r}}^{(i)})\cos(\phi_{\vec{r}}^{(i)}), \sin(\theta_{\vec{r}}^{(i)})\sin(\phi_{\vec{r}}^{(i)}), \cos(\theta_{\vec{r}}^{(i)})\right), \tag{13}$$

$\phi_{\vec{r}}^{(i)}$ and $\theta_{\vec{r}}^{(i)}$ are functions of space-time. Under Z_2 and Z_2^T symmetry, $\theta^{(i)}$ and $\phi^{(i)}$ transform as

$$\begin{aligned}
Z_2 &: \ \theta^{(i)} \to \theta^{(i)}, \\
& \ \ \ \phi^{(1)} \to \phi^{(1)}, \ \ \phi^{(i)} \to \phi^{(i)} + \pi, \ (i = 2, 3); \\
Z_2^T &: \ \theta^{(i)} \to \pi - \theta^{(i)}, \\
& \ \ \ \phi^{(i)} \to \phi^{(i)} + \pi, \ (i = 1, 2), \ \ \phi^{(3)} \to \phi^{(3)}.
\end{aligned} \tag{14}$$

First of all, since $\theta^{(i)}$ have the same transformation for all i, we can turn on strong coupling between the three NLSMs to make $\theta^{(1)} = \theta^{(2)} = \theta^{(3)} = \theta$. We can also turn on couplings to make $\phi^{(3)} = \phi^{(1)} + \phi^{(2)}$. Now $\vec{n}^{(3)}$ becomes

$$\begin{aligned}
n_1^{(3)} &= \sin(\theta)\cos(\phi^{(1)} + \phi^{(2)}), \\
n_2^{(3)} &= \sin(\theta)\sin(\phi^{(1)} + \phi^{(2)}), \\
n_3^{(3)} &= \cos(\theta).
\end{aligned} \tag{15}$$

It is straightforward to prove that the topological number of $\vec{n}^{(3)}$ in 1+1d space-time is the sum of topological numbers of $\vec{n}^{(1)}$ and $\vec{n}^{(2)}$. More explicitly, an instanton of $\vec{n}^{(a)}$ is a domain wall of $n_3^{(a)}$ decorated with a vortex of $\phi^{(a)}$. As we explained above, with appropriate coupling between these vectors, we can make $\theta^{(1)} = \theta^{(2)} = \theta^{(3)} = \theta$, and $\phi^{(3)} = \phi^{(1)} + \phi^{(2)}$. Thus a domain wall of $n_3^{(3)}$ is also a domain wall of $n_3^{(1)}$ and $n_3^{(2)}$, while the vortex number of $\phi^{(3)}$ is the sum of vortex number of $\phi^{(1)}$ and $\phi^{(2)}$. Thus the $\Theta-$term of $\vec{n}^{(3)}$ reduces to the sum of $\Theta-$terms of $\vec{n}^{(1)}$ and $\vec{n}^{(2)}$. In this example we have shown that NLSMs (1) and (2) in Eq. 12 can "merge" into NLSM (3). Thus the three NLSMs defined with transformations (1), (2) and (3) are not independent from each other.

Also, for either NLSM (1) or (2) in Eq. 12, we can show that $\Theta^{(i)} = 0$ and 4π can be connected to each other without a bulk transition (using the same method as the previous subsection). Then eventually the 1d SPT phase with $Z_2 \times Z_2^T$ symmetry is parametrized by two independent $\Theta-$terms, the fixed point values of $\Theta^{(1)}$ and $\Theta^{(2)}$ can be either 0 or 2π, thus this SPT phase has a $(\mathbb{Z}_2)^2$ classification, which is consistent with the classification using group cohomology. NLSMs with transformations (1), (2) are two "root phases" of 1d SPT phases with $Z_2 \times Z_2^T$ symmetry. All the other SPT phases can be constructed with these two root phases.

For most SPT phases, we can construct the NLSMs using the smallest representation (fundamental representation) of the symmetry groups G, because usually (but not always!) NLSMs constructed using higher representations can reduce to constructions with the fundamental representation with a different Θ. For example, the 1d SPT phase with $U(1) \rtimes Z_2$ symmetry can be described by Eq. 1 with the following transformation

$$\begin{aligned}
U(1) &: \ (n_1 + in_2) \to e^{i\theta}(n_1 + in_2), \ \ n_3 \to n_3, \\
Z_2 &: \ n_1 \to n_1, \ \ n_{2,3} \to -n_{2,3},
\end{aligned} \tag{16}$$

namely $B \sim (n_1 + in_2)$ is a charge-1 boson under the U(1) rotation, and the edge state of this SPT phase carries charge-1/2 of boson B. We can also construct an O(3) NLSM using charge-2 boson $B' \sim (n_1' + in_2') \sim (n_1 + in_2)^2$ that transforms as $B' \to B' e^{2i\alpha}$, then mathematically we can demonstrate that the NLSM with $\Theta = 2\pi$ for order parameter $\vec{n}' = (n_1', n_2', n_3)$ reduces to a NLSM of $\vec{n}$ with $\Theta = 4\pi$, hence it is a trivial phase.

More explicitly, let us take unit vector $\vec{n} = (\sin(\theta)\cos(\phi), \sin(\theta)\sin(\phi), \cos(\theta))$, and vector $\vec{n}' = (\sin(\theta)\cos(2\phi), \sin(\theta)\sin(2\phi), \cos(\theta))$, then we can show that when $\vec{n}$ has topological number 1 in 1+1d space-time, $\vec{n}'$ would have topological number 2. This means that if there is a $\Theta-$term for $\vec{n}'$ with $\Theta = 2\pi$, it is equivalent to a $\Theta-$term for $\vec{n}$ with $\Theta = 4\pi$.

Physically, the edge state of NLSM of $\vec{n}'$ with $\Theta = 2\pi$ carries a half-charge of B', which is still a charge-1 object, so it can be screened by another charge-1 boson B. Hence in this case NLSM constructed using charge-2 boson B' would be trivial.

However, later we will also show that when the symmetry group involves Z_m with even integer $m > 2$, then using higher representations of Z_m we can construct SPT phases that *cannot* be obtained from the fundamental representation of Z_m.

F. Boundary topological order of 3d SPT phases

The $(d-1)-$dimensional boundary of a $d-$dimensional SPT phase must be either degenerate or gapless. When $d = 3$, its 2d boundary can spontaneously break the symmetry, or have a topological order[21]. We can use the bulk field theory Eq. 3 to derive the quantum numbers of the anyons at the boundary.

Let us take the 3d SPT phase with $Z_2 \times Z_2^T$ symmetry as an example. One of the SPT phases has the following transformations:

$$\begin{aligned} Z_2 &: \; n_a \to -n_a (a = 1, \cdots 4), \quad n_5 \to n_5; \\ Z_2^T &: \; \vec{n} \to -\vec{n}. \end{aligned} \tag{17}$$

The 2+1d boundary of the system is described by a 2+1d O(5) NLSM with a Wess-Zumino-Witten (WZW) term at level $k = 1$:

$$\begin{aligned} S &= \int d^2x d\tau \; \frac{1}{g}(\partial_\mu \vec{n})^2 \\ &+ \int_0^1 du \; \frac{i2\pi}{\Omega_4} \epsilon_{abcde} n^a \partial_x n^b \partial_y n^c \partial_z n^d \partial_\tau n^e, \end{aligned} \tag{18}$$

where $\vec{n}(x, \tau, u)$ satisfies $\vec{n}(x, \tau, 0) = (0, 0, 0, 0, 1)$ and $\vec{n}(x, \tau, 1) = \vec{n}(x, \tau)$. If the time-reversal symmetry is preserved, namely $\langle n_5 \rangle = 0$, we can integrate out n_5, and Eq. 18 reduces to a 2+1d O(4) NLSM with $\Theta = \pi$:

$$S = \int d^2x d\tau \; \frac{1}{g}(\partial_\mu \vec{n})^2 + \frac{i\pi}{\Omega_3} \epsilon_{abcd} n^a \partial_\tau n^b \partial_x n^c \partial_y n^d. \tag{19}$$

In Eq. 19 $\Theta = \pi$ is protected by time-reversal symmetry.

In the following we will argue that the topological terms in Eq. 18 and Eq. 19 guarantee that the $2d$ boundary cannot be gapped without degeneracy. One particularly interesting possibility of the boundary is a phase with 2d Z_2 topological order[21]. A 2d Z_2 topological phase has e and m excitations that have mutual semion statistics[53]. The semion statistics can be directly read off from Eq. 19: if we define complex boson fields $z_1 = n_1 + in_2$ and $z_2 = n_3 + in_4$, then the $\Theta-$term in Eq. 19 implies that a vortex of (n_3, n_4) carries half charge of z_1, while a vortex of (n_1, n_2) carries half charge of z_2, thus vortices of z_1 and z_2 are bosons with mutual semion statistics. This statistics survives after z_1 and z_2 are disordered by condensing the *double vortex* (vortex with vorticity 4π) of either z_1 or z_2 at the boundary, then the disordered phase must inherit the statistics and become a Z_2 topological phase[21]. The vortices of (n_1, n_2) and (n_3, n_4) become the e and m excitations respectively. Normally a vortex defect is discussed in systems with a U(1) global symmetry. We do not assume such U(1) global symmetry in our case, this symmetry reduction is unimportant in the Z_2 topological phase.

At the vortex core of (n_3, n_4), namely the m excitation, Eq. 18 reduces to a $0+1d$ O(3) NLSM with a WZW term at level 1[54]:

$$\mathcal{S}_m = \int d\tau \frac{1}{g}(\partial_\tau \vec{N})^2 + \int_0^1 du \frac{i2\pi}{8\pi} \epsilon_{abc} \epsilon_{\mu\nu} N^a \partial_\mu N^b \partial_\nu N^c \tag{20}$$

where $\vec{N} \sim (n_1, n_2, n_5)$. This 0+1d field theory describes a single particle moving on a 2d sphere with a magnetic monopole at the origin. It is well known that if there is a SO(3) symmetry for $\vec{N}$, then the ground state of this 0d problem has two fold degeneracy, with two orthogonal solutions

$$\begin{aligned} u_m &= \cos(\theta/2) e^{i\phi/2}, \quad v_m = \sin(\theta/2) e^{-i\phi/2}, \\ \vec{N} &= (\sin(\theta)\cos(\phi), \sin(\theta)\sin(\phi), \cos(\theta)). \end{aligned} \tag{21}$$

Likewise, the vortex of (n_1, n_2) (e excitation) also carries a doublet (u_e, v_e). Under the Z_2 transformation, $\phi \to \phi + \pi$, thus $u_{e,m}$ and $v_{e,m}$ carry charge $\pm 1/2$ of the Z_2 symmetry, namely under the Z_2 transformation:

$$Z_2 : U_{e,m} \to i\sigma^z U_{e,m}, \tag{22}$$

where $U_{e,m} = (u_{e,m}, v_{e,m})^t$.

Under time-reversal transformation $\mathcal{T}$, $\vec{N} \to -\vec{N}$, $\theta \to \theta + \pi$. Thus the e and m doublets transform as

$$Z_2^T : U_{e,m} \to i\sigma^y U_{e,m}, \tag{23}$$

thus the e and m anyons at the boundary carry projective representation of Z_2^T which satisfies $\mathcal{T}^2 = -1$.

Based on this Z_2 topological order, we can derive the phase diagram around the Z_2 topological order, and show that this boundary cannot be gapped without degeneracy. For example, starting with a $2d$ Z_2 topological order, one can condense either e or m excitation and kill the topological degeneracy. However, because $U_{e,m}$ transform nontrivially under the symmetry group, condensate of either e or m will always spontaneously break certain symmetry and lead to degeneracy. For example, the condensate of e excitation has nonzero expectation value of $(n_3, n_4, n_5) \sim U_e^\dagger \vec{\sigma} U_e$, which necessarily spontaneously breaks the Z_2 or Z_2^T symmetry.

We also note that one bulk BSPT state can have different boundary states, which depends on the details of the boundary Hamiltonian. Recently a different boundary topological order of BSPT state was derived in Ref. 55, but the bulk state is the same as ours.

G. Rule of classification

With all these preparations, we are ready to lay out the rules of our classification:

1. In $d-$dimensional space, all the SPT phases discussed in this paper are described by a $(d+1)-$dimensional O$(d+2)$ NLSM with a $\Theta-$term. The O$(d+2)$ vector field $\vec{n}$ is an order parameter, namely it must carry a nontrivial representation of the given symmetry. In other words, no component of the vector field transforms completely trivially under the symmetry, because otherwise it is allowed to turn on a strong linear "Zeeman" term to the trivial component, and then the system will become a trivial direct product state.

2. The classification is given by all the possible *independent* symmetry transformations on vector order parameter $\vec{n}$ that keep the $\Theta-$term invariant, for *arbitrary* value of Θ. Independent transformations mean that any NLSM defined with one transformation cannot be obtained by "merging" two (or more) other NLSMs defined with other transformations. SPT phases constructed using independent NLSMs are called "root phases". All the other SPT phases can be constructed with these root phases.

3. With a given symmetry, and given transformation of $\vec{n}$, if $\Theta = 2\pi k$ and $\Theta = 0$ can be connected without a bulk transition, this transformation will contribute classification $\mathbb{Z}_k$; otherwise the transformation will contribute classification $\mathbb{Z}$.

Using the rule and strategy discussed in this section, we can obtain the classification of all SPT phases in all dimensions. In this paper we will systematically study SPT phases in one, two and three spatial dimensions with symmetries Z_2^T, Z_2, $Z_2 \times Z_2$, $Z_2 \times Z_2^T$, $U(1)$, $U(1) \times Z_2$, $U(1) \rtimes Z_2$, $U(1) \times Z_2^T$, $U(1) \rtimes Z_2^T$, Z_m, $Z_m \times Z_2$, $Z_m \rtimes Z_2$, $Z_m \times Z_2^T$, $Z_m \rtimes Z_2^T$, $SO(3)$, $SO(3) \times Z_2^T$, $Z_2 \times Z_2 \times Z_2$. The final classification of the SPT phases we study in this paper is *completely identical* to the classification based on group cohomology[1,2].

III. 1D SPT PHASE WITH $Z_2 \times Z_2 \times Z_2^T$ SYMMETRY

Before we discuss our full classification, let us carefully discuss 1d SPT phases with $Z_2 \times Z_2 \times Z_2^T$ symmetry as an example. These SPT phases were discussed very thoroughly in Ref. 56. There are in total 16 different phases (including the trivial phase). The goal of this section is to show that all these phases can be described by the *same* equation Eq. 1 with certain transformation of $\vec{n}$, and the projective representation of the boundary states given in Ref. 56 can be derived explicitly using Eq. 6.

For the consistency of notation in this paper, R_z and R_x in Ref. 56 will be labelled Z_2^A and Z_2^B here. Let us consider one example, namely Eq. 1 with the following transformation:

$$\begin{aligned} Z_2^A &: \ n_{1,2} \to -n_{1,2}, \quad n_3 \to n_3; \\ Z_2^B &: \ n_{2,3} \to -n_{2,3}, \quad n_1 \to n_1; \\ Z_2^T &: \ n_2 \to -n_2, \quad n_{1,3} \to n_{1,3}. \end{aligned} \tag{24}$$

Now let us parametrize $\vec{n}$ as

$$\vec{n} = (\sin\theta\cos\phi, \ \sin\theta\sin\phi, \ \cos\theta), \tag{25}$$

then θ and ϕ transform as

$$\begin{aligned} Z_2^A &: \ \theta \to \theta, \quad \phi \to \phi + \pi, \\ Z_2^B &: \ \theta \to \pi - \theta, \quad \phi \to -\phi, \\ Z_2^T &: \ \theta \to \theta, \quad \phi \to -\phi. \end{aligned} \tag{26}$$

These transformations lead to the following projective transformation of edge state Eq. 6:

$$\begin{aligned} Z_2^A &: \ U \to i\sigma^z U, \\ Z_2^B &: \ U \to \sigma^x U, \\ Z_2^T &: \ U \to U. \end{aligned} \tag{27}$$

Thus this NLSM corresponds to phase E_5 in Ref. 56.

The 16 phases in Ref. 56 correspond to the following transformations of O(3) vector $\vec{n}$:

$$\begin{aligned} E_0 &: \ \text{Trivial phase, } \Theta = 0; \\ E_0' &: \ Z_2^A, Z_2^B : \vec{n} \to \vec{n}, \quad Z_2^T : \vec{n} \to -\vec{n}; \\ E_1 &: \ Z_2^A : \vec{n} \to \vec{n}, \\ & \quad\ Z_2^B : n_{1,2} \to -n_{1,2}, \quad n_3 \to n_3, \\ & \quad\ Z_2^T : \vec{n} \to -\vec{n}, \\ E_1' &: \ Z_2^A : \vec{n} \to \vec{n}, \\ & \quad\ Z_2^B : n_{1,2} \to -n_{1,2}, \quad n_3 \to n_3, \\ & \quad\ Z_2^T : n_{1,2} \to n_{1,2}, \quad n_3 \to -n_3; \\ E_3 &: \ Z_2^B : \vec{n} \to \vec{n}, \\ & \quad\ Z_2^A : n_{1,2} \to -n_{1,2}, \quad n_3 \to n_3, \\ & \quad\ Z_2^T : \vec{n} \to -\vec{n}, \\ E_3' &: \ Z_2^B : \vec{n} \to \vec{n}, \\ & \quad\ Z_2^A : n_{1,2} \to -n_{1,2}, \quad n_3 \to n_3, \\ & \quad\ Z_2^T : n_{1,2} \to n_{1,2}, \quad n_3 \to -n_3; \end{aligned}$$

$$
\begin{aligned}
E_5 \;&:\; Z_2^A : n_{1,2} \to -n_{1,2}, \quad n_3 \to n_3; \\
& \quad Z_2^B : n_{2,3} \to -n_{2,3}, \quad n_1 \to n_1; \\
& \quad Z_2^T : n_2 \to -n_2, \quad n_{1,3} \to n_{1,3}; \\
E_5' \;&:\; E_5 \oplus E_0'; \\
E_7 \;&:\; Z_2^A : n_{1,2} \to -n_{1,2}, \quad n_3 \to n_3, \\
& \quad Z_2^B : n_{1,2} \to -n_{1,2}, \quad n_3 \to n_3, \\
& \quad Z_2^T : n_{1,2} \to n_{1,2}, \quad n_3 \to -n_3; \\
E_7' \;&:\; Z_2^A : n_{1,2} \to -n_{1,2}, \quad n_3 \to n_3, \\
& \quad Z_2^B : n_{1,2} \to -n_{1,2}, \quad n_3 \to n_3, \\
& \quad Z_2^T : \vec{n} \to -\vec{n}; \\
E_9 \;&:\; Z_2^A : n_{1,2} \to -n_{1,2}, \quad n_3 \to n_3; \\
& \quad Z_2^B : n_{2,3} \to -n_{2,3}, \quad n_1 \to n_1; \\
& \quad Z_2^T : n_3 \to -n_3, \quad n_{1,2} \to n_{1,2}; \\
E_9' \;&:\; E_9 \oplus E_0', \\
E_{11} \;&:\; Z_2^A : n_{1,2} \to -n_{1,2}, \quad n_3 \to n_3; \\
& \quad Z_2^B : n_{2,3} \to -n_{2,3}, \quad n_1 \to n_1; \\
& \quad Z_2^T : n_1 \to -n_1, \quad n_{2,3} \to n_{2,3}; \\
E_{11}' \;&:\; E_{11} \oplus E_0'; \\
E_{13} \;&:\; Z_2^A : n_{1,2} \to -n_{1,2}, \quad n_3 \to n_3; \\
& \quad Z_2^B : n_{2,3} \to -n_{2,3}, \quad n_1 \to n_1; \\
& \quad Z_2^T : \vec{n} \to -\vec{n}; \\
E_{13}' \;&:\; E_{13} \oplus E_0'. \qquad (28)
\end{aligned}
$$

All the phases except for the trivial phase E_0 have $\Theta = 2\pi$ in Eq. 1. Here $E_5 \oplus E_0'$ means it is a spin ladder with symmetry allowed weak interchain couplings, and the two chains are E_5 phase and E_0' phase respectively. For all the 16 phases above, we can compute the projective representations of the boundary states using Eq. 6, and they all precisely match with the results in Ref. 56.

IV. FULL CLASSIFICATION OF SPT PHASES

A. SPT phases with Z_2 symmetry

In 1d and 3d, there is no Z_2 symmetry transformation that we can assign vector $\vec{n}$ that makes the actions Eq. 1 and Eq. 3 invariant, thus there is no SPT phase in 1d and 3d with Z_2 symmetry. However, in 2d there is obviously one and only one way to assign the Z_2 symmetry:

$$Z_2 : (n_1, n_2, n_3, n_4) \to -(n_1, n_2, n_3, n_4). \qquad (29)$$

Then when $\Theta = 2\pi$ this 2+1d O(4) NLSM describes the Z_2 SPT phase studied in Ref. 16. Using the method in section IIC, one can show that with the transformation Eq. 29, the 2+1d O(4) NLSM Eq. 2 with $\Theta = 4\pi$ is equivalent to $\Theta = 0$, thus the classification in 2d is $\mathbb{Z}_2$.

In Ref. 24, the authors also used this NLSM to derive the ground state wave function of the SPT phase:

$$|\Psi\rangle = \sum (-1)^{dw} |C\rangle, \qquad (30)$$

where $|C\rangle$ standards for an arbitrary Ising field configuration, while dw is the number of Ising domain walls of this configuration. This wave function was also derived in Ref. 16 with an exactly soluble model for this SPT phase.

The classification of SPT phases with Z_2 symmetry is:

$$1d : \mathbb{Z}_1, \quad 2d : \mathbb{Z}_2, \quad 3d : \mathbb{Z}_1. \qquad (31)$$

Here $\mathbb{Z}_1$ means there is only one trivial state, and $\mathbb{Z}_2$ means there is one trivial state and one nontrivial SPT state.

B. SPT phases with Z_2^T symmetry

In 2d, there is no way to assign Z_2^T symmetry to the O(4) NLSM order parameter in Eq. 2 to make the $\Theta-$term invariant, thus there is no bosonic SPT phase in 2d with Z_2^T symmetry. In 1d and 3d, there is only one way to assign the Z_2^T symmetry to vector $\vec{n}$:

$$Z_2^T : \vec{n} \to -\vec{n}, \qquad (32)$$

and $\Theta = 0$ and $\Theta = 4\pi$ are equivalent. Thus in both 1d and 3d, the classification is $\mathbb{Z}_2$. Notice that time-reversal is an antiunitary transformation, thus $i \to -i$ under Z_2^T; also since our NLSMs are defined in Euclidean space-time, the Euclidean time $\tau = it$ is invariant under Z_2^T.

Using the method in section II.F, one can demonstrate that the boundary of the $3d$ SPT state with Z_2^T symmetry is a $2d$ Z_2 topological order, whose both e and m excitations are Kramers doublet, i.e. the so called $eTmT$ state.

The classification of SPT phases with Z_2^T symmetry is:

$$1d : \mathbb{Z}_2, \quad 2d : \mathbb{Z}_1, \quad 3d : \mathbb{Z}_2. \qquad (33)$$

Now it is understood that in $3d$ there is bosonic SPT state with Z_2^T symmetry that is beyond the group cohomology classification[21], and there is a explicit lattice construction for such state[57]. This state is also beyond our current NLSM description. However, a generalized field theory which involves both the NLSM and Chern-Simons theory can describe at least a large class of BSPT states beyond group cohomology. This will be discussed in a different paper[44].

C. SPT phases with $U(1)$ symmetry

In 1d and 3d, there is no way to assign U(1) symmetry to vector $\vec{n}$ that keeps the entire Lagrangian invariant. But in 2d, bosonic SPT phase with U(1) symmetry was discussed in Ref. 17, and its field theory is given by Eq. 2. And since in this case we cannot connect $\Theta = 2\pi k$ and $\Theta = 0$ without a bulk transition, the classification is $\mathbb{Z}$.

The classification of SPT phases with $U(1)$ symmetry is:

$$1d : \mathbb{Z}_1, \quad 2d : \mathbb{Z}, \quad 3d : \mathbb{Z}_1. \tag{34}$$

D. SPT phases with $U(1) \rtimes Z_2$ symmetry

$U(1) \rtimes Z_2$ is a subgroup of SO(3). In 1d, there is only one way of assigning the symmetry to vector $\vec{n}$ that keeps the entire Lagrangian invariant:

$$\begin{aligned} U(1) &: (n_1 + in_2) \to e^{i\theta}(n_1 + in_2), \quad n_3 \to n_3, \\ Z_2 &: n_1 \to n_1, \quad n_{2,3} \to -n_{2,3}. \end{aligned} \tag{35}$$

Here Z_2 is a particle-hole transformation of rotor/boson field $b \sim n_1 + in_2$. n_3 can be viewed as the boson density, which changes sign under particle-hole transformation. One can check that the U(1) and Z_2 symmetry defined above do not commute with each other. The boundary state of this 1d SPT phase is given in Eq. 6. Under U(1) and Z_2 transformation, the boundary doublet U transforms as

$$U(1) : U \to e^{i\theta\sigma^z/2}U, \quad Z_2 : U \to \sigma^x U. \tag{36}$$

In 3d, there is also only one way of assigning the symmetry to the O(5) vector:

$$\begin{aligned} U(1) &: (n_1 + in_2) \to e^{i\theta}(n_1 + in_2), \quad n_b \to n_b, \ b = 3,4,5; \\ Z_2 &: n_1 \to n_1, \quad n_b, \to -n_b, \quad b = 2, \cdots 5. \end{aligned} \tag{37}$$

In both 1d and 3d, $\Theta = 4\pi$ is equivalent to $\Theta = 0$, thus in both 1d and 3d the classification is $\mathbb{Z}_2$.

In 2d, there are two independent ways of assigning $U(1) \rtimes Z_2$ transformations to the O(4) vector $\vec{n}$:

$$\begin{aligned} (1) &: U(1) : (n_1 + in_2) \to e^{i\theta}(n_1 + in_2), \\ &\quad (n_3 + in_4) \to e^{i\theta}(n_3 + in_4); \\ &\quad Z_2 : n_1, n_3 \to n_1, n_3, \quad n_2, n_4 \to -n_2, -n_4; \\ (2) &: U(1) : \vec{n} \to \vec{n}, \quad Z_2 : \vec{n} \to -\vec{n}. \end{aligned} \tag{38}$$

The transformation (1) contributes $\mathbb{Z}$ classification, while transformation (2) contributes $\mathbb{Z}_2$ classification, *i.e.* in 2d the classification is $\mathbb{Z} \times \mathbb{Z}_2$. *The final classification of SPT phases with $U(1) \rtimes Z_2$ symmetry is*:

$$1d : \mathbb{Z}_2, \quad 2d : \mathbb{Z} \times \mathbb{Z}_2, \quad 3d : \mathbb{Z}_2. \tag{39}$$

E. SPT phases with $U(1) \times Z_2$ symmetry

In both 1d and 3d, there is no way of assigning $U(1) \times Z_2$ transformations to vector $\vec{n}$ that keeps the Θ term invariant. But in 2d, we can construct three root phases:

$$\begin{aligned} (1) &: U(1) : (n_1 + in_2) \to e^{i\theta}(n_1 + in_2), \\ &\quad (n_3 + in_4) \to e^{i\theta}(n_3 + in_4); \\ &\quad Z_2 : \vec{n} \to \vec{n}; \\ (2) &: U(1) : \vec{n} \to \vec{n}, \quad Z_2 : \vec{n} \to -\vec{n}; \\ (3) &: U(1) : (n_1 + in_2) \to e^{i\theta}(n_1 + in_2), \\ &\quad n_{3,4} \to n_{3,4}; \\ &\quad Z_2 : n_{1,2} \to n_{1,2}, \quad n_{3,4} \to -n_{3,4}. \end{aligned} \tag{40}$$

The first transformation contributes classification $\mathbb{Z}$, while transformations (2) and (3) both contribute classification $\mathbb{Z}_2$, *thus the final classification of SPT phases with $U(1) \times Z_2$ symmetry is*:

$$1d : \mathbb{Z}_1, \quad 2d : \mathbb{Z} \times (\mathbb{Z}_2)^2, \quad 3d : \mathbb{Z}_1. \tag{41}$$

F. SPT phases with $U(1) \rtimes Z_2^T$ symmetry

A boson operator b with $U(1) \rtimes Z_2^T$ symmetry transforms as $b \to b$ under Z_2^T. In 1d, the only $U(1) \rtimes Z_2^T$ symmetry transformation that keeps Eq. 1 invariant is the same transformation as Z_2^T SPT phase, namely vector $\vec{n}$ does not transform under $U(1)$, but changes sign under Z_2^T.

In 2d, the only transformation that keeps Eq. 2 invariant is

$$\begin{aligned} U(1) &: (n_1 + in_2) \to e^{i\theta}(n_1 + in_2), \ n_{3,4} \to n_{3,4}; \\ Z_2^T &: n_1 \to n_1, \quad n_a \to -n_a (a = 2,3,4), \end{aligned} \tag{42}$$

and this NLSM gives classification $\mathbb{Z}_2$.

The NLSMs for $U(1) \rtimes Z_2^T$ SPT phases in 3d have been discussed in Ref. 21, and in 3d the classification is $(\mathbb{Z}_2)^2$. *Thus the final classification of SPT phases with* $U(1) \rtimes Z_2^T$ *symmetry is*:

$$1d : \mathbb{Z}_2, \quad 2d : \mathbb{Z}_2, \quad 3d : (\mathbb{Z}_2)^2. \tag{43}$$

G. SPT phases with $U(1) \times Z_2^T$ symmetry

In 1d, there are two independent transformations that keep Eq. 1 invariant:

$$\begin{aligned}
(1) \ &: \ U(1) : (n_1 + in_2) \rightarrow e^{i\theta}(n_1 + in_2), \quad n_3 \rightarrow n_3; \\
&\quad Z_2^T : n_{1,2} \rightarrow n_{1,2}, \quad n_3 \rightarrow -n_3, \\
(2) \ &: \ U(1) : \vec{n} \rightarrow \vec{n}, \\
&\quad Z_2^T : \vec{n} \rightarrow -\vec{n}.
\end{aligned} \tag{44}$$

In 2d there is no $U(1) \times Z_2^T$ transformation that keeps Eq. 2 invariant. In 3d the NLSMs for $U(1) \times Z_2^T$ SPT phases were discussed in Ref. 21. *The final classification of SPT phases with* $U(1) \times Z_2^T$ *symmetry is*:

$$1d : (\mathbb{Z}_2)^2, \quad 2d : \mathbb{Z}_1, \quad 3d : (\mathbb{Z}_2)^3. \tag{45}$$

H. SPT phases with $Z_2 \times Z_2$ symmetry

In 1d, there is only one $Z_2 \times Z_2$ transformation that keeps Eq. 1 invariant:

$$\begin{aligned}
Z_2^A \ &: \ n_{1,2} \rightarrow -n_{1,2}, \quad n_3 \rightarrow n_3, \\
Z_2^B \ &: \ n_1 \rightarrow n_1, \quad n_{2,3} \rightarrow -n_{2,3}.
\end{aligned} \tag{46}$$

The boundary state U defined in Eq. 6 transforms as

$$Z_2^A : U \rightarrow i\sigma^z U, \quad Z_2^B : U \rightarrow \sigma^x U. \tag{47}$$

Thus Z_2^A and Z_2^B no longer commute with each other at the boundary.

In 2d, there are three independent $Z_2 \times Z_2$ transformations (three different root phases):

$$\begin{aligned}
(1) \ &: \ Z_2^A : \vec{n} \rightarrow -\vec{n}, \quad Z_2^B : \vec{n} \rightarrow \vec{n}; \\
(2) \ &: \ Z_2^A : \vec{n} \rightarrow \vec{n}, \quad Z_2^B : \vec{n} \rightarrow -\vec{n}; \\
(3) \ &: \ Z_2^A : n_{1,2} \rightarrow -n_{1,2}, \quad n_{3,4} \rightarrow n_{3,4}; \\
&\quad Z_2^B : n_{1,2} \rightarrow n_{1,2}, \quad n_{3,4} \rightarrow -n_{3,4}.
\end{aligned} \tag{48}$$

In 3d, there are also two independent $Z_2 \times Z_2$ transformations that keep Eq. 3 invariant (two root phases):

$$\begin{aligned}
(1) \ &: \ Z_2^A : n_{1,2} \rightarrow -n_{1,2}, \quad n_a \rightarrow n_a (a = 3, 4, 5); \\
&\quad Z_2^B : n_1, \rightarrow n_1, \quad n_a \rightarrow -n_a (a = 2, \cdots 5); \\
(2) \ &: \ Z_2^B : n_{1,2} \rightarrow -n_{1,2}, \quad n_a \rightarrow n_a (a = 3, 4, 5); \\
&\quad Z_2^A : n_1, \rightarrow n_1, \quad n_a \rightarrow -n_a (a = 2, \cdots 5).
\end{aligned} \tag{49}$$

As we discussed in section II.F, the boundary of these 3d SPT phases can have 2d Z_2 topological order. A 2d Z_2 topological phase has e and m anyon excitations, and these anyons correspond to vortices of certain components of order parameter $\vec{n}$. If the e and m anyons correspond to vortices of (n_3, n_4) and (n_1, n_2) respectively, then according to Eq. 20, the e excitation corresponds to a $0 + 1d$ O(3) WZW model for vector (n_1, n_2, n_5), and the m excitation corresponds to a $0 + 1d$ WZW model for vector (n_3, n_4, n_5). The boundary anyons of phase (1) transform as:

$$\begin{aligned}
(1) \ &: \ Z_2^A : U_e \rightarrow i\sigma^z U_e, \quad U_m \rightarrow U_m; \\
&\quad Z_2^B : U_e \rightarrow \sigma^x U_e, \quad U_m \rightarrow i\sigma^y U_m^*.
\end{aligned} \tag{50}$$

Notice that under Z_2^B, a vortex of (n_1, n_2) becomes an antivortex, thus the transformation of U_m under Z_2^B involves a complex conjugation. The transformation of boundary anyons of phase (2) is the same as Eq. 50 after interchanging Z_2^A and Z_2^B.

The final classification of SPT phases with $Z_2 \times Z_2$ *symmetry is*:

$$1d : \mathbb{Z}_2, \quad 2d : (\mathbb{Z}_2)^3, \quad 3d : (\mathbb{Z}_2)^2. \tag{51}$$

I. SPT phases with $Z_2 \times Z_2^T$ symmetry

In 1d and 3d, the SPT phases with $Z_2 \times Z_2^T$ symmetry are simply SPT phases with $U(1) \times Z_2^T$ symmetry after reducing U(1) to its subgroup Z_2. The classification is the same as the $U(1) \times Z_2^T$ SPT phases discussed in the previous subsection. In 2d, there are two different root phases that correspond to the following transformations:

$$\begin{aligned}
(1) \ &: \ Z_2 : n_{1,2} \rightarrow -n_{1,2}, \quad n_{3,4} \rightarrow n_{3,4}, \\
&\quad Z_2^T : n_1 \rightarrow n_1, \quad n_a \rightarrow -n_a (a = 2, 3, 4); \\
(2) \ &: \ Z_2 : \vec{n} \rightarrow -\vec{n}, \\
&\quad Z_2^T : n_1 \rightarrow n_1, \quad n_a \rightarrow -n_a (a = 2, 3, 4).
\end{aligned} \tag{52}$$

The final classification of SPT phases with $Z_2 \times Z_2^T$ *symmetry is*:

$$1d : (\mathbb{Z}_2)^2, \quad 2d : (\mathbb{Z}_2)^2, \quad 3d : (\mathbb{Z}_2)^3. \tag{53}$$

J. SPT phases with Z_m symmetry

In 1d and 3d, there are no nontrivial Z_m transformations that can keep Eq. 1 and Eq. 3 invariant. In 2d, we can construct the following root phase:

$$
\begin{aligned}
Z_m \ &: \ (n_1 + in_2) \to e^{i2\pi k/m}(n_1 + in_2); \\
&(n_3 + in_4) \to e^{i2\pi k/m}(n_3 + in_4), \\
&k = 0, \cdots m-1 \qquad (54)
\end{aligned}
$$

Using the method in section II, we can demonstrate that with these transformations, Eq. 2 with $\Theta = 2\pi m$ and $\Theta = 0$ are equivalent to each other, thus the classification is $\mathbb{Z}_m$ in 2d.

The final classification of SPT phases with Z_m symmetry is:

$$
1d : \mathbb{Z}_1, \quad 2d : \mathbb{Z}_m, \quad 3d : \mathbb{Z}_1. \qquad (55)
$$

K. SPT phases with $Z_m \rtimes Z_2$ symmetry

In 1d, there is one SPT phase with $U(1) \rtimes Z_2$ symmetry. Naively one would expect that when U(1) is broken down to Z_m, this SPT phase survives and becomes a SPT phase with $Z_m \rtimes Z_2$ symmetry. However, this statement is only true for even m, and when m is odd the $U(1) \rtimes Z_2$ SPT phase becomes trivial once U(1) is broken down to Z_m.

The 1d $U(1) \rtimes Z_2$ SPT phase is described by a 1d O(3) NLSM of vector $\vec{n}$ with $\Theta = 2\pi$, and $B \sim (n_1 + in_2)$ is a charge-1 boson under the U(1) rotation. Because the classification of 1d $U(1) \rtimes Z_2$ SPT phase is $\mathbb{Z}_2$, $\Theta = 2\pi$ is equivalent to $\Theta = 2\pi m$ for odd m. As we discussed in section IID, this NLSM with $\Theta = 2\pi m$ is equivalent to another NLSM defined with $\vec{n}'$ and $\Theta = 2\pi$, where $B' \sim (n'_1 + in'_2) \sim (n_1 + in_2)^m$ is a charge-m boson. Under Z_2 transformation, $n'_1 \to n'_1$, $n'_2 \to -n'_2$.

Now let us break U(1) down to its subgroup Z_m. B' transforms trivially under Z_m, thus we are allowed to turn on a Zeeman term $\mathrm{Re}[B'] \sim n'_1$ which fully polarizes n'_1 and kills the SPT phase. Thus the original $U(1) \rtimes Z_2$ SPT phase is instable under U(1) to Z_m breaking with odd m.

The discussion above is very abstract, let us understand this result physically, and we will take $m = 3$ as an example. With a full SO(3) symmetry and $\Theta = 2\pi$ in the bulk, the ground state of the boundary is a spin-1/2 doublet in Eq. 6. The excited states of the boundary include a spin-3/2 quartet. When $\Theta = 6\pi$ in the bulk, the boundary ground state is a spin-3/2 quartet. The spin-3/2 and spin-1/2 states can have a boundary transition (level crossing at the boundary) without closing the bulk gap, thus $\Theta = 2\pi$ and 6π are equivalent in the bulk. Now let us take $\Theta = 6\pi$ in the bulk, and break the SO(3) down to $Z_3 \rtimes Z_2$. Then we are allowed to turn on a perturbation $\cos(3\phi)$ at the boundary (which precisely corresponds to the Zeeman coupling $\mathrm{Re}[B'] \sim n'_1$ discussed in the previous paragraph), which will mix and split the two states $S^z = \pm 3/2$ at the boundary, and the boundary ground state can become nondegenerate. Thus when m is odd, the $U(1) \rtimes Z_2$ SPT phase does not survive the symmetry breaking from U(1) to Z_m.

The same situation occurs in 2d and 3d. There is a 3d SPT phase with $U(1) \rtimes Z_2$ symmetry, but once we break the U(1) down to Z_m, this SPT phase does not survive when m is odd. When m is even, besides the phase deduced from $U(1) \rtimes Z_2$ SPT phase, one can construct another root phase:

$$
\begin{aligned}
Z_2 \ &: \ n_{1,2} \to -n_{1,2}, \quad n_a \to n_a \ (a = 3, 4, 5); \\
Z_m \ &: \ n_1, \to n_1, \quad n_a \to (-1)^k n_a \ (a = 2, \cdots 5), \\
&k = 0, \cdots m-1. \qquad (56)
\end{aligned}
$$

Here $n_a (a = 2, \cdots 5)$ still carries a nontrivial representation of Z_m for even integer m. n_a with $a = 3, 4, 5$ can be viewed as the real parts of charge-$m/2$ bosons, while n_2 is the imaginary part of such charge-$m/2$ boson. This construction does not apply for odd m.

In 2d, for arbitrary $m > 1$, the $U(1) \rtimes Z_2$ SPT phases survive under $U(1)$ to Z_m symmetry breaking. With even m, another root phase can be constructed

$$
\begin{aligned}
Z_m \ &: \ n_{1,2} \to (-1)^k n_{1,2}, \quad n_{3,4} \to n_{3,4}; \\
Z_2 \ &: \ n_{1,2} \to n_{1,2}, \quad n_{3,4} \to -n_{3,4}, \\
&k = 0, \cdots m-1. \qquad (57)
\end{aligned}
$$

Here n_1 and n_2 are both the real parts of the charge-$m/2$ bosons.

The final classification of SPT phases with $Z_m \rtimes Z_2$ symmetry is:

$$
1d : \mathbb{Z}_{(2,m)}, \quad 2d : \mathbb{Z}_m \times \mathbb{Z}_2 \times \mathbb{Z}_{(2,m)}, \quad 3d : (\mathbb{Z}_{(2,m)})^2 (58)
$$

L. SPT phases with $Z_m \times Z_2$ symmetry

The case $m = 2$ has already been discussed. When $m > 2$, one would naively expect these SPT phases can be interpreted as $U(1) \times Z_2$ SPT phases after breaking U(1) to its Z_m subgroup, but again this is not entirely correct. In 1d there is no SPT phase with $U(1) \times Z_2$ symmetry, simply because we cannot find a nontrivial transformation of $\vec{n}$ under $U(1) \times Z_2$ that keeps Eq. 1 invariant. But when m is an even number, we can construct one SPT phase with $Z_m \times Z_2$ symmetry using Eq. 1:

$$
\begin{aligned}
Z_m \ &: \ n_{1,2} \to (-1)^k n_{1,2}, \quad n_3 \to n_3, \\
Z_2 \ &: \ n_1 \to n_1, \quad n_{2,3} \to -n_{2,3}, \\
&k = 0, \cdots m-1. \qquad (59)
\end{aligned}
$$

The Z_m and Z_2 transformations on $\vec{n}$ commute with each other.

Again this construction applies to even integer m only. The boundary states of this 1d SPT phase have the following transformations:

$$Z_m \ : \ U \to (i\sigma^z)^k U, \quad Z_2 : U \to \sigma^x U;$$
$$k = 0, \cdots m-1. \qquad (60)$$

Thus the boundary states carry projective representations of $Z_m \times Z_2$, and the transformations of Z_m and Z_2 do not commute.

Similar situations occur in 3d. In 3d, we can construct two root phases for even m, even though there is no SPT phase with $U(1) \times Z_2$ symmetry in 3d :

$$(1) \ : \ Z_m : n_{1,2} \to (-1)^k n_{1,2}, \quad n_a \to n_a (a = 3, 4, 5);$$
$$Z_2 : n_1, \to n_1, \quad n_a \to -n_a \ (a = 2, \cdots 5);$$
$$(2) \ : \ Z_2 : n_{1,2} \to -n_{1,2}, \quad n_a \to n_a (a = 3, 4, 5);$$
$$Z_m : n_1, \to n_1, \quad n_a \to (-1)^k n_a (a = 2, \cdots 5);$$
$$k = 0, \cdots m-1. \qquad (61)$$

The boundary of these 3d SPT phases can have 2d Z_2 topological order. If the e and m anyons correspond to vortices of (n_3, n_4) and (n_1, n_2) respectively, then the boundary anyons of phase (1) transform as:

$$(1) \ : \ Z_m : U_e \to (i\sigma^z)^k U_e, \quad U_m \to U_m;$$
$$Z_2 : U_e \to \sigma^x U_e, \quad U_m \to i\sigma^y U_m^*. \qquad (62)$$

The transformation of boundary anyons of phase (2) can be derived in the same way.

In 2d all the $Z_m \times Z_2$ SPT phases can be deduced from $U(1) \times Z_2$ SPT phases, by breaking U(1) down to its Z_m subgroup. Thus cases (1), (2) and (3) in Eq. 40 seem to reduce to SPT phases with $Z_m \times Z_2$ symmetry after breaking U(1) down to Z_m. However, case (3) in Eq. 40 becomes the trivial phase when m is odd. In case (3) of $U(1) \times Z_2$ SPT phase (Eq. 40), the NLSM is constructed with a charge-1 boson $B \sim (n_1 + in_2)$, and because case (3) contributes classification $\mathbb{Z}_2$, $\Theta = 2\pi m$ is equivalent to $\Theta = 2\pi$ for odd m. Also, the NLSM with $\Theta = 2\pi m$ is equivalent to the NLSM with $\Theta = 2\pi$ constructed using a charge-m boson $B' \sim (n'_1 + in'_2) \sim (n_1 + in_2)^m$. Now let us break the U(1) symmetry down to Z_m. Because B' is invariant under Z_m and Z_2, we can turn on a linear Zeeman term that polarizes $\mathrm{Re}[B'] \sim n'_1$, and destroy the boundary states. Thus the NLSM constructed with the charge-m boson B' is trivial once we break U(1) down to Z_m. This implies that when m is odd, case (3) in Eq. 40 becomes a trivial phase once U(1) is broken down to Z_m.

The final classification of SPT phases with $Z_m \times Z_2$ symmetry is:

$$1d : \mathbb{Z}_{(2,m)}, \quad 2d : \mathbb{Z}_m \times \mathbb{Z}_2 \times \mathbb{Z}_{(2,m)}, \quad 3d : (\mathbb{Z}_{(2,m)})^2 \qquad (63)$$

M. SPT phases with $Z_m \rtimes Z_2^T$ symmetry

Again, the situation depends on the parity of m. If m is odd, then in 1d and 3d the only SPT phase is the SPT phase with Z_2^T only. In 2d and 3d the $U(1) \rtimes Z_2^T$ SPT phases (except for the one with Z_2^T symmetry only) do not survive when U(1) is broken down to Z_m with odd m. The reason is similar to what we discussed in the previous two subsections.

When m is even, then in 1d besides the Haldane phase with Z_2^T symmetry, we can construct another SPT phase:

$$Z_m \ : \ n_{1,2} \to (-1)^k n_{1,2}, \quad n_3 \to n_3,$$
$$k = 0, \cdots m-1;$$
$$Z_2^T \ : \ \vec{n} \to -\vec{n}. \qquad (64)$$

Here n_1 and n_2 are both imaginary parts of charge-$m/2$ bosons. The boundary state is a Kramers doublet and transforms as

$$Z_m \ : \ U \to (i\sigma^z)^k U, \quad Z_2^T : U \to i\sigma^y U;$$
$$k = 0, \cdots m-1. \qquad (65)$$

In 2d, we can construct two different root phases:

$$(1) \quad Z_m : (n_1 + in_2) \to (n_1 + in_2) e^{i2\pi k/m},$$
$$n_3, n_4 \to n_3, n_4;$$
$$Z_2^T : n_1 \to n_1, \quad n_a \to -n_a (a = 2, 3, 4);$$
$$(2) \quad Z_m : \vec{n} \to (-1)^k \vec{n};$$
$$Z_2^T : n_1 \to n_1, \quad n_a \to -n_a (a = 2, 3, 4);$$
$$k = 0, \cdots m-1. \qquad (66)$$

Phase (1) is the same phase as the 2d $U(1) \rtimes Z_2^T$ SPT phase, after breaking U(1) to Z_m; phase (2) is a new phase, where n_1 is the real part of a charge-$m/2$ boson, while $n_{2,3,4}$ are the imaginary parts of such charge-$m/2$ bosons.

Using similar methods, we can construct three root phases in 3d for even m. Two of the phases can be deduced from the 3d $U(1) \rtimes Z_2^T$ SPT phases. The third root phase has the following transformation:

$$Z_m \ : \ n_{1,2} \to (-1)^k n_{1,2}, \quad n_a \to n_a (a = 3, 4, 5);$$
$$Z_2^T \ : \ \vec{n} \to -\vec{n};$$
$$k = 0, \cdots m-1. \qquad (67)$$

Both n_1 and n_2 are imaginary parts of charge-$m/2$ bosons.

Just like the 3d SPT phase with $U(1) \rtimes Z_2^T$ symmetry, the 2d boundary of the 3d $Z_m \rtimes Z_2^T$ SPT phase described by Eq. 67 can have a Z_2 topological order with electric and magnetic anyons. The electric and magnetic anyons are both Kramers doublet, and only one of them has a nontrivial transformation under Z_m: $Z_m : U \to (i\sigma^z)^k U$, $(k = 0, \cdots m-1)$.

The final classification of SPT phases with $Z_m \rtimes Z_2^T$ symmetry is:

$$1d : \mathbb{Z}_2 \times \mathbb{Z}_{(2,m)}, \quad 2d : (\mathbb{Z}_{(2,m)})^2, \quad 3d : \mathbb{Z}_2 \times (\mathbb{Z}_{(2,m)})^2. \quad (68)$$

N. SPT phases with $Z_m \times Z_2^T$ symmetry

In 1d and 3d, the SPT phases with $Z_m \times Z_2^T$ symmetry can all be deduced from $U(1) \times Z_2^T$ symmetry by breaking U(1) down to Z_m. Again, when m is odd, some of the SPT phases become trivial, for the same reason as what we discussed before.

In 2d there is no SPT phase with $U(1) \times Z_2^T$ symmetry, but when m is even we can construct two root phases, which *cannot* be deduced from $U(1) \times Z_2^T$ SPT phases:

$$\begin{aligned}
(1) &: Z_m : \vec{n} \to (-1)^k \vec{n}; \\
& Z_2^T : n_1 \to n_1, \quad n_a \to -n_a (a = 2, 3, 4); \\
(2) &: Z_m : n_{1,2} \to (-1)^k n_{1,2}, \quad n_{3,4} \to n_{3,4}; \\
& Z_2^T : n_1 \to n_1, \quad n_a \to -n_a (a = 2, 3, 4); \\
& k = 0, \cdots m - 1. \qquad (69)
\end{aligned}$$

The final classification of SPT phases with $Z_m \times Z_2^T$ symmetry is:

$$1d : \mathbb{Z}_2 \times \mathbb{Z}_{(2,m)}, \quad 2d : (\mathbb{Z}_{(2,m)})^2, \quad 3d : \mathbb{Z}_2 \times (\mathbb{Z}_{(2,m)})^2. \quad (70)$$

O. SPT phases with $SO(3)$ symmetry

In 1d, the SO(3) symmetry leads to the Haldane phase, which is described by Eq. 1 with $\Theta = 2\pi$. In 3d, there is no way to assign SO(3) symmetry to the five-component vector $\vec{n}$ which makes the $\Theta-$term invariant, thus there is no 3d SPT phase with SO(3) symmetry.

In 2d, Ref. 19 has given a nice way of describing SPT phase with SO(3) symmetry, which is a principal chiral model defined with group elements $SO(3)$. We will argue *without* proof that the SO(3) principal chiral model in Ref. 19 can be formally rewritten as the O(4) NLSM Eq. 2, because we can represent every group element G_{ab} (3×3 orthogonal matrix) as a SU(2) matrix $\mathcal{Z}$:

$$G_{ab} = \frac{1}{2} \mathrm{tr}[\mathcal{Z}^\dagger \sigma^a \mathcal{Z} \sigma^b], \qquad (71)$$

and the SU(2) matrix $\mathcal{Z}$ is equivalent to an O(4) vector $\vec{n}$ with unit length: $\mathcal{Z} = n^4 I_{2\times 2} + i\vec{n} \cdot \vec{\sigma}$. We propose that the minimal SO(3) SPT phase discussed in Ref. 19 can be effectively described by Eq. 2 with $\Theta = 8\pi$:

$$\begin{aligned}
\mathcal{S}_{2d} &= \int d^2x d\tau \ \frac{1}{g} (\partial_\mu \vec{n})^2 + \frac{i8\pi}{12\pi^2} \epsilon_{abcd} \epsilon_{\mu\nu\rho} n^a \partial_\mu n^b \partial_\nu n^c \partial_\rho n^d \\
&= \int d^2x d\tau \ \frac{1}{g} \mathrm{tr}[\partial_\mu \mathcal{Z}^\dagger \partial_\mu \mathcal{Z}] + \frac{i8\pi}{24\pi^2} \mathrm{tr}[(\mathcal{Z}^\dagger d\mathcal{Z})^3]. \qquad (72)
\end{aligned}$$

Physically, Eq. 72 with $\Theta = 8\pi$ gives SU(2) Hall conductivity $\sigma_{SU(2)} = 8$, or equivalently SO(3) Hall conductivity $\sigma_{SO(3)} = 2$, which is the same as the principal chiral model in Ref. 19. Mathematically, when field $\mathcal{Z}$ has a instanton number $\int d^3x \ \mathrm{tr}[(\mathcal{Z}^\dagger d\mathcal{Z})^3]/(24\pi^2) = +1$ in the 2+1d space-time, the SO(3) matrix field G_{ab} defined in Eq. 71 will have instanton number $\int d^3x \ \mathrm{tr}[(G^{-1}dG)^3]/(24\pi^2) = +4$. This factor of 4 is precisely why $\Theta = 8\pi$ in Eq. 72.

In order to represent G_{ab} as $\mathcal{Z}$, we need to introduce a Z_2 gauge field that couples to $\mathcal{Z}$, because $\mathcal{Z}$ is a "fractional" representation of G_{ab}, and G_{ab} is invariant under gauge transformation $\mathcal{Z} \to -\mathcal{Z}$. In the language of lattice gauge theory, our statement in the previous paragraph implies that one of the possible confined phases of this Z_2 gauge field is trivial in the bulk without any extra symmetry breaking or topological degeneracy, which awaits further analysis.

The final classification of SPT phases with $SO(3)$ symmetry is:

$$1d : \mathbb{Z}_2, \quad 2d : \mathbb{Z}, \quad 3d : \mathbb{Z}_1. \qquad (73)$$

P. SPT phases with $SO(3) \times Z_2^T$ symmetry

In 1d, there are two different SPT root phases with $SO(3) \times Z_2^T$ symmetry, which correspond to the following transformations of O(3) vector $\vec{n}$:

$$\begin{aligned}
(1) &: SO(3) : n_a \to G_{ab} n_b, \quad Z_2^T : \vec{n} \to -\vec{n}; \\
(2) &: SO(3) : \vec{n} \to \vec{n}, \quad Z_2^T : \vec{n} \to -\vec{n}. \qquad (74)
\end{aligned}$$

In 2d, the SPT phases with $SO(3) \times Z_2^T$ symmetry were discussed in Ref. 23, and it is described by Eq. 2 with transformation

$$\begin{aligned}
SO(3) &: n_a \to G_{ab} n_b (a, b = 1, 2, 3), \quad n_4 \to n_4; \\
Z_2^T &: n_a \to n_a (a = 1, 2, 3), \quad n_4 \to -n_4. \qquad (75)
\end{aligned}$$

In 3d, there are three root phases for $SO(3) \times Z_2^T$ SPT phases, two of which have the following field theory:

$$\begin{aligned}
(1) &: SO(3) : \vec{n} \to \vec{n}, \quad Z_2^T : \vec{n} \to -\vec{n}; \\
(2) &: SO(3) : n_a \to G_{ab} n_b (a, b = 1, 2, 3), \quad n_{4,5} \to n_{4,5} \\
& Z_2^T : \vec{n} \to -\vec{n}; \qquad (76)
\end{aligned}$$

phase (1) is simply the SPT phase with Z_2^T symmetry only. After we break the SO(3) symmetry down to its inplane O(2) subgroup, phase (2) will reduce to a SPT phase with $U(1) \times Z_2^T$ symmetry discussed in Ref. 21, which is a phase whose bulk vortex line is a 1d Haldane phase with Z_2^T symmetry.

Besides the two phases discussed above, there should be another root phase (3) that will reduce to the $U(1) \times Z_2^T$ SPT phase whose boundary is a bosonic quantum Hall state with Hall conductivity ± 1, when time-reversal symmetry is broken at the boundary[21]. In the next two paragraphs we will argue *without* proof that this third root phase can be described by Eq. 3 with the following definition and transformation of O(5) vector order parameter $\vec{n}$:

$$\begin{aligned} (3) \ : \ & \mathcal{Z} = n^4 I_{2\times 2} + \sum_{a=1}^{3} i n_a \sigma^a, \\ & Z_2^T : \mathcal{Z} \to i\sigma^y \mathcal{Z}, \quad n_5 \to -n_5; \\ & \Theta = 8\pi \text{ in bulk.} \end{aligned} \tag{77}$$

Here $\mathcal{Z}$ is still the "fractional" representation of SO(3) matrix G_{ab} introduced in Eq. 71. If we break the Z_2^T symmetry at the boundary of phase (3), the boundary becomes a 2d SO(3) SPT phase with SO(3) Hall conductivity ± 1 (when SO(3) is broken to U(1), the boundary becomes a bosonic integer quantum Hall state with Hall conductivity ± 1), thus it cannot be realized in a pure 2d bosonic system without degeneracy.

In principle $\mathcal{Z}$ is still coupled to a Z_2 gauge field. We propose that the confined phase of this Z_2 gauge field is the desired $SO(3) \times Z_2^T$ SPT phase. In the confined phase of a 3d Z_2 gauge field, the vison loops of the Z_2 gauge field proliferate. Since the Z_2 gauge field is coupled to the fractional field $\mathcal{Z}$, a vison loop of this Z_2 gauge field is bound with a vortex loop of SO(3) matrix field G_{ab}[58], which is defined based on homotopy group $\pi_1[SO(3)] = \mathbb{Z}_2$, thus the confined phase of the Z_2 gauge field is a phase where the SO(3) vortex loops proliferate. If we reduce the SO(3) symmetry down to its inplane U(1) symmetry, the vison loop reduces to the vortex loop of the U(1) phase. When a bulk vortex (vison) loop ends at the boundary, it becomes a 2d vortex (vison). This 2d vortex is a fermion, because according to the previous paragraph, once the Z_2^T is broken at the boundary, the boundary becomes a boson quantum Hall state with Hall conductivity ± 1. This is consistent with the results for $U(1) \times Z_2^T$ SPT phase discussed in Ref. 21,24,28. Thus the SPT phase described by Eq. 77 is a phase where SO(3) vortex loops proliferate, and the SO(3) vortices at the boundary are fermions.

The final classification of SPT phases with $SO(3) \times Z_2^T$ symmetry is:

$$1d : (\mathbb{Z}_2)^2, \quad 2d : \mathbb{Z}_2, \quad 3d : (\mathbb{Z}_2)^3. \tag{78}$$

Q. SPT phases with $Z_2 \times Z_2 \times Z_2$ symmetry

In 1d, we can construct three different root phases:

$$\begin{aligned} (1) \ : \ & Z_2^A : n_{1,2} \to -n_{1,2}, \quad n_3 \to n_3; \\ & Z_2^B : n_1 \to n_1, \quad n_{2,3} \to -n_{2,3}; \\ & Z_2^C : \vec{n} \to \vec{n}; \\ (2) \ : \ & Z_2^B : n_{1,2} \to -n_{1,2}, \quad n_3 \to n_3; \\ & Z_2^C : n_1 \to n_1, \quad n_{2,3} \to -n_{2,3}; \\ & Z_2^A : \vec{n} \to \vec{n}; \\ (3) \ : \ & Z_2^C : n_{1,2} \to -n_{1,2}, \quad n_3 \to n_3; \\ & Z_2^A : n_1 \to n_1, \quad n_{2,3} \to -n_{2,3}; \\ & Z_2^B : \vec{n} \to \vec{n}. \end{aligned} \tag{79}$$

In 2d there are seven different root phases:

$$\begin{aligned} (1) \ : \ & Z_2^A : \vec{n} \to -\vec{n}, \quad Z_2^B, Z_2^C : \vec{n} \to \vec{n}; \\ (2) \ : \ & Z_2^B : \vec{n} \to -\vec{n}, \quad Z_2^C, Z_2^A : \vec{n} \to \vec{n}; \\ (3) \ : \ & Z_2^C : \vec{n} \to -\vec{n}, \quad Z_2^A, Z_2^B : \vec{n} \to \vec{n}; \\ (4) \ : \ & Z_2^A : n_{1,2} \to -n_{1,2}, \quad n_{3,4} \to n_{3,4}; \\ & Z_2^B : n_{1,2} \to n_{1,2}, \quad n_{3,4} \to -n_{3,4}; \\ & Z_2^C : \vec{n} \to \vec{n}; \\ (5) \ : \ & Z_2^A : n_{1,2} \to -n_{1,2}, \quad n_{3,4} \to n_{3,4}; \\ & Z_2^C : n_{1,2} \to n_{1,2}, \quad n_{3,4} \to -n_{3,4}; \\ & Z_2^B : \vec{n} \to \vec{n}; \\ (6) \ : \ & Z_2^A : n_{1,2} \to -n_{1,2}, \quad n_{3,4} \to n_{3,4}; \\ & Z_2^B : n_{1,3} \to -n_{1,3}, \quad n_{2,4} \to n_{2,4}; \\ & Z_2^C : n_{1,4} \to -n_{1,4}, \quad n_{2,3} \to n_{2,3}; \\ (7) \ : \ & Z_2^A : n_{1,2} \to -n_{1,2}, \quad n_{3,4} \to n_{3,4}; \\ & Z_2^B : n_{3,4} \to -n_{3,4}, \quad n_{1,2} \to n_{1,2}; \\ & Z_2^C : n_{2,3} \to -n_{2,3}, \quad n_{1,4} \to n_{1,4}. \end{aligned} \tag{80}$$

In 3d there are six different root phases:

$$\begin{aligned} (1) \ : \ & Z_2^A : n_{1,2} \to -n_{1,2}, \quad n_a \to n_a, (a = 3, 4, 5); \\ & Z_2^B : n_1 \to n_1, \quad n_a \to -n_a, (a = 2, \cdots 5); \end{aligned}$$

$$Z_2^C : \vec{n} \to \vec{n};$$

$$(2) \ : \ Z_2^B : n_{1,2} \to -n_{1,2}, \quad n_a \to n_a, (a = 3,4,5);$$
$$Z_2^A : n_1 \to n_1, \quad n_a \to -n_a, (a = 2, \cdots 5);$$
$$Z_2^C : \vec{n} \to \vec{n};$$

$$(3) \ : \ Z_2^B : n_{1,2} \to -n_{1,2}, \quad n_a \to n_a, (a = 3,4,5);$$
$$Z_2^C : n_1 \to n_1, \quad n_a \to -n_a, (a = 2, \cdots 5);$$
$$Z_2^A : \vec{n} \to \vec{n};$$

$$(4) \ : \ Z_2^C : n_{1,2} \to -n_{1,2}, \quad n_a \to n_a, (a = 3,4,5);$$
$$Z_2^B : n_1 \to n_1, \quad n_a \to -n_a, (a = 2, \cdots 5);$$
$$Z_2^A : \vec{n} \to \vec{n};$$

$$(5) \ : \ Z_2^A : n_{1,2} \to -n_{1,2}, \quad n_a \to n_a, (a = 3,4,5);$$
$$Z_2^C : n_1 \to n_1, \quad n_a \to -n_a, (a = 2, \cdots 5);$$
$$Z_2^B : \vec{n} \to \vec{n};$$

$$(6) \ : \ Z_2^C : n_{1,2} \to -n_{1,2}, \quad n_a \to n_a, (a = 3,4,5);$$
$$Z_2^A : n_1 \to n_1, \quad n_a \to -n_a, (a = 2, \cdots 5);$$
$$Z_2^B : \vec{n} \to \vec{n};$$

$$(7) \ : \ Z_2^A : n_{1,2} \to -n_{1,2}, \quad n_{3,4,5} \to n_{3,4,5};$$
$$Z_2^B : n_{2,3} \to -n_{2,3}, \quad n_{1,4,5} \to n_{1,4,5};$$
$$Z_2^C : n_{4,5} \to -n_{4,5}, \quad n_{1,2,3} \to n_{1,2,3};$$

$$(8) \ : \ Z_2^A : n_{1,2} \to -n_{1,2}, \quad n_{3,4,5} \to n_{3,4,5};$$
$$Z_2^C : n_{2,3} \to -n_{2,3}, \quad n_{1,4,5} \to n_{1,4,5};$$
$$Z_2^B : n_{4,5} \to -n_{4,5}, \quad n_{1,2,3} \to n_{1,2,3}. \qquad (81)$$

All the other SPT phases can be constructed with these root phases above.

The final classification of SPT phases with $Z_2 \times Z_2 \times Z_2$ symmetry is:

$$1d : (\mathbb{Z}_2)^3, \quad 2d : (\mathbb{Z}_2)^7, \quad 3d : (\mathbb{Z}_2)^8. \qquad (82)$$

V. SUMMARY AND COMMENTS

In this work we systematically classified and described bosonic SPT phases with a large set of physically relevant symmetries for all physical dimensions. We have demonstrated that all the SPT phases discussed in this paper can be described by three universal NLSMs Eq. 1, 2 and 3, and the classification of these SPT phases based on NLSMs is completely identical to the group cohomology classification[1,2]. However, we have not built the general connection between these two classifications, and it is likely that SPT phases with some other symmetry groups (for example symmetry much larger than $\mathrm{O}(d+2)$) can no longer be described by these three NLSMs any more. In Ref. 22,23, SPT phases that involve a large symmetry group PSU(N)= $SU(N)/Z_N$ were discussed, and in these systems it was necessary to introduce NLSMs with a larger target manifold. But it is likely that all the SPT phases with arbitrary symmetry groups (continuous or discontinuous) can be described by a NLSM with certain continuous target manifold.

As we already mentioned, now it is clear that there is a series of BSPT states beyond the group cohomology classification, and a generalized field theory description for such states will be given in Ref. 44. Our NLSM can also be very conveniently generalized to the cases that involve lattice symmetry such as inversion, as was discussed in Ref. 59, as long as we require our order parameter $\vec{n}$ transform nontrivially under lattice symmetry. We leave a thorough study of SPT states involving lattice symmetry to future studies.

Recently it was pointed out that after the $3d$ SPT state is coupled to gauge field, the gauge defects, which in $3d$ can be loop excitations, can have a novel loop braiding statistics[60]. In a recent work we showed that this loop statistics can also be computed using our NLSM field theory discussed in this work[61].

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